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Axisymmetric Tokamak Control**

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Model Reduction for Axisymmetric Tokamak Control

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Abstract

We deal with the problem of reducing a complicated electromagnetic passive structure model coupled to a linear plasma response model to a size that allows rapid calculations of gains for plasma position and shape control. We find that model reduction through eigenmode decomposition does not reproduce the input-to-output relationship of the system, unless one has a good idea of which eigenmodes are important. Hankel singular mode decomposition, on the other hand, provides an orthogonal basis for the system response, where the modes are ordered by their importance to the input-to-output relationship. A perturbed equilibrium plasma response model is used together with an electromagnetic model of the Alcator C-MOD passive structure to assess the performance of different model reduction schemes. We find that between 10 and 20 modes are required to give an adequate representation of the passive system. Emphasis is placed on keeping the reduction process independent of the parameters of the plasma we are trying to control.

I. Introduction

The development of tokamak experiments in the past twenty years indicates a steady trend towards higher toroidal field and higher plasma current. In order for the toroidal field magnets to withstand the mechanical stresses associated with producing a large magnetic field, they have to rest against large pieces of structural material. A vacuum vessel containing a plasma, which carries a large toroidal current has to be able to withstand the mechanical stresses due to the large eddy currents which can arise when the plasma moves or the currents in the shaping and ohmic heating coils change. It is clear then that the vacuum vessel has to be thick in order to sustain these stresses. Insulating breaks, which would weaken it, are probably ruled out. Since, in an experimental tokamak, one would like to examine a wide variety of plasma shapes, a further complication is introduced by a vacuum vessel that is not conformal to the plasma, i.e., its distance from the plasma edge varies significantly with poloidal angle. It is evident, that, in modern tokamaks, one cannot avoid using large amounts of conducting structure which interacts with the coils and the plasma to a non-negligible degree. Accurate modelling of the electromagnetic coupling of this structure to the coils, the plasma and the magnetic diagnostic measurements is essential for the analysis of dynamic control of the position and shape of the plasma. The axisymmetric effects of the structure and the vacuum vessel can be modelled by a set of toroidally symmetric elements of finite cross section. However, complex structures lead to a large number of elements and a system that is computationally too cumbersome for rapid multi-input, multi-output (MIMO) control calculations. It is desirable to reduce the system to a smaller size which still describes the important modes of its behaviour. After all, the number of degrees of freedom of the conductor/plasma system should be of the order of the number of active coils and not of the order of the number of elements used in modelling the passive conductors. With a complicated structure, however, it may

not be possible to use physical intuition alone as a guide to model reduction. Therefore, we examine here the utility of powerful, yet unintuitive methods developed in the field of control theory in recent years. We also explore the degree to which the model can be reduced without compromising its accuracy.

In past work in the field of tokamak control, the trend has been either to oversimplify or not to simplify at all. In the ISX-B tokamak,¹ where the vacuum vessel had two toroidal breaks, the vessel was successfully modelled as a single circuit carrying toroidal current with an $m=1$ poloidal distribution. In the DIII-D tokamak, it was found both theoretically² and experimentally³ that only one eigenmode of the vacuum vessel response was enough to calculate gains that control the vertical instability. However, this degree of simplification may not be generally attainable and almost certainly will not yield quantitatively accurate predictions of the dynamic behaviour. In ASDEX-Upgrade,⁴ the passive coils inside the vacuum vessel are the main sources of passive stabilisation. The vacuum vessel is modelled as a set of 60 toroidal filaments. This model is subjected to eigenmode analysis and only a small number of modes with small numbers of current reversals is kept. By contrast, Hofmann et al., in Refs. 5 and 6, tried to keep their control calculations independent of plasma parameters, and they used the large MHD transport code TSC⁷ to simulate plasma time evolution and optimize feedback gains. In TSC, the vacuum vessel is modelled as a set of filaments. No attempt is made to reduce the model.

In section II, we describe how filament plasma models and linear, quasistatic, axisymmetric MHD models can be put into linear MIMO state space form for control calculations. In section III, we present two methods of model reduction, and in section IV, we use the perturbational equilibrium model of Humphreys¹⁰ and apply these methods to Alcator C-MOD.

II. Models

In order to exploit the many recent achievements of MIMO linear state space control theory, we have to have a linearised model for the response of the system consisting of the plasma and the conductors around it. To arrive at such a model several assumptions must be made. If the only tools we have to control the plasma are the ohmic heating and poloidal field coils, we can only affect toroidally symmetric modes of the plasma, so we are justified in confining ourselves to considering axisymmetric behaviour. If we suppose that the response of the plasma is governed by the ideal MHD momentum equation, two time scales are of interest: the Alfvén time of the plasma and the L/R time of the conductors around it. If the first is much shorter than the second (and usually it is by about 3 orders of magnitude), we are justified in neglecting the inertia term in the momentum equation. Then, the plasma is supposed to be in equilibrium at each time and the conductors determine how it moves from one equilibrium to the next. A set of toroidal conductors is governed by circuit equations which describe the evolution of the poloidal flux at the locations of the conductors:

$$\mathbf{M}\dot{\vec{I}} + \mathbf{R}\vec{I} = \vec{V} \quad (1)$$

where \mathbf{M} is the inductance matrix (including mutual and self inductances), \mathbf{R} is the diagonal resistance matrix for the conductors, and \vec{V} is the vector of voltages applied to the conductors. \vec{I} is a vector containing the currents flowing in the conductors. We can choose the state of the plasma at each point in time to be described by the poloidal flux it creates at the conductor locations. Then, including a linearised plasma response would amount to adding to \mathbf{M} some matrix \mathbf{X} accounting for the coupling between conductors mediated by the plasma¹⁰:

$$\mathbf{M}\dot{\vec{I}} + \mathbf{R}\vec{I} + \mathbf{X}\dot{\vec{I}} = \vec{V} \quad (2)$$

\vec{I} is then the state vector of the plasma/conductor system.

Several linear models for the plasma have been devised. The simplest one is to replace the plasma by a single toroidal filament.² The next step is to use several toroidal filaments for the plasma in order to simulate a distribution of toroidal current in the plasma.⁸ One can also determine the linearised plasma response by perturbing the conductor currents that give a certain base equilibrium of interest and considering the plasma to be always in an equilibrium which is a linear combination of the set of perturbed equilibria. This approach was introduced in Ref. 9 and was extended in Ref. 10 to include passive conductor response and approximate flux conservation. A more rigorous approach based on the energy principle (but still neglecting plasma inertia) is used in Ref. 11.

The aim of this paper is not to evaluate these plasma models or to suggest a new one, but rather to make use of the fact that all these methods can be put into the standard linear control theory state space equation form:

$$\dot{\vec{x}} = \mathbf{A}\vec{x} + \mathbf{B}\vec{u} \quad (3)$$

where $\vec{x} = \vec{I}$, $\mathbf{A} = -(\mathbf{M} + \mathbf{X})^{-1}\mathbf{R}$, $\mathbf{B} = (\mathbf{M} + \mathbf{X})^{-1}$ and $\vec{u} = \vec{V}$ is the input vector. Since the state vector \vec{x} can usually not be measured, we also need another equation which relates the state and input vectors to the quantities that can be measured (the magnetic diagnostics, for example). This is the output equation:

$$\vec{y} = \mathbf{C}\vec{x} + \mathbf{D}\vec{u} \quad (4)$$

where \vec{y} is called the output vector. Many techniques for choosing \vec{u} to ensure satisfactory system response have been developed in the recent years which we could benefit from.

III. Model Reduction

A. Methods of Model Reduction

We employ two methods for the reduction of the standard control problem consisting of the state equation (Eq. 3) and the output equation (Eq. 4), where the state vector is of size n_x , the output vector is of size n_y and the input vector is of size n_u : eigenmode decomposition and Hankel singular mode (HSM) decomposition. In each of these methods, two transformation matrices, \mathbf{T}_l and \mathbf{T}_r are calculated so that the model reduction can be represented as the transformation:

$$\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} \rightarrow \begin{bmatrix} \mathbf{T}_l \mathbf{A} \mathbf{T}_r & \mathbf{T}_l \mathbf{B} \\ \mathbf{C} \mathbf{T}_r & \mathbf{D} \end{bmatrix} \quad (5)$$

The transformed model in Eq. 5 has the same number of inputs and outputs as the original system but a smaller number of internal states.

