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ON DISSIPATIVE MODE-COUPLING IN ICRF MINORITY HEATING

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DISSIPATIVE MODE-COUPLING IN ION-CYCLOTRON RESONANCE MINORITY HEATING

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Coupled wave-equations and the corresponding wave energy-flow conservation law describing the fast magnetosonic and ion-Bernstein waves are derived for minority heating in the ion-cyclotron range of frequencies. This fourth-order full-wave system is subsequently reduced in order through representation by means of two, completely decoupled, second-order systems. One is a second-order equation for the fast wave in which the Bernstein mode is treated as a driven response. The second are coupled-mode equations for amplitudes varying slowly under the influence of coupling, inhomogeneity, and dissipation. The coupled-mode equations are approximately solved for both high-field and low-field incidence to give the transmission and mode-conversion coefficients in closed form for arbitrary wave-numbers parallel to the magnetic field. Good agreement with fourth-order calculations is obtained.

I INTRODUCTION

Rf heating of a high-temperature plasma typically occurs through the transformation of the incident wave to a kinetic mode which is dissipated on the charged particles by Landau and/or cyclotron damping. Such is the case, for example, of heating by the fast Alfvén wave (FAW) in the ion-cyclotron range of frequencies (ICRF)[1-4]. As the incident wave propagates into the hot core of the plasma, such a transformation can take place gradually and this is describable by the usual geometric optics formalism. However, near resonances a more abrupt and fundamental change of mode structure (e.g. polarization, orientation of group velocity, etc.) can occur. There is then a singular layer in the propagation and the geometric optics is in applicable. This process of transformation we term mode-conversion. Dissipation and mode-conversion may become locally intertwined and a good part of the incident power may also be reflected. The FAW by itself is essentially an undamped hydrodynamic wave. However, in the vicinity of ion-cyclotron resonances the FAW couples to an ion-Bernstein wave (IBW) which is a kinetic mode. Through this coupling some of the incident of power may thus be dissipated.

ICRF heating experiments[5-7] have been very successful and a number of existing as well as planned future tokamaks are committed to the installation of powerful ($\gtrsim$10 MW) ICRF systems. Correspondingly, there is an intense ongoing theoretical and computational effort whose goal is the understanding of the coupling process, and the prediction of the fraction of incident power dissipated.

The starting point for such studies is the Vlasov-Maxwell equations, from which a set of coupled wave-equations is derived for the two waves in question[8-11]. Special effort and methods are required for the numerical intergration of these equations in order to overcome difficulties associated with the presence of exponentially-growing spurious signals appearing on the low-field side of the ion-cyclotron resonances where the IBW is evanescent. This, and the need to go beyond slab models to toroidal geometry, are other aspects of the problem giving motivation to a particular effort aimed at obtaining analytic solutions of the full fourth-order system[12], or the reduction of the system to simpler decoupled second-order equations[13-17].

Relevant to the issue of reduction, a major trend in coupled-mode theory has been the work on representation of coupled waves by second-order[18-24], or even first-order[25,26], equations from which the transmission coefficient could be easily derived. Except for the last reference, which only determines the transmission, these theories are presently limited to non-dissipative systems.

In the present work we remove this restriction and develop a framework allowing to specifically determine conversion coefficients. We first formulate the full-wave coupled mode equations for minority heating, and derive first-order coupled equations for amplitudes varying slowly under the effect of plasma inhomogeneity, coupling, and cyclotron damping. The equations are solved analytically and the transmission and conversion coefficients are given for an arbitrary oblique angle of incidence (i.e. $n_\parallel \neq 0$). In high-field (HF) incidence of the fast wave (FAW) this then completely resolves the problem of order reduction and of wave-energy transfer. In low-field (LF) incidence the transmission is the same as in HF incidence, but the IBW cutoff is now accessible and reflection can occur. Hence all four wave-branches, forward- and backward- propagating, are present but order- reduction by means of decoupling is possible. In the scheme we propose, the LF conversion coefficient is expressed in terms of the HF results and the FAW reflection coefficient, which itself is obtained from a different second-order approximation of the full wave-equations.

II WAVE EQUATIONS AND CONSERVATION LAW

We start from the standard 2x2 zero-electron-mass Vlasov-Maxwell ICRF local dispersion relation

$$\begin{vmatrix} \varepsilon_{xx} - n_\parallel^2 & \varepsilon_{xy} \\ -\varepsilon_{xy} & \varepsilon_{yy} - n^2 \end{vmatrix} = 0, \qquad (1)$$

where $n = kc/\omega$, $\varepsilon_{\alpha\beta}$ are the dielectric tensor elements[1,14] and $v_{Ti}^2\, k_\parallel^2/\omega_{ci}^2 \ll 1$ is assumed ($k_\parallel \equiv k_z$). We assume slab geometry in which the tokamak toroidal magnetic field is directed along the z-coordinate, and its gradient along x. The gradient scale length is R_0, the Tokamak major radius.

In a usual ICRF heating situation $v_{Ti}^2 k_\perp^2/\omega_{ci}^2$ is much less than unity, and typically the highest cyclotron resonance with the lowest ordering in $\rho_i^2 k_\perp^2$ which falls into the plasma cross-section is the majority (or single-species) second-harmonic. We therefore expand the dielectric tensor elements $\varepsilon_{\alpha\beta}$ to first order in $v_{Ti}^2 k_\perp^2/\omega_{ci}^2$:

$$\begin{aligned} \varepsilon_{xx} &\simeq \varepsilon_{yy} \simeq \varepsilon_0 + \varepsilon_1 n_\perp^2 \\ \varepsilon_{xy} &\simeq i\,(g + \varepsilon_1 n_\perp^2)\,, \end{aligned} \qquad (2)$$

where we retain $\rho_i^2 k_\perp^2$ terms only coming from the majority second-harmonic. For D(H) minority heating (of which pure second-harmonic heating is a special case) this gives

$$\frac{c_A^2}{c^2}\,\varepsilon_0 = -\frac{1}{3} + \frac{Z_2^2}{4}\,\frac{\eta}{\sqrt{\beta_2}N_\parallel}\,Z(a_2\xi) \equiv K_0 + N_\parallel^2 \quad , \tag{3}$$

$$\varepsilon_1 = \frac{Z_1^2}{4\,N_\parallel}\,\sqrt{\beta_1}\;Z(a_1\,\xi) \equiv K_1 \quad , \tag{4}$$

$$\frac{c_A^2}{c^2}\,g = K_0 + N_\parallel^2 - \frac{1}{3} \,, \tag{5}$$

where we have introduced quantities normalized to the Alfven velocity c_A

$$\xi = x\omega/c_A \;,\; N = nc_A/c = kc_A/\omega \tag{6}$$

and Z is the plasma dispersion function. Further, $Z_{1,2} = 1$ is the ion charge and

$$\sqrt{\beta_{1,2}} = \sqrt{2}\,\frac{v_{T1,2}}{c_A} \;,\; c_A^2 = \frac{B_0^2}{\mu_0\,n_1 m_1} \;,\; a_{1,2} = \frac{1}{N_\parallel\,\sqrt{\beta_{1,2}}\,R_A} \quad , \tag{7}$$

where $R_A = R_0\,\omega/c_A$. The majority ion population is labeled 1, the minority 2, and we have situated the fundamental minority resonance at the origin $\xi = 0$.

