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ALTERNATE REPRESENTATION OF THE DIELECTRIC TENSOR FOR A RELATIVISTIC MAGNETIZED PLASMA IN THERMAL EQUILIBRIUM

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An alternate representation of the dielectric tensor $\varepsilon_{ij}(\mathbf{k},\omega)$ for a relativistic magnetized plasma in thermal equilibrium is presented. This representation involves an infinite series expansion in powers of $c^2k_\perp^2/\alpha\omega_c^2$, as well as an asymptotic expansion for large $c^2k_\perp^2/\alpha\omega_c^2$. Here, $\omega_c = eB_0/mc$ is the nonrelativistic cyclotron frequency, $k_\perp$ is the wavenumber perpendicular to the magnetic field $B_0\hat{\mathbf{e}}_z$, and α is the dimensionless parameter defined by $\alpha = mc^2/\kappa_B T$. The present work generalizes Shkarofsky's representation [1966, *Phys. Fluids* **9**, 561]. Moreover, unlike Trubnikov's formal result [1958, in *Plasma Physics and the Problem of Controlled Thermonuclear Reactions*, Pergamon, New York] in which the $k_\perp$- and k_z-dependences of $\varepsilon_{ij}(\mathbf{k},\omega)$ are inexorably coupled, the present representation naturally separates the $k_\perp$- and k_z-dependences of $\varepsilon_{ij}(\mathbf{k},\omega)$. As an application, the general expression is simplified for the case of a weakly relativistic plasma, and the dispersion relation is obtained for electromagnetic waves including first-order relativistic effects. The method developed in this paper can be used for other nonthermal distributions.

1. INTRODUCTION

The relativistic dielectric tensor $\varepsilon_{ij}(\mathbf{k},\omega)$ for a magnetized plasma in thermal equilibrium was first derived by Trubnikov (1958). Trubnikov's formulation leads to an elegant representation of $\varepsilon_{ij}(\mathbf{k},\omega)$ (see Appendix A), but unfortunately the results are very difficult to simplify in limiting regimes of physical interest. For the case of weakly relativistic plasma ($\alpha = mc^2/\kappa_B T >> 1$), Dnestrovskii, *et al.* (1964) have simplified Trubnikov's result. However, their analysis is applicable only to the case of electromagnetic waves propagating exactly perpendicular to the applied magnetic field $\mathbf{B}_0 = B_0\hat{\mathbf{e}}_z$. Generalization of Dnestrovskii *et al.*'s method to the case of arbitrary angle of propagation was carried out by Shkarofsky (1966). Since Shkarofsky's work, similar methods have been used for a wide range of applications (Bornatici *et al.* 1983; Lee & Wu 1980; Pritchett 1984; Robinson 1986; Wong, Wu & Gaffey 1985) to weakly relativistic plasma both in thermal equilibrium and for other choices of distribution function. Other attempts have been made to simplify Trubnikov's original expression (Shkarofsky 1986), or to express the dielectric tensor $\varepsilon_{ij}(\mathbf{k},\omega)$ in a more tractable form (Airoldi & Orefice 1982).

In the present analysis, we obtain an alternate expression for the dielectric tensor $\varepsilon_{ij}(\mathbf{k},\omega)$ utilizing a representation that naturally separates the k_z-dependence and the $k_\perp$-dependence in an infinite series expansion in powers of $c^2k_\perp^2/\alpha\omega_c^2$. An asymptotic expansion of $\varepsilon_{ij}(\mathbf{k},\omega)$ for large values of $c^2k_\perp^2/\alpha\omega_c^2$ is also presented. The series expansion in powers of $c^2k_\perp^2/\alpha\omega_c^2$ is exact, and is a generalization of Shkarofsky's result obtained for the weakly relativistic regime. Moreover, unlike Trubnikov's expression, which cannot be directly manipulated to give an expansion for large values of $c^2k_\perp^2 / \alpha\omega_c^2$, the alternative formulation presented here leads to a large-$c^2k_\perp^2 / \alpha\omega_c^2$ asymptotic expansion of $\varepsilon_{ij}(\mathbf{k},\omega)$.

In the present analysis, we make use of the following notation

$$\alpha = m\,c^2 / \kappa_B T\ ,$$
$$\omega_p^2 = 4\pi \hat{n}\, e^2 / m\ ,$$
$$\omega_c = e\,B_0 / m\,c\ . \qquad (1)$$

Here, mc^2 is the electron rest mass energy, $-e$ is the electron charge, c is the speed of light *in vacuo*, T is the electron temperature, κ_B is the Boltzmann constant, $\hat{n}$=const. is the ambient electron density, and $\mathbf{B}_0 = B_0\hat{\mathbf{e}}_z$ is the uniform applied magnetic field. For simplicity, the positive ions are treated as an infinitely massive background providing overall charge neutrality. The final expressions for $\varepsilon_{ij}(\mathbf{k},\omega)$ are readily generalized to the case of several active plasma components.

The organization of this paper is the following. In Sec. 2 and Appendix B, the expressions for $\varepsilon_{ij}(\mathbf{k},\omega)$ are derived for a relativistic plasma in thermal equilibrium. These elements are simplified in the weakly relativistic regime in Sec. 3. As an application of the formal result, the dispersion relation for transverse and longitudinal electromagnetic waves in a relativistic plasma is obtained including first-order relativistic effects. For completeness, Trubnikov's expression for $\varepsilon_{ij}(\mathbf{k},\omega)$ is presented in Appendix A.

