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Two-Dimensional Kinetic Modeling of a Tokamak Scrape-Off Layer with Recycling

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Abstract

The two-dimensional kinetic model of the scrape-off layer recently suggested by Catto and Hazeltine [Submitted to Physics of Plasmas] is extended by allowing for ion recycling at the limiter or divertor plates by an effective ion reflection coefficient. The model describes the balance between radial diffusion and streaming along the magnetic field, and the structure of the resulting scrape-off-layer is found to depend on the reflection coefficient. The particle and heat loads on the limiter or divertor plates are calculated, as well as the boundary conditions on the density and temperature gradients of the core plasma. By assuming that electrons have a Maxwell-Boltzmann distribution, the radial variation of the plasma potential is determined, and the potential of the Debye sheaths formed at the limiter or divertor plates is estimated.

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I. INTRODUCTION

The physics of the edge plasma in a tokamak is very intricate since a large number of effects come into play.[1] Densities and temperatures vary by several orders of magnitude, neutral particles are abundant, making ionization, recombination and charge-exchange processes important, and the interaction between the plasma and the wall is of significance. It is therefore very difficult to construct reasonably simple mathematical models of the edge plasma. Either one has to resort to computer simulation of complex systems of equations, or substantial simplifications must be made. Although the latter approach does not give results that are accurate enough for a detailed comparison with experiments, it has the advantage of giving physical insight, providing qualitatively useful information and simple limits that can be used to check large codes. This is the philosophy of, e.g., the two-dimensional kinetic models for the scrape-off-layer (SOL) suggested by Hinton and Hazeltine[2], and, more recently, by Catto and Hazeltine[3]. The latter model describes the streaming of the plasma along the magnetic field lines and the diffusion across the field, and is, probably, the simplest two-dimensional kinetic model possible. Analytic tractability is achieved by only considering one particle species, neglecting collisions, and assuming perfectly absorbing walls. However, within the same mathematical framework it is possible to describe a more realistic wall-particle interaction by modeling the effect of recycling as a effective reflection coefficient. The motivation for this approach is that an ion hitting the wall usually recombines and returns as a slightly less energetic neutral particle, which subsequently is ionized.[1] In addition, we are able to estimate the potential of the Debye sheaths formed near the limiter or divertor plates. These topics are the subject of the present paper, which is organized as follows. In Sec. II, we formulate the model equation, identical to that in Ref.[3], but introduce more general boundary conditions allowing particles to be reflected at the limiter or divertor plates. The solution of the resulting boundary value problem is derived in Sec. III, and is simplified in Sec. IV. In Sec. V, the matching to the core plasma is discussed, and the particle and heat loads on the wall are calculated. The physics of the electron population is treated in Sec. VI. By equating the ion and electron loss rates, the Debye sheath potential is calculated. Since the fluxes of ions and electrons turn out to have different radial distributions, we conclude that a current must flow in the limiter or divertor plates, a fact which is indeed observed in experiments. Finally, in Sec. VII our conclusions are summarized.

II. KINETIC MODEL FOR THE IONS

In describing the SOL plasma, we use the simple kinetic model developed in Ref.[3], describing ions from the core plasma diffusing into the SOL, and flowing along the magnetic field lines into the limiter or the divertor plates. However, we modify the boundary conditions to allow for the reflection of particles. Since the radial width of the SOL is very small, it is reasonable to assume that only the terms describing radial diffusion and motion along the magnetic field lines need to be retained in the kinetic equation, i.e.

$$v_{//} \frac{\partial f}{\partial s} = D \frac{\partial^2 f}{\partial r^2}. \qquad (1)$$

Here, f is the ion distribution function, $v_{//}$ the parallel velocity, s the coordinate along the magnetic field lines, and D denotes the diffusion coefficient in the radial direction r. We have neglected any parallel electric field, as well as magnetic trapping and drift motion, but a radial electric field is admissible. The diffusion may be either classical or anomalous. In the former case, only those parts of the collision operator leading to radial diffusion are retained in (1); in the latter case, collisions are ignored altogether.

It is convenient to introduce the normalized coordinates

$$x \equiv (r - r_0)(|v_{//}|/LD)^{1/2}, \qquad (2)$$
$$y \equiv s / L,$$

where r_0 is the radius of the last closed flux surface, and L is the connection length, i.e. the characteristic length of a magnetic field line connecting the last closed flux surface to the limiter or divertor plates. In these coordinates, the kinetic equation (1) becomes

$$\sigma \frac{\partial f_\sigma}{\partial y} = \frac{\partial^2 f_\sigma}{\partial x^2}, \qquad (3)$$

where $\sigma \equiv v_{//}/|v_{//}|$, and f_+ and f_- refer to the co- and counterpassing populations, respectively.