In terms of the functions K_0 and K_1 introduced in (3) and (4), the approximate electrostatic ion-Bernstein dispersion relation $\varepsilon_{xx} - n_\parallel^2 = 0$ is

$$K_0 + K_1 N_\perp^2 = 0 \ , \tag{8}$$

and the dispersion relation (1) becomes the quartic

$$N_\perp^4 K_1 + N_\perp^2 (K_0 - 2\lambda_N K_1) - 2\lambda_N K_0 + \lambda_N^2 = 0 \ , \tag{9}$$

$$\lambda_N = 1/3 - N_\parallel^2 \ .$$

Equation (9) describes the fast and ion-Bernstein waves, whose coupling at this point is implicit, but can be made explicit by straightforward factorization. We find

$$(2\lambda_N - N_\perp^2)(K_0 + K_1 N_\perp^2) = \lambda_N^2 \ , \tag{10}$$

which preserves the form (8) of the kinetic IBW and identifies a true coupling constant λ_N^2 independent of the operator $N_\perp^2$. To this extent the factorization is unique and allows us to immediately write down the corresponding coupled second-order differential equations [$(\ldots)' \equiv d/d\xi$]

$$F'' + 2\lambda_N F = \lambda_N \Phi \tag{11a}$$

$$(K_1 \Phi')' - K_0 \Phi = - \lambda_N F \ , \tag{11b}$$

where F and Φ are respectively the normalized FAW and IBW field amplitudes.

It was shown in Ref. 17 that the given form of the IBW differential operator is the appropriate representation of (8) compatible with criteria established by Berk and Book[27] for slowly-varying media.

The type of coupling described by Eqs. (11) can be illustrated by the local dispersion relation (9) or (10), through plots of Re $N_\perp$ versus ξ. This is schematically shown in Fig. 1, where first we have plotted Re $N_\perp^2$ for both waves to explicitly demonstrate the coupling and show the IBW cutoff. In Fig. 1b and 1c we then indicate the powerflow pattern in respectively high-field and low-field incidence of the fast wave.

The energy-flux conservation law associated with the system (11) is

$$[\mathrm{Im}\ (F^*F' - \Phi^*K_1\Phi')]' = -\ \Phi\Phi^*\mathrm{Im}K_0 - \Phi'\Phi'^*\mathrm{Im}K_1\ , \tag{12}$$

easily obtained from the two equations through conjugate manipulations. Exchange terms depending on λ_N vanish on account of λ_N being real. It is almost obvious that Im (F^*F') must be the fast-wave Poynting flux and that Im $(\Phi^*K_1\Phi')$ is the IBW kinetic flux. On the HF side Re K_1 is positive, signifying that the two waves have oppositely-oriented group velocities. On the low-field side, only the fast wave propagates. The global power conservation law is obtained by integrating (12) between the asymptotic HF and LF sides where the waves are well-represented by eikonals of the form σ exp $(i\int N_\perp d\xi)$, σ being a power transfer factor. In low-field incidence, for example, we obtain

$$1 - \tau\tau^* - \rho\rho^* - \mathrm{Re}\ K_1\big|_{HF}\ \mu\mu^* = P_{diss}\ , \tag{13}$$

where for the incident wave we have taken $\sigma = 1$, and associated τ with the transmitted wave, ρ with the reflected wave, and μ with the IBW. In high-field incidence $\rho = 0$.

The power transmission, reflection, and mode-conversion coefficients are respectively

$$T = \tau\tau^* \ , \ R = \rho\rho^* \ , \ C = \mu\mu^* \ \ \mathrm{Re}\ K_1\big|_{HF} \ , \tag{14}$$

and P_{diss} is the integral of the right-hand side of (12).

Before we now proceed with the reduction of Eqs. (11) to a coupled second-order system, we note that a very useful second-order equation for the fast wave was derived in Refs. 13 and 14 by treating the IBW as a wave driven by the fast wave at the coupling wave-number $N_\perp = N_c$. The present treatment offers yet another way of obtaining the same second-order equation . First, the IBW response to the FAW can be here obtained in explicit form from Eq. (11b), by successive approximations of the solution Φ. As the zeroth-order iteration for Φ we simply take the eikonal solution $\exp(iN_c\xi)$ in the coupling region, as required by the stationary-phase argument of the next section. This gives $(K_1\Phi')' \simeq -K_1N_c{}^2\Phi$, and on expressing Φ from (11b) in terms of F we immediately obtain the fast-wave approximation[14]

$$F'' + (2\lambda_N - \frac{\lambda_N^2}{K_0 + N_c^2K_1})\ F = 0 \ , \tag{15}$$

from which T and R can be obtained. The information needed to complement the reduced coupled-mode equations is R, while T is redundant but can be used to check the compatibility of the two systems.

III REDUCED COUPLED-MODE EQUATIONS

We wish to reduce the system (11) to a set of two first-order equations, describing the spatial evolution of wave-amplitudes due to plasma inhomogeneity, coupling, and dissipation. We do so on assumption that the systematic evolution of the amplitudes is slow on the eikonal (i.e. fast) scale, permitting separation of the fast and slow scales in the sense that the wave-function is written as a product of the eikonal exponential and a slowly-varying amplitude.

Before doing this, however, it is useful to realize that the net slow effect is due to two different causes, and this should be taken into account as far as the perturbation-asymptotic ordering is concerned. The inhomogeneity is due to $\omega_{ci}(x)$ and is manifested through the dependence of K_0 and K_1 on ω_{ci}. The effect, as is well-known from WKB theory, is the appearance of a slowly-varying flux preserving factor. This can be established separately. Further, the systematic depletion due to dissipation arises because of non-Hermitian terms in the dispersion tensor (i.e. Im K_0 and Im K_1). Finally, there is a spatial variation due to coupling, i.e. due to transfer of energy from the fast wave to the Bernstein wave. We suggest that, at least in the case of interest here, variations due to dissipation and coupling have to be ordered equivalently, the reason being not only that both depend on the existence of the kinetic mode, but that the coupling itself is already a form of dissipation of the fast wave, the result of which is an irreversible reduction in its amplitude, no matter what thereafter happens with the kinetic mode. This is clearly manifested by the Budden case of $n_\parallel = 0$, (the "cold" ion-ion hybrid resonance) where the "missing" energy can be immediately recovered as either dissipated if we remove singularity by a small amount of damping, or as mode-converted if we add a warm correction to introduce another mode.