The simplest approach to model reduction is via eigenmode decomposition. The left and right eigenvectors of \mathbf{A} , \vec{w}_i and \vec{v}_i , and its eigenvalues λ_i for $i = 1, \dots, n_x$ satisfy the equation

$$\mathbf{A} = \mathbf{V} \mathbf{\Lambda} \mathbf{W} \quad (6)$$

where \mathbf{V} is a matrix with \vec{v}_i 's as its columns, \mathbf{W} is a matrix with \vec{w}_i^H 's (superscript H stands for Hermitian conjugate) as its rows, $\mathbf{\Lambda} = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_{n_x})$, and $\mathbf{W} = \mathbf{V}^{-1}$. If we consider certain modes to be more important than others (one could favour unstable and slowly damped modes over fast damped modes for example), \mathbf{T}_l would have as rows the \vec{w}_i^H 's corresponding to the important modes, and \mathbf{T}_r would have as columns their \vec{v}_i 's.

The concept of singular values of a matrix has been used very successfully in all areas of control theory lately, and one might expect it to appear here as well. Note, however, that, for a real symmetric matrix, the singular values are equal to the eigenvalues. \mathbf{M} is a symmetric matrix and the plasma response is usually

only a perturbation from this symmetry. Discarding small singular value modes is, therefore, equivalent to discarding the slow eigenmodes.

As opposed to the above method, which is concerned with the properties of the response matrix \mathbf{A} alone, model reduction in terms of Hankel singular values focuses on the input-to-output behaviour of the complete system described by Eqs. 3 and 4. The solution to these equations is:

$$\vec{y}(t) = \mathbf{C} \exp[\mathbf{A}(t - t_0)] \vec{x}(t = t_0) + \int_{t_0}^t \mathbf{C} \exp[\mathbf{A}(t - \tau)] \mathbf{B} \vec{u}(\tau) d\tau + \mathbf{D} \vec{u}(t) \quad (7)$$

We define the controllability grammian as:

$$\mathbf{P} \equiv \int_0^{\Delta t} \exp(\mathbf{A}t) \mathbf{B} \mathbf{B}^H \exp(\mathbf{A}^H t) dt \quad (8)$$

and the observability grammian as:

$$\mathbf{Q} \equiv \int_0^{\Delta t} \exp(\mathbf{A}^H t) \mathbf{C}^H \mathbf{C} \exp(\mathbf{A}t) dt \quad (9)$$

From the formulation of the formal solution in Eq. 7 one can show,¹² that, when \mathbf{P} is non-singular, it is possible to go from any initial state to any final state in a finite time interval Δt using the inputs \vec{u} . Also, when \mathbf{Q} is non-singular, it is possible to determine $\vec{x}(t)$ by using the measurements \vec{y} over a finite interval Δt after t . As $\Delta t \rightarrow \infty$, \mathbf{P} and \mathbf{Q} satisfy the Lyapunov equations¹²:

$$\mathbf{A} \mathbf{P} + \mathbf{P} \mathbf{A}^H + \mathbf{B} \mathbf{B}^H = \mathbf{0} \quad (10)$$

$$\mathbf{A}^H \mathbf{Q} + \mathbf{Q} \mathbf{A} + \mathbf{C}^H \mathbf{C} = \mathbf{0} \quad (11)$$

The Hankel singular values (HSV's) of the system $[\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}]$ are defined as:

$$\sigma_{Hi}([\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}]) \equiv \sqrt{\lambda_i(\mathbf{P} \mathbf{Q})} \quad (12)$$

where $\lambda_i(\mathbf{P} \mathbf{Q})$ is the i 'th eigenvalue of $\mathbf{P} \mathbf{Q}$. The HSV's are the singular values of the mapping from past inputs to future outputs (see appendix).

It is worthwhile to note that HSV's, as well as eigenvalues, are invariant under state space transformation, which is a necessary property for an input-to-output figure of merit. If we define a new state space $\vec{z} \equiv \mathbf{T}\vec{x}$, where \mathbf{T} is non-singular, the new state equation is

$$\dot{\vec{z}} = \mathbf{TAT}^{-1}\vec{z} + \mathbf{TB}\vec{u}$$

and the output equation becomes

$$\vec{y} = \mathbf{CT}^{-1}\vec{z} + \mathbf{D}\vec{u}$$

while the controllability and observability grammians, \mathbf{P} and \mathbf{Q} , become $\hat{\mathbf{P}} = \mathbf{TPT}^H$ and $\hat{\mathbf{Q}} = (\mathbf{T}^H)^{-1}\mathbf{QT}^{-1}$ respectively and their product becomes \mathbf{TPQT}^{-1} , thereby yielding the same eigenvalues and HSV's as \mathbf{PQ} .

Furthermore, \mathbf{P} and \mathbf{Q} are both real symmetric matrices, so that there exists a real matrix \mathbf{R} such that $\mathbf{Q} = \mathbf{R}^H\mathbf{R}$ and $\mathbf{RPR}^H = \mathbf{U}^H\mathbf{\Sigma}^2\mathbf{U}$ where \mathbf{U} is a unitary matrix and $\mathbf{\Sigma} = \text{diag}(\sigma_{H1}, \sigma_{H2}, \dots, \sigma_{Hn_x})$. If we choose $\mathbf{T} \equiv \mathbf{T}_{BAL} = -\mathbf{\Sigma}^{-1/2}\mathbf{U}^H\mathbf{R}$, we get $\hat{\mathbf{P}} = \hat{\mathbf{Q}} = \mathbf{\Sigma}$. This is known as a balancing transformation. If we partition the transformed matrices,

$$\begin{bmatrix} \hat{\mathbf{A}} & \hat{\mathbf{B}} \\ \hat{\mathbf{C}} & \hat{\mathbf{D}} \end{bmatrix} \equiv \begin{bmatrix} \mathbf{T}_{BAL}\mathbf{AT}_{BAL}^{-1} & \mathbf{T}_{BAL}\mathbf{B} \\ \mathbf{CT}_{BAL}^{-1} & \mathbf{D} \end{bmatrix} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} & \mathbf{B}_1 \\ \mathbf{A}_{21} & \mathbf{A}_{22} & \mathbf{B}_2 \\ \mathbf{C}_1 & \mathbf{C}_2 & \mathbf{D} \end{bmatrix}$$

where the subscript 1 refers to the largest k HSV's and the subscript 2 refers to the smallest $n_x - k$ HSV's, we get a reduced system $[\mathbf{A}_{11}, \mathbf{B}_1, \mathbf{C}_1, \mathbf{D}]$. This method of model reduction was proposed by Moore.¹⁴ Glover¹³ showed that the frequency domain transfer function matrix of this reduced system, $\hat{\mathbf{G}}(i\omega) \equiv \hat{\mathbf{C}}(i\omega\mathbf{I} - \hat{\mathbf{A}})\hat{\mathbf{B}} + \hat{\mathbf{D}}$, differs from the transfer function matrix of the full system, $\mathbf{G}(i\omega) \equiv \mathbf{C}(i\omega\mathbf{I} - \mathbf{A})\mathbf{B} + \mathbf{D}$, by the following maximum error:

$$\|(\mathbf{G}(i\omega) - \hat{\mathbf{G}}(i\omega))\|_{\infty} \leq 2 \sum_{i=k+1}^{n_x} \sigma_{Hi} \quad (13)$$

where the infinity norm signifies the largest singular value of a matrix.

