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ABSTRACT

The self-consistent interaction of an electron beam with an electromagnetic wiggler and a radiation field is analysed. The equations are derived from a lagrangian action principle, which includes the relativistic particle lagrangians and the electromagnetic field lagrangian. The action is then specialized for the FEL to include wiggler, radiation, and space charge waves, and is averaged over the wiggler period. The approximations are made in the action, rather than in the equations of motion, thereby guaranteeing the self-consistency of the system. Equations are derived for the depletion and diffraction of the wiggler field. Using a one-dimensional approximation, pump depletion is examined for pulsed and steady-state electron beams. A three-dimensional analysis shows that the wiggler diffraction is dominated by the nonresonant interaction of the electrons with the wiggler wave. This is studied in detail for a specific electron pulse shape. It is concluded that the depletion and diffraction should not substantially degrade the FEL interaction.

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I. INTRODUCTION

Free electron laser (FEL) physics has advanced rapidly over recent years [1-10]. Oscillators have lased from visible to submillimeter wavelengths. High-power and high-efficiency operation have been demonstrated experimentally at millimeter wavelengths and experiments have been in good agreement with theory.

The utility of the FEL for the scientific community will depend not only on the power and frequency content of the optical pulses which it can produce, but also on the cost and size of the laser. At present, unfortunately, an FEL operating at visible wavelengths requires electron beam energies of approximately 50 MeV or greater. The development of short wavelength ($\lambda_w \sim 1\,\mathrm{mm}$) wigglers will allow for optical lasers which require only 10 MeV electron beams. Various proposals for developing short wavelength magnetostatic wigglers and the associated electron beam technology are being studied [11-12]. Another possible short wavelength wiggler, which is under active investigation [13-23], is an electromagnetic wave. Such a wiggler may be realized [21-23], for example, by using a gyrotron [24-26] to power a superconducting cavity. In this configuration, the electrons influence both the wiggler and optical fields. New phenomena, wiggler depletion and diffraction, can be studied through a formalism similar to that which has been developed for the FEL.

The advantages and disadvantages of electromagnetic wigglers for use in the free electron laser (FEL) have been examined previously [22]. For a fixed output frequency, the short wavelengths obtainable with electromagnetic wigglers allow for lower electron beam voltages and, consequently, a more compact FEL. The magnetostatic wiggler has transverse gradients which scale exponentially in $|\mathbf{x}_\perp|/\lambda_w$. Unless the electron beam radius scales with the magnetostatic wiggler wavelength, the performance of the FEL can be limited by these gradients. If the gap between the wiggler magnets (or coils) is held fixed, then the amplitude of the wiggler field on axis decreases exponentially as the wiggler wavelength is decreased. These two factors, reduced field strength and severe transverse gradients, along with fabrication difficulties, motivate the examination electromagnetic wigglers for compact FEL designs.

Theoretical models of the FEL [27–34], based on averaging the particle and field equations over an optical period, have been compared, after appropriate modifications, with experiment. Measurements of the nonlinear evolution of the wave amplitude and phase in a Compton [5,35] and a collective [8] (Raman) FEL have obtained good agreement with theory.

The free electron laser interaction produces both gain and phase shift. This phase shift corresponds to a beam refractive index, and the optical mode can be guided by the bunched electron beam. It has been shown [31–34] theoretically and numerically that the self-consistent wave phase shift induced by the FEL interaction can compensate for the diffraction of the light. Thus the electron beam acts like an optical fiber which can support a guided electromagnetic mode.

The measurements of wave phase shifts are evidence of the refractive index produced by the FEL, and thus lend support to the optical guiding theory. More recently, direct measurements [36,37] of the electric field have been made.

The FEL resonance condition is

$$\omega_s = \frac{\omega_w + k_w\beta_\|}{1-\beta_\|} = \frac{2\gamma^2}{1+a_w^2}(\omega_w + k_w)\,, \tag{1}$$

where ω_w and k_w are the wiggler wavenumber and frequency, $c = 1$, $a_w = eB_w/mk_w$ is the dimensionless wiggler vector potential, $\beta_\|$ is the parallel electron velocity, and $\gamma = 1/(1-\vec{\beta}\cdot\vec{\beta})^{1/2} \gg 1$. By allowing for $k_w \neq \omega_w$, our model includes waveguide or cavity modes as well as free space wiggler waves; the terms proportional to ω_w arise from electron oscillations in the wiggler electric field and those proportional to k_w result from oscillations in the wiggler magnetic field. With $\omega_w = 0$, Equation (1) reduces to the well-known FEL resonance condition.

The exponential scaling of a magnetostatic wiggler field amplitude with wiggler wavelength λ_w and a gap g between the magnets is, in planar geometry, $|B_w| \sim$

$B_{max}e^{-k_w g}$. As a consequence of this scaling, the gap must be reduced with the wavelength. This results in constraints on the electron beam transport and in the degradation of the FEL interaction as $k_w r_e$ increases (r_e is the electron beam radius). With an electromagnetic wiggler the beam pipe may be kept at a fixed radius without an increase of transverse gradients or an exponential loss of field amplitude as λ_w is decreased. The transverse variation of the wiggler wave is determined, to lowest order, by the vacuum dispersion equation $k_\perp^2 = \omega_w^2 - k_w^2$. A primary advantage of electromagnetic wigglers is this decoupling of the transverse spatial scale from the wiggler wavelength.

One potential problem with the electromagnetic wiggler is pump depletion[22,23]. Every photon produced in the optical wave must be created through the Compton backscattering of a photon in the pump wave. The total photon flux in both waves, $N_s - N_w$, is conserved. Since photons have energy $\hbar\omega$, the power depleted from the pump, P_D, is given by $P_D = P_s\omega_w/\omega_s$, where P_s is the power produced at the optical frequency, ω_s. Recent estimates [22,23], using conservation of total power over the entire interaction region, show that pump depletion, although small, is more important when high efficiency and high power are desired.

Utilizing a Lagrangian formalism and an action principle, we derive, in Section II, three-dimensional equations for the self-consistent evolution of the electron beam and both the optical and wiggler waves. The pump depletion is examined in Section III, for both long and short electron pulses, in a one-dimensional approximation. The longitudinal variation of the wiggler field amplitude is found to be quite different in these two cases. For a long electron pulse, in which the optical and wiggler field amplitudes are assumed time independent, the wiggler field is depleted rapidly at the end of the interaction region. Except in the region corresponding to the final e-folding of the optical wave, the estimate of the field amplitude obtained by total power conservation can be used. In the final e-folding, the total power conservation method yields an underestimate of the wiggler wave strength. For an intense short electron pulse, such as those produced by a high brightness photo-cathodes[38], the wiggler wave suffers depletion within the pulse.

In Section IV, the diffraction experienced by the wiggler wave as it interacts with the electron beam is studied. There are wiggler phase shifts generated by the oscillating electrons. An analysis of our self-consistent equations of motion shows, in fact, that the FEL-induced phase shift (which guides the optical wave) has little influence on the transverse variation of the wiggler field. This model also includes the phase shift due to the nonresonant interaction of the beam and with the wiggler wave. This effect corresponds to the term $\omega_{p0}^2/\gamma\omega^2$ in the dispersion relation for an electromagnetic wave propagating through an electron plasma. It can be neglected for studies of the optical wave propagation, but has a dominant role in the details of the transverse profile of the wiggler wave.

A simplified model of the wiggler wave profile is solved in Section V, yielding estimates for the transverse profile of the wiggler wave. For a short beam (less than a Rayleigh range of the electromagnetic wiggler wave calculated with the electron beam radius), the change in k_w is seen to be given approximately by the one-dimensional phase shift $\omega_{p0}^2/\omega_w\gamma$. For a long pulse, the transverse phase shift is largest over a length of one wiggler Rayleigh range from the front of the pulse. It is seen that transverse gradients in k_w are compensated for by corresponding gradients in ω_w when the electron density is assumed to be a function of $vt - z$ only. Thus FEL resonance condition is unaffected by the diffraction.

II. LAGRANGIAN THEORY

In this section, we write the action principle for particles and fields, in the system of coordinates [28] where the spatial variable z (which defines the 'parallel' direction) plays the role of the time-like variable. Self-consistent equations for the particles and for the electromagnetic fields are obtained from the extremization of the system action S, specialized to the free electron laser problem.

A. General formalism.

Consider a system of relativistic particles (electrons), interacting with an electromagnetic field. Such a system is described by an action functional [39,40], formed from the sum of the individual particle actions S_p, and of the electromagnetic field action, S_M. The dynamics of a particle will be given in terms of the evolution in z of its phase space coordinates, namely the perpendicular position, $\mathbf{r}_\perp(z)$, the perpendicular momentum, $\mathbf{p}_\perp(z)$, the time variable, $s(z)$, and the particle energy, $h(z)$. The corresponding hamiltonian is the parallel momentum, equal to

$$P(\mathbf{r}_\perp, s, \mathbf{p}_\perp, h, z) = \left[(h - \phi(\mathbf{r}_\perp, s, z))^2 - 1 - |\mathbf{p}_\perp - \mathbf{A}_\perp(\mathbf{r}_\perp, s, z)|^2\right]^{1/2} + A_z(\mathbf{r}_\perp, s, z), \tag{2}$$

where $\mathbf{A}_\perp(\mathbf{x}_\perp, t, z)$, $\phi(\mathbf{x}_\perp, t, z)$, and $\mathbf{A}_\perp(\mathbf{x}_\perp, t, z)$ are the electromagnetic potentials, and where we have chosen units so that velocity of light, electron mass and charge, are equal to unity. With these conventions, the particle action for phase space trajectories is $S_p = \int(Pdz + \mathbf{p}_\perp \cdot d\mathbf{r}_\perp - hds)$, or explicitly,

$$S_p = \int dz \Big\{ P(\mathbf{r}_\perp(z), s(z), \mathbf{p}_\perp(z), h(z), z) + \mathbf{p}_\perp(z) \cdot \mathbf{r}'_\perp(z) - h(z)s'(z) \Big\}, \tag{3}$$

where the prime denotes derivation with respect to z. The condition that the variation δS_p vanishes for all variations of phase space trajectories generates the equations of evolution for the particle.