We shall now proceed. It is a straightforward matter to separate out the variation due to plasma inhomogeneity and we deal with this first. Recalling that in the IBW propagation region Re $K_1 > 0$, we introduce

$$B = \Phi K_1^{\frac{1}{4}} . \tag{16}$$

This transformation eliminates first-order terms in (11a), and we neglect terms $O(\varepsilon^2)$ in comparison with terms $O(1)$, where we have ordered $K_1^{\frac{1}{2}} \sim \varepsilon$. This gives

$$F'' + 2\lambda_N F = \lambda_N B/K_1^{\frac{1}{4}} \tag{17a}$$

$$B'' - \frac{K_0}{K_1} B = - \lambda_N F/K_1^{\frac{1}{4}} . \tag{17b}$$

we now write

$$F = \tilde{F}(\delta\xi)\, w_F(\xi) , \; B = \tilde{B}(\delta\xi)\, w_B(\xi) , \tag{18}$$

where w_F and w_B are the undamped and uncoupled eikonal solutions and where the slow variation δ is ordered with the dissipative term Im (K_0/K_1)

and the coupling. The uncoupled fast wave is obtained in the asymptotic limit of $|a_{1,2}\,\xi| \gg 1$, in which $K_1 \to 0$ and $K_0 \sim -1/3 - N_\parallel^2$, wherefrom

$$w_F = \exp(iN_c\xi)\ ,\ N_c^2 = \frac{\lambda_N(1 + N_\parallel^2)}{1/3 + N_\parallel^2}\ . \tag{19}$$

The IB eikonal is

$$w_B = \exp(i\int N_B d\xi)\ ,\ N_B^2 = -\,\mathrm{Re}\,\frac{K_0}{K_1}\ . \tag{20}$$

We now substitute (18) into (17), and to first order in δ we obtain the reduced system

$$\tilde{F}' = -\,i\,\frac{\lambda_N}{2K_1^{\frac{1}{2}} N_c}\,\tilde{B}\,\frac{w_B}{w_F} \tag{21a}$$

$$\tilde{B}' = i\,\frac{\lambda_N}{2K_1^{\frac{1}{2}} N_B}\,\tilde{F}\,\frac{w_F}{w_B} + \frac{\mathrm{Im}\,(K_0/K_1)}{2N_B}\,\tilde{B}\ . \tag{21b}$$

The transformation

$$\tilde{F} = f\ ,\ \tilde{B} = be^{D(\xi)}\ ,\ D(\xi) = \int_{-\infty}^{\xi} \frac{\mathrm{Im}\,(K_0/K_1)}{2N_B}\,d\xi' \tag{22}$$

then gives the more symmetric form

$$f' = -i \frac{\lambda_N}{2K_1^{1/2} N_c} e^{i\psi(\xi)} e^{D(\xi)} b \quad , \tag{23a}$$

$$b' = i \frac{\lambda_N}{2K_1^{1/2} N_B} e^{-i\psi(\xi)} e^{-D(\xi)} f \quad , \tag{23b}$$

$$\psi = \int_{-\infty}^{\xi} (N_B(\xi') - N_c)\, d\xi' \ .$$

The problem of integrating the system (23) is greatly simplified if we make use of the fact that the phase ψ is stationary (i.e. $\psi' = 0$) at the point, say ξ_c, where the two eikonal wavenumbers match, $N_B = N_c$. The principal contribution to the amplitudes f and b thus comes from the neighborhood of $\xi = \xi_c$, and we can approximate (23) by

$$f' = -i\lambda_c\, e^{i\Psi_c}\, e^{\frac{1}{2}i\Psi_c''(\xi - \xi_c)^2}\, e^{D_c}\, b \quad , \tag{24a}$$

$$b' = i\lambda_c\, e^{-i\Psi_c}\, e^{-\frac{1}{2}i\Psi_c''(\xi - \xi_c)^2}\, e^{-D_c}\, f \ , \tag{24b}$$

where $D_c = D(\xi_c)$,

$$\lambda_c^2 = \frac{\lambda_N^2}{4N_c^2 K_{1c}} \ , \ \Psi_c'' \simeq - \frac{(1/3 + N_\parallel^2)}{N_c \beta_1 R_A} \ , \tag{25}$$

and ξ_c is the solution of

$$N_c^2 = -\,\mathrm{Re}\,(K_0/K_1)\ . \tag{26}$$

This is the final form of the coupled-mode equations, which now can be solved analytically. There are two things to note. First, it is easy to verify that in the dissipationless case the symmetrized system (24) automatically conserves powerflow. Second, the asymptotic solutions of Eqs. (24) are directly the wave energy-flow factors. This can be immediately seen if we reconstruct the wave amplitudes F and Φ through (22), (18), and (16), and substitute these into the conservation law (12). With this in mind, we now solve Eqs. (24) to obtain the transmission and conversion coefficients.

IV TRANSMISSION AND MODE-CONVERSION

In this section we establish the transmission and conversion properties of the fast wave on the basis of asymptotic solutions of Eq. (24). We do so by transforming Eqs. (24) to Weber's equation, whose asymptotics is well-established[28], and use (24) itself only to impose the appropriate boundary conditions. In this respect we recall that $\xi \to -\infty$ is the high-field side, and $\xi \to \infty$ is the low-field side.

Let us first derive Weber's equation associated with (24). To begin, we easily obtain

$$f'' + i\kappa\,(\xi - \xi_c)\,f' - \lambda_c^2\,f = 0 \qquad (27a)$$

$$b'' - i\kappa\,(\xi - \xi_c)\,b' - \lambda_c^2\,b = 0\ , \qquad (27b)$$

where we have introduced a positive dephasing rate $\kappa = -\Psi_C'''$. The transformations

$$f = D_{n_f}\,\exp(z^2/4),\ b = D_{n_b}\,\exp(-z^2/4), \qquad (28)$$

where

$$z = \sqrt{\kappa}\,(\xi - \xi_c)\,e^{i(\ell\pi - \frac{\pi}{4})};\ \ell = 0 \text{ or } 1, \qquad (29)$$

then immediately give Weber's equation

$$\frac{d^2 D_n}{dz^2} + (n + \frac{1}{2} - z^2/4)\, D_n = 0 \; . \tag{30}$$

The amplitudes f and b given by (28) are distinguished by the respective index of the parabolic cylinder function D_n:

$$n_f = -i \frac{\lambda_c^2}{\kappa} \; , \; n_b = n_f - 1 \; . \tag{31}$$

The importance of carefully manipulating expressions with the index n cannot be over-emphasized. The index determines the leading asymptotic behavior of D_n, and via the Stokes phenomenon the transmission properties. The asymptotic analysis we carry out here is merely an application of textbook material to a more sophisticated example, so we only present its essential steps.

