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A THEORY OF FAST WAVE ABSORPTION, TRANSMISSION
AND REFLECTION IN THE ION CYCLOTRON RANGE OF FREQUENCIES.

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Abstract

A second-order differential equation for the fast wave propagating in a hot, two ion species plasma is obtained. This second-order approximation is obtained unambiguously and allows the wave amplitude to be identified with one of the electric field components. The approximation, is based on replacing the coupling to the ion Bernstein wave by a localized perturbation of the fast wave. For the case of perpendicular propagation, the second order equation reduces to Budden's equation giving the well known transmission coefficient for both two ion hybrid and second harmonic resonance. The equation includes the effect of simultaneous minority fundamental and majority second harmonic cyclotron damping. The solutions of the second order equation as a function of n_{\parallel} give absorption transmission and reflection coefficients which agree well with the results based on models giving higher order differential equations and solved by means of much more complex numerical codes.

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I. INTRODUCTION

Plasma heating in tokamaks by means of RF power coupled to the fast-wave in the ion cyclotron range of frequencies is now a well established method.⁽¹⁻³⁾ In spite of the success of this method the physics, underlying the absorption, transmission and reflection of the incident (i.e. fast) wave in this frequency range is probably more involved than for any other RF method. These phenomena, which occur in the core of the plasma, involve both mode-conversion by coupling to ion-Bernstein waves and dissipation through cyclotron damping on the various ion species and Landau as well as transit time damping on electrons: in general these effects occur simultaneously.

In a global sense (i.e. over the cross section of the plasma core the incident fast-wave power is partially reflected (for low field side incidence) and partially transmitted; the remaining power goes to mode-conversion and dissipation (ion cyclotron and electron Landau damping). On either side of the central core of the plasma the fast-wave has a simple description given, approximately, by a cold-plasma model. On the other hand, inside the central core of the plasma, where mode conversion and dissipation take place, the dynamics must be described by the much more complex Vlasov kinetic plasma model. A first simplification is therefore obtained by seeking a description of the fast wave transmission and reflection in the presence of the combined effects of mode-conversion and dissipation. As far as the fast-wave is concerned, at the boundaries of the central core where its incident, reflected and transmitted power flows are well defined, the combined effects of mode-conversion and dissipation appear as power "absorbed" within these boundaries. In this manner the description of the fast-wave entails establishing a second order ordinary differential equation which contains an appropriate representation of this power "absorbed" inside its boundaries and which matches asymptotically to fast-wave propagation outside these boundaries. Such a simplification, in addition to reducing the order of the wave-propagation equations would also benefit considerations of scaling.

In this paper we shall attempt to construct such a global description of fast wave heating in a two ion species plasma. The physical effects we must include in this analysis are of course coupling to ion-Bernstein waves and the collisionless damping at the fundamental resonance of the minority species and at the second harmonic resonance of the majority species. We shall ignore the rotational transform (poloidal magnetic field) and electron Landau damping. These two effects are related and we shall comment further on their neglect later.

The plan of the paper is as follows. In Section II we discuss the reduction of the general electromagnetic Vlasov-Maxwell dispersion relation to the approximate form on which our theory will be based. Section III gives an account of the manipulation of the approximate dispersion relation for the case of pure second harmonic into coupled mode form which explicitly exhibits the coupling of the fast wave to the ion Bernstein wave and the presence of ion cyclotron damping. The desired second order approximation is then obtained which we call the fast-wave approximation. In Section IV we generalize the fast-wave approximation to the case of a two ion species plasma and by considering the resulting fast-wave refractive index show how it includes the effects of coupling to the ion Bernstein wave as well as ion cyclotron damping. In Section V, with the aid of this fast-wave approximation, we generate a second order full wave equation for the fast-wave. We show that in the limit of $k_{\parallel} \rightarrow 0$ this reduces to Budden's equation giving the correct transmission coefficient for both second harmonic and ion-ion hybrid cases. In Section VI we obtain a conservation relation from the wave equation and show how this enables us to identify the power 'absorbed' (i.e. the combined effects of mode-conversion and dissipation). In Section VII we give some results of our theory, comparing with other more involved codes, for various situations of current experimental interest. Finally the details of the numerical scheme for solving the second order ordinary differential equation are given in an appendix.