2. DIELECTRIC TENSOR FOR RELATIVISTIC MAGNETIZED PLASMA IN THERMAL EQUILIBRIUM

In this section, we obtain formal expressions for the elements of the dielectric tensor. The positive ions are assumed to form an infinitely massive, neutralizing background. In equilibrium ($\partial/\partial t = 0$), it is assumed that the electrons are distributed according to the relativistic thermal equilibrium distribution

$$f_0(\mathbf{u}) = \alpha \exp(-\alpha\gamma)/4\pi K_2(\alpha), \tag{2}$$

where $\alpha = mc^2/\kappa_B T$ is defined in (1),

$$\gamma = (1 + \mathbf{u}^2)^{1/2} \tag{3}$$

is the relativistic mass factor,

$$\mathbf{u} = \mathbf{p}/mc \tag{4}$$

is the normalized momentum, and $\mathbf{p}$ is the mechanical momentum. In (2), $K_n(\alpha)$ is the modified Bessel function of the second kind of order n.

Following a standard analysis of the linearized Vlasov-Maxwell equations (Trubnikov 1958; Davidson 1983), the dielectric tensor $\varepsilon_{ij}(\mathbf{k},\omega)$ for the thermal equilibrium distribution in (2) can be expressed as

$$\varepsilon_{ij}(\mathbf{k},\omega) = \delta_{ij} + i\frac{\omega_p^2}{\omega^2}\frac{\alpha^2}{4\pi K_2(\alpha)}\int_0^\infty d\tau \int \frac{d^3\mathbf{u}}{\gamma}$$

$$\times u_i D_j \exp[-(\alpha - i\tau)\gamma] \tag{5}$$

$$\times \exp\left\{-i\frac{ck_\perp}{\omega_c}u_\perp\left[\sin\left(\phi + \frac{\omega_c}{\omega}\tau\right) - \sin\phi\right] - i\frac{ck_z}{\omega}u_z\tau\right\}.$$

Here, the indices i,j denote x,y,z, and D_x, D_y and D_z are defined by

$$D_x = u_x \cos(\omega_c\tau/\omega) - u_y \sin(\omega_c\tau/\omega),$$

$$D_y = u_y \cos(\omega_c \tau / \omega) + u_x \sin(\omega_c \tau / \omega) \,, \tag{6}$$

$$D_z = u_z \ .$$

Moreover, making use of the standard Bessel function identity

$$\exp(z \sin\theta) = \sum_{n=-\infty}^{\infty} J_n(z) \exp(in\theta) \,, \tag{7}$$

where $J_n(z)$ is the Bessel function of the first kind of order n, the expression for $\varepsilon_{ij}(\mathbf{k},\omega)$ in (5) can be expressed as

$$\varepsilon_{ij}(\mathbf{k},\omega) = \delta_{ij} + i \frac{\omega_p^2}{\omega^2} \frac{\alpha^2}{4\pi K_2(\alpha)} \sum_{s=-\infty}^{\infty} \int_0^{\infty} d\tau \exp\left(-is\frac{\omega_c}{\omega}\tau\right)$$

$$\times \int \frac{d^3\mathbf{u}}{\gamma} V_i^s V_j^{*s} \exp\left[-(\alpha - i\tau)\gamma - i\frac{ck_z}{\omega} u_z \tau\right] . \tag{8}$$

In (8), $\mathbf{V}^s$ is defined by

$$\mathbf{V}^s = \left[u_\perp \frac{s J_s(b)}{b}, -iu_\perp J_s'(b), u_z J_s(b)\right] , \tag{9}$$

and the argument b of the Bessel function $J_s(b)$ is given by

$$b = ck_\perp u_\perp / \omega_c \,. \tag{10}$$

In obtaining (5), it is assumed that Im $\omega > 0$ and $\mathbf{k} = k_\perp \hat{\mathbf{e}}_x + k_z \hat{\mathbf{e}}_z$. Moreover, $J_s'(b) = (d/db) J_s(b)$ in (8) .

In the nonrelativistic regime, the relativistic factors γ in (5) and (8) are all set equal to unity, except when γ appears in the combination $\alpha\gamma \cong \alpha(1 + u^2/2)$. Moreover, the modified Bessel function $K_2(\alpha)$ is replaced by its asymptotic form $K_2(\alpha) \cong (\pi/2\alpha)^{1/2} \exp(-\alpha)$. In the nonrelativistic regime, it is customary to carry out the orbit integral over τ first in (8), and the resulting expression leads to an infinite summation over gyroharmonics (Davidson 1983). In the present relativistic treatment, however, the summation over gyroharmonics is not a particularly useful representation, because of the relativistic mass dependence of the gyrofrequency. For sufficiently energetic electrons, the individual harmonic structure is smeared by relativistic effects. In the present formalism, we first carry out the momentum integration over **u** in (5) and (8), and the resulting elements of the dielectric tensor are expressed as a series expansion in powers of the dimensionless parameter $\lambda = c^2k_\perp^2 / \alpha\omega_c^2 = k_\perp^2 \kappa_B T / m\omega_c^2$, or in terms of an asymptotic expansion for large values of $\lambda = c^2k_\perp^2 / \alpha\omega_c^2$.

A. Series Expansion in Powers of $\lambda = c^2k_\perp^2 / \alpha\omega_c^2$

For present purposes, we begin with the expression for $\varepsilon_{ij}(\mathbf{k},\omega)$ in (8). The elements of the tensor $V_i^s V_j^{*s}$ are proportional to $J_s^2(b)$, $J_s(b)J_s'(b)$ or $[J_s'(b)]^2$, and use is made of the following series representations

$$J_s^2(b) = \sum_{m=0}^{\infty} A_m^s\, b^{2s+2m} ,$$

$$J_s(b)\, J_s'(b) = \sum_{m=0}^{\infty} B_m^s\, b^{2s+2m-1} ,$$

$$[J_s'(b)]^2 = \sum_{m=0}^{\infty} C_m^s\, b^{2s+2m-2} , \qquad (11)$$

where

$$A_m^s = \frac{(-1)^m (2s+2m)!}{2^{2s+2m}\, m!\, (2s+m)!\, [(s+m)!]^2} ,$$

$$B_m^s = (s+m)\, A_m^s ,$$

$$C_m^s = \left[(s+m)^2 - \frac{(2s+m)\,m}{2s+2m-1}\right] A_m^s . \qquad (12)$$

In (11), care must be taken when applying the series expansions to the case where $s = 0$. In this case, we make use of $J_0(b)\, J_0'(b) = -J_0(b)\, J_1(b)$ and $[J_0'(b)]^2 = J_1^2(b)$ to express