We adopt a Vlazov description of the electron gas, and neglect accordingly discreteness effects. In the Lagrangian formalism, such a continuous system is best treated by defining a reference state, D, whose points η label the 'particles'. A measure dN is defined on this reference state, and represents the 'number' of particles in some neighborhood of the point η. For instance, the particle density n and the plasma frequency square ω_{p0}^2 are

$$n(\mathbf{x}_\perp, z, t) \equiv \omega_{p0}^2(\mathbf{x}_\perp, z, t)/4\pi = \int dN \delta(\mathbf{x}_\perp - \mathbf{r}_\perp(\eta, z))\delta(t - s(\eta, z))s'(\eta, z). \tag{4}$$

The total action is

$$S = \int dN S_p(\eta) + S_M, \tag{5}$$

where

$$S_M = (1/8\pi) \int d\mathbf{x}_\perp dz dt \Big(|\mathbf{E}(\mathbf{x}_\perp, z, t)|^2 - |\mathbf{B}(\mathbf{x}_\perp, z, t)|^2 \Big) \tag{6}$$

is the electromagnetic field action, to be expressed in terms of the electromagnetic potentials. The coupling between particle dynamic and field evolution occurs through the field-dependent terms in S_p. It turns out to be convenient to choose the (rather unusual) gauge for the potentials

$$A_z(\mathbf{x}_\perp, z, t) = 0, \tag{7}$$

which is very similar to the radiation gauge, since the z coordinate plays in this formalism the privileged role of the evolution variable. The action S is now to be considered as a functional of the independent fields $\mathbf{r}_\perp(\eta, z)$, $\mathbf{p}_\perp(\eta, z)$, $s(\eta, z)$, $\gamma(\eta, z)$, $\mathbf{A}_\perp(\mathbf{x}_\perp, z, t)$, and $\phi(\mathbf{x}_\perp, z, t)$, where, instead of h, we adopt the new particle variable $\gamma = h - \phi$. The variation of S with respect to any of those fields must vanish, which leads to the relativistic equations of motions for the particles, and to Maxwell equations for the electromagnetic field.

B. Free electron laser model.

The formulation above is exact. Some simplifications of the action functionals are appropriate for free electron laser [41]. In particular, the particles are highly relativistic ($\gamma \gg 1$), and there are generally few electromagnetic modes present.

For large γ, the parallel momentum equation (2) is expanded, and becomes, with the gauge equation (7),

$$P = \gamma \left(1 - \frac{1 + |\mathbf{p}_\perp - \mathbf{A}_\perp|^2}{2\gamma^2} \right) + O(\gamma^{-3}) \,, \tag{8}$$

which can then be replaced in the particle action Eq. (3).

We restrict the electromagnetic field to the sum of three components: the wiggler field ($\mathbf{A}_w$, ϕ_w), possibly tapered, the radiation field ($\mathbf{A}_s$, ϕ_s), and the space charge wave ($\mathbf{A}_{sc}$, ϕ_{sc}). For the wiggler and radiation fields, it is appropriate to use an eikonal form

$$\mathbf{A}_w = (1/\sqrt{2}) \left[\mathbf{a}_w(\mathbf{x}_\perp, t, z) \exp(i\theta_w(t, z)) + c.c. \right] \,, \tag{9}$$

$$\mathbf{A}_s = (1/\sqrt{2}) \left[\mathbf{a}_s(\mathbf{x}_\perp, t, z) \exp(i\theta_s(t, z)) + c.c. \right] \,, \tag{10}$$

which separate the fast variation in the phases, and the slow spatial variation due to tapering, field amplification and transverse variation in the amplitudes. Locally, wavenumbers and frequencies are defined by

$$d\theta_w = k_w dz + \omega_w dt \,, \tag{11}$$

$$d\theta_s = k_s dz - \omega_s dt \,. \tag{12}$$

The potentials ϕ_w and ϕ_s must be written similarly. It turns out however that the equations which are obtained from the variation of ϕ_w and ϕ_s allow for the trivial solution $\phi_w = 0$, $\phi_s = 0$ (i.e., $E_\parallel = 0$). In this analysis, we will adopt the TE polarization, and simplify the presentation by imposing it in the action already at this point. The representation of the space charge wave is not eikonal, but uses the fact that space charge wave is nearly stationary (if side-bands can be neglected) in the ponderomotive frame, moving with the bunched electrons. It is therefore given the form

$$\phi_{sc} = \phi_{sc}(\Theta, \mathbf{x}_\perp, z) \qquad \text{and} \qquad \mathbf{A}_{sc} = \mathbf{A}_{sc}(\Theta, \mathbf{x}_\perp, z)\,, \tag{13}$$

where the phase of the ponderomotive field is defined as

$$\Theta(t,z) = \theta_w(t,z) + \theta_s(t,z)\,, \tag{14}$$

from which

$$d\Theta = (k_w + k_s)dz + (\omega_w - \omega_s)dt\,. \tag{15}$$

After these substitutions, the action still includes betatron oscillations due to field inhomogeneities. For tractability however, we neglect here betatron oscillations and perpendicular particle motion due to space charge wave. Perpendicular electron motion is thus dominated by oscillation in both wiggler and radiation fields. It is easy to see that these approximations amount to replacing in the particle lagrangian the squared velocity term $|\mathbf{p}_\perp - \mathbf{A}_\perp|^2$ by the expression $|\mathbf{A}_w + \mathbf{A}_s|^2$, i.e., the squared velocity due to oscillations.

The particle action is now

$$S_p = \int dz \Big\{ (\gamma - \gamma s' - \phi_{sc} s') - (1/2\gamma)\,\mathrm{Re}\big[1 + |\mathbf{a}_w|^2 + |\mathbf{a}_s|^2 + 2\mathbf{a}_w \cdot \mathbf{a}_s \exp(i\Theta) + 2\mathbf{a}_w{}^* \cdot \mathbf{a}_s \exp(-i\theta_w + i\theta_s)\big] \Big\}\,, \tag{16}$$

where the fields are evaluated at the particle position. The resonance condition is

$$d\Theta(s(z),z)/dz = (k_w + k_s) + (\omega_w - \omega_s)s' = 0\,. \tag{17}$$

Neglecting now the rapid oscillations along z, $\langle \exp(-i\theta_w + i\theta_s)\rangle \approx 0$, one gets the final form of the particle action :

$$S_p = \int dz \Big\{ \gamma - \gamma s' - \phi_{sc} s' - (1/2\gamma)\,\mathrm{Re}\big[1 + |\mathbf{a}_w|^2 + |\mathbf{a}_s|^2 + 2\mathbf{a}_w \cdot \mathbf{a}_s \exp(i\Theta)\big] \Big\}\,. \tag{18}$$

When the fields are substituted in the field action, and when rapid oscillations due to the different wavenumbers are averaged out, the field action turns out to be the sum of three terms,

$$S_M = (S_M)_w + (S_M)_s + (S_M)_{sc}\,, \tag{19}$$

quadratic in the wiggler field, radiation field, and space charge field respectively :

$$(S_M)_w = (1/8\pi)\,\mathrm{Re}\int d\mathbf{x}_\perp\, dt\, dz\Big\{(\omega_w^2 - k_w^2)|\mathbf{a}_w|^2 - |\nabla_\perp \times \mathbf{a}_w|^2 \\ - 2ik_w\mathbf{a}_w\cdot\frac{\partial \mathbf{a}_w^*}{\partial z} + 2i\omega_w\mathbf{a}_w\cdot\frac{\partial \mathbf{a}_w^*}{\partial t}\Big\}, \tag{20}$$

$$(S_M)_s = (1/8\pi)\,\mathrm{Re}\int d\mathbf{x}_\perp\, dt\, dz\Big\{(\omega_s^2 - k_s^2)|\mathbf{a}_s|^2 - |\nabla_\perp \times \mathbf{a}_s|^2 \\ - 2ik_s\mathbf{a}_s\cdot\frac{\partial \mathbf{a}_s^*}{\partial z} - 2i\omega_s\mathbf{a}_s\cdot\frac{\partial \mathbf{a}_s^*}{\partial t}\Big\}, \tag{21}$$

and

$$(S_M)_{sc} = (1/8\pi)\int d\mathbf{x}_\perp\, dt\, dz\Big\{\Big|(\omega_w - \omega_s)\frac{\partial \mathbf{A}_{sc}}{\partial \Theta} + \nabla_\perp\phi_{sc}\Big|^2 \\ + (k_w + k_s)^2\left(\left|\frac{\partial \phi_{sc}}{\partial \Theta}\right|^2 - \left|\frac{\partial \mathbf{A}_{sc}}{\partial \Theta}\right|^2\right) - |\nabla_\perp \times \mathbf{A}_{sc}|^2\Big\}. \tag{22}$$

The action describing FEL's is therefore formed from Eq. (5, 18-22).