\mathbf{T}_{BAL} is not necessarily an orthogonal matrix, and the above balancing transformation can be badly conditioned when the system is nearly unobservable or uncontrollable, i.e., \mathbf{P} or \mathbf{Q} are close to singular. Safonov and Chiang¹⁵ proposed the following set of transformation matrices that yield exactly the same $\hat{\mathbf{G}}(i\omega)$ as the truncation of the above balanced realization of the full model: For every real matrix with real eigenvalues, such as \mathbf{PQ} , there is a real orthogonal matrix \mathbf{V} such that $\mathbf{V}^T\mathbf{PQV}$ is an upper triangular matrix with the diagonal consisting of the eigenvalues of \mathbf{PQ} — see Golub and van Loan¹⁶ — which is known as the Schur form of \mathbf{PQ} . Two Schur forms of \mathbf{PQ} in which its eigenvalues appear on the diagonal in ascending or descending order can be realized using orthogonal, real transformations $\mathbf{V}_A = [\mathbf{V}_{A2} \mid \mathbf{V}_{A1}]$ and $\mathbf{V}_D = [\mathbf{V}_{D1} \mid \mathbf{V}_{D2}]$ respectively, where, again, the subscript 1 refers to the largest k HSV's and the subscript 2 refers to the smallest $n_x - k$ HSV's. Note, that \mathbf{V}_A and \mathbf{V}_D are orthogonal eigenspaces of \mathbf{PQ} . Next, a new matrix, \mathbf{E} , is formed and decomposed according to its singular values:

$$\mathbf{E} \equiv \mathbf{V}_{A1}^T \mathbf{V}_{D1} \equiv \mathbf{U}_E \Sigma_E \mathbf{V}_E^T$$

It can be shown¹⁵ that the transformation matrices

$$\mathbf{T}_l = \Sigma_E^{-1/2} \mathbf{U}_E^T \mathbf{V}_{A1}^T$$

$$\mathbf{T}_r = \mathbf{V}_{D1} \mathbf{V}_E \Sigma_E^{-1/2}$$

produce the same reduced-model transfer function matrix as Moore's¹⁴ balance-and-truncate approach. What has been gained by opting for these not so intuitive \mathbf{T}_l and \mathbf{T}_r is an algorithm which works even if the full system is close to unobservable or uncontrollable. This is the technique we use here, in the form of a MATLAB application.¹⁷

B. Partitioning the Model

The equation yielding the magnetic diagnostic signals (poloidal flux loop signals, $\vec{\psi}$, and poloidal field coil signals, \vec{B}_p) due to currents flowing in the conductors around the plasma is:

$$\begin{bmatrix} \vec{\psi} \\ \vec{B}_p \end{bmatrix} = \begin{bmatrix} \mathbf{N} \\ \mathbf{G} \end{bmatrix} \vec{I} \quad (14)$$

where \mathbf{N} is the mutual inductance matrix between the toroidal flux loops and the conductors and \mathbf{G} is the matrix of Green's functions between conductors and poloidal field coil locations integrated over the cross sectional area of the conductors (assuming a uniform current density is flowing through the conductors).

One can then transform Eqs. 2 and 14 into state and output equations as in Eqs. 3 and 4 and use the model reduction methods mentioned above. We have to go through the computationally tedious process of model reduction, however, for each equilibrium we wish to investigate, because the plasma response matrix, \mathbf{X} , depends on the equilibrium. We should like to have a reduced model of the vacuum vessel/structure without a plasma so that model reduction would only have to be carried out once. We want to keep the active coils complete in our reduced model but reduce the total size to manageable proportions. In general, the passive current system consists of approximately nested sets of conductors. The set closest to the plasma is generally a representation of the vacuum vessel. Further out, will be the mechanical structure. As we shall show, it can be advantageous to partition the model and treat the “vacuum vessel” and “structure” separately. This partitioning can be done intuitively for the examples we discuss. In what follows, we use the subscript v to refer to the vacuum vessel, s to refer to the steel structure around the vacuum vessel, c to refer to the active coils, g to refer to either vacuum vessel or structure elements for unpartitioned (“composite”) models, and r to refer to the reduced space.

If we consider a composite model, keeping the vacuum vessel and the structure together, we can write the circuit equation for the vacuum vessel/structure without a plasma as

$$\mathbf{M}_{gg}\dot{\vec{I}}_g + \mathbf{M}_{gc}\dot{\vec{I}}_c + \mathbf{R}_{gg}\vec{I}_g = \mathbf{0}, \quad (15)$$

and rewrite this in state equation form as

$$\dot{\vec{I}}_g = -\mathbf{M}_{gg}^{-1}\mathbf{R}_{gg}\vec{I}_g - \mathbf{M}_{gg}^{-1}\mathbf{M}_{gc}\dot{\vec{I}}_c$$

which, together with an appropriate output equation, lends itself to any of the order reduction schemes mentioned earlier, resulting in the two transformation matrices, \mathbf{T}_l and \mathbf{T}_r . This reduction can then be applied to the full model including the plasma response. Then, an approximate reduced model is:

$$\begin{aligned} & \begin{bmatrix} \mathbf{T}_l(\mathbf{M}_{gg} + \mathbf{X}_{gg})\mathbf{T}_r & \mathbf{T}_l(\mathbf{M}_{gc} + \mathbf{X}_{gc}) \\ (\mathbf{M}_{cg} + \mathbf{X}_{cg})\mathbf{T}_r & (\mathbf{M}_{cc} + \mathbf{X}_{cc}) \end{bmatrix} \begin{bmatrix} \dot{\vec{I}}_{gr} \\ \dot{\vec{I}}_c \end{bmatrix} + \\ & + \begin{bmatrix} \mathbf{T}_l\mathbf{R}_g\mathbf{T}_r & \mathbf{0} \\ \mathbf{0} & \mathbf{R}_{cc} \end{bmatrix} \begin{bmatrix} \vec{I}_{gr} \\ \vec{I}_c \end{bmatrix} = \begin{bmatrix} \vec{0} \\ \vec{V}_c \end{bmatrix} \end{aligned} \quad (16)$$

When we use a composite model of the vacuum vessel and the structure, it is possible that the order reduction process will keep some irrelevant modes of one and neglect important modes of the other, thereby forcing us to keep more modes than necessary to get a good reduced model. This is the case, for example, when one tries to reduce the model of the vacuum vessel and the structure for Alcator C-MOD by eigenmode decomposition. The structure elements are thick pieces of conductor and give rise to a large number of slowly damped modes (large L/R time) so that, if we choose to keep only the slow modes, we almost end up neglecting the vacuum vessel altogether. A better approach is to reduce the vacuum vessel and the structure models separately and then add the coil and plasma response. We can write one circuit equation for the vacuum vessel without plasma,

$$\mathbf{M}_{vv}\dot{\vec{I}}_v + \mathbf{M}_{vs}\dot{\vec{I}}_s + \mathbf{M}_{vc}\dot{\vec{I}}_c + \mathbf{R}_{vv}\vec{I}_v = \mathbf{0}, \quad (17)$$

and one for the structure,

$$\mathbf{M}_{ss}\dot{\vec{I}}_s + \mathbf{M}_{sv}\dot{\vec{I}}_v + \mathbf{M}_{sc}\dot{\vec{I}}_c + \mathbf{R}_{ss}\vec{I}_s = \mathbf{0}, \quad (18)$$

and then we can reduce the order of each one of these as we did above for Eq. 15 to obtain transformation matrices \mathbf{T}_{vl} and \mathbf{T}_{vr} for the vacuum vessel and \mathbf{T}_{sl} and \mathbf{T}_{sr} for the structure. Adding the plasma response, we get the following approximate reduced system:

$$\begin{bmatrix} \mathbf{M}_{11} & \mathbf{M}_{12} & \mathbf{M}_{13} \\ \mathbf{M}_{21} & \mathbf{M}_{22} & \mathbf{M}_{23} \\ \mathbf{M}_{31} & \mathbf{M}_{32} & \mathbf{M}_{33} \end{bmatrix} \begin{bmatrix} \dot{\vec{I}}_{vr} \\ \dot{\vec{I}}_{sr} \\ \dot{\vec{I}}_c \end{bmatrix} + \begin{bmatrix} \mathbf{R}_{11} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{R}_{22} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{R}_{33} \end{bmatrix} \begin{bmatrix} \vec{I}_{vr} \\ \vec{I}_{sr} \\ \vec{I}_c \end{bmatrix} = \begin{bmatrix} \vec{0} \\ \vec{0} \\ \vec{V}_c \end{bmatrix} \quad (19)$$

where

$$\mathbf{M}_{11} \equiv \mathbf{T}_{vl}(\mathbf{M}_{vv} + \mathbf{X}_{vv})\mathbf{T}_{vr}$$

$$\mathbf{M}_{12} \equiv \mathbf{T}_{vl}(\mathbf{M}_{vs} + \mathbf{X}_{vs})\mathbf{T}_{sr}$$

$$\mathbf{M}_{13} \equiv \mathbf{T}_{vl}(\mathbf{M}_{vc} + \mathbf{X}_{vc})$$

$$\mathbf{M}_{21} \equiv \mathbf{T}_{sl}(\mathbf{M}_{sv} + \mathbf{X}_{sv})\mathbf{T}_{vr}$$