The leading asymptotic behavior of D_n is given by the connection formula

$$D_n \sim z^n \, e^{-z^2/4} \; ; \qquad |\arg z| < \frac{3\pi}{4} \tag{32a}$$

$$D_n \sim z^n \, e^{-z^2/4} - \frac{\sqrt{2\pi}}{\Gamma(-n)} \, e^{in\pi} \, z^{-n-1} \, e^{z^2/4} \; ; \; \frac{\pi}{4} < \arg z < \frac{5\pi}{4} \quad , \tag{32b}$$

which we can write in symbolic form

$$D_n \sim D_n^{(T)}$$
$$D_n \sim D_n^{(INC)} + D_n^{(C)} , \tag{33}$$

from which it is obvious which branches in (32) correspond to the incident, transmitted, and converted wave-amplitudes.

Let us first consider HF incidence of the fast wave. Since the transmitted wave is on $\xi > 0$, we take $\ell = 0$ in (27), so that arg $z = -\pi/4$ on the transmitted side and arg $z = 3\pi/4$ on the incidence side, as indicated in Fig. 2. We substitute these phases into (32) and divide through by exp $(in_f\, 3\pi/4)$, the coefficient at the incident branch. We find the transmission factor $\tau = \exp(-in_f\pi)$, and the power transmission coefficient is

$$T = \tau\tau^* = \exp(-2\pi\ \mathrm{Re}\ \lambda_c^2/\kappa) . \tag{34}$$

In order to obtain the associated conversion coefficient, we have to relate the fast wave branch $D_{n_f}^{(C)}$ to the Bernstein amplitude via Eq. (24a). To leading order in $1/z$ we easily find

$$\frac{df}{d\xi} = - \frac{\sqrt{2\pi}\ \kappa}{\Gamma(-n_f)} e^{in_f\pi} e^{z^2/4} z^{-n_f} e^{-i\frac{\pi}{4}} , \tag{35}$$

so that the Bernstein amplitude is

$$b = e^{-i(\phi_c + \frac{3\pi}{4})} e^{-z^2/4} \frac{\sqrt{\kappa}}{\lambda_c} e^{-D_c} \frac{\sqrt{2\pi}}{\Gamma(-n_f)} e^{-i\frac{\pi}{2} n_f} e^{z^2/4} |z|^{-n_f} \tag{36}$$

The expression standing at the asymptotic eigenfunction $|z|^{-n_f} \exp(z^2/4)$ is the conversion factor γ ,

$$\gamma_{HF} = e^{-i(\phi_c + \frac{3\pi}{4})} \frac{\sqrt{\kappa}}{\lambda_c} \frac{\sqrt{2\pi}}{\Gamma(-n_f)} e^{-i\frac{\pi}{2} n_f} e^{-D_c} , \tag{37}$$

and the conversion coefficient thus is

$$C_{HF} = \gamma_{HF}\gamma_{HF}^{*} = \frac{2\pi\kappa}{|\lambda_c|^2} \frac{e^{-\frac{\pi}{\kappa} \mathrm{Re}\, \lambda_c^2} e^{-2D_c}}{|\Gamma(-n_f)|^2} . \tag{38}$$

Without dissipation, Eq. (38) becomes, as expected,

$$C_{HF} \to 1 - \exp(-2\pi\lambda_c^2/\kappa) = 1 - T, \tag{39}$$

where for $N_{\parallel} \to 0$ we easily find

$$T \to \exp\left[-\frac{\pi}{2}\frac{R_0\omega}{c_A}\left(\frac{\eta}{2} + \frac{v_{T1}^2}{c_A^2}\right)\right] . \tag{40}$$

The same result in this limit was also obtained by other methods[14,26].

This resolves the general problem of transmission in systems described by equations of the type (11), as well as the particular problem of transmission-conversion in HF incidence of the fast wave. However, in ICRF heating, low-field incidence of the fast wave is the more interesting configuration, and in this case the results given so far are not sufficient to provide a complete picture of wave-power transfer. We will discuss this in the next section.

V CONVERSION OF THE FAST WAVE IN LOW FIELD INCIDENCE

In LF incidence of the fast wave there are two coupling events such as discussed in the previous sections, plus reflection at the IBW cutoff, as schematically illustrated in Fig. 1c. In the first coupling event, the fast wave is partially transmitted and partially converted to an IB wave going to cutoff. The reflected IB wave partially transmits to the HF side, becoming what we call the mode-converted branch, and partially converts to form the reflected fast wave.

In the absence of dissipation, once T is established the problem can be completely resolved[20] on the grounds that the power coming out of the cutoff must be 1 - T. Then $C_{LF} = T(1 - T)$ and $R = (1 - T)^2$. This procedure, with T calculated from the reduced, coupled-mode equations, gives exact agreement with fourth-order theories[8,12]. With dissipation, particularly in the given case with cyclotron resonances close to the cutoff, the reflection is bound to be modified by dissipation, and the scheme outlined above cannot be applied.

Despite this complication, a simple reduction scheme still exists for the given case. First, we find R by means of the second-order equation (15). Next, using R as a boundary condition for the reflected fast wave, we will consider its coupling to the corresponding Bernstein branch and find the relation

$$C_{HF}C_{LF} = T\,R\,\exp\,(-\,4D_c)\ . \qquad (41)$$

This then resolves the problem of conversion in LF incidence.

We now proceed. The transmission side is now $\xi > 0$, so in (29) we take $\ell = 1$, and the phases we substitute into (32) are $\arg z = -\pi/4$ for $\xi < 0$, and $\arg z = 3\pi/4$ for $\xi > 0$ (Fig. 1). Consider first the transmission of the incident fast wave. We get $\tau = \exp(-i\pi n_f)$, the same result as in HF incidence.

Next consider the transmission of the IB wave from the cutoff to the high-field side. As we already emphasized, in this particular problem the incident power is generally unknown but this is not a constraint since we assume that R is given and only one boundary condition in the connection formula (32) need to be specified.