C. Free electron laser equations.

The particle equations are obtained by varying S_p Eq. (18) with respect to $s(z)$ and $\gamma(z)$, taking into account the fact that the fields are evaluated at the particle position, i.e., at $t = s(z)$. One gets, using $\delta\Theta = (\omega_w - \omega_s)\delta s$,

$$\delta S_p = \int dz \Big\{ (1 - s') + (1/2\gamma^2)\, \mathrm{Re}\left[1 + |\mathbf{a}_w|^2 + |\mathbf{a}_s|^2 + 2\mathbf{a}_w \cdot \mathbf{a}_s \exp(i\Theta)\right] \Big\} \delta\gamma$$
$$+ \int dz \Big\{ \gamma' + (k_w + k_s)\frac{\partial \phi_{sc}}{\partial \Theta} + \frac{\partial \phi_{sc}}{\partial z} + (1/\gamma)(\omega_w - \omega_s)\, \mathrm{Im}\left[\mathbf{a}_w \cdot \mathbf{a}_s \exp(i\Theta)\right]$$
$$- (1/2\gamma)\, \mathrm{Re}\left[\frac{\partial}{\partial t}\left(|\mathbf{a}_s|^2 + |\mathbf{a}_w|^2\right) + 2\frac{\partial}{\partial t}(\mathbf{a}_w \cdot \mathbf{a}_s)\exp(i\Theta) \right] \Big\} \delta s. \tag{23}$$

Extremization of the action over particle trajectories leads to the equations for the particle phase

$$s' = 1 + (1/2\gamma^2)\, \mathrm{Re}\left[1 + |\mathbf{a}_w|^2 + |\mathbf{a}_s|^2 + 2\mathbf{a}_w \cdot \mathbf{a}_s \exp(i\Theta)\right], \tag{24}$$

and for the particle energy

$$\gamma' = (1/\gamma)(\omega_s - \omega_w)\, \mathrm{Im}\left[\mathbf{a}_w \cdot \mathbf{a}_s \exp(i\Theta)\right] - (k_w + k_s)\frac{\partial \phi_{sc}}{\partial \Theta} - \frac{\partial \phi_{sc}}{\partial z}$$
$$+ (1/2\gamma)\, \mathrm{Re}\left[\frac{\partial}{\partial t}\left(|\mathbf{a}_s|^2 + |\mathbf{a}_w|^2\right) + 2\frac{\partial}{\partial t}(\mathbf{a}_w \cdot \mathbf{a}_s)\exp(i\Theta) \right]. \tag{25}$$

The field equations, on the other hand, are obtained by variation of S_M and S_p (19, 18) with respect to the electromagnetic potentials.

First, vary the wiggler field by $\delta\mathbf{a}_w$. One gets the following expressions,

$$\delta S_M = (1/4\pi)\, \mathrm{Re} \int d\mathbf{x}_\perp\, dt\, dz \Big\{ (\omega_w^2 - k_w^2)\mathbf{a}_w \cdot \delta\mathbf{a}_w^* - \nabla_\perp \times (\nabla_\perp \times \mathbf{a}_w) \cdot \delta\mathbf{a}_w^*$$
$$+ ik_w \frac{\partial}{\partial z}\mathbf{a}_w \cdot \delta\mathbf{a}_w^* + i\frac{\partial}{\partial z}(k_w \mathbf{a}_w) \cdot \delta\mathbf{a}_w^* - i\omega_w \frac{\partial}{\partial t}\mathbf{a}_w \cdot \delta\mathbf{a}_w^* - i\frac{\partial}{\partial t}(\omega_w \mathbf{a}_w) \cdot \delta\mathbf{a}_w^* \Big\}, \tag{26}$$

and

$$\int dN\,\delta S_p = \int d\mathbf{x}_\perp\,dt\,dz\,\mathrm{Re}\int dN(-1/\gamma)\delta(\mathbf{x}_\perp - \mathbf{r}_\perp)\delta(t-s) \times [\mathbf{a}_w + \mathbf{a}_s^* \exp(-i\Theta)]\cdot\delta\mathbf{a}_w^* . \tag{27}$$

Stationnarity of the sum of Eq.(26) and (27) generates the paraxial equation for $\mathbf{a}_w$,

$$\begin{aligned}(\omega_w^2 - k_w^2)\mathbf{a}_w &- \nabla_\perp \times (\nabla_\perp \times \mathbf{a}_w) + ik_w\frac{\partial}{\partial z}\mathbf{a}_w + i\frac{\partial}{\partial z}(k_w\mathbf{a}_w) \\ &- i\omega_w\frac{\partial}{\partial t}\mathbf{a}_w - i\frac{\partial}{\partial t}(\omega_w\mathbf{a}_w) \\ &= \omega_{p0}^2(\mathbf{x}_\perp, t, z)\left[\left\langle\frac{1}{\gamma s'}\right\rangle\mathbf{a}_w + \left\langle\frac{\exp(-i\Theta)}{\gamma s'}\right\rangle\mathbf{a}_s^*\right].\end{aligned} \tag{28}$$

Then, vary the radiation field by $\delta\mathbf{a}_s$. The equations for the radiation field are entirely similar to the previous ones, and one gets:

$$\begin{aligned}(\omega_s^2 - k_s^2)\mathbf{a}_s &- \nabla_\perp \times (\nabla_\perp \times \mathbf{a}_s) + ik_s\frac{\partial}{\partial z}\mathbf{a}_s + i\frac{\partial}{\partial z}(k_s\mathbf{a}_s) \\ &+ i\omega_s\frac{\partial}{\partial t}\mathbf{a}_s + i\frac{\partial}{\partial t}(\omega_s\mathbf{a}_s) \\ &= \omega_{p0}^2(\mathbf{x}_\perp, t, z)\left[\left\langle\frac{1}{\gamma s'}\right\rangle\mathbf{a}_s + \left\langle\frac{\exp(-i\Theta)}{\gamma s'}\right\rangle\mathbf{a}_w^*\right].\end{aligned} \tag{29}$$

Finally, variation of the action with respect to the space charge wave field is composed of

$$\begin{aligned}\delta S_M = (1/4\pi)\int d\mathbf{x}_\perp\,dt\,dz&\Big\{(\omega_w - \omega_s)\nabla_\perp\phi_{sc}\cdot\frac{\partial\delta\mathbf{A}_{sc}}{\partial\Theta} \\ &+ \left[(\omega_w - \omega_s)^2 - (k_w + k_s)^2\right]\frac{\partial\mathbf{A}_{sc}}{\partial\Theta}\cdot\frac{\partial\delta\mathbf{A}_{sc}}{\partial\Theta} - \nabla_\perp \times (\nabla_\perp \times \mathbf{A}_{sc})\cdot\delta\mathbf{A}_{sc}\Big\} \\ + (1/4\pi)\int d\mathbf{x}_\perp\,dt\,dz&\Big\{-\nabla_\perp\cdot\left[(\omega_w - \omega_s)\frac{\partial\mathbf{A}_{sc}}{\partial\Theta} + \nabla_\perp\phi_{sc}\right]\delta\phi_{sc} \\ &+ (k_w + k_s)^2\frac{\partial\phi_{sc}}{\partial\Theta}\frac{\partial\delta\phi_{sc}}{\partial\Theta}\Big\},\end{aligned} \tag{30}$$

and of

$$\int dN\,\delta S_p = \int dN\,dz\Big\{-s'\delta\phi_{sc}\Big\} = -(1/4\pi)\int d\mathbf{x}_\perp\,dt\,dz\,\omega_{p0}^2\,\delta\phi_{sc}\,. \tag{31}$$

The equations for the space charge fields are therefore

$$[(k_w+k_s)^2-(\omega_w-\omega_s)^2]\frac{\partial^2\mathbf{A}_{sc}}{\partial\Theta^2} - \nabla_\perp\times(\nabla_\perp\times\mathbf{A}_{sc}) = (\omega_w-\omega_s)\nabla_\perp\frac{\partial\delta\phi_{sc}}{\partial\Theta}\,, \tag{32}$$

and

$$(k_w+k_s)^2\frac{\partial^2\phi_{sc}}{\partial\Theta^2} + \nabla_\perp\cdot\left[\nabla_\perp\phi_{sc} + (\omega_w-\omega_s)\frac{\partial\mathbf{A}_{sc}}{\partial\Theta}\right] = -\omega_{p0}^2\,. \tag{33}$$

Note that $\delta\mathbf{A}_{sc}$ does not appear in δS_p, only in the vacuum field action δS_M : the space charge wave is 'electrostatic' in the ponderomotive frame moving with velocity $(\omega_w - \omega_s)/(k_w + k_s)$, and $\mathbf{A}_{sc}$ only results from a Lorentz transformation to the 'laboratory' frame, when one takes into account the gauge $A_z = 0$.