$$\mathbf{M}_{22} \equiv \mathbf{T}_{sl}(\mathbf{M}_{ss} + \mathbf{X}_{ss})\mathbf{T}_{sr}$$

$$\mathbf{M}_{23} \equiv \mathbf{T}_{sl}(\mathbf{M}_{sc} + \mathbf{X}_{sc})$$

$$\mathbf{M}_{31} \equiv (\mathbf{M}_{cv} + \mathbf{X}_{cv})\mathbf{T}_{vr}$$

$$\mathbf{M}_{32} \equiv (\mathbf{M}_{cs} + \mathbf{X}_{cs})\mathbf{T}_{sr}$$

$$\mathbf{M}_{33} \equiv (\mathbf{M}_{cc} + \mathbf{X}_{cc})$$

$$\mathbf{R}_{11} \equiv \mathbf{T}_{vl}\mathbf{R}_{vv}\mathbf{T}_{vr}$$

$$\mathbf{R}_{22} \equiv \mathbf{T}_{sl}\mathbf{R}_{ss}\mathbf{T}_{sr}$$

$$\mathbf{R}_{33} \equiv \mathbf{R}_{cc}$$

These reduction schemes are not expected to work as well as the reduction of the combined plasma/coils/vessel/structure system. One thing we can do to improve their performance in capturing some of the plasma behaviour is to include the response of a generic plasma in the reduction of the composite or the separate vessel/structure system. This would amount to adding to all \mathbf{M} -matrices in Eqs. 15, 17, and 18 the corresponding \mathbf{X} -matrices for the generic plasma.

IV. Application on Alcator C-MOD

A. The Perturbational Plasma Response Model

For evaluation of the model reduction in application to Alcator C-MOD, we use a perturbed equilibrium model of the plasma response. The circuit equation for a set of conductors including the vacuum vessel and structure around the plasma and the active coils is then, according to Humphreys¹⁰:

$$\mathbf{M}\vec{I} + \mathbf{R}\vec{I} + \mathbf{X}_1\vec{I} + \mathbf{X}_2\vec{I} = \vec{V} \quad (20)$$

where \mathbf{X}_1 represents the coupling between conductors due to plasma response alone when the plasma current density, $J(\psi)$, stays a constant function of the poloidal flux ψ :

$$\mathbf{X}_1 \equiv \left. \frac{\partial \vec{\psi}_p}{\partial \vec{I}} \right|_{J(\psi)} \quad (21)$$

In principle, the current in each conductor will have to be perturbed individually from the base equilibrium to get \mathbf{X}_1 . However, if the number of important modes is less than the number of active coils — in other words, for Alcator C-MOD, if the rank of \mathbf{X}_1 is less than 13 — a combination of vacuum vessel/structure currents can be represented as an equivalent combination of active coil currents \vec{I}_c , which create the same flux on some chosen set of points, and only the active coil currents need be perturbed thereby saving computational effort.

\mathbf{X}_2 is a correction to \mathbf{X}_1 which allows the plasma current density to vary, when moving from one equilibrium to another, so that poloidal flux is approximately conserved. Namely, if we perturb the total plasma current I_p and some $J(\psi)$ profile parameter α from the base equilibrium, we can require two other quantities to remain constant such as some definition of flux on the axis and flux on the edge of

the plasma, $\vec{\psi}_x$

$$\frac{\partial \vec{\psi}_x}{\partial \vec{I}} \vec{I} + \frac{\partial \vec{\psi}_x}{\partial \vec{p}} \vec{p} = \vec{0} \quad (22)$$

where $\vec{p} \equiv \begin{bmatrix} I_p \\ \alpha \end{bmatrix}$. Note that in this formulation the state of the plasma/conductor system at any time is assumed to be describable by a set of conductor currents. Any other plasma model with this property could equally well have been used.

B. Results

We represent the Alcator C-MOD vacuum vessel by 94 elements and the structure by 96 elements as shown in Fig. 1.

These are elements of finite thickness, where a uniform current density is assumed to be flowing. \mathbf{M} , \mathbf{R} , \mathbf{N} , and \mathbf{G} were computed based on geometry and materials properties using the SOLDESIGN code.¹⁸

As an example to test the techniques described in the previous section we choose a typical expected high performance Alcator C-MOD plasma. A different slightly more elongated equilibrium was chosen as the generic plasma. Parameters describing these equilibria are shown in Table 1.

Two figures of merit were chosen for the performance of the different model reduction techniques:

- How well the vertical instability mode eigenvalue of the full model is reproduced.
- The relative maximum error in the transfer function matrix as a function of frequency defined by:

$$\epsilon_r(\omega) \equiv \frac{\|\mathbf{G}(i\omega) - \hat{\mathbf{G}}(i\omega)\|_\infty}{\|\mathbf{G}(i\omega)\|_\infty}$$

Fig. 2 shows $\epsilon_r(\omega)$ for a reduction of the combined plasma/coils/vessel/structure model by eigenmode and HSM decomposition. Note how badly eigenmode reduction keeping the unstable and the 39 slowest modes reproduces the input-to-output relationship. Reduction to the same number of modes by HSM gives errors that are smaller by several orders of magnitude. With eigenmode decomposition, we have no guidance as to which modes influence the outputs. It is obviously not just the slowest modes in this case.

Fig. 3 shows $\epsilon_r(\omega)$ for eigenmode reduction where the plasma and coil response were reduced by acting on them with the transformation matrices calculated when reducing the composite (upper plot) or the separate (lower plot) vessel/structure model as described in section III. Note how using more modes in the first case does not noticeably decrease the error. We observe that in this case, no unstable mode appears. This is because the (slow) modes we have kept are due to the structure, and the vacuum vessel has effectively been ignored. Consequently, the plasma becomes vertically unstable on the ideal MHD timescale. Our massless plasma assumption cannot handle such instabilities with growth rates of the order of the Alfvén frequency. When we split the vessel from the structure, thereby making sure that some modes due to the vessel are included, we are able both, to reduce the error by keeping more vessel modes, and to reproduce the unstable mode.

Fig. 4 shows the same for HSM reduction. Together with Eq. 15 and Eqns. 17 and 18, we used as output equations the parts of Eq. 14 relating the currents in the respective passive elements to the magnetic diagnostic signals. This proved to give better results than using an identity as output equation, i.e., using the state vector as output vector. Note how the error is reduced in the composite vessel/structure case (upper plot) when the number of modes kept is increased. An unstable eigenmode is reproduced, provided we keep at least 20 vessel/structure modes.

We see that $\epsilon_r(\omega)$ curves for different number of modes kept do not intersect, so

we abbreviate the presentation of results hereafter by considering only one frequency.

Fig. 5 once more shows how the error in eigenmode reduction stays unaffected as the number of modes kept is increased for the composite vessel/structure system. In contrast, HSM reduction shows a decrease in error if more than 20 modes are kept. In both cases, the reduction with a generic plasma response yields smaller error for the same number of modes kept.

Fig. 6 shows the difference in unstable eigenvalue between reduced and full models for composite vessel/structure reduction.

Figures 7 and 8 show ϵ_r and unstable eigenvalue error for the reduction of the *separate* vessel/structure model by eigenmode and HSM decomposition with and without the generic plasma response. Note how eigenmode and HSM reduction perform comparably. Also note how the error decreases if we keep more than 10 vessel modes (20 vessel/structure modes total). The generic plasma helps in both cases, but it does not help as much in the eigenmode reduction as in the HSM reduction.

V. Conclusions

We have described and investigated two types of general model reduction schemes, based on eigenmodes or Hankel singular modes respectively. In application to the axisymmetric electromagnetic model of Alcator C-MOD, we find that two additional factors are also of importance, namely whether or not a plasma is included in the model during reduction, and whether the passive elements can be partitioned in such a way as to guarantee retaining the important modes of the vacuum vessel.

Reduction of the entire system using the Hankel singular modes can be achieved down to dimension 40 with negligible error and to dimension 10 with probably acceptable accuracy. In contrast, retaining even 40 of the slowest eigenmodes leads to large errors in the system response. Plainly, case-by-case analysis of a specific complete system, for example to study optimum feedback control algorithms, will benefit greatly from model reduction using the HSM approach. The eigenmode decomposition is unsuccessful in its direct form.