In the connection formula (32), the index n is then $n_b = n_f - 1$, the conversion factor is associated with the Bernstein branch $D_n^{(T)}$, and the boundary condition is imposed on the reflected fast wave f related through Eq. (24b) to the Bernstein branch $D_n^{(C)}$. From the second part of (32b) we have

$$D_{n_b}^{(C)} = -\frac{\sqrt{2\pi}}{\Gamma(-n_b)}\, e^{in_b\pi}\, z^{-n_b-1}\, e^{z^2/4} . \tag{42}$$

Using this in (28), the leading asymptotic behavior of $db/d\xi$ is

$$\frac{db}{d\xi} = -\sqrt{\kappa}\, e^{i\frac{3\pi}{4}}\, e^{-z^2/4}\, \frac{n_b + 1}{z}\, D_{n_b}^{(C)} , \tag{43}$$

and so using (24b) the reflected fast wave branch is

$$f = - e^{i\psi_c} e^{z^2/4} \frac{\lambda_c}{\sqrt{\kappa}} \frac{\sqrt{2\pi}}{\Gamma(-n_b)} e^{in_b \frac{\pi}{4}} e^{D_c} |z|^{-n_b-2} e^{z^2/4} . \tag{44}$$

To obtain the conversion factor we multiply the connection formula (32) by an arbitrary constant σ, and impose the boundary condition on (44), i.e. $f = \rho|z|^{-n_b-2} \exp(z^2/4)$, where $\rho\rho^* = R$ is the reflection coefficient. This gives

$$\rho = - \sigma \frac{\lambda_c}{\sqrt{\kappa}} e^{i\psi_c} e^{z^2/4} \frac{\sqrt{2\pi}}{\Gamma(-n_b)} e^{in_b \frac{\pi}{4}} e^{D_c} . \tag{45}$$

On the other hand, the conversion factor, which is the coefficient in (32a), is

$$\gamma_{LF} = \sigma e^{-i\frac{\pi}{4} n_b} . \tag{46}$$

Since ρ is assumed as known from the solution of the fast-wave approximation (15), we can solve for σ from (45) and finally obtain

$$C_{LF} = \gamma_{LF} \gamma_{LF}^* = R e^{-\frac{\pi}{\kappa} \mathrm{Re}\, \lambda_c^2} e^{-2D_c} \frac{|\lambda_c|^2}{2\pi\kappa} |\Gamma(-n_f)|^2 . \tag{47}$$

Equation (41) easily follows.

In the dissipationless limit (47) becomes

$$C_{LF} \to \frac{RT}{1 - T} , \tag{48}$$

and since in that limit $R = (1 - T)^2$, we obtain, as expected, $C_{LF} \to T(1 - T)$.

It is now worthwhile noting that when

$$\mathrm{Re}\ \lambda_c^2 \gg |\mathrm{Im}\ \lambda_c^2| , \tag{49}$$

which is the case for moderate values of $k_\parallel$, then the coefficients (38) and (47) reduce to the very simple forms

$$C_{HF} = (1 - T) \exp (- 2D_c) \tag{50}$$

and

$$C_{LF} = \frac{RT}{1 - T} \exp (- 2D_c) , \tag{51}$$

where from (22) with $\xi = \xi_c$

$$D_c = \tfrac{1}{2}\int_{-\infty}^{\xi_c} \frac{\mathrm{Im}\ (K_0/K_1)}{(- \mathrm{Re}\ K_0/K_1)^{\frac{1}{2}}}\ d\xi , \tag{52}$$

and ξ_c is given by Eq. (26).

VI EXAMPLES

As mentioned before, to get the reflection coefficient R we numerically integrate Eq. (15), the details of which are presented in Ref. 14. In order to evaluate the other coefficients, we must determine the position ξ_c of the stationary-phase, or coupling, point, and the related parameters λ_c^2, $\kappa = -\psi_c''$, and D_c. First, it can be shown that to order $O\,(1/a_1^3\,\xi_c^3)$, the dephasing rate κ is independent of ξ_c and given by (25). The remaining parameters are best determined numerically. A first good guess for ξ_c from (26) is its small - $N_\parallel$ limit

$$\xi_c \rightarrow -\frac{1}{4}\,\frac{R_A}{1/3 + N_\parallel^2}\,(\eta + \beta_1\,N_c^2)\;. \tag{53}$$

Finally, in (52) it typically suffices to integrate from a lower limit of about $\xi = -5$.

We now give some examples. In Fig. 3 we compare our results for a PLT-type plasma with results from the full-wave boundary-layer code of Imre and Weitzner[29]. The respective numbers are given in Tables I and II, where we have also included (Table II) results from the reduction schemes of Refs. 17 and 26.

We see that while in most cases there is good agreement between the results obtained by the various methods, there also are some substantial differences. Thus, for example, the transmission coefficients calculated by any of the methods agree very well. Quite striking is the agreement between T from Francis et al.[26] and T from the fast-wave approximation. The origin of this agreement probably lies in that both methods treat the ion-Bernstein wave as a resonant plasma response to excitation by the fast wave. We wish to point out that in the asymptotic limit $|a_1 \xi_c| \gg 1$ our transmission coefficient (34) becomes

$$T = \exp\left[-\frac{\pi}{4}\frac{R_A N_c^3}{(1 + N_\parallel^2)^2}(\eta + \beta_1 N_c^2)\right] , \tag{54}$$

which is exactly the coefficient given by Francis et al.[26].

We would now like to draw attention to the conversion efficiencies C_{HF} and C_{LF}. While C_{HF} from Eq. (38) agrees with the results of Table I quite well, the agreement for C_{LF} from (46) is not so good (and the same can be said for C_{LF} from Ref. 17) for larger values of $k_\parallel$. At the moment we cannot offer an explanation for these discrepancies.

The next example, minority heating in a CIT-type plasma, is shown in Fig. 4. The LF incidence conversion coefficient C_{LF} remained less than 1% over the entire range of $k_\parallel$ and is therefore omitted. Likewise, also transmission remains very small. In LF incidence the incident power is therefore either reflected or dissipated. The crossover point is $k_\parallel \simeq 5$, beyond which strong dissipation occurs.

VII SUMMARY AND DISCUSSION

We have derived a set of coupled first-order equations (24) describing the systematic spatial evolution of wave-amplitudes under the effect of plasma inhomogeneity, cyclotron damping, and coupling. The equations are solved analytically and the transmission and conversion coefficients are obtained in closed form.

In ICRF minority heating by means of the fast wave incident from the high-field side, the analytic results from coupled-mode theory can be immediately applied to get the fraction of power dissipated. In the case of low-field incidence, the fast wave reflection coefficient is first calculated from the completely independent second-order equation (15), and this data is then used as a boundary condition for a coupling problem from which the conversion coefficient is established. Compared with this, the reduction schemes described in Refs. 16 and 17 require the numerical solution of two second-order systems even in HF incidence.