To eliminate $\mathbf{A}_{sc}$ from the equations, take the Fourier transform in Θ. With

$$\omega_{p0}^2 = \sum_m(\omega_{p0}^2)_m\exp(im\Theta) \tag{34}$$

and

$$\phi_{sc} = \sum_m(\phi_{sc})_m\exp(im\Theta)\,, \tag{35}$$

Eq. (32) becomes

$$\mathbf{A}_{sc} = \sum_{m\neq 0}\chi_m\nabla_\perp(\phi_{sc})_m\exp(im\Theta)\,, \tag{36}$$

where

$$\chi_m = \frac{-i(\omega_w-\omega_s)}{m[(k_w+k_s)^2-(\omega_w-\omega_s)^2]} \qquad \text{where } m\neq 0\,. \tag{37}$$

Substitution in Eq. (33) gives

$$m^2(\phi_{sc})_m - \frac{\nabla_\perp^2(\phi_{sc})_m}{(k_w+k_s)^2-(\omega_w-\omega_s)^2} = \frac{(\omega_{p0}^2)_m(\mathbf{x}_\perp)}{(k_w+k_s)^2}\,, \tag{38}$$

where the denominator can also be written as

$$[(k_w+k_s)^2-(\omega_w-\omega_s)^2]^{-1} = \gamma_\parallel^2(k_w+k_s)^{-2}\,. \tag{39}$$

When $\omega_w = 0$, the space charge equation (38) reduces to its expression for conventional FEL [8,42].

III. PUMP DEPLETION

Pump depletion occurs because of scattering of wiggler field photons by the electrons. It is calculated here in the one-dimensional approximation.

A. One-dimensional Equations

The one-dimensional equations are obtained by setting $\nabla_\perp = 0$ in equations (28, 29); in this section, the wiggler and signal field are chosen to be linearly polarized TE modes. Because of the uniformity in the perpendicular directions, it is possible to choose the wavevectors appropriately (k_s and k_w deviate slightly from the vacuum values ω_s and ω_w), so that the paraxial equations simplify as follows, for the radiation field,

$$2ik_s \frac{\partial}{\partial z}\mathbf{a}_s + 2i\omega_s \frac{\partial}{\partial t}\mathbf{a}_s = \omega_{p0}^2 \left\langle \frac{\exp(-i\Theta)}{\gamma s'} \right\rangle \mathbf{a}_w^* , \tag{40}$$

and for the wiggler field,

$$2ik_w \frac{\partial}{\partial z}\mathbf{a}_w - 2i\omega_w \frac{\partial}{\partial t}\mathbf{a}_w = \omega_{p0}^2 \left\langle \frac{\exp(-i\Theta)}{\gamma s'} \right\rangle \mathbf{a}_s^* . \tag{41}$$

B. Steady State Beam

For a time-invariant electron beam, consider stationary solutions, and set $\partial/\partial t = 0$ in the previous equations. It can be verified directly that usual conservation properties are satisfied.

Photon conservation is obtained by taking the imaginary part of, symbolically, $\mathbf{a}_s^* \times (40) - \mathbf{a}_w^* \times (41)$. One gets the relation

$$\frac{\partial}{\partial z}(N_s - N_w) = 0\,, \tag{42}$$

where the radiation and wiggler field photon densities, N_s and N_w are proportional to $k_s|\mathbf{a}_s|^2$ and $k_w|\mathbf{a}_w|^2$. Equation (42) means that the divergence of photon fluxes (propagating in opposite directions) is zero, recovering thus the result that the FEL interaction verifies the conservation of photon number. The same property can conveniently be expressed in terms of radiation energy densities $P_s = N_s\omega_s$ and $P_w = N_w\omega_w$. It takes then the form

$$\frac{\partial}{\partial z}P_s = \frac{\omega_s}{\omega_w}\frac{\partial}{\partial z}P_w\,. \tag{43}$$

Energy conservation follows also from (40) and (41) : the imaginary part of $\mathbf{a}_s^*\omega_s \times (40) - \mathbf{a}_w^*\omega_w \times (41)$ yields

$$\frac{\partial}{\partial z}(P_s - P_w) = -\,\mathrm{Im}\left[\omega_{p0}^2(\omega_s - \omega_w)\left\langle\frac{\mathbf{a}_w\mathbf{a}_s\exp(i\Theta)}{\gamma s'}\right\rangle\right] = -\left\langle\frac{\omega_{p0}^2}{s'}\frac{d\gamma}{dz}\right\rangle, \tag{44}$$

where one has used the particles energy equations

$$\frac{d\gamma}{dz} = \frac{\omega_s - \omega_w}{\gamma}\,\mathrm{Im}\left(\mathbf{a}_w\mathbf{a}_s\exp(i\Theta)\right)\,. \tag{45}$$

Equation (44) shows that the divergence of the Poynting vector equals rate of energy input due to the FEL interaction.

It is now possible to evaluate the depletion of wiggler field for a steady state amplifier. Assume that the radiation field increases exponentially, and therefore that the photon number has the form

$$N_s = N_{sf}\exp[-2\Gamma(L - z)]\,; \tag{46}$$

photon conservation (42) implies that

$$N_w = N_{wf} - N_{sf}\exp(-2\Gamma(L-z)), \tag{47}$$

where N_{sf} and N_{wf} are the photon number at the end of the interaction region. As a result, the depletion rate is maximum at the end of the interaction region, and occurs on a length $(2\Gamma)^{-1} \ll L$, which can possibly be much shorter than the length of the whole device. This is illustrated in Fig. 1.

C. Pulsed Beam

An alternative picture emerges when the electron beam is emitted in short pulses of length L_p. As the pulse propagates and as the radiation field grows, the pump depletion increases along the device, to reach a maximum at the end of the interaction region. The wiggler equation (41) is simplified with the following assumptions: first, the pulse travels in the z-direction with uniform velocity v, and consequently $\omega_{p0}^2(z,t) = \omega_{p0}^2(vt-z)$; second, the FEL interaction strength for an unsaturated beam is estimated to be $\langle \exp(-i\Theta)/\gamma s' \rangle \propto 2ia_w^{(0)}a_s$; third, the radiation field grows exponentially and travels along z with no slippage with respect to the electron pulse, i.e., is also of the form $|\mathbf{a}_s|^2 = \exp(2\Gamma z) f(vt-z)$, where f is some function depending on the relative position with respect to the head of the pulse. Thus the wiggler equation becomes

$$2ik_w \frac{\partial \mathbf{a}_w}{\partial z} - 2i\omega_w \frac{\partial \mathbf{a}_w}{\partial t} = S(vt-z)a_w^{(0)} e^{2\Gamma z}, \tag{48}$$

where S depends on the pulse shape, and with boundary condition $\mathbf{a}_w = \mathbf{a}_{w0}$ for $z > vt$.

This equation can be solved in two limits. It is useful to perform a change of coordinates, from (z,t) to (z,ζ), where the relative coordinate $\zeta = vt - z$ is the distance with respect to the head of the pulse. In the new system of coordinates, Eq. (48) becomes

$$k_w \frac{\partial \mathbf{a}_w}{\partial z} - (k_w + v\omega_w)\frac{\partial \mathbf{a}_w}{\partial \zeta} = S(\zeta)e^{2\Gamma z}. \tag{49}$$

The first case is the limit of long pulse, when $\Gamma L_p \gg 1$ and when $\partial/\partial\zeta \ll \partial/\partial z$. Equation (49) reduces then to

$$2ik_w \frac{\partial \mathbf{a}_w}{\partial z} = S(\zeta) a_w^{(0)} e^{2\Gamma z}\,, \tag{50}$$

from which one recovers the steady state solution (47).

The second case is the limit of short pulse, when $\Gamma L_p \ll 1$. One has now $\partial/\partial z \ll \partial/\partial\zeta$, and the solution of (49) is

$$\mathbf{a}_w(z,\zeta) \approx a_w^{(0)} - a_w^{(0)} \frac{\exp(2\Gamma z)}{2i(k_w + v\omega_w)} \int_0^\zeta S(\zeta')d\zeta'\,. \tag{51}$$

It results that the depletion increases exponentially, like $\exp(2\Gamma z)$, as the pulse propagates. Is is maximum in the last e-folding, as shown on Fig. 2. In the three-dimensional analysis, we shall see how large depletion rate localized to one Rayleigh length reduces the field only locally, within the electron beam, therefore with increased result.

IV. PUMP DIFFRACTION

In this section the diffraction of the wiggler wave induced by the electron beam is studied by representing the wiggler wave with a slowly varying amplitude and phase. This permits examination of the physically important terms which govern the diffraction of the wiggler field.

The derivation begins by taking the dot product of Eq. (28) with $\mathbf{a}_w^*$ and finding

$$\begin{aligned}(\omega_w^2 - k_w^2)|\mathbf{a}_w|^2 - \mathbf{a}_w^* \cdot \nabla_\perp \times (\nabla_\perp \times \mathbf{a}_w) + ik_w \mathbf{a}_w^* \cdot \frac{\partial}{\partial z}\mathbf{a}_w + i\mathbf{a}_w^* \cdot \frac{\partial}{\partial z}(k_w \mathbf{a}_w)\\ - i\omega_w \mathbf{a}_w^* \cdot \frac{\partial}{\partial t}\mathbf{a}_w - i\mathbf{a}_w^* \cdot \frac{\partial}{\partial t}(\omega_w \mathbf{a}_w)\\ = \omega_{p0}^2(\mathbf{x}_\perp, t, z)\left[\left\langle \frac{1}{\gamma s'}\right\rangle |\mathbf{a}_w|^2 + \left\langle \frac{\exp(-i\Theta)}{\gamma s'}\right\rangle \mathbf{a}_w^* \cdot \mathbf{a}_s^*\right].\end{aligned} \tag{52}$$

Multiplication of the imaginary part of Eq. (52) by ω_w results in

$$\nabla_\perp \cdot \mathrm{Im}[\mathbf{a}_w^* \times (\nabla_\perp \times \mathbf{a}_w)\omega_w] + \mathrm{Re}[k_w\omega_w \mathbf{a}_w^* \cdot \frac{\partial}{\partial z}\mathbf{a}_w + \omega_w \mathbf{a}_w^* \cdot \frac{\partial}{\partial z}(k_w \mathbf{a}_w)]$$
$$- \mathrm{Re}[\omega_w^2 \mathbf{a}_w^* \cdot \frac{\partial}{\partial t}\mathbf{a}_w + \omega_w \mathbf{a}_w^* \cdot \frac{\partial}{\partial t}(\omega_w \mathbf{a}_w)]$$
$$= -\omega_{p0}^2(\mathbf{x}_\perp, t, z)\omega_w \,\mathrm{Im}\left\langle \frac{\exp(i\Theta)}{\gamma s'} \right\rangle \mathbf{a}_w \cdot \mathbf{a}_s \,. \tag{53}$$

Equation (53) can be interpreted as a relation between the divergence of the Poynting vector, derivatives of the energy density, and the current sources. The first term of Eq. (53) is the divergence of the radial Poynting flux, and the second and third terms correspond to $k_w\partial/\partial z - \omega_w\partial/\partial t$ acting on the energy density. On the right hand side of Eq. (53) are the synchronous current sources.