II. THE APPROXIMATE DISPERSION RELATION.

For the central regions of the plasma we content ourselves with a slab model and consider only one direction of inhomogeneity into which the fast-wave is launched i.e. across the magnetic field direction. We start from the local electromagnetic dispersion relation. Assuming a wave vector $\underline{k} = (k_{\perp}, 0, k_{\parallel})$ where \perp and \parallel are, respectively, perpendicular and parallel to the applied magnetic field \underline{B}_0 , Maxwell's equations give

$$[n^2 \underline{\underline{I}} - \underline{\underline{nn}} - \underline{\underline{\epsilon}}(\omega, \underline{k})] \underline{\underline{E}}(\omega, \underline{k}) = 0 \quad (1)$$

Here, $\underline{n} \equiv c\underline{k}/\omega$, $\underline{\underline{\epsilon}}$ is the (Vlasov) permittivity tensor of the plasma and $\underline{\underline{E}}$ is the electric field vector.

The full electromagnetic dispersion relation resulting from Eq.(1) is

$$\begin{aligned} & \epsilon_{yy} n_{\perp}^4 - \{ \epsilon_{\parallel} (\epsilon_{yy} - n_{\parallel}^2) + \epsilon_{yy} (\epsilon_{xx} - n_{\parallel}^2) + \epsilon_{xy}^2 \} n_{\perp}^2 \\ & + \epsilon_{\parallel} \{ (\epsilon_{xx} - n_{\parallel}^2) (\epsilon_{yy} - n_{\parallel}^2) + \epsilon_{xy}^2 \} \\ & = - 2 n_{\parallel} \epsilon_{yz} n_{\perp}^3 - \epsilon_{yz}^2 n_{\perp}^2 + 2 n_{\parallel} \{ \epsilon_{yz} (\epsilon_{xx} - n_{\parallel}^2) + \epsilon_{xz} \epsilon_{xy} \} n_{\perp} \\ & + \epsilon_{yz}^2 (\epsilon_{xx} - n_{\parallel}^2) - \epsilon_{xz}^2 (\epsilon_{yy} - n_{\parallel}^2) + 2 \epsilon_{xz} \epsilon_{yz} \epsilon_{xy}, \end{aligned} \quad (2)$$

where ϵ_{ij} are the elements of $\underline{\underline{\epsilon}}(\omega, \underline{k})$.

The dispersion relation has been written in the above form since in the limit of a cold plasma the right-hand side vanishes. For wave frequencies in the ion cyclotron range we have compared the order of magnitude of the majority ion terms on the right-hand side with the corresponding cold terms on the left-hand side. The right-hand side is negligible if $(v_{Ti} n_{\parallel}/c)^2 \ll 1$, a condition which is very well satisfied for existing tokamaks.

Let us now expand ϵ_{xx} , ϵ_{yy} and ϵ_{xy} to first order in $k_{\perp}^2 v_{Ti}^2/\omega_{ci}^2$ which from

here on we call the FLR ("finite Larmor radius") correction term. We then find

$$\epsilon_{xx} = \epsilon_{\perp} - \alpha n_{\perp}^2 \quad (3)$$

$$\epsilon_{yy} = \epsilon_{\perp} - \alpha n_{\perp}^2 \quad (4)$$

$$\epsilon_{xy} = ig - i \beta n_{\perp}^2, \quad (5)$$

where only the FLR terms coming from the majority second harmonic have been retained. The quantities ϵ_{\perp} , g , α and β are:

$$\epsilon_{\perp} = \sum_{j=1,2} \frac{\omega_{pj}^2}{2\sqrt{2} \omega k_{\parallel} v_{Tj}} \{ Z(z_{1j}) + Z(z_{-1j}) \} \quad (6)$$

$$g = -\frac{\omega_{pe}^2}{\omega k_{\parallel}^2} + \sum_{j=1,2} \frac{\omega_{pj}^2}{2\sqrt{2} \omega k_{\parallel} v_{Tj}} \{ Z(z_{-1j}) - Z(z_{1j}) \} \quad (7)$$

$$\alpha = - \sum_{j=1,2} \frac{\omega_{pj}^2}{2\sqrt{2} \omega k_{\parallel} v_{Tj}} \frac{\omega^2}{\omega_{cj}^2} \frac{v_{Tj}^2}{c^2} \{ Z(z_{-2j}) + Z(z_{2j}) \} \quad (8)$$

$$\beta = - \sum_{j=1,2} \frac{\omega_{pj}^2}{2\sqrt{2} \omega k_{\parallel} v_{Tj}} \frac{\omega^2}{\omega_{cj}^2} \frac{v_{Tj}^2}{c^2} \{ Z(z_{-2j}) - Z(z_{2j}) \}, \quad (9)$$

where

$$z_{nj} \equiv (\omega + n\omega_{cj})/\sqrt{2} k_{\parallel} v_{Tj}, \text{ and } n = \pm 1, \pm 2. \quad (10)$$

The summation over $j=1,2$ gives the contribution of the two ion species where $j=1$ corresponds to the majority ions and $j=2$ to the minority species; $\omega_{cj} = eZ_j B_0/m_j$ are the ion cyclotron frequencies and $\Omega_e = eB_0/m_e$ is the electron cyclotron frequency, with e positive. In expressions (6) through (9) we can neglect the non-resonant terms $Z(z_{ij})$ and $Z(z_{zj})$ so that, in particular, $\alpha \approx \beta$.