The limiter or the divertor plates are taken to be situated at y=0 and y=1. When they are hit by an ion, the latter is either absorbed, as assumed in Ref.[3], or it is reflected. If the probability of reflection is γ, the boundary condition at the plate y=1 is $f_-(x, y=1) = \gamma f_+(x, y=1)$. Because of the symmetry about y=1/2, we must have $f_-(x,y) = f_+(x, 1-y)$, and the boundary conditions for f_+ are thus

$$\gamma f_+(x, y=1) = f_+(x, y=0), \; x > 0,$$
$$f_+(x, y=1) = f_+(x, y=0), \; x < 0. \tag{4}$$

For notational simplicity, we suppress the velocity dependence of f_+ and γ. The second boundary condition describes the periodicity of the core plasma in the poloidal-angle-like variable y. In practice, the reflected particle usually comes back with less energy than the impacting ion.[1] On the other hand, the latter is accelerated in the Debye sheath, which is formed near the material surfaces. To some extent, these processes can be expected to cancel; we neglect them both for the sake of simplicity and analytic tractability. More importantly, the backscattered particle is usually neutral, and is subsequently ionized. In our analysis, we assume that the ionization takes place inside the SOL, close to the limiter or divertor plates, so that we can model this recycling by the effective ion reflection coefficient γ. When this is the case, the boundary conditions (4) are appropriate. As we shall see, the solution to the model equation (3) is quite insensitive to the exact value of γ. In Ref.[1], a graph of γ as a function of energy can be found. In the solution of the kinetic equation presented in Secs III and IV below, we allow for arbitrary velocity dependence of γ and D. In Secs V and VI, however, we take γ and D to be velocity independent for simplicity.

The distribution function should be small for x>>1, and in the opposite limit, -x>>1, we expect it to give a y-independent flux out of the core and into the SOL. Therefore, the boundary conditions at infinity are

$$f_+(x,y) \to 0, \; x \to +\infty,$$
$$f_+(x,y) \to \alpha + \beta x \,, \; x \to -\infty, \tag{5}$$

where α and β are (velocity-dependent) constants.

Close to the limiter or divertor plates, a Debye sheath is formed, as will be discussed in more detail in Sec.VI below. The thickness of the sheath is ignored since it is comparable to the Debye length (or perhaps an ion Larmor radius is the magnetic field lines intersect the divertor plates at a shallow enough angle), and it therefore is very small in comparison with the SOL width.

III. SOLUTION

The solution to the equation (3) with the boundary conditions (4),(5) can be solved by the Wiener-Hopf method[4], using Fourier transforms in the complex plane. For complex k, we define

$$F_+(k,y) \equiv \int_{-\infty}^{\infty} f_+(x,y)\exp(ikx)dx, \tag{6}$$

$$L(k) \equiv \int_{-\infty}^{0} f_+(x,1)\exp(ikx)dx, \tag{7}$$

$$U(k) \equiv \int_{0}^{\infty} f_+(x,1)\exp(ikx)dx. \tag{8}$$

Because of the boundary conditions (5), $L(k)$ and $U(k)$ are analytic in the lower and upper half planes, respectively. Moreover, anticipating that $|f_+(x,1)|/\exp(-\delta x) < \infty$ as $x \to \infty$, we conclude that $U(k)$ is also analytic in at least a narrow region below the real axis, i.e. for Im k>-δ. (Here and in the following, δ denotes any arbitrary, sufficiently small positive number). Furthermore, from the definitions (7),(8) it follows by integrating by parts that $L(k)$ and $U(k)$ are $O(|k|^{-1})$ as $|k| \to \infty$ in the respective regions of analyticity.