The Poynting vector can be simplified by expressing the complex fields $\mathbf{a}_w$, $\mathbf{a}_s$ in terms of real, slowly varying, amplitudes and phases. With a choice of helical polarization, the slowly varying amplitude and phase of the wiggler and signal fields are defined by

$$\mathbf{a}_w = \frac{a_w}{\sqrt{2}}(\hat{x} + i\hat{y})e^{i\phi_w} \,,$$
$$\mathbf{a}_s = \frac{a_s}{\sqrt{2}}(\hat{x} - i\hat{y})e^{i\phi_s} \,. \tag{54}$$

Using the relation $(\hat{x} + i\hat{y})(\hat{x} - i\hat{y}) = \mathrm{I}_\perp + i\hat{z} \times \mathrm{I}_\perp$, where $\mathrm{I}_\perp$ is the unit tensor perpendicular to $\hat{z}$, it can be shown that

$$\mathrm{Im}\left(\mathbf{a}_w^* \times (\nabla_\perp \times \mathbf{a}_w)\right) = \frac{1}{2}a_w^2 \nabla_\perp \phi_w + \frac{1}{4}\nabla_\perp \times (a_w^2 \hat{z}) \,. \tag{55}$$

Combining Eqs. (53) and (55), the amplitude of the wiggler wave is seen to satisfy

$$-\nabla_\perp \cdot (k_w a_w^2 \nabla_\perp \phi_w) - k_w^2 \frac{\partial}{\partial z}a_w^2 - k_w a_w^2 \frac{\partial}{\partial z}k_w + k_w\omega_w \frac{\partial}{\partial t}a_w^2 + k_w a_w^2 \frac{\partial}{\partial t}\omega_w$$
$$= \omega_{p0}^2(\mathbf{x}_\perp, t, z)k_w \left\langle \frac{\sin(\theta + \phi_w + \phi_s)}{\gamma s'} \right\rangle a_w a_s \,.$$

(56)

The real part of Eq. (52) yields an equation for the evolution of the wiggler phase. With the circular polarization, one finds

$$
\begin{aligned}
(\omega_w^2 - k_w^2)a_w^2 + \nabla_\perp \cdot \mathrm{Re}(\mathbf{a}_w^* \times (\nabla_\perp \times \mathbf{a}_w)) - |\nabla_\perp \times \mathbf{a}_w|^2 \\
- 2k_w a_w^2 \frac{\partial}{\partial z}\phi_w - 2\omega_w a_w^2 \frac{\partial}{\partial t}\phi_w \\
= \omega_{p0}^2(\mathbf{x}_\perp, t, z)\left[\left\langle \frac{1}{\gamma s'} \right\rangle a_w^2 + \left\langle \frac{\cos(\Theta + \phi_w + \phi_s)}{\gamma s'} \right\rangle a_w a_s \right].
\end{aligned}
\tag{57}
$$

With the identities

$$
|\nabla_\perp \times \mathbf{a}_w|^2 = \frac{1}{2}|\nabla_\perp a_w|^2 + \frac{1}{2}a_w^2|\nabla_\perp \phi_w|^2 + a_w(\nabla_\perp \phi_w \times \nabla_\perp a_w)\cdot \hat{z} \tag{58}
$$

and

$$
\nabla_\perp \cdot \mathrm{Re}(\mathbf{a}_w^* \times (\nabla_\perp \times \mathbf{a}_w)) = \frac{1}{2}|\nabla_\perp a_w|^2 + \frac{1}{2}a_w \nabla_\perp^2 a_w + a_w(\nabla_\perp \phi_w \times \nabla_\perp a_w)\cdot \hat{z}\,, \tag{59}
$$

Equation (57) can be rearranged to yield, after division by $2a_w^2 k_w^2$,

$$
\begin{aligned}
(\frac{1}{k_w}\frac{\partial}{\partial z} - \frac{\omega_w}{k_w}\frac{\partial}{\partial t})\phi_w = \frac{(\omega_w^2 - k_w^2)}{2k_w^2} - \frac{\omega_{p0}^2(\mathbf{x}_\perp, t, z)}{2k_w^2}\left\langle \frac{1}{\gamma s'} \right\rangle + \frac{\nabla_\perp^2 a_w}{4k_w^2 a_w} - \frac{|\nabla_\perp \phi_w|^2}{4k_w^2} \\
- \frac{\omega_{p0}^2(\mathbf{x}_\perp, t, z)}{2k_w^2}\left\langle \frac{\cos(\Theta + \phi_w + \phi_s)}{\gamma s'} \right\rangle \frac{a_s}{a_w}.
\end{aligned}
\tag{60}
$$

The equation for the slowly varying amplitude of the signal field can be derived from Eq. (29) in a similar manner to the derivation of Eq. (56). The result is

$$
\begin{aligned}
-\nabla_\perp \cdot (k_s a_s^2 \nabla_\perp \phi_s) - k_s^2 \frac{\partial}{\partial z} a_s^2 - k_s a_s^2 \frac{\partial}{\partial z} k_s - k_s \omega_s \frac{\partial}{\partial t} a_s^2 - k_s a_s^2 \frac{\partial}{\partial t}\omega_s \\
= \omega_{p0}^2(\mathbf{x}_\perp, t, z) k_s \left\langle \frac{\sin(\theta + \phi_w + \phi_s)}{\gamma s'} \right\rangle a_w a_s\,.
\end{aligned}
\tag{61}
$$

The phase equation for the signal field is similarly found to be

$$(\frac{1}{k_s}\frac{\partial}{\partial z}+\frac{\omega_s}{k_s}\frac{\partial}{\partial t})\phi_s=\frac{(\omega_s^2-k_s^2)}{2k_s^2}-\frac{\omega_{p0}^2(\mathbf{x}_\perp,t,z)}{2k_s^2}\left\langle\frac{1}{\gamma s'}\right\rangle+\frac{\nabla_\perp^2 a_s}{4k_s^2 a_s}-\frac{|\nabla_\perp\phi_s|^2}{4k_s^2}$$
$$-\frac{\omega_{p0}^2(\mathbf{x}_\perp,t,z)}{2k_s^2}\left\langle\frac{\cos(\Theta+\phi_w+\phi_s)}{\gamma s'}\right\rangle\frac{a_w}{a_s}. \tag{62}$$

When, as is usually the case, the wiggler and signal field have fixed input frequencies and wavelengths, the terms in $\partial\omega_{s,w}/\partial t$ or $\partial k_{s,w}/\partial z$ do not appear in the amplitude and phase equations.

In the following discussion of the order of magnitude of the current sources the beam is assumed to be steady-state. Therefore, any time variation of ϕ_s and ϕ_w is neglected.

The phase shift induced during the FEL interaction (the term proportional to $(a_w/a_s)\langle\cos(\theta+\phi_s+\phi_w)/\gamma s'\rangle$ in Eq. (62)) has been shown to produce optical guiding of the signal field. The FEL interaction causes a wave phase shift which slows the wave phase velocity inside the electron beam. The signal wave can then be regarded as propagating along an optical fiber whose index of refraction is that produced by the FEL interaction. The signal field has a axially invariant transverse profile, and thus does not diffract away from the electron beam. In addition to the FEL-induced phase shifts, which are driven by the resonant electrons which bunch in the ponderomotive potential, there is also a phase shift from the unbunched electrons. This phase shift (the term proportional to $\langle 1/\gamma s'\rangle$ in Eq. (62)) tends to diffract the signal away from the electron beam, but it is much smaller than the refractive phase shift which results from the FEL interaction.

Transverse wiggler field profile modifications also occur. From Eq. (57), the phase shifts of the wiggler wave due to the electron beam are

$$\left.\frac{1}{k_w}\frac{\partial}{\partial z}\phi_w\right|_{\mathrm{FEL}}=-\frac{\omega_{p0}^2(\mathbf{x}_\perp,t,z)}{2k_w^2}\left\langle\frac{\cos(\Theta+\phi_w+\phi_s)}{\gamma s'}\right\rangle\frac{a_s}{a_w} \tag{63}$$

for the FEL-induced phase shift, and

$$\left.\frac{1}{k_w}\frac{\partial}{\partial z}\phi_w\right|_{\text{e-beam}} = -\frac{\omega_{p0}^2(\mathbf{x}_\perp, t, z)}{2k_w^2}\left\langle\frac{1}{\gamma s'}\right\rangle . \tag{64}$$

for the usual electron beam phase shift.