An intuitive partitioning of the passive structure into separate vacuum vessel and structure allows one to obtain successful reduction using the eigenmode technique as well as HSM. However partitioning requires the use of more or less ad hoc judgement about which elements to include in which partition. It may not always be straightforward to make this judgement effectively. In our example, where partitioning is rather natural, we still need to retain between 10 and 20 vessel modes to obtain accuracy of 10% or better in the open-loop system response and unstable mode growth rate (30 when using HSM without generic plasma).

In reducing the passive elements alone, which is convenient because it allows the reduction to be done once and for all, it is very advantageous to include a generic plasma. This enables the HSM approach to obtain 10% accuracy with between 10 and 20 passive modes even in the unpartitioned model. Roughly twice as many are

required with no generic plasma.

The eigenmode reduction also benefits from the inclusion of a generic plasma. However, it obtains only about 20% accuracy without partitioning and this does not improve even adding up to 60 modes. This limited accuracy is likely to be even worse for larger differences between generic and actual plasmas. If a vertically stable generic plasma were chosen, for example, there would be little or no improvement over the no-plasma eigenmode reduction. What appears to happen is that the unstable generic plasma forces the inclusion of one mode dominated by the vessel (namely the unstable mode). This single vessel mode differs from the actual unstable mode (unless one is dealing with exactly the generic plasma) by enough to cause significant errors.

We conclude that accurate axisymmetric control modelling based on system reduction by selecting the slowest eigenmodes is possible in situations where retention of the important modes is guaranteed either by system simplicity or by appropriate partitioning. The more complex HSM reduction technique can handle situations where eigenmode reduction fails but it offers no clear quantitative advantage in situations to which eigenmode reduction is well suited. Neither technique gives a quantitatively accurate representation of the Alcator C-MOD with fewer than between 10 and 20 significant modes.

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Appendix A.

The HSV's are the singular values of the mapping from past inputs to future outputs. To see this, rewrite Eq. 7 for $\vec{x}(t = t_0) = \vec{0}$, $t_0 = -\infty$, $\vec{u}(t) = \vec{v}(-t)$ for $t < 0$, $\vec{u}(t) = 0$ for $t > 0$ and $\mathbf{D} = \mathbf{0}$:

$$\vec{y}(t) = \mathbf{C} \exp(\mathbf{A}t) \vec{x}_0 \equiv \Gamma(t) [\vec{v}(t)] \quad (\text{A1})$$

where

$$\vec{x}_0 \equiv \int_0^\infty \exp(\mathbf{A}\tau) \mathbf{B} \vec{v}(t) d\tau$$

$\Gamma(t)$ is a time dependent integral operator mapping the input for $t < 0$ to the output for $t > 0$. At the present, time $t = 0$, the singular values of $\Gamma(t = 0)$, σ_{Γ_i} , are defined by the eigenproblem:

$$\Gamma^H(t = 0) [\Gamma(t = 0) [\vec{v}(t)]] = \sigma_{\Gamma_i}^2 \vec{r}_i \quad (\text{A2})$$

where

$$\Gamma^H(t) [\vec{y}(t)] \equiv \int_0^\infty \mathbf{B}^H \exp[\mathbf{A}^H(t + \tau)] \mathbf{C}^H \vec{y}(\tau) d\tau$$

We also have that:

$$\Gamma^H(t = 0) [\Gamma(t = 0) [\vec{v}(t)]] = \mathbf{B}^H \exp(\mathbf{A}^H t) \mathbf{Q} \vec{x}_{0i} \quad (\text{A3})$$

where

$$\vec{x}_{0i} \equiv \int_0^\infty \exp(\mathbf{A}\tau) \mathbf{B} \vec{r}_i d\tau$$

Using Eqs. A2 and A3, we get (Glover¹³):

$$\mathbf{P} \mathbf{Q} \vec{x}_{0i} = \sigma_{\Gamma_i}^2 \vec{x}_{0i} \quad (\text{A4})$$

which is equivalent to Eq. 12, the definition of the HSV's.

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Quantity	example	generic	units
plasma current	3.01	3.01	MA
radial magnetic axis location	67.5	67.9	cm
vertical magnetic axis location	0.00	2.00	cm
minor radius	21.1	21.3	cm
elongation of 95% flux surface	1.58	1.70	
elongation of separatrix	1.69	1.85	
triangularity of 95% flux surface	.271	.379	
safety factor on axis	1.01	.973	
safety factor on 95% flux surface	2.08	2.53	
β_p	.197	.101	

Table 1: Essential characteristics of the example and generic equilibria used in this section.

Figures

FIG. 1. Model of Alcator C-MOD. The boxes with a '+' sign represent toroidally continuous elements. The empty boxes represent toroidally discontinuous elements that were left out of the model.

FIG. 2. $\epsilon_r(\omega)$ for two different model reduction methods. The full model is of length 200 (190 vessel/structure elements and 10 coils) and includes the response of a typical Alcator C-MOD plasma. The model reduced by eigenmode decomposition is of length 40. The two models reduced by Hankel singular mode decomposition are of length 10 (upper) and 40 (lower). The unstable mode eigenvalue is reproduced exactly in all cases.

FIG. 3. $\epsilon_r(\omega)$ for eigenmode decomposition. In the first plot, the 190-element vessel/structure model was reduced to seven different sizes ranging from 5 to 60. In the second plot, the 94-element vacuum vessel model was reduced to six different sizes ranging from 5 to 50 and the 96-element structure model was reduced to size 10. The coil and plasma response were added afterwards.

FIG. 4. $\epsilon_r(\omega)$ for Hankel singular mode decomposition. In the first plot, the 190-element vessel/structure model was reduced to seven different sizes ranging from 5 to 60. In the second plot, the 94-element vacuum vessel model was reduced to six different sizes ranging from 5 to 50 and the 96-element structure model was reduced to size 10. The coil and plasma response were added afterwards.

FIG. 5. ϵ_r at 10 Hz as a function of number of modes kept for eigenmode (eigen) and Hankel singular mode (HSM) reduction of the composite vessel/structure system with and without a generic plasma.

FIG. 6. Difference between reduced model and full model unstable eigenvalue (279.1 rad/sec) as a function of number of modes kept for eigenmode (eigen) and Hankel singular mode (HSM) reduction of the composite vessel/structure system with and without a generic plasma. Note that none of the reduced models obtained with eigenmode reduction without a generic plasma response give an unstable mode. The same holds for the first two models obtained by HSM reduction without a generic plasma.

FIG. 7. ϵ_r at 10 Hz as a function of number of vessel modes kept in addition to 10 structure modes for eigenmode (eigen) and Hankel singular mode (HSM) reduction of the separate 94-element vessel/ 96-element structure with and without a generic plasma.

FIG. 8. Difference between reduced model and full model unstable eigenvalue (279.1 rad/sec) as a function of number of vessel modes kept in addition to 10 structure

modes for eigenmode (eigen) and Hankel singular mode (HSM) reduction of the separate 94-element vessel/ 96-element structure with and without a generic plasma.