Complementing the obvious advantages of the given reduction scheme there are well-defined validity conditions. More specifically, as pointed out at the outset of section V, the coupled-mode equations in the absence of dissipation give an excellent representation of slow amplitude variations due to coupling. Furthermore, the separability of the eikonal and dissipative spatial scales which is the basic premise of the present theory is well-satisfied away from the cutoff and cyclotron resonances in the region between the coupling point ξ_c and the asymptotic HF side, where the Bernstein mode is described by the coupled-mode equations (24). Throughout the cutoff and cyclotron damping region WKB validity is violated, and there we use the approximate full-wave equation (15), which is not limited to weak dissipation. We thereby avoid the difficulty we see with a previous treatment[12] of the dissipative case, where to obtain the effect of dissipation on the incoming-reflected Bernstein mode, WKBJ solutions of the Bernstein wave-equation were used throughout the cutoff and cyclotron resonance region.

Having found the transmission, reflection, and mode-conversion coefficients, the dissipation per unit of incident fast-wave power can be obtained from the global conservation equation (13).

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FIGURES CAPTIONS

Fig. 1 Roots of the local dispersion relation (9) or (10). (a) Identification of asymptotic branches and branch points. (b) Powerflow in high-field incidence. (c) Powerflow in low-field incidence.

Fig. 2 Orientation of the independent variable z in Weber's equation (30) with respect to the spatial variable ξ, in HF and LF incidence of the FAW.

Fig. 3 Power transfer coefficients T, R, C_{HF}, and C_{LF} for PLT minority heating. Comparison of present results with results from Imre and Weitzner. (For details see Tables I and II).

Fig. 4 Power transfer coefficients T, R, C_{HF}, ($C_{LF} < 0.01$), and the powers dissipated D_{HF} and D_{LF}, for CIT minority heating.
$R_0 = 1.75$ m, f = 95 MHz, $B_0 = 7$ T, $T_0 = 14$ keV,
$n_e = 1.3 \times 10^{20}$ m^{-3}, D(H), $\eta = 0.05$.

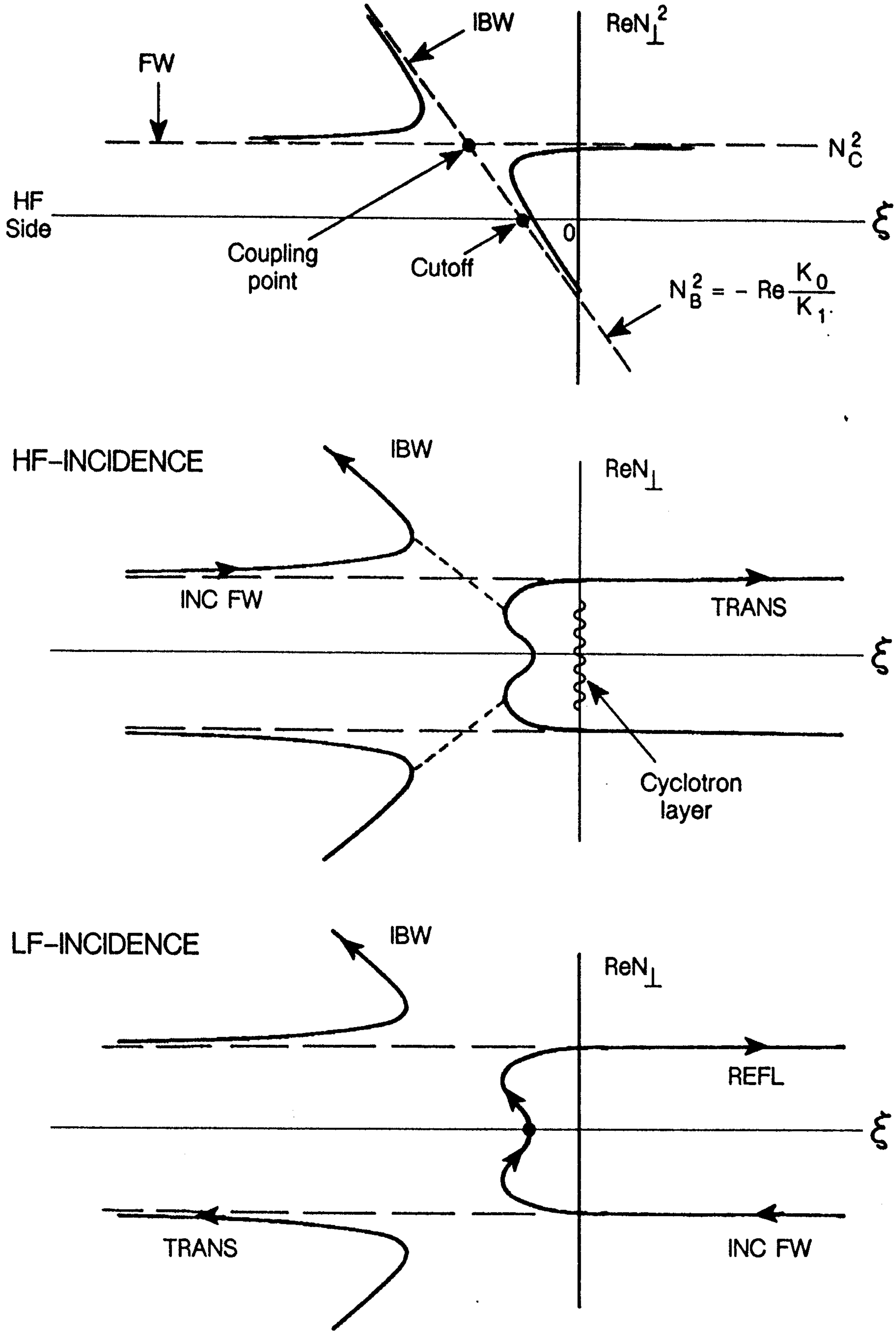

IBW
$ReN_\perp^2$
FW
N_C^2
HF
Side
ξ
0
Coupling
point
Cutoff
$N_B^2 = - Re \frac{K_0}{K_1}$
HF–INCIDENCE
IBW
$ReN_\perp$
INC FW
TRANS
ξ
Cyclotron
layer
LF–INCIDENCE
IBW
$ReN_\perp$
REFL
ξ
TRANS
INC FW