We now substitute Eqs.(3) - (5) into Eq.(2) giving the sixth order dispersion relation

$$\begin{aligned}
& (-\alpha + \beta^2 - \alpha^2) n_{\perp}^6 \\
& + \{ \epsilon_{\parallel} \alpha + \epsilon_{\perp} + \alpha(2\epsilon_{\perp} - n_{\parallel}^2) - 2g\beta + \epsilon_{\parallel}(\alpha^2 - \beta^2) \} n_{\perp}^4 \\
& + \{ -\epsilon_{\parallel}(\epsilon_{\perp} - n_{\perp}^2) - \epsilon_{\perp}(\epsilon_{\perp} - n_{\parallel}^2) + g^2 \\
& + \epsilon_{\parallel}(-2\epsilon_{\perp}\alpha + 2\alpha n_{\parallel}^2 + 2g\beta) \} n_{\perp}^2 \\
& + \epsilon_{\parallel}(\epsilon_{\perp}^2 - 2\epsilon_{\perp}n_{\parallel}^2 + n_{\parallel}^4 - g^2) = 0 \quad . \quad . \quad .
\end{aligned} \tag{11}$$

The above equation can be reduced to fourth order by neglecting electron inertia (i.e. keeping only terms in ϵ_{\parallel}). This is a good approximation for the ion cyclotron range of frequencies and has been widely used⁽⁴⁻⁶⁾.

Using this approximation we obtain the approximate dispersion relation

$$\begin{aligned}
& \alpha n_{\perp}^4 - \{ \epsilon_{\perp} - n_{\parallel}^2 + 2\alpha(\epsilon_{\perp} - n_{\parallel}^2) - 2\beta g \} n_{\perp}^2 \\
& + (\epsilon_{\perp} + g - n_{\parallel}^2)(\epsilon_{\perp} - g - n_{\parallel}^2) = 0 \quad . \quad . \quad .
\end{aligned} \tag{12}$$

This form of the ion cyclotron dispersion relation was obtained earlier by Brambilla⁽⁶⁾. He noted that the neglected root in going from Eq.(11) to (12) is not important in the central regions of a hot plasma where it corresponds to an evanescent wave. (In the cold plasma approximation this corresponds to neglecting the slow wave).

The dispersion relation (12) contains the fast magnetosonic (or compressional Alfvén) wave and the ion Bernstein wave. It is this dispersion relation that can be used as a basis for obtaining a full wave description of the fast wave in order to gain insight into the dependence of absorption, reflection and mode conversion on the various parameters.

Equation (12) is also similar to the approximate dispersion relation used by Jacquinet, McVey and Scharer⁽⁷⁾ to describe mode conversion in a two

ion species plasma. The difference is that Eq.(12) includes the effect of ion cyclotron damping whereas these previous authors specifically avoided this but included the effect of electron dissipation. However, this only gave a small correction to the behaviour which was dominated by mode conversion in the vicinity of the two-ion hybrid resonance.

III. COUPLED MODE FORM OF SINGLE-SPECIES SECOND-HARMONIC DISPERSION RELATION

We wish to obtain a second order differential equation description of the fast-wave in the two-ion and second harmonic resonance regions. We shall consider the case of a degenerate resonance, e.g. D(H) where the majority second harmonic resonance coincides with the minority fundamental.

First, we show how to re-write Eq.(12) in a more appropriate form which specifically displays the coupling between the fast wave and the ion Bernstein wave. We note that it is far easier to manipulate the local dispersion relation into a form having the desired structure and then generate the required full wave equation than to look for transformations and reductions of a more complicated (fourth or sixth order) full wave equation.