After Fourier transformation, the model equation (3) and the boundary conditions (4) give

$$F_+(k,y) = [L(k) + \gamma U(k)]\exp(-k^2 y), \tag{9}$$

$$V(k) \equiv -\frac{U(k)}{L(k)} = \frac{1-\exp(-k^2)}{1-\gamma\exp(-k^2)}. \tag{10}$$

In order to find the unknown functions $L(k)$ and $U(k)$, we write $V(k)$ as a quotient,

$$V(k) = V_l(k)\,/\,V_u(k), \tag{11}$$

of two other functions $V_l(k)$ and $V_u(k)$, which are analytic in the lower and upper half planes, respectively, and have the additional property of being $O(|k|)$ as $|k| \to \infty$. If this can be done, the function

$$C(k) \equiv \begin{cases} L(k)V_l(k) & ,\mathrm{Im}\,k < 0 \\ -U(k)V_u(k) & ,\mathrm{Im}\,k \geq -\delta \end{cases} \tag{12}$$

which is defined by either of these expressions when $-\delta \leq \mathrm{Im}\,k < 0$, is apparently analytic in the entire complex plane and bounded when $k \to \infty$. Therefore, by Liouvilles theorem[5], it must be equal to a constant, $C(k) = C$, and we have simply $L(k) = C / V_l(k)$ and $U(k) = -C / V_u(k)$.

The factorization (11) is not unique. Instead of following Baldwin, Cordey and Watson,[6] as was done in Ref. [3], we employ a different factorization[7]

$$\begin{aligned} V_l(k) &\equiv \frac{k^2}{k - ia}\exp[-q_l(k)], \\ V_u(k) &\equiv (k + ia)\exp[-q_u(k)], \end{aligned} \tag{13}$$

where a is an arbitrary positive number, and

$$q_{u,l}(k) \equiv \frac{1}{2\pi i}\int_{-\infty\mp i\delta}^{\infty\mp i\delta} \frac{q(z)\mathrm{d}z}{z - k}\,, \tag{14}$$

$$q(z) \equiv \ln\left(\frac{z^2 + a^2}{z^2}\frac{1 - \exp(-z^2)}{1 - \gamma\exp(-z^2)}\right). \tag{15}$$

Here and in the following, the upper sign is taken for $q_u(k)$ and the lower one for $q_l(k)$, so that the integration path is slightly below the real axis for $q_u(k)$, and slightly above it for $q_l(k)$. Since the logarithm is analytic in a strip around the real axis, it follows from Cauchy's theorem that $q_u(k) - q_l(k) = q(k)$. Together with the definitions (13), this proves the validity of (11).

This solves, in principle, the problem of finding $L(k)$ and $U(k)$, and the distribution function $f_+(x, y)$ can be obtained by taking the inverse Fourier transform of (6). From Eqs (9) we have

$$f_+(x,y) = \frac{1}{2\pi} \int_{-\infty-i\delta}^{\infty-i\delta} [L(k) + \gamma U(k)] \exp(-ikx - k^2 y) \, dk, \tag{16}$$

and by using Eqs (12)-(14), we obtain

$$f_+(x,y) = \frac{C(1-\gamma)}{2\pi} \int_{-\infty-i\delta}^{\infty-i\delta} \exp[-ikx - k^2 y + q_l(k)] \frac{k - ia}{k^2[1 - \gamma \exp(-k^2)]} dk \tag{17}$$

$$= \frac{C(1-\gamma)}{2\pi} \int_{-\infty-i\delta}^{\infty-i\delta} \exp[-ikx - k^2 y + q_u(k)] \frac{dk}{(k + ia)[1 - \exp(-k^2)]} \tag{18}$$

This completes the solution of the kinetic equation.

IV. LIMITING FORMS

Unfortunately, the solution (17),(18) for the distribution function we have obtained is not very explicit. It is, however, possible to find useful asymptotic expressions, which is the purpose of this Section.

First, we derive expressions for $q_{l,u}(k)$. Integrating (14) by parts gives

$$q_{u,l}(k) = -\frac{1}{\pi i} \int_{-\infty \mp i\delta}^{\infty \mp i\delta} z dz \left[\frac{1}{\exp(z^2) - 1} - \frac{\gamma}{\exp(z^2) - \gamma} - \frac{a^2}{z^2(z^2 + a^2)} \right] \ln(z - k) \tag{19}$$