FIGURE 1

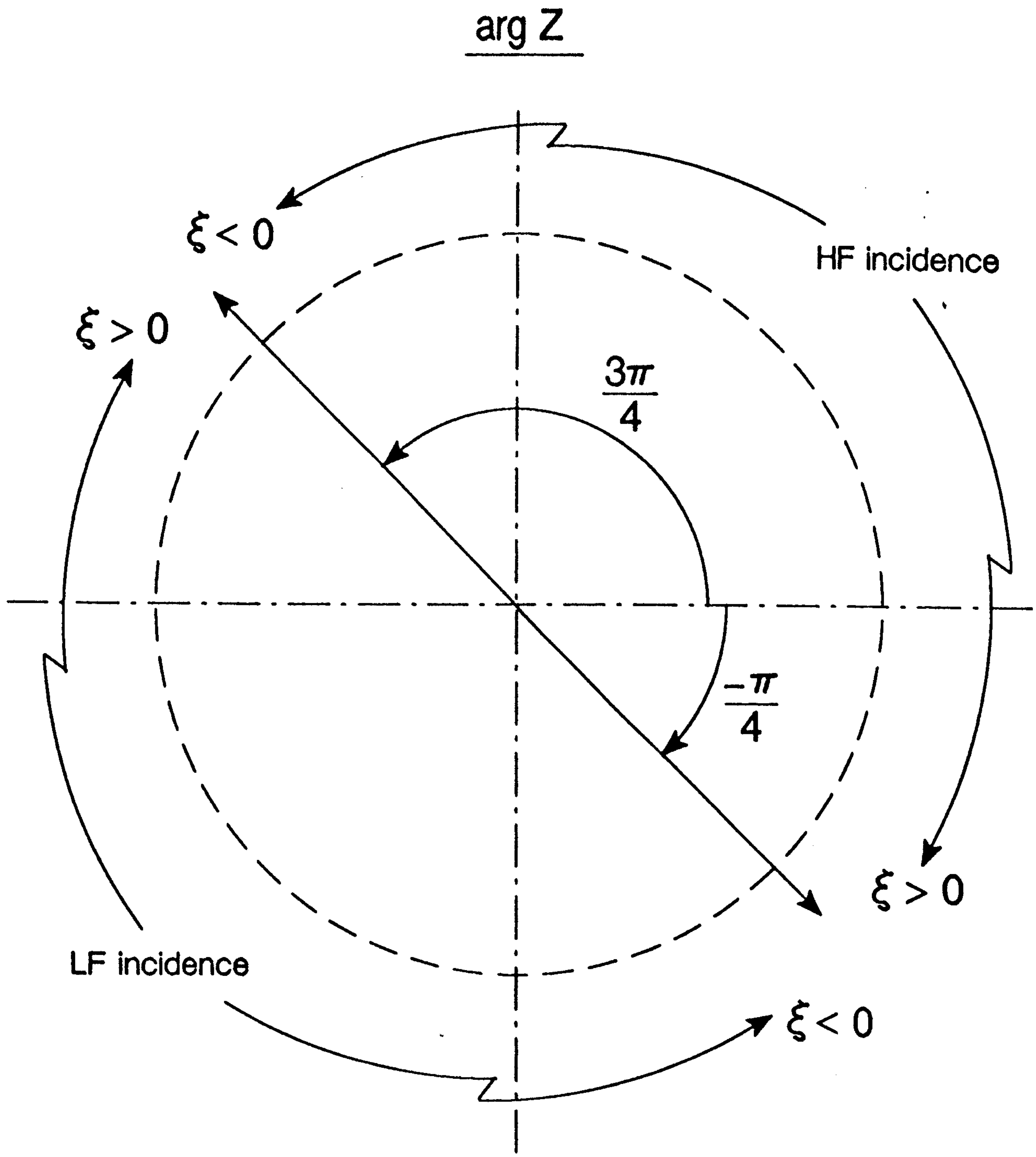


FIGURE 2

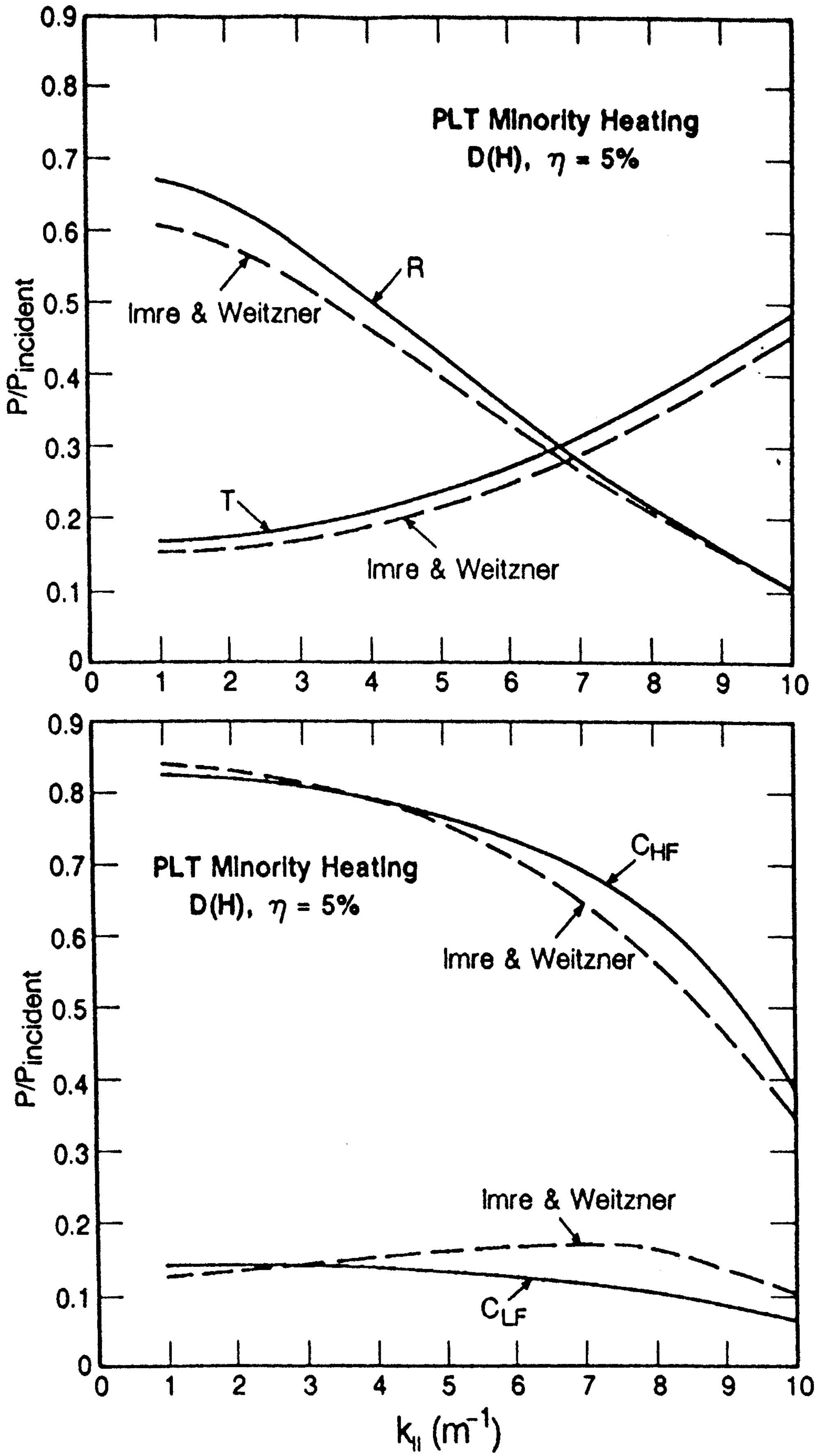
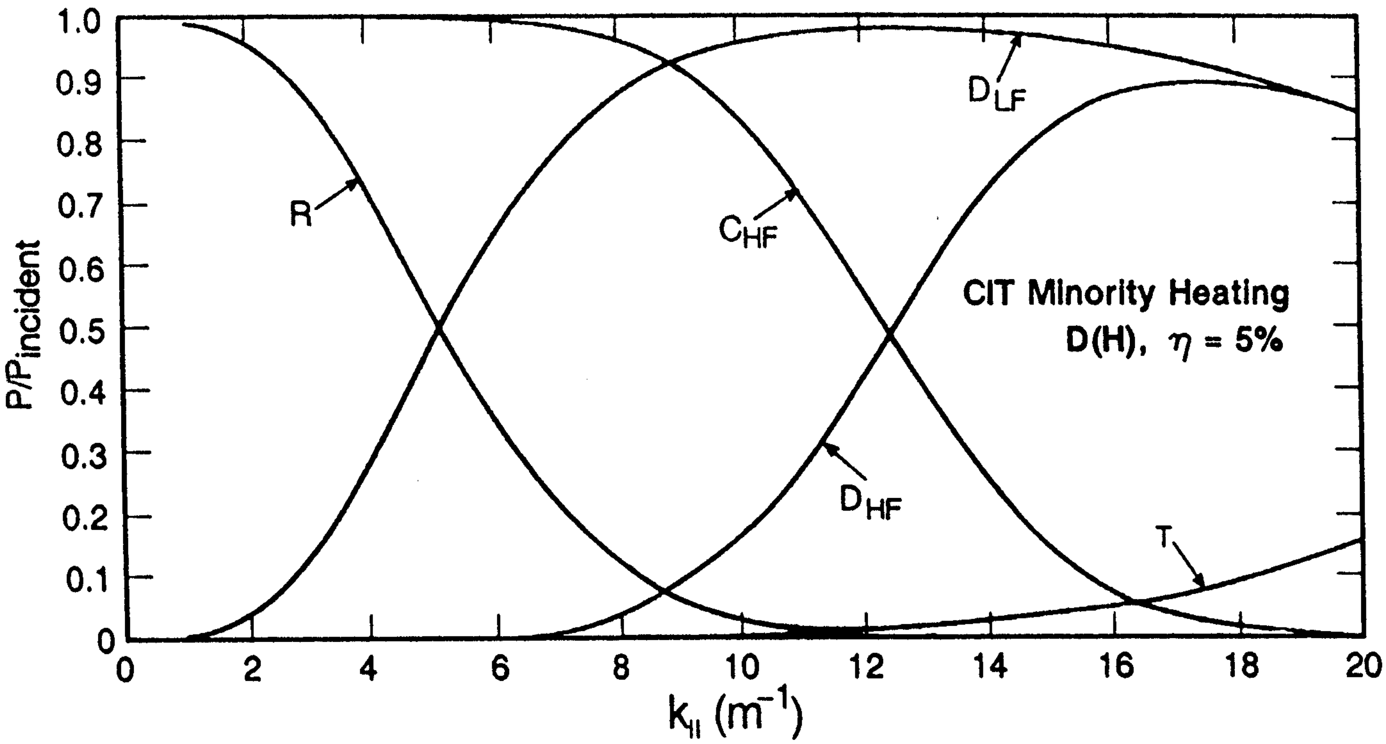

PLT Minority Heating
D(H), η = 5%
R
T
Imre & Weitzner
Imre & Weitzner
P/P$_{incident}$
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0 1 2 3 4 5 6 7 8 9 10
PLT Minority Heating
D(H), η = 5%
C_{HF}
Imre & Weitzner
Imre & Weitzner
C_{LF}
P/P$_{incident}$
$k_{\parallel}$ (m^{-1})

Figure 3



1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
P/P incident
0
2
4
6
8
10
12
14
16
18
20
k∥ (m⁻¹)
R
C HF
D LF
D HF
T
CIT Minority Heating
D(H), η = 5%

Figure 4

TABLE I

PLT HEATING . Results from code by Imre and Weitzner.

$R_o = 1.32m$, $f = 45.7$ MHz, $B_o = 3T$, $T_o = 2$ KeV, $n_e = 3 \times 10^{19} m^{-3}$, D (H), $\eta = 0.05$

$k_{\|\|} [m^{-1}]$	1	2	3	4	5	6	7	8	9	10
T_{HF}	15.3	16.1	17.4	19.3	22.0	25.2	29.3	34.2	39.9	46.2