$$J_0(b)\, J_0'(b) = \sum_{m=0}^{\infty} (m+2)\, A_m^1\, b^{2m+1}\,,$$

$$[J_0'(b)]^2 = \sum_{m=0}^{\infty} A_m^1\, b^{2m+2}\,. \tag{13}$$

Using (11)-(13), we rewrite (8) as

$$\varepsilon_{ij}(\mathbf{k},\omega) = \delta_{ij} + i\,\frac{\omega_p^2}{\omega^2}\,\frac{\alpha^2}{4\pi\, K_2(\alpha)} \int_0^\infty d\tau \int \frac{d^3\mathbf{u}}{\gamma}$$

$$\times \exp\left[-(\alpha - i\tau)\,\gamma - i\,\frac{ck_z}{\omega} u_z \tau\right] \tag{14}$$

$$\times \sum_{s=-\infty}^{\infty} \exp\left(-i\,\frac{s\omega_c}{\omega}\tau\right) \sum_{m=0}^{\infty} \left(\frac{c^2 k_\perp^2}{\omega_c^2}\right)^{|s|+m-1} u_\perp^{2|s|+2m}\, T_{ij}^{sm}\,.$$

Here, the elements T_{ij}^{sm} are defined by

$$T_{xx}^{sm} = s^2 A_m^{|s|}\,,$$

$$T_{yy}^{sm} = A_m^1\, \frac{c^4 k_\perp^4}{\omega_c^4} u_\perp^4\, \delta_{s0} + C_m^{|s|}\,(1 - \delta_{s0}),$$

$$T_{xy}^{sm} = -\,T_{yx}^{sm} = -\,i\, s\, B_m^{|s|}\,,$$

$$T_{xz}^{sm} = T_{zx}^{sm} = s\, A_m^{|s|}\, \frac{ck_\perp}{\omega_c}\, u_z\,, \tag{15}$$

$$T_{yz}^{sm} = -\,T_{zy}^{sm} = i\,(m+2)\, A_m^1\, \frac{c^3 k_\perp^3}{\omega_c^3}\, u_\perp^2 u_z\, \delta_{s0} - i\, B_m^{|s|}\, \frac{ck_\perp}{\omega_c}\, u_z\,(1 - \delta_{s0})\,,$$

$$T_{zz}^{sm} = A_m^{|s|}\, \frac{c^2 k_\perp^2}{\omega_c^2}\, u_z^2\,.$$

Note that the double summations $\sum_{s=-\infty}^{\infty} \sum_{m=0}^{\infty} \cdots$ in (14) can be expressed as

$$\sum_{s=-\infty}^{\infty} \sum_{m=0}^{\infty} \exp\left(-i\frac{s\omega_c}{\omega}\tau\right)\left(\frac{c^2k_\perp^2}{\omega_c^2}\right)^{|s|+m-1} u_\perp^{2|s|+2m}\, T_{ij}^{sm}$$

$$= \sum_{m=1}^{\infty} \left(\frac{c^2k_\perp^2}{\omega_c^2}\right)^{m-1} u_\perp^{2m}\left[T_{ij}^{0,m-1}\frac{\omega_c^2}{c^2k_\perp^2}\frac{1}{u_\perp^2} + \sum_{s=-m\,(s\neq 0)}^{\infty} \exp\left(-i\frac{s\omega_c}{\omega}\tau\right) T_{ij}^{s,\,m-|s|}\right]. \qquad (16)$$

The expression in (16), which involves a single infinite summation and a finite summation for each m, is substantially simpler than a representation in terms of double infinite summations, for obvious reasons.

The momentum integral required in the evaluation of the dielectric tensor ε_{ij} is typically of the form (see Appendix B)

$$\int \frac{d^3\mathbf{u}}{4\pi}\frac{u_\perp^{2q}}{\gamma}\exp\left(-\xi\gamma - i\eta u_z\right) = 2^q q!\,\frac{K_{q+1}\left[\left(\xi^2+\eta^2\right)^{1/2}\right]}{\left(\xi^2+\eta^2\right)^{(q+1)/2}}. \qquad (17)$$

Using the integral representation in (17) and its derivatives with respect to η, we can readily obtain the desired expression for the dielectric tensor ε_{ij} in terms of an infinite series expansion in powers of $\lambda = c^2k_\perp^2/\alpha\,\omega_c^2$. Before proceeding further, it is convenient to introduce the following notation

$$\lambda = c^2k_\perp^2/\alpha\,\omega_c^2\,,$$

$$n_z = ck_z/\omega\,,$$

$$z = \alpha\,\omega_c/\omega\,. \qquad (18)$$

Here, $\lambda = c^2k_\perp^2/\alpha\,\omega_c^2$ is proportional to the Larmor radius-squared of an electron with thermal energy $\kappa_B T$, $n_z = ck_z/\omega$ is the field-aligned component of the index of refraction, and $z = \alpha\, \omega_c / \omega$ is proportional to the ratio of the cyclotron frequency to the wave frequency. Furthermore, we introduce the functions ϕ_q^s and ϕ_q defined by

$$\varphi_q^s (\alpha ,n_z,z) = - i \int_0^\infty d\tau \, \exp(- i\, sz\tau) \frac{K_q(\alpha\zeta)}{\zeta^q} ,$$

$$\varphi_q (\alpha ,n_z) = \varphi_q^0 (\alpha ,n_z,z) = - i \int_0^\infty d\tau \, \frac{K_q(\alpha\zeta)}{\zeta^q} , \qquad (19)$$

where

$$\zeta^2 = (1 - i\tau)^2 + (n_z\tau)^2 . \qquad (20)$$

The desired series representation for ε_{ij} is then given by

$$\varepsilon_{ij} = \delta_{ij} - \frac{\omega_p^2}{\omega^2} \frac{\alpha}{K_2(\alpha)} \sum_{m=1}^{\infty} \lambda^{m-1} \sigma_{ij}^m . \qquad (21)$$