The last term in this integral is readily integrated by closing the integration contour in the upper or lower half plane,

$$\frac{1}{\pi i} \int_{-\infty \mp i\delta}^{\infty \mp i\delta} \frac{a^2 z dz}{z^2(z^2 + a^2)} \ln(z - k) = \pm \ln(\mp ia - k), \tag{20}$$

where, again, the upper signs are to be taken in the expression for $q_u(k)$, and the lower ones in that for $q_l(k)$. The first two terms in (19) can be handled by expanding the logarithm as $\ln(z-k) = \ln z - \Sigma_1^\infty n^{-1}(k/z)^n$. Then, noting that $\ln(-|z|) = \ln|z| \mp \pi i$, we obtain

$$\frac{1}{\pi i}\int_{-\infty\mp i\delta}^{\infty\mp i\delta}\frac{\gamma\ln z}{\exp(z^2)-\gamma}zdz = \mp\int_{-\infty}^{-\varepsilon}\frac{\gamma zdz}{\exp(z^2)-\gamma}+\frac{1}{\pi i}\int_{-\varepsilon}^{\varepsilon}\frac{\gamma z\ln zdz}{\exp(z^2)-\gamma}$$

$$=\begin{cases}\mp\ln(1-\gamma)^{1/2} & \gamma<1\\ \pi/2i & \gamma=1\end{cases}, \tag{21}$$

where the integral from -ε to ε is taken below the origin for $q_u(k)$, and above it for $q_l(k)$. As shown in the Appendix, the remaining integrals can be expressed in terms of Riemann's zeta function, $\zeta(s)$, and a slight generalization of it, the so-called polylogarithmic function $\phi(s,\gamma)$,

$$\frac{1}{\pi i}\int_{-\infty\mp i\delta}^{\infty\mp i\delta}\frac{\gamma z^{-n}}{\exp(z^2)-\gamma}zdz = \mp\frac{(\pm i)^n}{\Gamma(n/2)}\phi(1-n/2,\gamma), \tag{22}$$

where $\phi(s,1)=\zeta(s)$. Collecting our results, we obtain the following series representations for $q_{u,l}(k)$ for γ<1

$$q_{u,l}(k) = \pi i/2 \pm \ln(\mp ia-k)\mp\ln(1-\gamma)^{1/2}$$
$$\mp\sum_{n=1}^{\infty}\frac{(\pm ik)^n}{n\Gamma(n/2)}[\zeta(1-n/2)-\phi(1-n/2,\gamma)]. \tag{23}$$

It is possible to sum the even terms in the series by using the knowledge that $q_u(k)-q_l(k)=q(k)$. Using the definition of $q(k)$, Eq.(15), and evaluating the difference between $q_u(k)$ and $q_l(k)$ as given by (23) gives

$$q_{u,l}(k) = \pi i/2 \pm \ln(\mp ia-k)\pm\frac{1}{2}\ln\left[\frac{1-\exp(-k^2)}{k^2(1-\gamma\exp(-k^2))}\right]$$
$$-i\sum_{n=0}^{\infty}\frac{(-1)^n k^{2n+1}}{(2n+1)\Gamma(1/2+n)}[\zeta(1/2-n)-\phi(1/2-n,\gamma)]. \tag{24}$$

The expressions (23) and (24) are useful for small k. The power series converge for $|k|<\min[|\ln\gamma|^{1/2},(2\pi)^{1/2}]$.

We are now able to determine the asymptotic behavior of $f_+(x,y)$. For $x\to-\infty$, i.e. towards the core plasma, $f_+(x,y)$ should approach a linear function in x, according to our boundary condition (5). Indeed, if the path of integration in (18) is displaced into the upper half plane but still passes below the singularities where $1-\exp(-k^2)=0$, it is clear that the double pole at the origin will give the principal contribution to the integral if -x>>1. In other words,

$$f_+(x,y) \approx iC(1-\gamma)\frac{\mathrm{d}}{\mathrm{d}k}\frac{\exp[-ikx-k^2y+q_u(k)]}{k+ia}\bigg|_{k=0}, \tag{25}$$

when $x \to -\infty$. The behavior of $q_u(k)$ can be read off from Eq.(24), and gives

$$f_+(x,y) \approx \beta(\mathrm{v})(x-\eta) \quad , x \to -\infty, \tag{26}$$

where $\beta \equiv -iC(1-\gamma)^{1/2}$, and

$$\begin{aligned} &\eta \equiv -\alpha/\beta = \pi^{-1/2}[\phi(1/2,\gamma)-\zeta(1/2)], \\ &\phi(1/2,\gamma) = \sum_{n=1}^{\infty}\frac{\gamma^n}{n^{1/2}} \xrightarrow{\gamma\to 1} \pi^{1/2}/(1-\gamma)^{1/2}, \\ &\zeta(1/2) \approx -1.46. \end{aligned} \tag{27}$$

This proves that $f_+(x,y)$ indeed satisfies the boundary condition (5).