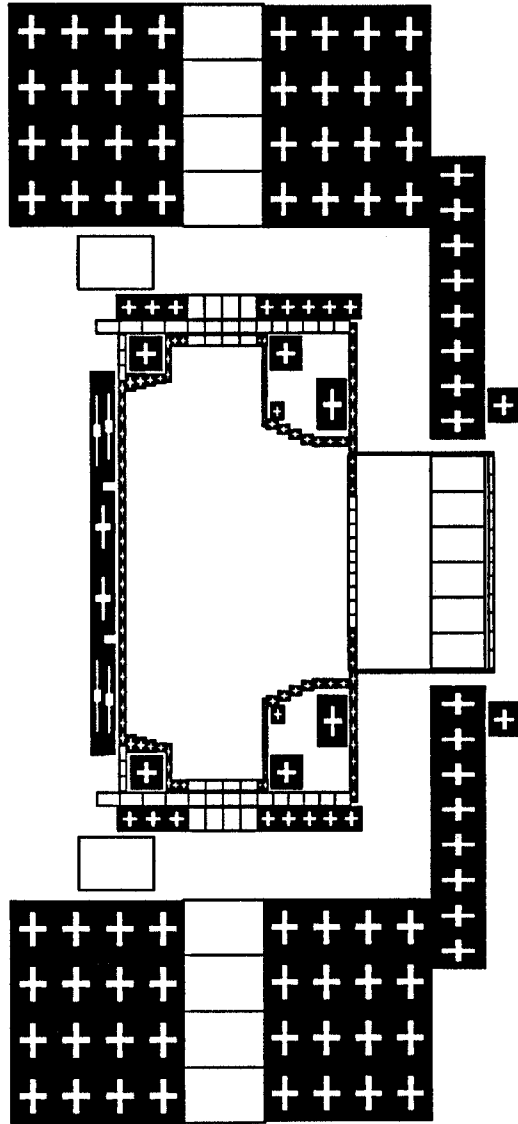


Fig. 1

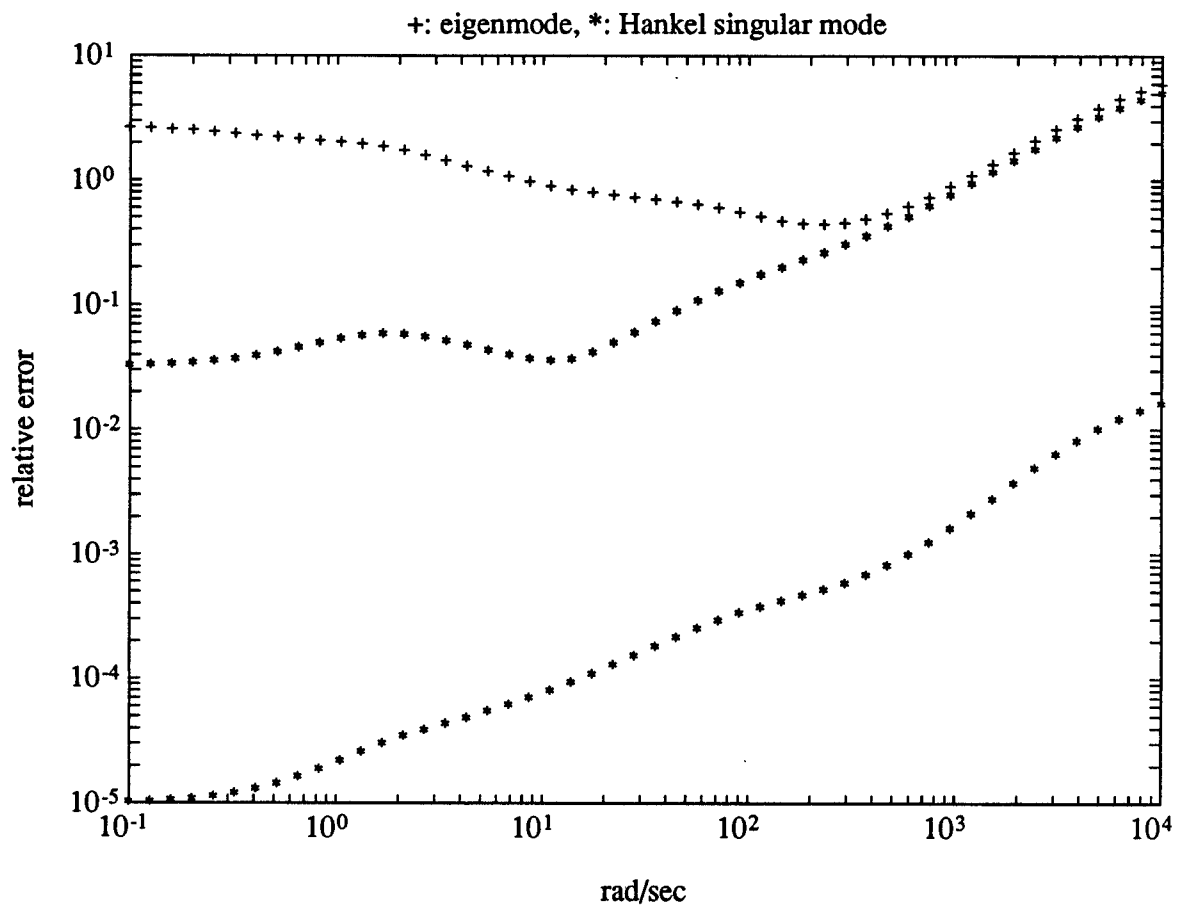


Fig. 2

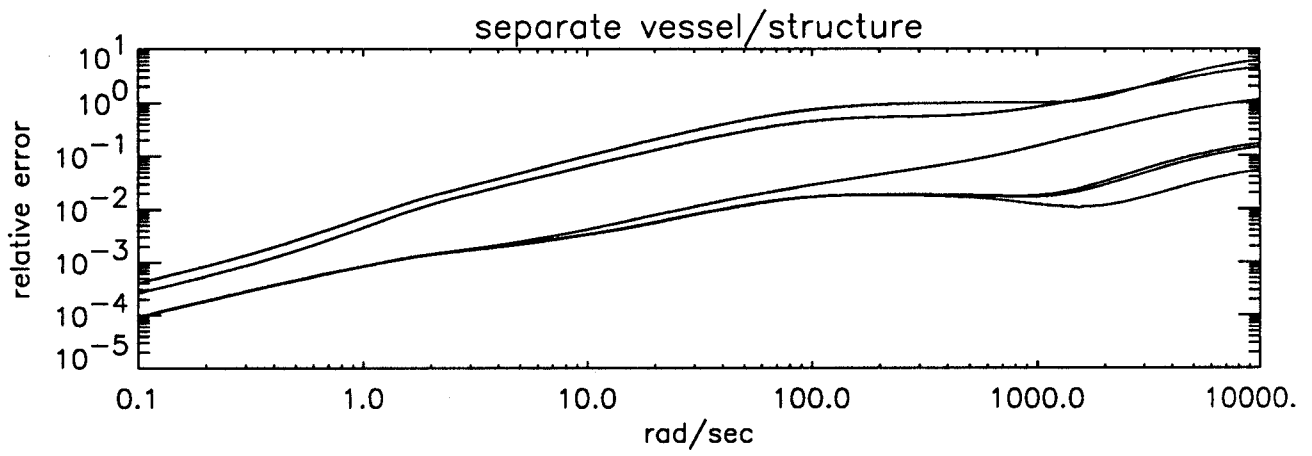
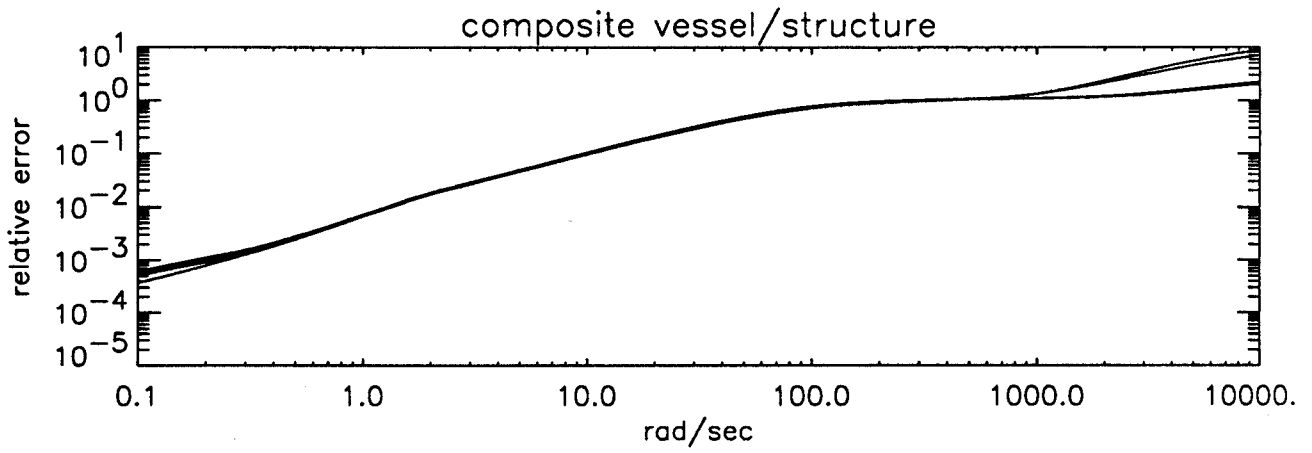