The ratio of the FEL-induced wiggler phase shift to the unbunched beam phase shift is approximately $a_s/a_w \ll 1$. Thus the FEL interaction, which would tend to guide the wiggler field, exerts a smaller influence on the transverse wave profile than the beam would experiences propagating, unbunched, through a an electromagnetic wave. This is in contrast to the signal field, where the ratio of the FEL-induced phase shift to the unbunched beam phase shift is approximately $a_w/a_s \gg 1$. Thus the wiggler field is defocused by the oscillating electron beam.

Any transverse variation of k_w across the electron beam might result in a loss of FEL gain. Indeed, a major motive for utilizing electromagnetic wigglers is the separation of the scale for transverse wiggler variation from λ_w and the beam radius, r_e. The electron beam induces a phase shift, $\phi_w(t, \mathbf{x}_\perp, z)$, with an amplitude that depends on the current density. When the electron density can be assumed to vary as $vt - z$, the wiggler phase shift becomes $\phi_w(vt - z)$, and its contribution to the evolution of the ponderomotive phase is seen to vanish, since $\partial\phi_w/\partial z + s'\partial\phi_w/\partial t = 0$, with $s' = 1/v$.

In the following section the wiggler field profile is solved for perturbatively. From the real and imaginary parts of the field perturbation the pump depletion and induced phase shifts are estimated.

The relative importance of the FEL bunching on the wiggler and signal fields can be seen by forming the ratio

$$\frac{\mathrm{Re}(n_w - 1)|_{\mathrm{FEL}}}{\mathrm{Re}(n_s - 1)|_{\mathrm{FEL}}} = \frac{\frac{1}{k_w}\frac{\partial}{\partial z}\phi_w|_{\mathrm{FEL}}}{\frac{1}{k_s}\frac{\partial}{\partial z}\phi_s|_{\mathrm{FEL}}} \approx \frac{a_s^2 k_s^2}{a_w^2 k_w^2} = \frac{B_s^2}{B_w^2} , \tag{65}$$

where n_w and n_s are the indices of refraction of, respectively the wiggler and signal fields. Thus the ratio of the indices of refraction scale as the ratio of energy densities, which can be of order unity. A better figure of merit for guiding properties of the beam is, however, the fiber parameter $V^2 = k^2 r_e^2(n^2 - 1)$. From Eq. (65),

$$\frac{\mathrm{Re}\, V_w^2|_{\mathrm{FEL}}}{\mathrm{Re}\, V_s^2|_{\mathrm{FEL}}} = \frac{a_s^2}{a_w^2} \ll 1 \,,$$

where V_w and V_s are the effective fiber parameters for, respectively the wiggler and signal fields. Comparing the dominant diffractive term for the wiggler, that from the unbunched electrons, with the usual optical guiding term for the signal field, one finds

$$\frac{\mathrm{Re}\, V_w^2|_{\mathrm{e-beam}}}{\mathrm{Re}\, V_s^2|_{\mathrm{FEL}}} = \frac{a_s}{a_w}$$

which is still small compared to unity. Overall, due to $k_w \ll k_s$, the transverse profile of the FEL signal wave is influenced by the electrons more than is the profile of the wiggler wave.

V. SIMPLIFIED MODEL OF PUMP DIFFRACTION

A complete study of the transverse wiggler field profile would require the numerical solution of Eq. (28) for the various electron beam profiles and wiggler geometries. Physical insight may be gained by analytically solving a simpler form of Eq. (28) in which the FEL interaction is neglected. Representing the dielectric modification due to the unbunched electrons with the coefficient $S(vt - z, \mathbf{x}_\perp, z) \approx \omega_{p0}^2/\langle \gamma s' \rangle$, and assuming vacuum propagation ($\omega_w = k_w$) and no field tapering, one has

$$-\nabla_\perp \times (\nabla_\perp \times \mathbf{a}_w) + 2ik_w \frac{\partial}{\partial z}\mathbf{a}_w - 2i\omega_w \frac{\partial}{\partial t}\mathbf{a}_w = S(\zeta, \mathbf{x}_\perp, z)\mathbf{a}_w \,, \qquad (66)$$

where $\zeta = vt - z$ is the distance relative to the head of the pulse. Equation (66) is simplified further by the choice of linear polarization, the approximation $\nabla_\perp \cdot \mathbf{a}_w = 0$, and the expansion $\mathbf{a}_w = a_w^{(0)} + a_w^{(1)}(\zeta, \mathbf{x}_\perp, z) + \ldots$, where the zero'th order wiggler field, $a_w^{(0)}$, has been assumed to be a plane wave. With the above assumptions, the solution for $\mathbf{a}_w^{(n)}$, the n^{th} term in the expansion of $\mathbf{a}_w$, can be found iteratively, by approximating the right side of the equation by $S\,\mathbf{a}_w^{(n-1)}$. To first order,

$$\nabla_\perp^2 a_w^{(1)} + 2ik_w \frac{\partial}{\partial z} a_w^{(1)} - 2i(k_w + v\omega_w)\frac{\partial}{\partial \zeta} a_w^{(1)} = S(\zeta, \mathbf{x}_\perp, z) a_w^{(0)} , \qquad (67)$$

with the wiggler field perturbation vanishing in front of the beam, i.e., $a_w^{(1)} = 0$ for $\zeta \leq 0$. Equation (67) may be solved exactly by Laplace transform method, for a profile of the form

$$S(\zeta, \mathbf{x}_\perp, z) = S_0(\mathbf{x}_\perp) H(\zeta) \mathrm{erfc}(\alpha^{1/2}/2\zeta^{1/2}) , \qquad (68)$$

where erfc is the error function, $H(\zeta)$ is the Heavyside step function, and where α determines the rise time of the electron pulse.

The calculation may be found in Appendix A for the analysis in a slab geometry, where there is no explicit y-dependence in the sources or fields. The results of Appendix A are summarized below.

With a radial profile

$$S_0(x) = \begin{cases} S_0 , & \text{if} \quad |x| < r_e \ ; \\ 0 , & \text{if} \quad |x| > r_e \ , \end{cases} \qquad (69)$$

one calculates the first order correction to the wiggler field $a_w^{(1)}$ at the center and edge of the beam, $x = 0$ and $x = r_e$. The diffraction of the wiggler field causes both a reduction in field amplitude within the beam, given by $\mathrm{Re}\, a_w^{(1)}/a_w^{(0)}$, and a radially dependent phase shift, approximately equal to $\mathrm{Im}\, a_w^{(1)}/a_w^{(0)}$. The results are shown on Fig. 3.

For distances far back in the pulse, asymptotic solutions are found in Appendix A. It is shown that

$$\frac{\partial}{\partial \zeta} a_w^{(1)}(x, \zeta \to \infty) = 0 , \qquad (70)$$

i.e., that the wavenumber perturbation in the beam decreases tends towards zero. Although the total phase shift and amplitude reduction keep on increasing,

$$a_w^{(1)}(x, \zeta \to \infty) \to \infty , \qquad (71)$$

the relative phase shift across the beam tends to zero, while the amplitude reduction across the beam remains finite,

$$a_w^{(1)}(r_e, \zeta \to \infty) - a_w^{(1)}(0, \zeta \to \infty) \to \frac{S_0 a_w^{(0)} r_e^2}{2} \, . \tag{72}$$

Since

$$\frac{\partial}{\partial z} a_w^{(1)}(x, t, z) = -\frac{\partial}{\partial \zeta} a_w^{(1)}(x, \zeta) \, , \tag{73}$$

equation (70) shows that far back in the pulse the wiggler adjusts to the electron beam in such a way that these gradients vanish. For pulses with $L_p \gg \zeta_0 = (k_w + v\omega_w) r_e^2$, the variation of ϕ_w across the beam is insignificant except at the front of the pulse $(\zeta < \zeta_0)$.

Even for high brightness accelerators, we estimate that this amplitude reduction is small, less than 0.5%.

VI. CONCLUSIONS

An action principle formalism has been developed to study the self-consistent evolution of the wiggler and signal fields. The equations are derived from a total lagrangian, which includes the relativistic particle lagrangians and the electromagnetic field lagrangian. The action is then specialized for the FEL to include wiggler, radiation, and space charge waves, and is averaged over the wiggler period. This method requires only one averaging, while in the standard treatment each of the equations of motion must be averaged separately. This formulation is compact and automatically self-consistent. The averaged action can then be varied to yield the equations of motion for the particles and fields.

The self-consistent interaction with the wiggler wave is found to complicate the FEL dynamics. The pump can suffer depletion, reducing a_w, and it can suffer diffraction, generating an effective energy spread. In both instances the FEL performance is not substantially degraded.

Other questions remain to be studied. Many electromagnetic wiggler designs envision using a high Q cavity. Each time the beam of an RF linac passes through the cavity, the field undergoes a slight phase shift. The cavity is thus momentarily detuned, but must recover before the next RF pulse arrives. There are severe constraints on the mode purity in the cavity, and the periodic phase shift and depletion induced when the beam propagates through the wiggler may provide a mechanism for mode coupling. In the analysis in Secs. III-V, the depletion and diffraction were evaluated separately. This restriction can eliminated by numerically solving the coupled equations of motion for the slow variation of the wiggler and signal fields.

The pump depletion and diffraction during the FEL interaction is a rich set a phenomena which needs further investigation as part of the quest for compact, inexpensive free electron lasers.