Here, making use of (14)-(20), the quantity σ_{ij}^m is defined by

$$\sigma_{xx}^m = \sum_{s=-m}^{m} s^2 a_s^m \varphi_{m+1}^s ,$$

$$\sigma_{yy}^m = 2\lambda (m + 1) a_1^m \varphi_{m+2} + {\sum_{s=-m}^{m}}' \left(m^2 a_s^m + a_s^{m-1}\right) \varphi_{m+1}^s ,$$

$$\sigma_{xy}^m = - \sigma_{yx}^m = - im \sum_{s=-m}^{m} s\, a_s^m \, \varphi_{m+1}^s , \qquad (22)$$

$$\sigma_{xz}^m = \sigma_{zx}^m = \frac{c^2 k_\perp k_z}{\omega\omega_c} {\sum_{s=-m}^{m}}' a_s^m \frac{\partial}{\partial z} \varphi_{m+2}^s ,$$

$$\sigma_{yz}^m = - \sigma_{zy}^m$$

$$= i\frac{c^2 k_\perp k_z}{\omega\omega_c}\left[\frac{(m+1)}{\alpha} a_1^m \left(1 - \frac{1 - n_z^2}{n_z}\frac{\partial}{\partial n_z}\right)\varphi_{m+1} - m \sum_{s=-m}^{m}{}' \frac{a_s^m}{s}\frac{\partial}{\partial z}\varphi_{m+2}^s\right],$$

$$\sigma_{zz}^m = \left(1 + n_z\frac{\partial}{\partial n_z}\right)\left[\frac{1}{\alpha}\frac{(m+1)}{m} a_1^m \ \varphi_{m+2} + \lambda \sum_{s=-m}^{m}{}' a_s^m \ \varphi_{m+2}^s\right].$$

In (22), the notation $\sum_s{}'$ denotes $\sum_s (s \neq 0)$, and the coefficients a_s^m are defined by

$$a_s^m = \frac{(-1)^{m+s}(2m-1)!!}{(m-s)!\,(m+s)!}, \quad 0 < s < m. \tag{23}$$

For example, for $m = 1$, $a_1^1 = 1$; for $m = 2$, $a_1^2 = -1/2$, and $a_2^2 = 1/8$; for $m = 3$, $a_1^3 = 5/16$, $a_2^3 = -1/8$, and $a_3^3 = 1/48$; ... etc. This shows that the series representation of ε_{ij} given by (21) converges rapidly. For example, if we are investigating the behavior of a mode $(\omega, \mathbf{k})$ which satisfies the condition for small Larmor radius $\lambda << 1$, then it is necessary to retain only the first few terms in the infinite series $\sum_{m=1}^{\infty} \cdots$.

B. Asymptotic Expansion for Large $\lambda = c^2 k_\perp^2 / \alpha\omega_c^2$

For $\lambda >> 1$, it is useful to start with the representation in (5). The momentum integral in (5) can be carried out in closed form, and the resulting expression for the dielectric tensor has been derived previously by Trubnikov (1958). However, for many practical purposes, Trubnikov's result is not very simple to apply, even though it has many elegant features. For this reason, the present paper focuses on the derivation of simplified expressions for ε_{ij} , which are particularly useful in regimes where $\lambda << 1$ (previous subsection) or $\lambda >> 1$ (this subsection). Moreover, the techniques developed here can be generalized readily to the case of nonthermal distribution functions, for which closed-form simplifications of the Trubnikov integrals are generally not tractable.

Returning to (5), we define the following quantities

$$\begin{aligned} \xi &= \alpha\,(1 - i\tau)\,, \\ \eta_x &= (ck_\perp/\omega_c)\sin(z\tau)\,, \\ \eta_y &= (ck_\perp/\omega_c)\,[\cos(z\tau) - 1]\,, \\ \eta_z &= \alpha\, n_z \tau\,, \end{aligned} \tag{24}$$

where $z = \alpha\omega_c/\omega$ and $n_z = ck_z/\omega$ are defined in (18). Using (24), it can be shown that (5) can be expressed as

$$\varepsilon_{ij} = \delta_{ij} - i\,\frac{\omega_p^2}{\omega^2}\frac{\alpha^3}{K_2(\alpha)}\int_0^\infty d\tau\, T_{ij}\int \frac{d^3\mathbf{u}}{4\pi\,\gamma}\exp\left(-\xi\gamma - i\,\boldsymbol{\eta}.\mathbf{u}\right), \tag{25}$$

where the elements of T_{ij} are the differential operators defined by

$$\begin{aligned} T_{xx} &= \cos(z\tau)\frac{\partial^2}{\partial\eta_x^2} - \sin(z\tau)\frac{\partial^2}{\partial\eta_x\partial\eta_y}\,, \\ T_{yy} &= \cos(z\tau)\frac{\partial^2}{\partial\eta_y^2} + \sin(z\tau)\frac{\partial^2}{\partial\eta_x\partial\eta_y}\,, \\ T_{xy} = -T_{yx} &= \sin(z\tau)\frac{\partial^2}{\partial\eta_x^2} + \cos(z\tau)\frac{\partial^2}{\partial\eta_x\partial\eta_y}\,, \\ T_{xz} = T_{zx} &= \frac{\partial^2}{\partial\eta_x\partial\eta_z}\,, \\ T_{yz} = -T_{zy} &= \frac{\partial^2}{\partial\eta_y\partial\eta_z}\,, \end{aligned} \tag{26}$$