The key to this problem is the method of description of the second harmonic terms. We therefore consider first the special case of pure second harmonic heating. Once we see how to do this case it will become clear how the method can be extended to the degenerate two-ion species case and the hybrid resonance. We are guided by the fact that as a wave approaches resonance the electrostatic component of the electric field increases. This corresponds to the coupling of the fast wave to the electrostatic ion Bernstein wave. In order to exhibit this behaviour we retain the thermal terms on the left hand side of the equation which describe the propagation of the electrostatic ion Bernstein wave. The remaining thermal terms are responsible for the coupling. Thus, for $\omega = 2\omega_{ci}$, $\beta = \alpha$ and we may write Eq.(12) in the form

$$\begin{aligned}
& \alpha n_{\perp}^4 - (\epsilon_{\perp} - n_{\parallel}^2) n_{\perp}^2 + (\epsilon_{\perp} + g - n_{\parallel}^2) (\epsilon_{\perp} - g - n_{\parallel}^2) \\
& = 2 \alpha (\epsilon_{\perp} - g - n_{\parallel}^2) n_{\perp}^2 .
\end{aligned} \tag{13}$$

Assuming $n_{\parallel}^2 \ll c^2/c_A^2$ we have

$$\epsilon_{\perp} + g \approx -\frac{c^2}{c_A^2} \tag{14}$$

$$\epsilon_{\perp} - g \approx \frac{c^2}{3 c_A^2} \tag{15}$$

and

$$\epsilon_{\perp} \approx -\frac{c^2}{3 c_A^2} , \tag{16}$$

where these quantities are local values referring to the central regions of the plasma and $c_A^2 \equiv B_0^2/n_0 m_i \mu_0$.

Substituting (14) - (16) into Eq.(13) and dividing throughout by ϵ_{\perp} , we obtain

$$-c^2 \frac{k_{\perp}^2}{\omega^2} + \frac{c^2}{c_A^2} - 3 \alpha c_A^2 \frac{k_{\perp}^2}{\omega^2} c^2 \frac{k_{\perp}^2}{\omega^2} = -2 \alpha c^2 \frac{k_{\perp}^2}{\omega^2} . \tag{17}$$

Multiplying Eq.(17) by c_A^2/c^2 and re-arranging, we have

$$(1 + 3 \alpha c_A^2 \frac{k_{\perp}^2}{\omega^2}) (1 - c_A^2 \frac{k_{\perp}^2}{\omega^2}) = \alpha c_A^2 \frac{k_{\perp}^2}{\omega^2} . \tag{18}$$

If we now consider the limit $n_{\parallel} = 0$ we may write α as

$$\alpha \approx \frac{\omega^2}{(\omega^2 - 4 \omega_{ci}^2)} \frac{v_{Ti}^2}{c_A^2} \quad , \quad (19)$$

and recast (18) in a form showing explicitly the coupling of modes

$$(\omega^2 - 4 \omega_{ci}^2 + 3 k_{\perp}^2 v_{Ti}^2) (\omega^2 - c_A^2 k_{\perp}^2) = \omega^2 k_{\perp}^2 v_{Ti}^2 \quad . \quad (20)$$

Equation (20) is now in the desired form since the first bracket on the left-hand side is the electrostatic ion Bernstein wave and the second bracket is the fast Alfvén wave; the right-hand side of (20) gives their coupling by finite temperature.

Now return to Eq.(18). Equation (20) was obtained from (18) by taking $n_{\parallel} = 0$. However, Eq.(13) may still be written in the form given by (18) when $n_{\parallel} \neq 0$. Under these conditions

$$\alpha = - \frac{\omega_{pi}^2}{2^{1/2} \omega k_{\parallel} v_{Ti}} \frac{\omega^2}{\omega_{ci}^2} \frac{v_{Ti}^2}{c^2} Z(z_{-2i}) \quad . \quad (21)$$

Since we are interested in the fast wave solution we now write (18) as

$$1 - c_A^2 \frac{k_{\perp}^2}{\omega^2} = \frac{\alpha c_A^2 k_{\perp}^2 / \omega^2}{1 + 3 \alpha c_A^2 k_{\perp}^2 / \omega^2} \quad . \quad (22)$$

Treating the right-hand side of Eq.(22) as a thermal perturbation to the fast wave we put $(c_A k_{\perp} / \omega)^2 = 1$ on the right-hand side and thus find the approximate fast wave solution in the region of second harmonic resonance

$$1 - c_A^2 \frac{k_{\perp}^2}{\omega^2} \approx \frac{\alpha}{1 + 3 \alpha} \quad , \quad (23)$$

where we emphasize that α is now given by Eq.(21). Equation (23) is the desired second order approximation for the fast wave. The idea behind this approximation, namely, that the coupling to the ion Bernstein wave can be replaced by a localized perturbation to the fast-wave has

previously been used by Kay, Cairns and Lashmore-Davies⁽⁸⁾. It should be noted, however, that in this reference all the second order terms appear on the left hand side of the equation whereas in equation (22) the thermal perturbation also contains second order terms. Kay et al⁽⁹⁾ have independently obtained the result given in Eq.(23) in another context.