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APPENDIX

In this appendix, we solve the wiggler field equation, when a pulsed electron beam propagates and diffracts the electromagnetic wave. Equation (67) is, in the slab approximation and in the short pulse limit $(\partial/\partial z = 0)$,

$$\frac{\partial^2}{\partial x^2} a_w^{(1)} - 2i(k_w + v\omega_w)\frac{\partial}{\partial \zeta} a_w^{(1)} = S(\zeta, x) a_w^{(0)} , \tag{74}$$

where the pulse is an error function rising on a scale α in the axial direction, and a step function radially bound to r_e,

$$S(\zeta, x) = S_0(x) H(\zeta) \mathrm{erfc}\left(\frac{\alpha^{1/2}}{2\zeta^{1/2}}\right) , \tag{75}$$

$$S_0(x) = \begin{cases} S_0 , & \text{if} \quad |x| < r_e \ ; \\ 0 , & \text{if} \quad |x| > r_e \ . \end{cases} \tag{76}$$

The boundary condition is such that the wiggler field is undisturbed ahead of the electron pulse, i.e., $a_w^{(1)} = 0$ for $\zeta < 0$.

The solution proceeds by Laplace transform in ζ, naming p the new Laplace variable. Then $a_w^{(1)}(\zeta, x)$ is transformed into $\tilde{a}(p, x)$, and Eq. (74) into

$$\frac{\partial^2}{\partial x^2} \tilde{a} - 2i(k_w + v\omega_w) p \tilde{a} = S_0(x) a_w^{(0)} \frac{1}{p} e^{-\sqrt{\alpha p}} . \tag{77}$$

This differential equation in x can be solved, with the condition that $\tilde{a}(p, x)$ vanishes for large x. One finds

$$\tilde{a}(p, x) = -\frac{S_0 a_w^{(0)}}{p\lambda^2} \left(1 - e^{-\lambda r_e} \mathrm{ch}(\lambda x)\right) e^{-\sqrt{\alpha p}} , \tag{78}$$

for $|x| < r_e$, and

$$\tilde{a}(p, x) = -\frac{S_0 a_w^{(0)}}{p\lambda^2} \mathrm{sh}(\lambda r_e) e^{-\lambda |x|} e^{-\sqrt{\alpha p}} , \tag{79}$$

for $|x| > r_e$, where one has defined $\lambda(p) = \left[2i(k_w + v\omega_w)p\right]^{1/2}$ with $\mathrm{Re}(\lambda) \geq 0$.

In particular, at the beam center $(x = 0)$ and edge $(x = r_e)$, the solution gives

$$\tilde{a}(p, 0) = -\frac{S_0 a_w^{(0)}}{p\lambda^2} \left(1 - e^{-\lambda r_e}\right) e^{-\sqrt{\alpha p}} , \tag{80}$$

and

$$\tilde{a}(p, r_e) = -\frac{S_0 a_w^{(0)}}{p\lambda^2}\left(\frac{1 - e^{-2\lambda r_e}}{2}\right) e^{-\sqrt{\alpha p}}\,. \tag{81}$$

Replacing λ by its value and performing the inverse Laplace transform yields

$$a_w^{(1)}|_{x=0} = \frac{-S_0 a_w^{(0)}}{2i(k_w + v\omega_w)} \int \frac{dp}{2i\pi}\, e^{p\zeta} \left(\frac{e^{-\sqrt{\alpha p}}}{p^2} - \frac{e^{-\left(\sqrt{\alpha}+\sqrt{2i\zeta_0}\right)\sqrt{p}}}{p^2}\right) \tag{82}$$

at $x = 0$ and

$$a_w^{(1)}|_{x=r_e} = \frac{-S_0 a_w^{(0)}}{2i(k_w + v\omega_w)} \int \frac{dp}{2i\pi}\, e^{p\zeta} \left(\frac{e^{-\sqrt{\alpha p}}}{2p^2} - \frac{e^{-\left(\sqrt{\alpha}+2\sqrt{2i\zeta_0}\right)\sqrt{p}}}{2p^2}\right) \tag{83}$$

at $x = r_e$, where $\zeta_0 = (k_w + v\omega_w)r_e^2$ equals, for $v = c$ and $k_w = \omega_w$, four times the Rayleigh range of the wiggler wave computed for a focused spot size equal to the electron beam radius.

The $1/p^2$ can be eliminated by differentiation and the integrals may be performed to yield

$$\frac{\partial^2}{\partial\zeta^2} a_w^{(1)}|_{x=0} = \frac{-S_0 a_w^{(0)}\sqrt{\alpha}}{4i(k_w + v\omega_w)\sqrt{\pi}\zeta^{3/2}} \times \left(e^{-\alpha/4\zeta} - \frac{\sqrt{\alpha}+\sqrt{2i\zeta_0}}{\sqrt{\alpha}} e^{-\left(\sqrt{\alpha}+\sqrt{2i\zeta_0}\right)^2/4\zeta}\right) \tag{84}$$

and

$$\frac{\partial^2}{\partial\zeta^2} a_w^{(1)}|_{x=r_e} = \frac{-S_0 a_w^{(0)}\sqrt{\alpha}}{4i(k_w + v\omega_w)\sqrt{\pi}\zeta^{3/2}} \times \left(e^{-\alpha/4\zeta} - \frac{\sqrt{\alpha}+2\sqrt{2i\zeta_0}}{\sqrt{\alpha}} e^{-\left(\sqrt{\alpha}+2\sqrt{2i\zeta_0}\right)^2/4\zeta}\right). \tag{85}$$

The solution of these equations is shown on Fig. 3.

Asymptotic values are obtained using the following property of Laplace transforms:

$$\lim_{p\to 0^+} pf(p) = F|_{\zeta=\infty}\,, \tag{86}$$

where $f(p)$ is the Laplace transform of $F(\zeta)$. Applied to $F = a_w^{(1)}$, $F = a_w^{(1)}|_{x=r_e} - a_w^{(1)}|_{x=0}$, and $F = \partial a_w^{(1)}/\partial\zeta$ respectively, this relation yields

$$a_w^{(1)}(x, \zeta \to \infty) = \lim_{p\to 0} p\tilde{a} = \infty\,, \tag{87}$$

$$a_w^{(1)}(r_e, \zeta \to \infty) - a_w^{(1)}(0, \zeta \to \infty) = \lim_{p \to 0} p(\tilde{a}|_{x=r_e} - \tilde{a}|_{x=0}) = \frac{S_0 a_w^{(0)} r_e^2}{2}, \tag{88}$$

and

$$\frac{\partial}{\partial \zeta} a_w^{(1)}(x, \zeta \to \infty) = \lim_{p \to 0} p^2 \tilde{a} = 0. \tag{89}$$

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FIGURE CAPTIONS

FIG. 1. Wiggler field amplitude versus position in a steady state, one-dimensional model of the FEL; the depletion occurs over the last Γ^{-1} of the interaction region, where Γ is the signal growth rate.

FIG. 2. Wiggler field amplitude versus position in a pulsed one-dimensional model of the FEL; the electron pulse and wiggler profile are shown at two different times t_1 and t_2, as the pulse propagates through the device. Maximum depletion occurs at the end of the interaction region.

FIG. 3. Wiggler field perturbation in a three-dimensional model of the FEL. The electron pulse shape is shown in Fig. 3a, where ζ measures the distance behind the head of the pulse and is normalized to the pulse rising length α. Phase shift $\mathrm{Im}\, a_w^{(1)}/a_w^{(0)}$, normalized to $\alpha S_0 r_e^2/2\zeta_0$, as a function of ζ is plotted in Fig. 3b, at the beam center (1) and beam edge (2). On the same figure, the amplitude reduction factor $\mathrm{Re}\, a_w^{(1)}/a_w^{(0)}$ is identically normalized, and is plotted at the beam center (3) and beam edge (4). In Fig. 3b, the beam radius is chosen such that $\zeta_0 = 2\alpha$.