$$T_{zz} = \frac{\partial^2}{\partial \eta_z^2}.$$

Again note that the u-integration in (25) can be carried out in closed analytical form. Indeed, the the u-integration in (25) is a special case of the more general result in (17). As mentioned earlier, if the u-integration in (25) is carried out by means of (17), then we recover the well-known result of Trubnikov. However, it is not straightforward to understand the properties of the resulting integral in the parameter regime corresponding to large values of $\lambda = c^2k_\perp^2/\alpha\omega_c^2$. For this reason, we reformulate the u-integration in order to obtain an asymptotic expression valid for large λ, or equivalently, valid for large values of $\eta_\perp = (\eta_x^2 + \eta_y^2)^{1/2}$. This reformulation is not exact, but it gives a satisfactory asymptotic result. We proceed by expressing

$$\int \frac{d^3\mathbf{u}}{4\pi\gamma} \exp\left(-\xi\gamma - i\,\eta_\perp u_\perp - i\,\eta_z u_z\right) = \frac{1}{2}\int_{-\infty}^{\infty} dx\ (1+x^2)^{1/2} \exp(-i\,\eta_z x)$$

$$\times \int_0^{\infty} dy\ \frac{y}{(1+y^2)^{1/2}} \exp\left[-\xi\,(1+x^2)^{1/2}(1+y^2)^{1/2} - i\,\eta_\perp (1+x^2)^{1/2} y\right]$$

$$\cong -\frac{1}{\eta_\perp^2}\left\{ K_0\left(\sqrt{\xi^2+\eta_z^2}\right) + \frac{3}{\eta_\perp^2}\int_\xi^\infty d\xi' \left[\int_{\xi'}^\infty d\xi''\ K_0\left(\sqrt{\xi''^2+\eta_z^2}\right) + \xi K_0\left(\sqrt{\xi'^2+\eta_z^2}\right)\right] + \cdots \right\}. \quad (27)$$

To leading order, after some straightforward algebraic manipulation, we obtain the following approximate expression for the dielectric tensor

$$\varepsilon_{ij} = \delta_{ij} + i\,\frac{1}{\lambda^2}\frac{\omega_p^2}{\omega^2}\frac{\alpha}{2K_2(\alpha)}\int_0^\infty d\tau \frac{M_{ij}(\tau)}{[1-\cos(z\tau)]^2}. \quad (28)$$

Here, $\lambda = c^2k_\perp^2/\alpha\omega_c^2$ and $z = \alpha\omega_c/\omega$, and the elements of the tensor $M_{ij}(\tau)$ are defined by

$$
\begin{aligned}
M_{xx} &= [2 + \cos(z\tau)]\, K_0(\alpha\zeta) , \\
M_{yy} &= -[2 - \cos(z\tau)]\, K_0(\alpha\zeta) , \\
M_{xy} &= - M_{yx} = \sin(z\tau)\, K_0(\alpha\zeta) , \\
M_{xz} &= M_{zx} = \frac{c^2 k_\perp k_z}{\omega\,\omega_c}\tau \sin(z\tau) \frac{K_1(\alpha\zeta)}{\zeta} , \\
M_{yz} &= - M_{zy} = - \frac{c^2 k_\perp k_z}{\omega\,\omega_c}\tau\, [1 - \cos(z\tau)] \frac{K_1(\alpha\zeta)}{\zeta} , \\
M_{zz} &= - \frac{1}{2\alpha}\left[\frac{K_1(\alpha\zeta)}{\zeta} - \alpha\,(n_z\tau)^2 \frac{K_2(\alpha\zeta)}{\zeta^2}\right] ,
\end{aligned}
\tag{29}
$$

where $n_z = ck_z/\omega$ and $\zeta^2 = (1 - i\tau)^2 + (n_z\tau)^2$.

Note that (28) and (29) can also be expressed in terms of the function $\varphi_q^s(\alpha, n, z)$ defined in (19). Upon making use of the series expansions of $\cos(zt)$ and $\sin(zt)$, we obtain

$$
\varepsilon_{ij} = \delta_{ij} - \frac{1}{\lambda^2}\frac{\omega_p^2}{\omega^2}\frac{\alpha}{K_2(\alpha)}\sum_{m=1}^{\infty} m\, M_{ij} , \tag{30}
$$

where

$$
\begin{aligned}
M_{xx} &= m^2\, \varphi_0^m , \\
M_{yy} &= \frac{(4 - m^2)}{3}\, \varphi_0^m , \\
M_{xy} &= - M_{yx} = - i\, m\, \varphi_0^m , \\
M_{xz} &= M_{zx} = \frac{c^2 k_\perp k_z}{\omega\,\omega_c}\frac{\partial}{\partial z}\varphi_1^m , \\
M_{yz} &= - M_{zy} = \frac{i}{m}\frac{c^2 k_\perp k_z}{\omega\,\omega_c}\frac{\partial}{\partial z}\varphi_1^m , \\
M_{zz} &= - \frac{(m^2 - 1)}{6}\left(\frac{\varphi_1^m}{\alpha} + \frac{\partial^2}{\partial z^2}\varphi_2^m\right) .
\end{aligned}
\tag{31}
$$

To summarize, in this section we have presented an alternate representation of the elements of the dielec‘-ic tensor for a relativistic plasma in thermal equilibrium. The electrons have been treated as the only active component; however the present analysis can be extended to a multispecies plasma in a straightforward manner. The alternate representation developed in this paper involves a series expansion which is useful for small values of $\lambda = c^2 k_\perp^2 / \alpha \omega_c^2 < 1$ [see (21) in Sec. 2.A], and an asymptotic expansion valid for large values of $\lambda = c^2 k_\perp^2 / \alpha \omega_c^2 >> 1$ [see (28) or (30) in Sec. 2.B]. Although the present formalism is equivalent to Trubnikov's approach (Appendix A), the expressions in (21), (28) and (30) are tractable in various limits ($\lambda > 1$, $\lambda < 1$, $\alpha > 1$ or $\alpha < 1$), whereas Trubnikov's results do not simplify in a straightforward manner. It should be noted that Trubnikov's result can be manipulated to give a series expansion in λ in the weakly relativistic regime ($\alpha > 1$). By comparison, the present formalism leads to an expansion in λ which is valid for all values of α . Moreover, the large-λ asymptotic expression in (30) is particularly useful because Trubnikov's formal result cannot be simplified directly in this regime. We also note that the present technique can be applied to other nonthermal distributions (Davidson & Yoon 1989) for which the closed expression for integrals of the Trubnikov-type are generally not tractable.