Fig. 3

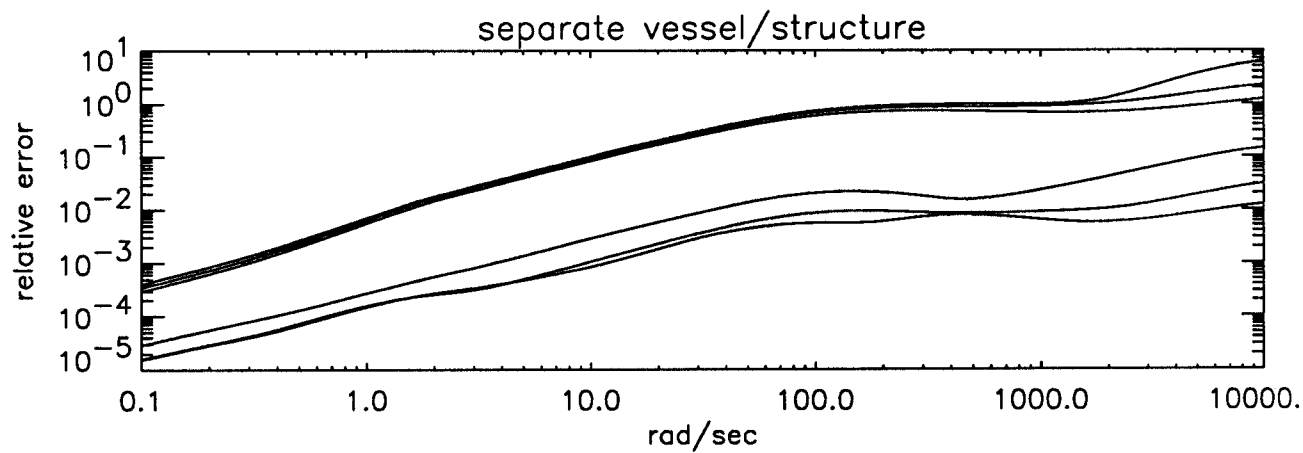
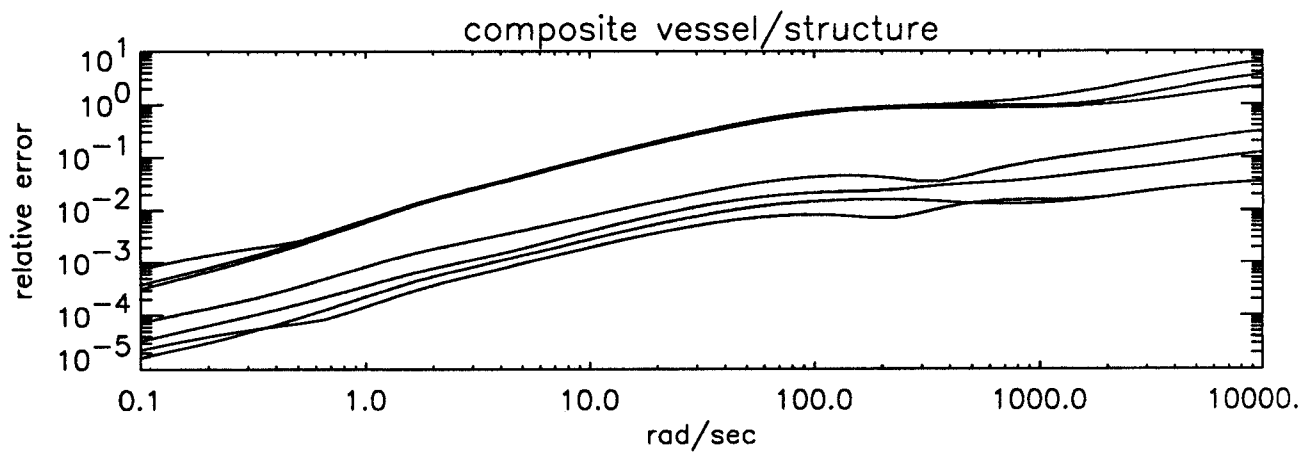