In the opposite limit, $x \to +\infty$, it is convenient to use Eq.(17), deforming the contour of integration into the lower half plane, still making it pass above the singularities

$$k_n = \pm(-|\ln\gamma|+2\pi ni)^{1/2} \tag{28}$$

satisfying $1-\gamma\exp(-k_n^2)=0$. The distribution function is equal to the sum of the residues at the poles,

$$f_+(x,y) = \beta(\mathrm{v})(1-\gamma)^{1/2}\sum_{n=-\infty}^{\infty}\frac{k_n-ia}{2k_n^3}\exp[-ik_nx-k_n^2y+q_l(k_n)] \tag{29}$$

The pole closest to the real axis, $k_0=-i|\ln\gamma|^{1/2}$, will of course make the strongest contribution. For large x, we therefore have

$$f_+(x,y) \approx -\frac{\beta(\mathrm{v})(1-\gamma)}{2|\ln\gamma|^{3/2}}\gamma^{-y}\exp[-x|\ln\gamma|^{1/2}+S(\gamma)] \qquad (30)$$

where we have made use of Eq.(23). $S(\gamma)$ denotes the infinite sum appearing in (23) taken (with the lower signs) at $k = k_0$. The spatial behavior of the distribution function inside the SOL, $f_+(x,y) \propto \gamma^{-y}\exp(-x|\ln\gamma|^{1/2})$ is in remarkable contrast to the case when $\gamma = 0$ and $f_+(x,y) \propto (y^{1/2}/x)\exp(-x^2/4y)$, as shown in Ref.[3]. Note that $|\ln\gamma|^{1/2}$ is an extremely slowly varying function of γ, and is of the order of unity for almost any value of γ. Therefore, even a very small reflection coefficient has a profound influence of the outer parts of the SOL, but its exact value is quite unimportant.

Still, when γ is very (unrealistically) small, $|\ln\gamma|^{1/2} >> 1$, the approximation (30) breaks down, and the contributions from the other poles need to be taken into account. In this case, the sum in (29) can be replaced by an integral over n since a large number of poles have nearly the same imaginary part and thus make comparable contributions to the sum. In addition, since all k_n are large, $q_l(k_n)$ is small [cf. Eq.(14)], and we recover the $\gamma = 0$ result[3]

$$f_+(x,y) \approx \frac{\beta(\mathrm{v})}{2\pi i}\int_{-\infty-i\delta}^{\infty-i\delta}\exp(-ikx-k^2y)k^{-1}\mathrm{d}k = -\frac{\beta(\mathrm{v})y^{1/2}}{\pi^{1/2}x}\exp\left(-\frac{x^2}{4y}\right). \qquad (31)$$

V. MATCHING TO THE CORE

The still arbitrary function $\beta(\mathrm{v})$ is determined by the conditions in the core plasma. In order to rigorously derive an expression for $\beta(\mathrm{v})$, the solution to the kinetic equation in the SOL should be matched to that in the main plasma, for which it thus is necessary to have an accurate model. However, considering the $x \to -\infty$ form obtained above (26),

$$f_+(x,y) \approx \beta(\mathrm{v})[(r-r_0)(|v_{//}|/LD)^{1/2} - \eta], \qquad (32)$$

it is reasonable to assume that $\beta(\mathrm{v})$ should be chosen as a Maxwellian, as was done in Ref.[3],

$$\beta(\mathrm{v}) = -(n_0/\eta)f_M(v) \equiv -(n_0/\eta)\left(M_i/2\pi T_0\right)^{3/2}\exp(-M_i v^2/2T_0), \qquad (33)$$

where n_0 and T_0 denote the edge density and temperature, and M_i the ion mass. It should be pointed out that (33) is by no means the only possible choice consistent with the assumptions we have made. Alternatively, one could, e.g., choose $\beta(\mathbf{v}) \propto f_M(v)/|v_{//}|^{1/2}$. This has the advantage of making the distribution function approach a Maxwellian towards the core (if D is velocity independent), but introduces a singularity at $v_{//} = 0$. In addition, the total number of ions in the SOL becomes infinite, as follows by integrating the distribution function over the velocity space and over $r > r_0$, i.e. x>0. This is somewhat unsatisfying, especially when calculating the Debye sheath potential as in Sec.VI below. A rigorous matching to the core is beyond the scope of this work, and we therefore adhere to the choice (33) used by Catto and Hazeltine[3]. The generalization of the results given below to other choices of $\beta(\mathbf{v})$ is straightforward.