3. DISPERSION RELATION IN THE WEAKLY RELATIVISTIC REGIME

In this section, we take the weakly relativistic limit of the results obtained in Sec. 2, which is characterized by the inequality $\alpha = mc^2 / k_B T >> 1$. In the weakly relativistic regime it is possible to express the various quantities in the definition of ε_{ij} in terms of functions which are mathematically tractable. For example, following Dnestrovskii *et al.* (1964) and Shkarofsky (1966), one can approximate the function φ_q^s by the Shkarofsky function

$$\frac{1}{K_2(\alpha)}\,\varphi_q^s(\alpha,n_z,z) \to F_{q+1/2}\left[\alpha\left(1-\frac{s\omega_c}{\omega}\right),\ \frac{\alpha}{2}\frac{c^2k_z^2}{\omega^2}\right], \qquad (32)$$

where $F_{q+1/2}$ is defined by

$$F_{q+1/2}(z,\mu) = -\,i\exp(-\mu)\int_0^{\infty}\frac{d\tau}{(1-i\tau)^{q+1/2}}\exp\left[i(z-\mu)\tau+\frac{\mu}{(1-i\tau)}\right] \qquad (33)$$

with $\mathrm{Im}(z-\mu) > 0$. The Shkarofsky function is related to the Fried-Conte plasma dispersion function. This approach is extensively used in various physical applications, and it is particularly useful in studies of the detailed properties of the absorption of plasma waves near the gyrofrequency. In the present section, however, we concentrate on the dispersive behavior of electromagnetic waves with weak dissipation, including first-order relativistic effects. For present purposes, the function φ_q^s is approximated by (Imre 1962)

$$\frac{\alpha}{K_2(\alpha)}\,\varphi_q^s(\alpha,n_z,z) \to \frac{1}{1-sz}\left[1+\frac{q^2-4}{2\alpha}-\frac{q+1/2}{\alpha}\,\frac{1}{1-sz}+\frac{n_z^2}{\alpha}\,\frac{1}{(1-sz)^2}\right], \qquad (34)$$

which is valid provided

$$\alpha \gg 1, \text{ and } \left|1-c^2k_z^2/\omega^2\right| < \alpha\ . \qquad (35)$$

With this approximation, we obtain the following approximate expressions for the dielectric tensor in the weakly relativistic regime :

$$\varepsilon_{xx} = 1-\frac{\omega_p^2}{\omega^2-\omega_c^2}\left(1-\frac{5}{2\alpha}\,\frac{\omega^2+\omega_c^2}{\omega^2-\omega_c^2}\right)$$

$$-\frac{c^2k_z^2}{\alpha}\,\frac{\omega_p^2\left(\omega^2+3\omega_c^2\right)}{\left(\omega^2-\omega_c^2\right)^3}+\frac{3c^2k_\perp^2}{\alpha}\,\frac{\omega_p^2}{\left(\omega^2-\omega_c^2\right)\left(\omega^2-4\omega_c^2\right)},$$

$$\varepsilon_{yy} = \varepsilon_{xx} + \frac{2}{\alpha}\frac{c^2k_\perp^2}{\omega^2}\frac{\omega_p^2}{\omega^2},$$

$$\varepsilon_{xy} = -\,\varepsilon_{yx} = -\,i\,\frac{\omega_c}{\omega}\frac{\omega_p^2}{\omega^2 - \omega_c^2}\left(1 - \frac{5}{2\alpha}\frac{\omega^2}{\omega^2 - \omega_c^2}\right)$$

$$-\,i\,\frac{c^2k_z^2}{\alpha}\frac{\omega_p^2\left(3\omega^2 + \omega_c^2\right)}{\left(\omega^2 - \omega_c^2\right)^3} - 6i\,\frac{c^2k_\perp^2}{\alpha}\frac{\omega_p^2}{\left(\omega^2 - \omega_c^2\right)\left(\omega^2 - 4\omega_c^2\right)^2}, \tag{36}$$

$$\varepsilon_{xz} = \varepsilon_{zx} = -\,\frac{2}{\alpha}c^2k_\perp k_z\frac{\omega_p^2}{\left(\omega^2 - \omega_c^2\right)^2},$$

$$\varepsilon_{yz} = -\,\varepsilon_{zy} = \frac{2i}{\alpha}c^2k_\perp k_z\frac{\omega_c}{\omega}\frac{\omega_p^2}{\left(\omega^2 - \omega_c^2\right)^2},$$

$$\varepsilon_{zz} = 1 - \frac{\omega_p^2}{\omega^2}\left(1 - \frac{5}{2\alpha}\right) - \frac{3c^2k_z^2}{\alpha}\frac{\omega_p^2}{\omega^4} - \frac{c^2k_\perp^2}{\alpha}\frac{\omega_p^2}{\omega^2\left(\omega^2 - \omega_c^2\right)^2}.$$

Using (36), the dispersion relation for electromagnetic waves in a weakly relativistic plasma is given by

$$D\,(\mathbf{k}\,,\omega\,) = 0 = \det\left[\varepsilon_{ij} - \frac{c^2k^2}{\omega^2}\left(\delta_{ij} - \frac{k_ik_j}{k^2}\right)\right]. \tag{37}$$