We shall say more on the physical significance of Eq.(23) in section IV. Let us now generalize this result to a two species plasma and consider a more direct way of obtaining it.

IV. THE FAST WAVE APPROXIMATION

Neglecting electron inertia at the outset, Eq.(1) then takes the much simpler form

$$(n_{\parallel}^2 - \epsilon_{xx}) E_x - \epsilon_{xy} E_y = 0 \quad (24)$$

$$\epsilon_{xy} E_x + (n^2 - \epsilon_{yy}) E_y = 0 \quad (25)$$

The dispersion relation resulting from Eqs.(24) and (25) is

$$n_{\perp}^2 = \frac{(\epsilon_{xx} - n_{\parallel}^2)(\epsilon_{yy} - n_{\parallel}^2) + \epsilon_{xy}^2}{\epsilon_{xx} - n_{\parallel}^2} \quad (26)$$

Substituting Eqs.(3) - (5) into Eq.(26) we again obtain the fourth order dispersion relation given in Eq.(12). We also note that the denominator in Eq.(26) represents the electrostatic ion Bernstein wave. However, we now keep the dispersion relation in the form shown in Eq.(26). We first observe that in the limit of a cold plasma Eq.(26) is the well known approximate fast wave solution of the dispersion relation. From the analysis in Section III we see that we can continue to use Eq.(26) as the approximate fast wave solution, even in the case of a hot plasma, simply by substituting the zero Larmor radius value

$$n_{\perp}^2 = \frac{(\epsilon_{\perp} - n_{\parallel}^2)^2 - g^2}{\epsilon_{\perp} - n_{\parallel}^2} \equiv (n_{\perp}^2)_A \quad (27)$$

into the FLR correction terms in Eq.(26). Equation (27) is the generalisation of the approximation $c_A^2 k_{\perp}^2/\omega^2 = 1$ to the case of arbitrary n_{\parallel} . Thus, substituting (27) into the FLR corrections in Eq.(26) we obtain the fast wave approximation appropriate to a hot, two-ion resonance case [as in a D(H) plasma]:

$$n_{\perp}^2 = \frac{(\epsilon_{\perp} - g - n_{\parallel}^2) [\epsilon_{\perp} + g - n_{\parallel}^2 - 2 \alpha (n_{\perp}^2)_A]}{\epsilon_{\perp} - n_{\parallel}^2 - \alpha (n_{\perp}^2)_A}, \quad (28)$$

where for n_{\perp}^2 in the FLR terms we have taken its zero Larmor radius value $(n_{\perp}^2)_A$, given by Eq.(27).

Equation (28) is the generalization of Eq.(23) to the case of a two species plasma for arbitrary n_{\parallel} . The interpretation of Eq.(28) is again similar to the one given for Eq.(23). The coupling of the fast-wave to a propagating ion Bernstein wave has been represented as a coupling to a localized perturbation of the fast-wave⁽¹²⁾. By ensuring that the approximate dispersion relation retained the correct structure we have preserved the most important characteristics of this coupling, i.e. the coupling region occurs on the high field side of the degenerate minority fundamental and second harmonic majority resonances and not at the resonance itself.

This feature arises from the ion Bernstein wave which only propagates on the high field side of the second harmonic resonance. Although the propagation of the ion Bernstein wave has been neglected in the fast-wave approximation its effect is still included as a non-propagating i.e. localized response which can become resonant as $n_{\parallel} \rightarrow 0$. It will be shown below that the fast wave approximation can give rise to two critical points. One of these is clearly a wave resonance and is associated with mode conversion and the other is a particle resonance which can give rise to strong cyclotron damping. These two critical points correspond to the

two coupling points which occur in the more accurate model when the propagation of the slow wave is included. The localized perturbation to the fast-wave can thus be interpreted as a non-propagating wave which can be strongly or weakly damped depending on the value of n_{\parallel} . It has been shown previously^(11,12) that the energy lost by a fast wave in the coupling region does not depend on whether the slow mode is treated as propagating or non-propagating. We emphasize that the approximate fast-wave solution given by Eq.(28) contains the full effects of fundamental (minority) and second harmonic (majority) cyclotron damping.