The ion density n_i and temperature T_i are obtained by integrating the distribution function over velocity space, keeping r (and not x) constant. With the choice (33) for $\beta(\mathbf{v})$, we find for -x>>1

$$\begin{aligned}\begin{pmatrix} n_i \\ 3n_iT_i/2 \end{pmatrix} &\equiv \int_{v_{//}>0} \begin{pmatrix} 1 \\ M_iv^2/2 \end{pmatrix} [f_+(x,y) + f_+(x,1-y)] d^3v \Big|_{r\ \text{const.}} \\ &\approx \begin{pmatrix} n_0 \\ 3n_0T_0/2 \end{pmatrix} \left[1 + \begin{pmatrix} 1 \\ 7/6 \end{pmatrix} \frac{\Gamma(3/4)}{\eta(\gamma)} \left(\frac{2T_0}{M_i} \right)^{1/4} \frac{r_0 - r}{(\pi LD)^{1/2}} \right], \end{aligned} \tag{34}$$

where $\eta(\gamma)$ is defined by Eq.(27). This determines the boundary conditions on the solution to the core plasma equation. For simplicity, we have assumed that γ and D are independent of velocity, and we continue to do so throughout the rest of this paper.

The fluxes of particles and heat to the limiter or divertor plates are calculated as in Ref. [3]. Locally, they are equal to

$$\frac{d}{dr}\begin{pmatrix} \Gamma_i \\ P_i \end{pmatrix} \equiv (1-\gamma)(B_p / B) \int_{v_{//}>0} f_+(x,1) v_{//} \begin{pmatrix} 1 \\ M_iv^2/2 \end{pmatrix} d^3v \Big|_{r\ \text{const.}} \tag{35}$$

per unit length in the toroidal direction for each surface (y=0 and y=1) facing the plasma. B_p is the poloidal magnetic field, and 1-γ is the probability that an impacting ion is absorbed. Integrating over r and using $\int_0^\infty f_+(x,1)\mathrm{d}r = \int_0^\infty f_+(x,1)(LD/|v_{//}|)^{1/2}\mathrm{d}x = (LD/|v_{//}|)^{1/2}U(0)$, gives

$$\begin{pmatrix}\Gamma_i\\ P_i\end{pmatrix} \equiv 2^{-3/4}\pi^{-1/2}(LD)^{1/2}\Gamma\left(\frac{3}{4}\right)\frac{B_p}{B}\left(\frac{T_0}{M_i}\right)^{1/4}\frac{n_0}{\eta(\gamma)}\begin{pmatrix}1\\ 7T_0/4\end{pmatrix}. \tag{36}$$

The total particle and heat loads spread over both plates is obtained by multiplying (36) by 4πR with R the major radius.

VI. THE ELECTRONS

The SOL width is determined by the ions, and quasineutrality is maintained by the electron population. Close to the limiter or divertor plates, a Debye sheath is formed, which reflects all but the very fastest electrons, and most electrons thus bounce back and forth between the plates. In Refs [2,3], the customary assumption was made that the electrons have a Maxwell-Boltzmann distribution,

$$\begin{aligned} f_e(v) &= n_e\left(M_e / 2\pi T_e\right)^{3/2}\exp(-M_e v^2 / 2T_e) \\ n_e &= Zn_i = Zn_0\exp[e(\Phi-\Phi_0)/T_e] \quad , \end{aligned} \tag{37}$$

where n_e is the density of the electrons, T_e their temperature, Z the ion charge number, and Φ_0 an edge potential. Apparently, this assumption cannot hold in the outer parts of the SOL since the ion density goes to zero there, which would force Φ to become negative infinite. However, it is reasonable to assume that (37) holds locally, in the vicinity of the last closed flux surface. This allows us to calculate the electrostatic potential Φ in this region, and to estimate the sheath potential.

For -x>>1, the ion density n_i has already been calculated in (34), and the electrostatic potential becomes

$$\Phi \approx \Phi_0 + \frac{T_e}{e}\ln\left[1+\frac{\Gamma(3/4)}{\eta(\gamma)}\left(\frac{2T_i}{M_i}\right)^{1/4}\frac{r_0-r}{(\pi LD)^{1/2}}\right], \tag{38}$$

where the second term in the logarithm is dominant. Apparently, Φ decreases logarithmically towards the last closed flux surface.