The dispersion relation (37) can be solved exactly for $c^2k_z^2/\omega^2$ and $c^2k_\perp^2/\omega^2$ in the two limiting cases corresponding to electromagnetic waves propagating parallel to the magnetic field $B_0\hat{\mathbf{e}}_z$ and perpendicular to the magnetic field, respectively. For waves propagating exactly parallel to the magnetic field ($k_z \neq 0, k_\perp = 0$), three independent solutions are found, which correspond to right-hand and left-hand circularly polarized electromagnetic waves, and to longitudinal (electrostatic) plasma oscillations. For longitudinal plasma oscillations, the dispersion relation for $k_\perp = 0$ and $\alpha >> 1$ is given by

$$\omega^2 = \omega_p^2 \left(1 - \frac{5}{2\alpha} + \frac{3}{\alpha}\frac{c^2 k_z^2}{\omega_p^2}\right). \tag{38}$$

On the other hand, for right-hand and left-hand circularly polarized electromagnetic waves with $k_\perp = 0$, the index of refraction-squared $c^2 k_z^2/\omega^2$ is given by

$$\frac{c^2 k_z^2}{\omega^2} = \left[1 + \frac{1}{\alpha}\frac{\omega_p^2\,\omega}{(\omega \pm \omega_c)^3}\right]^{-1}\left[1 - \frac{\omega_p^2}{\omega\,(\omega \pm \omega_c)}\left(1 - \frac{5}{2\alpha}\frac{\omega}{\omega \pm \omega_c}\right)\right]. \tag{39}$$

Here, the upper (+) and lower (-) signs correspond to left-hand and right-hand circular polarizations, respectively, and $\alpha >> 1$ is assumed.

For waves propagating exactly perpendicular to the magnetic field ($k_\perp \neq 0,\, k_z = 0$), the solutions to (37) consist of the electromagnetic extraordinary mode (X-mode) and ordinary mode (O-mode) branches, and the electrostatic upper-hybrid mode. For $k_z = 0$ and $\alpha >> 1$, the O-mode branch ($\delta\mathbf{E}$ parallel to $B_0\hat{\mathbf{e}}_z$) is described by the dispersion relation

$$\frac{c^2 k_\perp^2}{\omega^2} = \left(1 + \frac{1}{\alpha}\frac{\omega_p^2}{\omega^2 - \omega_c^2}\right)^{-1}\left[1 - \frac{\omega_p^2}{\omega^2}\left(1 - \frac{5}{2\alpha}\right)\right]. \tag{40}$$

For the X-mode branch ($\delta\mathbf{E}$ perpendicular to $B_0\hat{\mathbf{e}}_z$), the dispersion relation for $k_z = 0$ including first-order relativistic effects ($\alpha >> 1$) is given by

$$\frac{c^2 k_\perp^2}{\omega^2} = \left\{1 + \frac{1}{\alpha}\frac{\omega_p^2}{\omega^2 - \omega_c^2}\,\frac{\omega^2\left(\omega^2 - \omega_H^2\right)^2 + 4\omega_c^2\left[\omega_p^2\left(\omega^2 - \omega_c^2\right) - 2\left(\omega^2 - \omega_H^2\right)^2\right]}{\left(\omega^2 - 4\omega_c^2\right)\left(\omega^2 - \omega_H^2\right)^2}\right\}^{-1}$$

$$\times \left\{1 - \frac{\omega_p^2}{\omega^2}\,\frac{\omega^2 - \omega_p^2}{\omega^2 - \omega_H^2}\left[1 - \frac{5}{2\alpha}\,\frac{\omega_p^2\left(\omega^2 + \omega_c^2\right)}{\left(\omega^2 - \omega_H^2\right)\left(\omega^2 - \omega_c^2\right)} - \frac{5}{2\alpha}\,\frac{\omega^2\left(\omega^2 + \omega_c^2 - 2\omega_p^2\right)}{\left(\omega^2 - \omega_p^2\right)\left(\omega^2 - \omega_c^2\right)}\right]\right\}. \tag{41}$$

Here, $\omega_H^2 = \omega_c^2 + \omega_p^2$ is the upper-hybrid frequency-squared. Finally, an electrostatic upper-hybrid mode also exists ($\delta \mathbf{E}$ parallel to $k_\perp \hat{\mathbf{e}}_x$). For the upper-hybrid oscillation the dispersion relation for $k_z = 0$ and $\alpha >> 1$ is given by

$$\omega^2 = \left(1 - \frac{5}{2\alpha}\right)\omega_H^2 - \frac{5}{2\alpha}\,\omega_c^2\left(1 - \frac{6}{5}\frac{\omega_p^2}{\omega_c^2}\frac{c^2 k_\perp^2}{\omega_H^2 - 4\omega_c^2}\right). \tag{42}$$

Finally, for general propagation angle, the dispersion relation for $\alpha >> 1$ is determined as follows. For the longitudinal branch, the oscillation frequency ω is determined in terms of k_z and $k_\perp$ from the dispersion relation

$$0 = \varepsilon_{xx}\frac{k_\perp^2}{k^2} + \varepsilon_{zz}\frac{k_z^2}{k^2} + 2\varepsilon_{xz}\frac{k_\perp k_z}{k^2}. \tag{43}$$

For the two branches with mixed polarization, the index of refraction-squared $n_\pm^2 = c^2(k_\perp^2 + k_z^2)/\omega^2$ can be expressed as

$$n_\pm^2 = 1 - \frac{2(A - B + C)}{2A - B \pm \left(B^2 - 4AC\right)^{1/2}}, \tag{44}$$

where

$$A = \varepsilon_{xx}\frac{k_\perp^2}{k^2} + \varepsilon_{zz}\frac{k_z^2}{k^2} + 2\varepsilon_{xz}\frac{k_\perp k_z}{k^2},$$

$$B = A\varepsilon_{yy} + \varepsilon_{xx}\varepsilon_{zz} + \varepsilon_{xy}\frac{k_\perp}{k}\left(\varepsilon_{xy}\frac{k_\perp}{k} + 2\varepsilon_{xz}\frac{k_z}{k}\right), \tag{45}$$

$$C = \left(\varepsilon_{xx}\varepsilon_{yy} + \varepsilon_{xy}^2\right)\varepsilon_{zz}.$$

In (45) we have neglected terms proportional to $1/\alpha^2$. In the above, the upper sign (+) corresponds to the branch that reduces to the left-hand circularly polarized wave (L-mode) in the case of parallel propagation, and to the O-mode in the case of perpendicular propagation. On the other hand, the lower sign (-) corresponds to the branch that reduces to the right-hand circularly polarized wave (R-mode) for parallel propagation, and to the X-mode for perpendicular propagation.