Let us now write out the approximation to the fast wave refractive index in a more explicit form. Using Eqs.(6) - (9) we have

$$\epsilon_{\perp} + g \approx -\frac{c^2}{c_A^2} + \frac{\omega_{p2}^2}{\sqrt{2} \omega k_{\parallel} v_{T2}} Z(z_{-12}) \quad (29)$$

$$\epsilon_{\perp} - g \approx \frac{c^2}{3 c_A^2} \quad (30)$$

$$\epsilon_{\perp} \approx -\frac{c^2}{3c_A^2} + \frac{\omega_{p2}^2}{2\sqrt{2} \omega k_{\parallel} v_{T2}} Z(z_{-12}) \quad (31)$$

$$\alpha \approx -\frac{\omega_{p1}^2}{2\sqrt{2} \omega k_{\parallel} v_{T1}} - \frac{\omega^2}{\omega_{c1}^2} \frac{v_{T1}^2}{c^2} Z(z_{-21}) \quad (32)$$

where $c_A^2 \equiv B_0^2/n_1 m_1 \mu_0$ now refers to the majority species. Only the resonant minority and second harmonic majority terms have been included with the cold plasma terms in Eqs.(29) - (32). Substituting (29) - (32) into Eq.(28) we obtain

$$N_{\perp}^2 = \frac{(1 - 3 N_{\parallel}^2) \left[1 + N_{\parallel}^2 - \frac{\eta c_A}{2\sqrt{2} N_{\parallel} v_{T2}} Z(a_2 \xi) - \frac{f v_{T1}}{\sqrt{2} N_{\parallel} c_A} Z(a_1 \xi) \right]}{\left[1 + 3 N_{\parallel}^2 - \frac{3\eta c_A}{4\sqrt{2} N_{\parallel} v_{T2}} Z(a_2 \xi) - \frac{3f v_{T1}}{2\sqrt{2} N_{\parallel} c_A} Z(a_1 \xi) \right]} \quad (33)$$

where, given $B = B_0(1 - x/R_0)$, with R_0 the tokamak major radius,

$$a_{1,2} \equiv \frac{c_A}{\sqrt{2} N_{\parallel} v_{T1,2}} \frac{1}{R_A} \quad (34)$$

$$R_A \equiv R_0 \omega / c_A \quad (35)$$

$$\xi \equiv \omega x / c_A \quad (36)$$

$$N_{\parallel, \perp} \equiv c_A k_{\parallel, \perp} / \omega \quad (37)$$

$$\eta \equiv n_2 / n_1 \quad (38)$$

and

$$f \equiv \frac{\tau(1 - 3 N_{\parallel}^2)(1 + N_{\parallel}^2)}{(1 + 3 N_{\parallel}^2)} \quad (39)$$

In order to amplify our remarks concerning the representation of the ion Bernstein wave as a non-propagating wave we now consider the resonance behaviour of the fast wave refractive index given by Eq.(33).

In general N_{\perp}^2 will be complex due to the effect of cyclotron damping. The refractive index will however, display resonance properties when the real part of the denominator in Eq.(33) vanishes and simultaneously the imaginary part is very small. The condition for the real part of the denominator to be zero is

$$1 + 3N_{\parallel}^2 - \frac{3\eta}{4\sqrt{2}} \frac{c_A}{N_{\parallel} v_{T2}} \operatorname{Re} Z(a_2 \xi) - \frac{3f}{2\sqrt{2}} \frac{v_{T1}}{N_{\parallel} c_A} \operatorname{Re} Z(a_1 \xi) = 0 \quad (40)$$

The first point to notice is that Eq.(40) can only be satisfied on the high field side of the resonance, as illustrated in Fig.1. Furthermore there will be two roots $\xi_{1,2}$ which, for large enough η or small enough N_{\parallel} will satisfy the conditions $a_{1,2} |\xi_1| \ll 1$, $a_{1,2} |\xi_2| \gg 1$.

Clearly, for $a_{1,2} |\xi_2| \gg 1$ the imaginary terms will be exponentially small and we may interpret the resulting resonance in the fast wave refractive index as mode conversion to a weakly damped, non-propagating mode. Using the asymptotic form of $Z(a_{1,2} \xi)$ we obtain $\xi_2 \approx -(3\eta/4)R_A$ which is the position of the hybrid resonance. This result applies to minority concentrations such that $\eta > v_{T1}^2/c_A^2$. In the case of pure second harmonic heating $\xi_2 = -3v_{T1}^2/2c_A^2$. To ensure that these solutions for ξ_2 satisfy the condition $a_{1,2} |\xi_2| \gg 1$ we require $\eta \gg 4\sqrt{2} N_{\parallel} v_{T2}/3c_A$ in the two species case or $N_{\parallel} \ll 3v_{T1}/2\sqrt{2} c_A$ for second harmonic heating. Under these conditions the fast-wave refractive index will exhibit strongly resonant behaviour indicating mode conversion in the region $\xi = \xi_2$.