C_{HF}	84.1	83.1	81.4	78.9	75.4	70.6	64.3	56.3	46.6	35.6
T_{LF}	15.4	16.2	17.5	19.5	22.1	25.4	29.5	34.4	40.0	46.4
C_{LF}	12.9	13.4	14.2	15.1	16.0	16.7	16.8	15.7	13.5	10.4
R	61.2	57.9	52.9	46.7	39.8	32.9	26.4	20.4	15.3	11.1

TABLE II

PLT HEATING , D(H), η = 0.05. Results from present theory, Ref. 17, and Ref. 26.

$k_{\|\|}[m^{-1}]$	1	2	3	4	5	6	7	8	9	10	
T_{LF}	17.1	17.9	19.3	21.3	24.0	27.4	31.6	36.6	42.3	48.6	From Eq. (15)
R	67.7	64.0	58.2	51.0	43.3	35.4	78.0	21.4	15.9	11.3	
T	17.1	17.8	19.1	21.0	23.2	26.1	29.5	33.5	38.4	44.5	From Eqs: (34)
C_{HF}	82.8	82.0	80.8	79.0	76.7	73.7	69.5	62.8	52.2	38.0	(38)
C_{LF}	14.0	13.9	13.8	13.5	13.1	12.5	11.5	10.2	8.4	6.2	(46)
C_{HF}	82.5	81.6	80.0	77.5	74.1	69.5	62.8	53.5	41.9	29.9	From method of Ref. 17
C_{LF}	14.1	14.7	15.6	16.8	18.3	19.7	20.8	20.5	18.5	15.1	
T	17.2	18.0	19.4	21.4	24.0	27.5	31.6	36.6	42.3	48.6	From Ref. 26, our Eq. (53)