Near the limiter or divertor plates, the density is

$$n_i(r,1) = (1+\gamma) \int_{v_{//}>0} f_+(x,y=1)\mathrm{d}^3v$$

$$= \frac{n_0}{\eta(\gamma)} \frac{2(1+\gamma)LD}{(r-r_0)^2} \left(\frac{M_i}{2\pi T_0}\right)^{1/2} \int_0^\infty g(x)\exp(-qx^4)x\mathrm{d}x, \quad (39)$$

where $g(x) \equiv f_+(x,y=1)/\beta(\mathrm{v})$ and $q \equiv (LD)^2(M_i\,/\,2T_i)/(r-r_0)^4$. For q<<1, i.e. far outside the last closed flux surface, the integral in (39) can be approximated by

$$\int_0^\infty g(x)\exp(-qx^4)x\mathrm{d}x \approx \int_0^\infty g(x)x\mathrm{d}x = U'(0)/i\beta(\mathrm{v}) = -\eta(\gamma)/(1-\gamma) \quad (40)$$

[cf. Eq.(8)], and therefore

$$n_i(r,1) \approx n_0 \frac{1+\gamma}{1-\gamma} \frac{LD}{(r-r_0)^2} \left(\frac{2M_i}{\pi T_i}\right)^{1/2}. \quad (41)$$

This shows that the SOL width is roughly

$$w \equiv \left[\frac{(1+\gamma)LD}{(1-\gamma)}\right]^{1/2} \left(\frac{2M_i}{\pi T_i}\right)^{1/4}. \quad (42)$$

Let us now turn to estimating the potential of the Debye sheath formed near the limiter or divertor plates, which is equal to the difference between the plasma potential Φ(r,1) and the plate potential Φ_p. Only electrons fast enough that $v_{//} > v_0 \equiv [2e(\Phi - \Phi_p)/M_e]^{1/2}$ penetrate the potential barrier and are lost to the plates. If the Maxwell-Boltzmann assumption (37) holds, the local flux of electrons to a plate is therefore

$$\mathrm{d}\Gamma_e/\mathrm{d}r = (B_p/B) \int_{v_{//}>v_0} f_e(r,1,v_{//})v_{//}\mathrm{d}^3v =$$
$$Zn_0(B_p/B)(T_e/2\pi M_e)^{1/2}\exp[e(\Phi_p - \Phi_0)/T_e] \quad (43)$$

Apparently, it does not vary with r. The reason for this is that the Maxwell-Boltzmann assumption (37) makes the sheath potential barrier decrease at the same rate as the density as a function of r, so that the electron flux remains constant. As we have already remarked, however, inside the SOL the Maxwell-Boltzmann assumption can only hold in the vicinity of the last closed flux surface, i.e. approximately for $r_0 \lesssim r \lesssim r_0+w$, where w is the SOL width defined in (42). For still larger radii, the plasma potential is expected to fall off less rapidly than suggested by (37), and should smoothly approach the plate potential as $r \rightarrow \infty$. The total electron flux is thus approximately

$$\Gamma_e \sim w\,\mathrm{d}\Gamma_e/\mathrm{d}\,r = Zn_0 \frac{B_p}{B}\left[\frac{(1+\gamma)LD}{(1-\gamma)}\frac{T_e}{\pi M_e}\right]^{1/2}\left(\frac{M_i}{2\pi T_i}\right)^{1/4} \exp[e(\Phi_p - \Phi_0)/T_e]. \quad (44)$$

The total charge flux must vanish in steady state. Therefore, Γ_e is equal to the ion charge Z multiplied by the number of ions hitting the limiter of divertor plates in unit time. Since most reflected ions come back as neutrals, all impacting ions deposit the charge Ze on the wall, not only the absorbed ones, and therefore the ion charge flux is $Z\Gamma_i/(1-\gamma)$. Equating this to the electron flux (44) and solving for Φ_p, we obtain

$$\Phi_0 - \Phi_p \sim \frac{T_e}{2e}\ln\left[\eta^2(\gamma)(1-\gamma^2)\frac{M_i T_e}{M_e T_i}\right], \quad (45)$$

where we have omitted numerical factors of the order of unity. It should be kept in mind that the estimate (45) is only a very approximate one. Still, it can be expected to give a reasonable estimate for the γ-dependence of Φ_p. In Ref.[2], an estimate for the sheath potential was obtained in a similar manner but neglecting the radial variation of the plasma potential.