To summarize, in this section we have simplified the dispersion relation for electromagnetic waves in weakly relativistic plasma including relativistic effects to first order in $1/\alpha$. The influence of relativistic effects on the dispersive characteristics can be important (for example) in the problem of synchrotron maser amplification of electromagnetic waves by energetic electrons (Yoon 1989).

4. CONCLUSIONS

In this paper, we have presented an alternate representation of the dielectric tensor $\varepsilon_{ij}(\mathbf{k},\omega)$ for a relativistic magnetized plasma in thermal equilibrium. The representation involves an infinite series expansion in powers of $c^2k_\perp^2/\alpha\omega_c^2$ [see (21) in Sec. 2.A], as well as an asymptotic expansion valid for large values of $c^2k_\perp^2/\alpha\omega_c^2$ [see (28) or (30) in Sec. 2.B]. As an application, the dispersion relation is simplified in Sec. 3 for electromagnetic waves propagating with weak dissipation in a weakly relativistic plasma. In Sec. 3, relativistic effects are included to first order in $1/\alpha$.

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APPENDIX A. TRUBNIKOV DIELECTRIC TENSOR

Trubnikov's expression for the dielectric tensor for a relativistic plasma in thermal equilibrium in a uniform, applied magnetic field is given by

$$\varepsilon_{ij} = \delta_{ij} + i\,\frac{\omega_p^2}{\omega^2}\frac{\alpha}{K_2(\alpha)}\int_0^\infty d\tau\left[\frac{K_2(\alpha\zeta)}{\zeta^2}T_{ij} - \frac{c^2k_\perp^2}{\alpha\omega_c^2}\frac{K_3(\alpha\zeta)}{\zeta^3}S_{ij}\right]. \qquad \text{(A.1)}$$

Here, $\alpha = mc^2/\kappa_B T$ is the ratio of the electron rest mass energy to the thermal energy, and ζ^2, T_{ij} and S_{ij} are defined by

$$\zeta^2 = (1 - i\tau)^2 + 2\,\frac{c^2k_\perp^2}{\alpha^2\,\omega_c^2}[1 - \cos(\alpha\omega_c\tau/\omega)] + \frac{c^2k_z^2}{\omega^2}\tau^2\,, \qquad \text{(A.2)}$$

$$T_{xx} = T_{yy} = \cos(\alpha\omega_c\tau\,/\omega)\,,\ T_{zz} = 1,$$
$$T_{xy} = -\,T_{yx} = -\sin(\alpha\omega_c\tau/\omega)\,,\ T_{xz} = T_{zx} = T_{yz} = T_{zy} = 0\,, \qquad \text{(A.3)}$$

and

$$S_{xx} = \sin^2(\alpha\omega_c\tau/\omega)\,,\ S_{yy} = -\,[1 - \cos(\alpha\omega_c\tau/\omega)]^2\,,$$
$$S_{xy} = -\,S_{yx} = -\sin(\alpha\omega_c\tau/\omega)\,[1 - \cos(\alpha\omega_c\tau/\omega)]\,,$$
$$S_{xz} = S_{zx} = (k_z/k_\perp)(\alpha\omega_c\tau/\omega)\sin(\alpha\omega_c\tau/\omega)\,, \qquad \text{(A.4)}$$
$$S_{yz} = -\,S_{zy} = (k_z/k_\perp)(\alpha\omega_c\tau/\omega)[1 - \cos(\alpha\omega_c\tau/\omega)]\,,$$
$$S_{zz} = (k_z/k_\perp)^2(\alpha\omega_c\tau/\omega)^2\,.$$

APPENDIX B. REDUCTION OF THE MOMENTUM INTEGRAL (17)

Consider the following integral

$$\int \frac{d^3\mathbf{u}}{4\pi\gamma} u_\perp^{2q} \exp\left(-\xi\gamma - i\eta u_z\right)$$

$$= \frac{1}{2}\int_0^\infty du \frac{u^{2q+2}}{\gamma} \exp\left(-\xi\gamma\right) \int_{-1}^{1} dx \, (1 - x^2)^q \exp\left(-i\eta u x\right), \tag{B.1}$$

where $x = \cos\theta$, and $\theta = \tan^{-1}(u_\perp / u_z)$ is the pitch angle. The x-integration in (B.1) can be expressed as

$$\int_{-1}^{1} dx \, (1 - x^2)^q \exp\left(-i\eta u x\right) = 2^{q+1} \, q! \, \frac{(-1)^q}{u^{2q}} \left(\frac{\partial}{\eta\,\partial\eta}\right)^q \frac{\sin(\eta u)}{\eta u}. \tag{B.2}$$

Substituting (B.2) to (B.1) gives the desired result used in (17), i.e.,

$$\int \frac{d^3\mathbf{u}}{4\pi\gamma} u_\perp^{2q} \exp\left(-\xi\gamma - i\eta u_z\right)$$

$$= 2^q \, q! \left(-\frac{\partial}{\eta\,\partial\eta}\right)^q \int_0^\infty du \frac{u^2}{\gamma} \exp\left(-\xi\gamma\right) \frac{\sin(\eta u)}{\eta u}$$

$$= 2^q \, q! \left(-\frac{\partial}{\eta\,\partial\eta}\right)^{q+1} K_0\left[\left(\xi^2+\eta^2\right)^{1/2}\right]$$

$$= 2^q \, q! \, \frac{K_{q+1}\left[\left(\xi^2+\eta^2\right)^{1/2}\right]}{\left(\xi^2+\eta^2\right)^{(q+1)/2}}. \tag{B.3}$$

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