Fig. 4

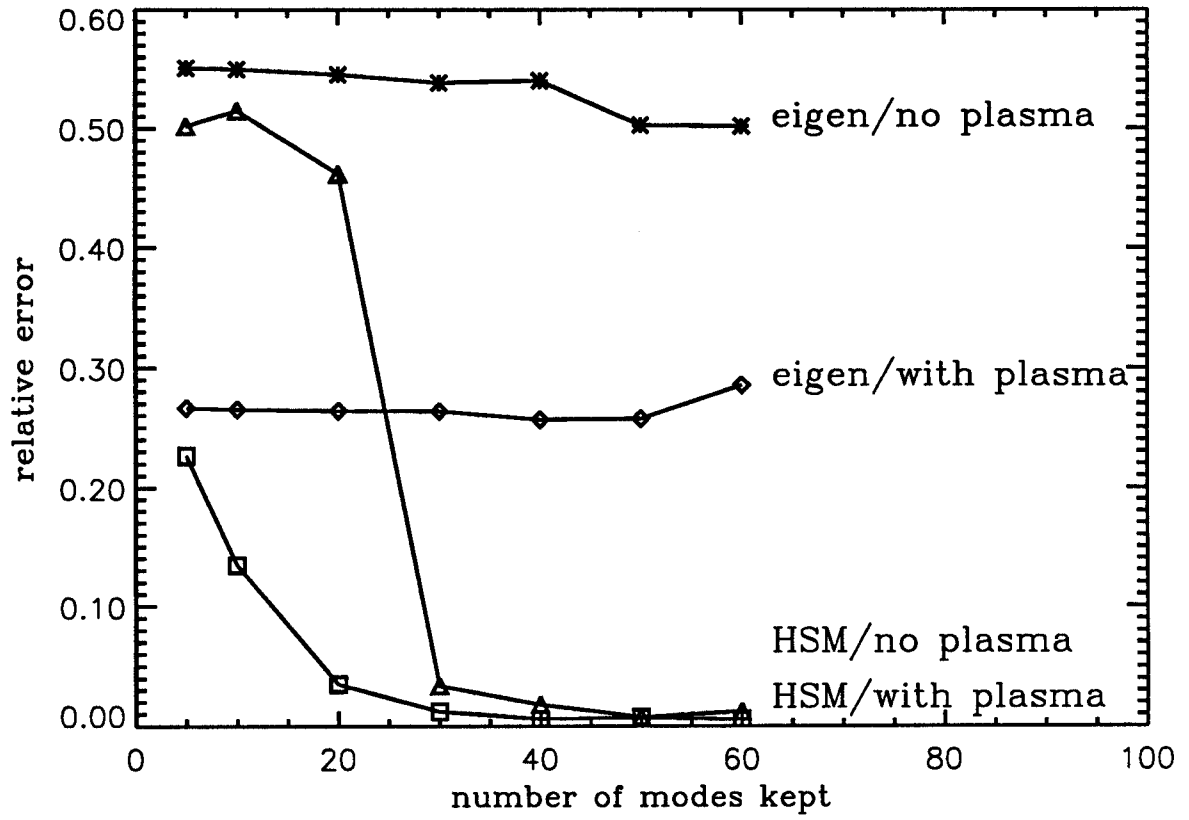


Fig. 5

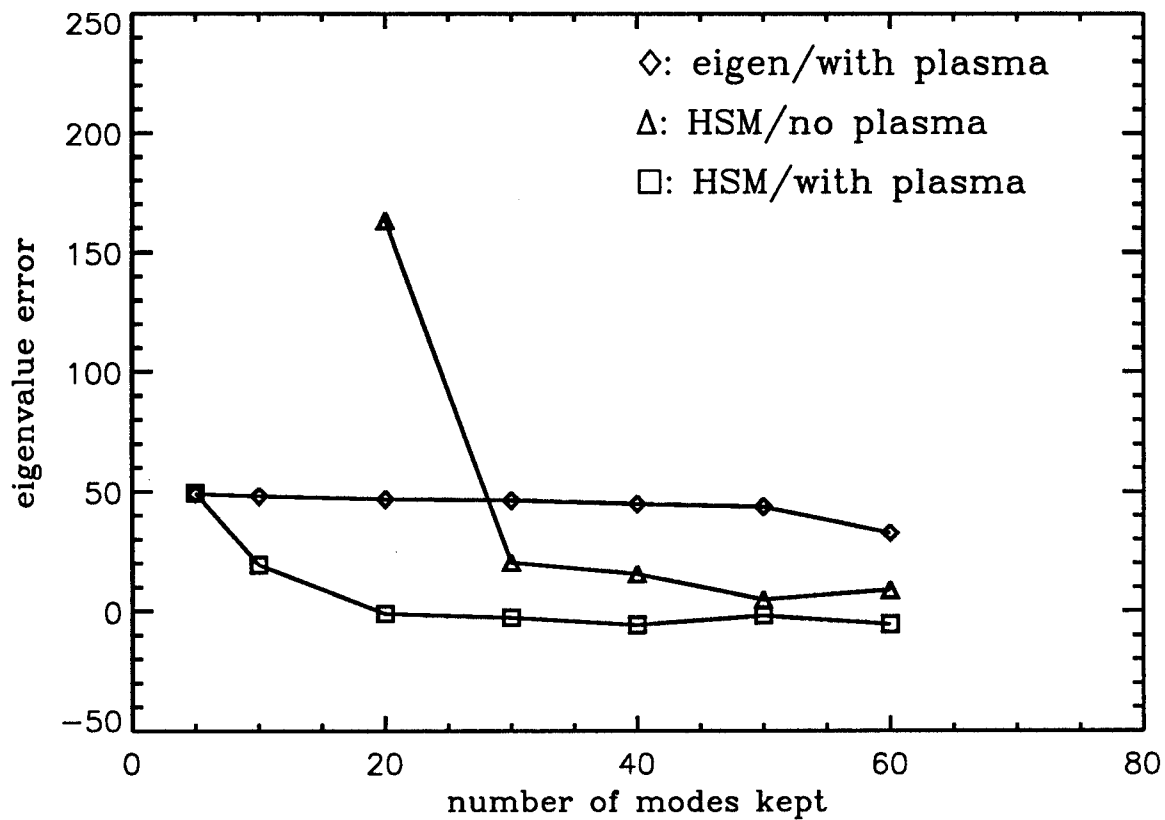


Fig. 6

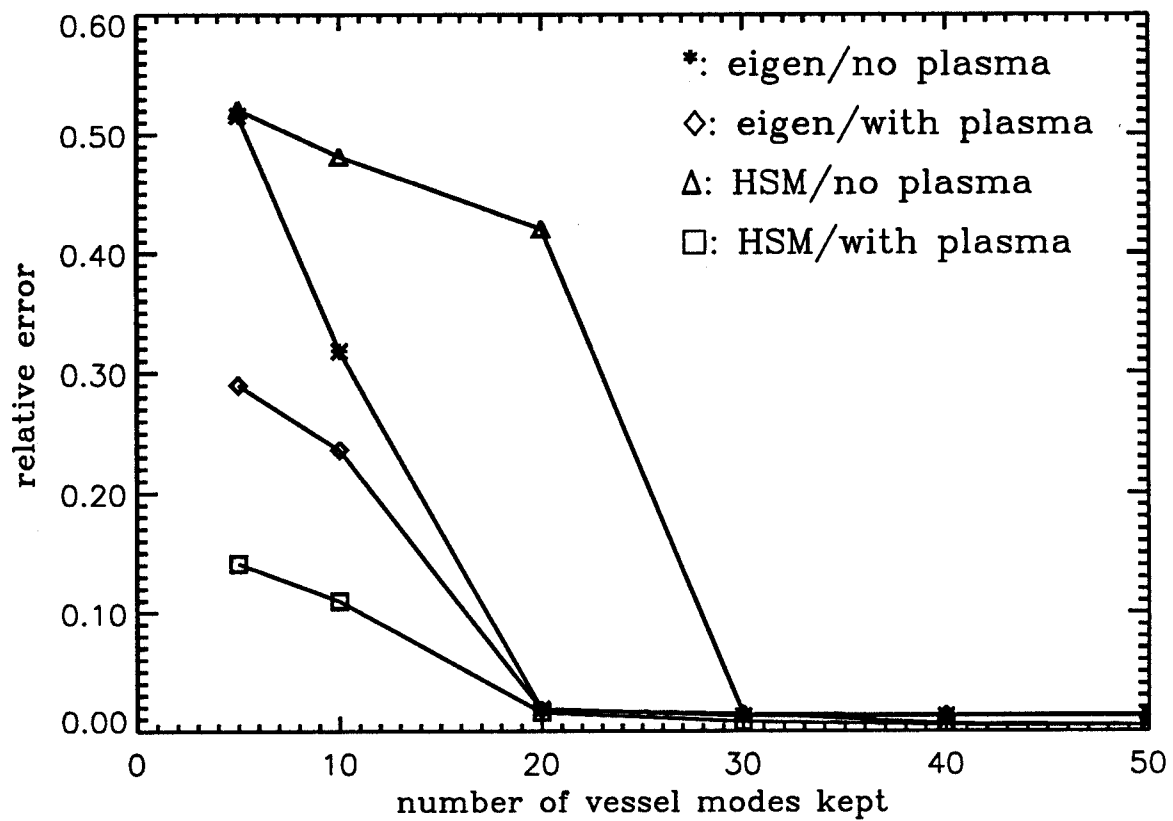


Fig. 7

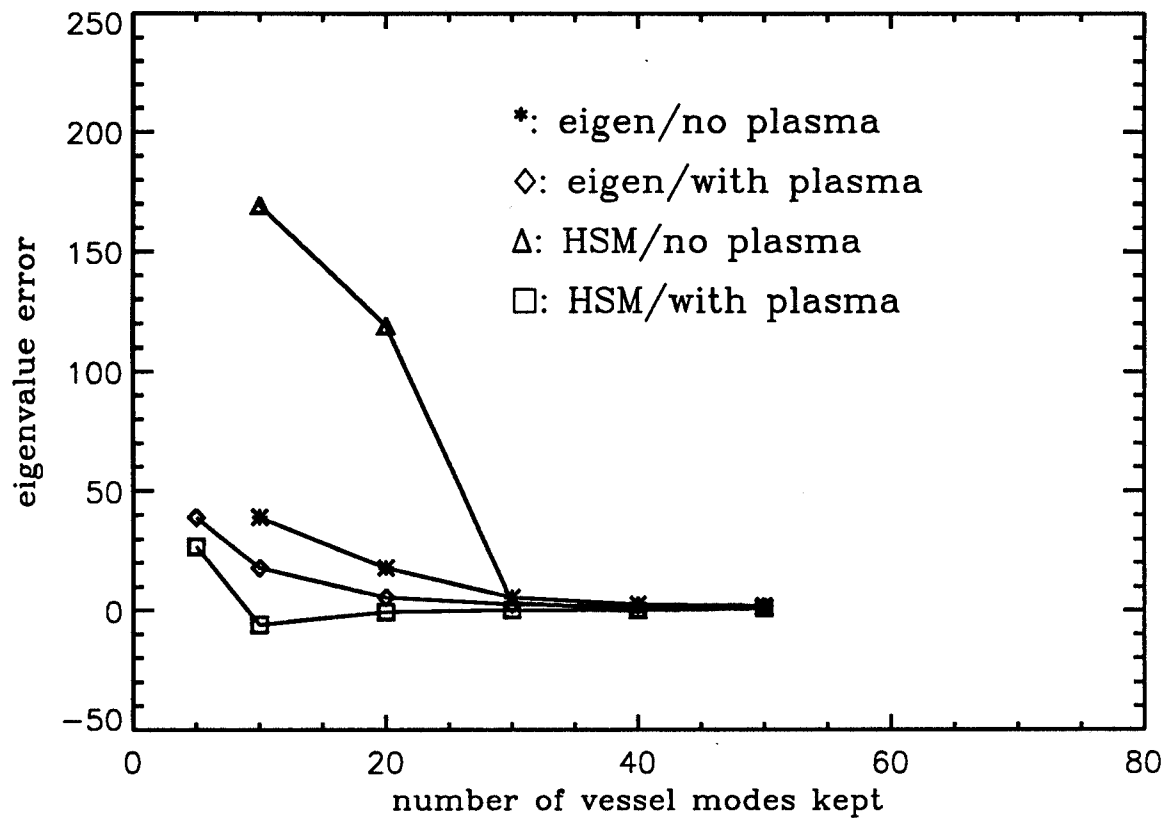


Fig. 8