The second root of Eq.(40) satisfying $a_{1,2} |\xi_1| \ll 1$ occurs in the region of minority fundamental and majority second harmonic cyclotron damping. This point is therefore associated with a particle resonance and can result in strong damping of the fast wave depending on the values of η , N_{\parallel} and the temperatures of the ion species. The damping in this region is evidently responsible for the reduction in the reflection coefficient for a wave incident from the low field side as N_{\parallel} is increased.

Now suppose that η is gradually reduced (for given N_{\parallel}) from values producing strongly resonant behaviour of the fast wave refractive index or, for a single ion species plasma N_{\parallel} is gradually increased. As a result the separation of the roots ξ_1 and ξ_2 gradually decreases, until, for some critical value $\eta = \eta_c$, $\xi_1 = \xi_2 = \xi_c$. The physical

significance of this condition is that the region of strong cyclotron damping overlaps the region of mode conversion. The critical value of η is given by noting that it occurs when $\text{Re}Z$ takes its maximum value which we approximate by unity giving

$$\eta_c = \frac{4/2}{3} \frac{N_{\parallel} v_{T2}}{c_A} (1 + 3N_{\parallel}^2) - 2 f \frac{v_{T1} v_{T2}}{c_A^2} \quad (41)$$

For a single species plasma the corresponding critical condition for second harmonic heating is given by putting $\eta = 0$ in Eq.(41) giving

$$N_{\parallel c} = \frac{3/2}{4} \frac{v_{T1}}{c_A} \quad (42)$$

For $\eta < \eta_c$ (or $N_{\parallel} > N_{\parallel c}$) there are no solutions of Eq.(40) and the fast-wave refractive index will no longer exhibit resonant behaviour. The resonance is 'smeared out' due to the effects of cyclotron damping and any mode conversion will result in rapid dissipation.

Thus, only for $\eta \gg \eta_c$ (or $N_{\parallel} \ll N_{\parallel c}$) would the ion Bernstein wave be able to carry energy away from the coupling region. As we have already emphasised, the propagation of the ion Bernstein wave is not included in the present fast wave approximation. What we have shown above is that most of the remaining features of the coupling to the ion Bernstein wave are preserved within the fast-wave approximation.

V. FULL WAVE DESCRIPTION OF THE FAST WAVE

We will now transform the fast-wave approximation represented by Eq.(33) into a differential equation by the usual device of replacing k_{\perp} by $-id/d\xi$. Since k_{\perp} only appears on the left-hand side of Eq.(33) there is no problem with regard to uniqueness of the resulting differential equation. The full wave equation in the D(H) resonance region is

$$\frac{d^2\phi}{d\xi^2} + \frac{(1 - 3N_{\parallel}^2) \left[1 + N_{\parallel}^2 - \frac{\eta}{2\sqrt{2}} \frac{c_A}{n_{\parallel} v_{T2}} Z(a_2 \xi) - \frac{f}{\sqrt{2}} \frac{v_{T1}}{N_{\parallel} c_A} Z(a_1 \xi) \right]}{\left[1 + 3N_{\parallel}^2 - \frac{3\eta}{4\sqrt{2}} \frac{c_A}{N_{\parallel} v_{T2}} Z(a_2 \xi) - \frac{3f}{2\sqrt{2}} \frac{v_{T1}}{N_{\parallel} c_A} Z(a_1 \xi) \right]} \phi = 0 \quad (43)$$

where ϕ represents the normalized amplitude of the fast-wave. On inspection of Eqs.(24) and (25) we see that once we have made the fast wave approximation by substituting $n_{\perp}^2 = (n_{\perp}^2)_A$ (Eq.(22)) in the thermal corrections we can identify the wave amplitude ϕ as $\phi \propto E_y$. As already noted, the wave equation for E_y is then obtained unambiguously. Once E_y has been obtained as the solution of Eq.(43), E_x can be calculated from either of Eqs.(24) or (25). Hence the field polarization across the resonance region can be found. Equation (43) is the central result of this paper and is the second order full wave description we have been seeking.

Equation (43) has been obtained for the case of D(H), i.e. a degenerate resonance minority. For a non-degenerate resonance minority the second harmonic resonance will be well separated from the hybrid resonance with the result that the fast-wave approximation will give a wave equation of the same form as Eq.(43) but with the last terms in the numerator and denominator missing i.e. those proportional to v_{T1} .