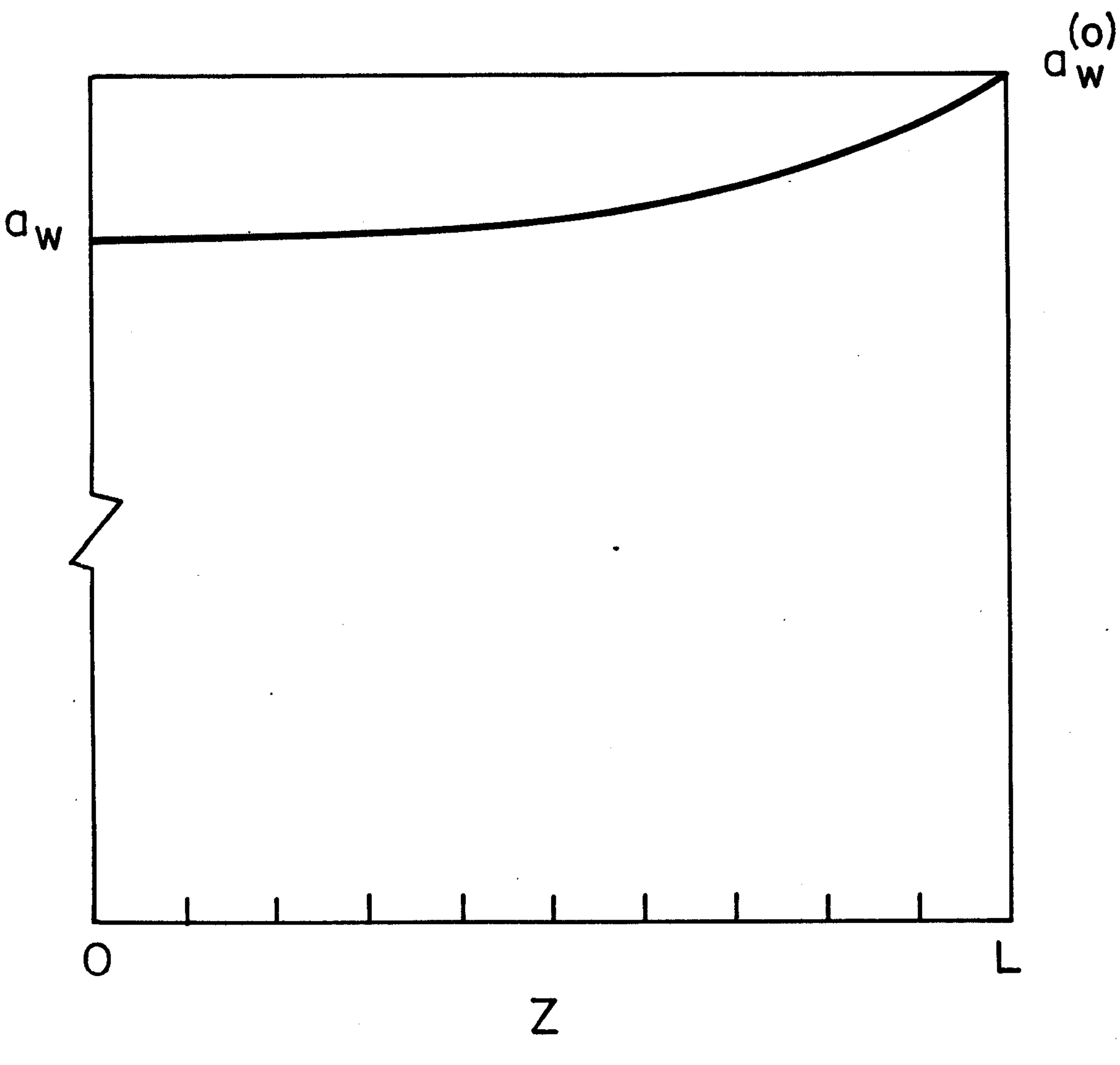

Fig. 1

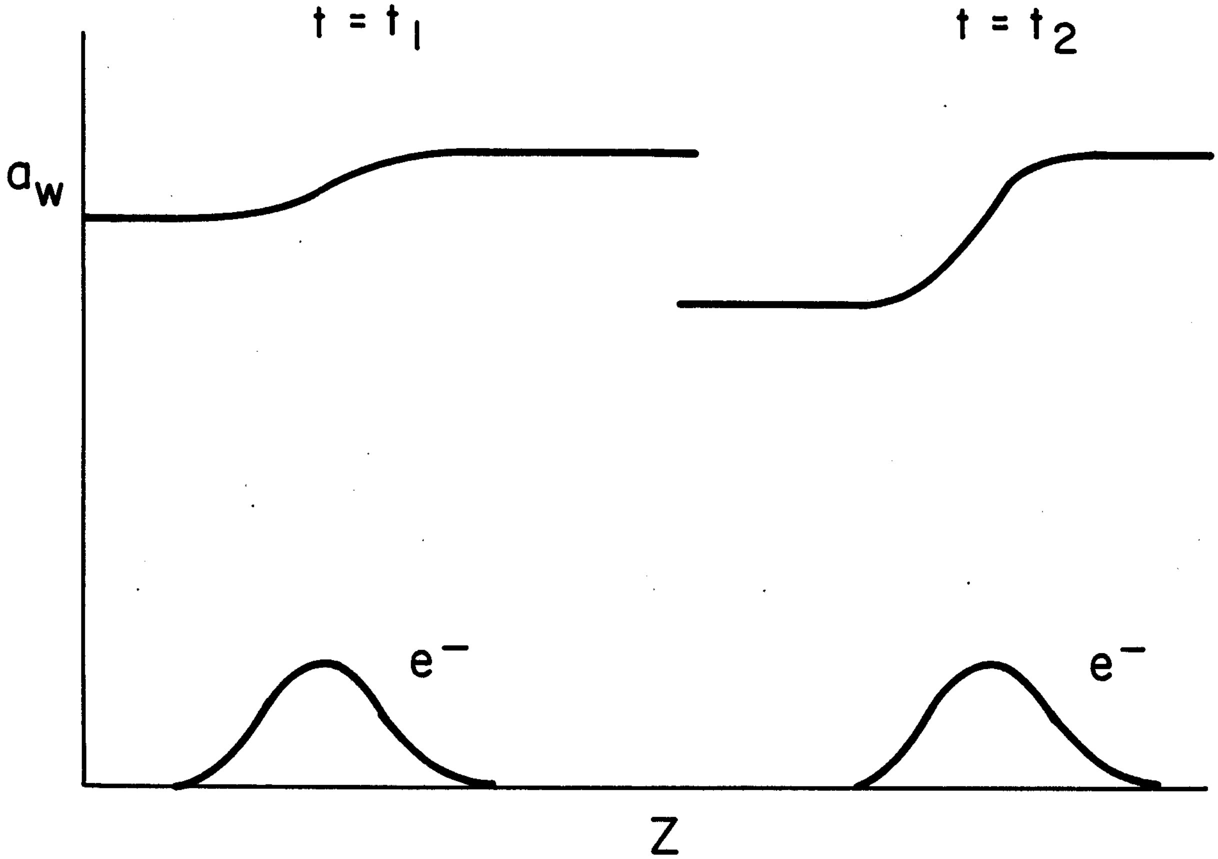

Fig. 2

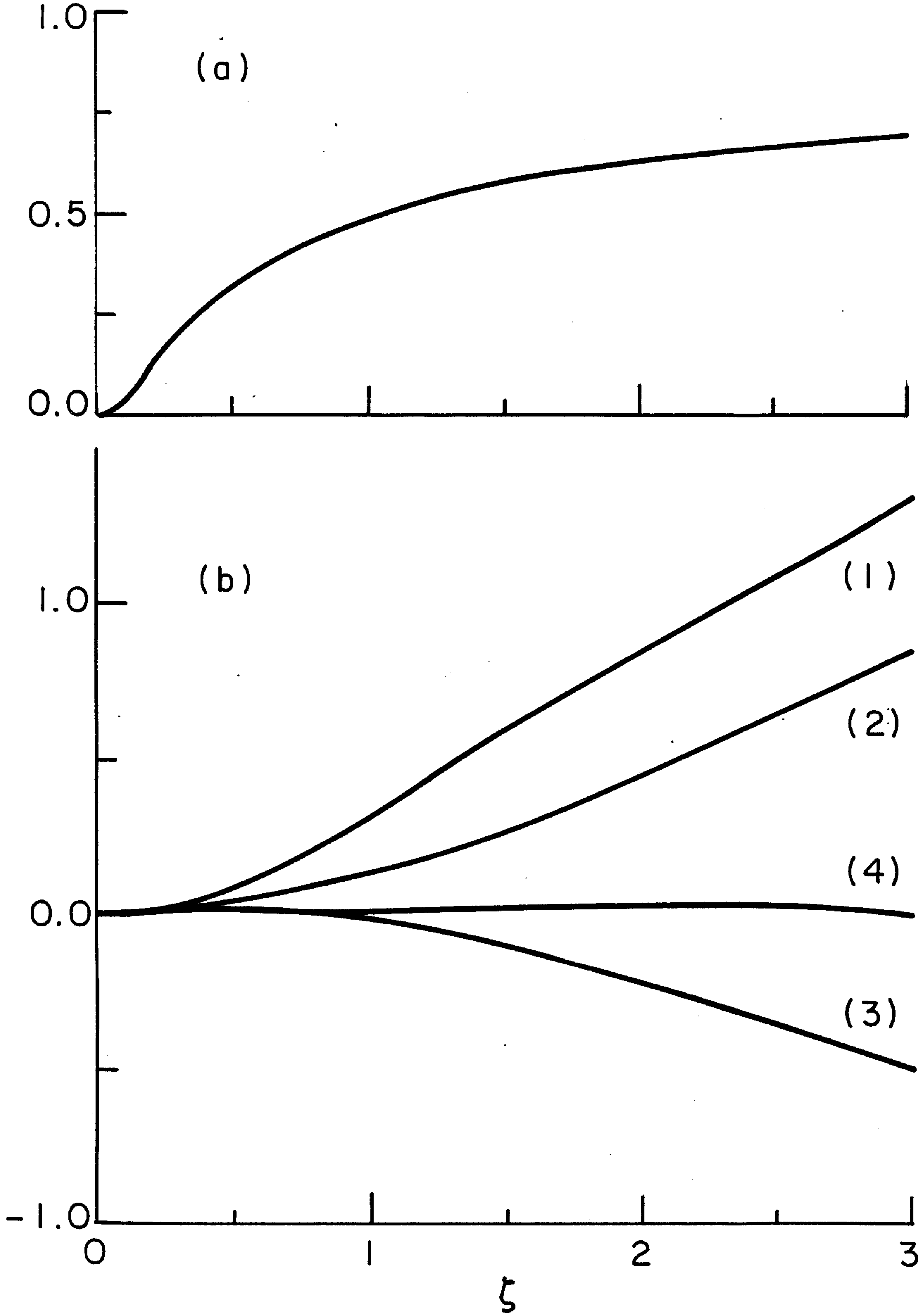

Fig. 3