When the potential (45) is established, the number of electrons hitting the plates is equal to the corresponding number of ions multiplied by Z. However, the electron and ion fluxes do not have the same radial distribution. This means that the local charge flux does not vanish, and must be compensated by a current flowing in the limiter or divertor plates. Since the ion flux is largest near $r = r_0$ and decreases for larger r, whereas the electron flux is approximately constant near $r = r_0$, the current is expected to be directed in the positive r-direction close to $r = r_0$.

VII. SUMMARY

The inclusion of an effective ion reflection coefficient γ to model recycling at the limiter or divertor plates leads to several quantitative modifications in the SOL structure. In the part of the SOL facing the core plasma, $r < r_0$, the boundary condition, i.e. the relation between the edge density and gradient, changes and this modifies the matching conditions to the core solution. The distribution function in the outer part of the SOL, $r > r_0$, is significantly affected at high energies by the presence of even a very small reflection coefficient. This is, of course, to be expected since the reflected particles diffuse in the radial direction whilst bouncing back and forth between the divertor plates. This leads to a broadening of the SOL of order $|\ln \gamma|^{-1/2}$ due to recycling [Cf. Eqs. (30) and (31)].

The density decreases as $(r - r_0)^{-2}$ a SOL width outside the last closed flux surface, and not exponentially, $\exp[-(r - r_0) / \lambda]$, as commonly assumed.[1] By assuming the electrons to have a Maxwell-Boltzmann distribution, it is possible to estimate the potential of the Debye sheath which is formed near the limiter or divertor plates. As usual, it depends only weakly (logarithmically) on the plasma parameters, and, in particular, as $(T_e/2e)\ln[\eta^2(\gamma)(1-\gamma^2)]$ on the reflection coefficient, where $\eta(\gamma)$ is defined in Eq. (27). Consequently, recycling reduces the sheath potential and thereby the ion acceleration to the wall. The particle and heat loads on the limiter or divertor plates have been calculated for the ion population. The different radial distributions of the particle fluxes give rise to a current flowing in the plates, which is predicted to flow outwards (i.e. in the positive r-direction) near the last closed flux surface.

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APPENDIX: THE POLYLOGARITHMIC FUNCTION

In this Appendix, we show how to express integrals of the type (22) in terms of the so-called polylogarithmic function[8] $\phi(s,x)$, defined by

$$\phi(s,x) \equiv \sum_{n=1}^{\infty} \frac{x^n}{n^s}, \qquad \text{(A1)}$$

whenever this sum converges. It does so for all $\mathrm{Re}\, s > 0$ if $|x| < 1$.For x=1, it reduces to the Riemann zeta function, and for s=1, $|x| < 1$ to the natural logarithm ln(1+x). When the sum (A1) does not converge, it is possible to obtain an alternative definition as follows. From the definition of the gamma function

$$n^{-s}\Gamma(s) = \int_0^{\infty} y^{s-1} \exp(-ny)\mathrm{d}y. \qquad \text{(A2)}$$

When (A1) converges, we therefore have

$$\phi(s,x) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{xy^{s-1}}{\exp(y) - x} \mathrm{d}y. \qquad \text{(A3)}$$

Now consider the integral

$$I \equiv \int_H \frac{(-z)^{s-1} x}{\exp(z) - x} \mathrm{d}z, \qquad \text{(A4)}$$

where $(-z)^{s-1} \equiv |z|^{s-1} \exp[(s-1)i \arg(-z)]$, $|\arg z| \le \pi$, and H is Hankel's contour[5], which starts at $z = +\infty$ on the real axis, encircles the origin once counter-clockwise, and returns to the real axis at $z = +\infty$. It is easily verified that if $\mathrm{Re}\, s > 0$, this integral is equal to

$$I \equiv -2i \sin(s\pi) \int_0^{\infty} \frac{xy^{s-1}}{\exp(y) - x} \mathrm{d}y. \qquad \text{(A5)}$$

This is done by letting H follow the real axis and encircle the origin very close to it. Finally, recalling (A3) and using $\Gamma(s)\Gamma(1-s) = \pi / \sin s\pi$, we find

$$\phi(s,x) = -\frac{\Gamma(1-s)}{2\pi i}\int_H \frac{(-z)^{s-1}x}{\exp(z)-x}\,\mathrm{d}z. \tag{A6}$$

This expression is analytic for all s except the positive integers s=1,2,... It therefore serves as a more general definition of $\phi(s,x)$ than the sum (A1). Still, the latter is a suitable definition when s is equal to a positive integer. Using (A6), it is straightforward to evaluate the integral (22) in Sec.IV.

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