We will conclude this section by considering Eq.(43) in the limit $N_{\parallel} \rightarrow 0$. Equation (43) then reduces to

$$\frac{d^2\phi}{d\xi^2} + \frac{\left(\xi + \frac{\eta}{2} R_A + \frac{v_{T1}^2}{c_A^2} R_A \right)}{\left(\xi + \frac{3\eta}{4} R_A + \frac{3}{2} \frac{v_{T1}^2}{c_A^2} R_A \right)} \phi = 0 \quad , \quad (44)$$

which will be recognized as Budden's⁽¹³⁾ equation. We may, therefore, immediately write down the power transmission coefficient⁽¹³⁾

$$T = \exp \left[- \pi \left(\frac{\eta}{4} + \frac{1}{2} \frac{v_{T1}^2}{c_A^2} \right) R_A \right] \quad (45)$$

which agrees with the well known results for the ion-ion hybrid resonance or the second harmonic in a pure plasma⁽¹⁴⁾. The reduction of the fourth order problem to Budden's equation has also been obtained by Chiu⁽¹⁵⁾ for the pure second harmonic for perpendicular propagation. Before describing the numerical solutions of Eq.(43) for arbitrary values of N_{\parallel} we first use Eq.(43) to obtain a conservation relation.

VI. CONSERVATION RELATION FOR THE FAST-WAVE

Let us write Eq.(43) in the form

$$\frac{d^2 \phi}{d\xi^2} + Q(\xi) \phi = 0 \quad (46)$$

where the complex potential $Q(\xi)$ is defined by the coefficient of ϕ in Eq.(43). By the usual procedure of multiplying Eq.(46) by ϕ^* and subtracting the complex conjugate of Eq.(46) multiplied by ϕ we obtain the following conservation relation

$$\frac{d}{d\xi} \left(\text{Im} \phi^* \frac{d\phi}{d\xi} \right) = - \phi \phi^* \text{Im} Q \quad (47)$$

Now assuming

$$\phi = \frac{e^{i\psi}}{Q^{1/4}} + \frac{\rho e^{-i\psi}}{Q^{1/4}} \quad (48)$$

in the asymptotic region on the incidence side and

$$\phi = \frac{\tau e^{i\psi}}{Q^{1/4}} \quad (49)$$

in the asymptotic region on the transmitted side, where $\psi = \int Q^{1/2} d\xi$, we may integrate Eq.(47) from $\xi = \xi_1$ to $\xi = \xi_2$ where $\xi_{1,2}$ are on the incidence and transmitted sides respectively. We obtain

$$T + R + \int_{\xi_1}^{\xi_2} \phi \phi^* \operatorname{Im} (Q) d\xi = 1 \quad , \quad (50)$$

where $T = \tau \tau^*$ and $R = \rho \rho^*$ are respectively the power transmission and reflection coefficients.

If we now integrate the real part of the complex Poynting theorem over the region ξ_1 to ξ_2 , at whose boundaries there are only fast-wave electromagnetic power flows, the integral in (51) is clearly the power "absorbed" from the incident fast wave; in detail, this power "absorbed" is in general made up of power dissipated and power mode-converted to the ion-Bernstein wave but, on the basis of only our global perturbation model of the fast wave, this distinction cannot be made explicit. The density of this power "absorbed" is given by the usual expression:

$$P_{\text{abs}} = \frac{1}{2} \operatorname{Re} (\tilde{\mathbf{E}}^* \cdot \tilde{\mathbf{J}}) = \frac{1}{2} \tilde{\mathbf{E}}^* \cdot \tilde{\boldsymbol{\sigma}}^{(h)} \cdot \tilde{\mathbf{E}} \quad , \quad (51)$$

where $\tilde{\boldsymbol{\sigma}}^{(h)}$ is the hermitian part of the conductivity tensor. For the zero electron inertia approximation this gives

$$P_{\text{abs}} = \frac{\omega \epsilon_0}{2} \left[|E_x|^2 \operatorname{Im} \epsilon_{xx} + |E_y|^2 \operatorname{Im} \epsilon_{yy} + 2(\operatorname{Re} \epsilon_{xy})(\operatorname{Im}(E_x^* E_y)) \right] \quad (52)$$

As we have already noted the fast-wave amplitude is identified with E_y . We therefore substitute for E_x from Eq.(24) into (52) and, after some algebra, obtain

$$P_{\text{abs}} = \frac{\omega \epsilon_0}{2} |E_y|^2 \operatorname{Im} (n_{\perp}^2) \quad . \quad (53)$$

We now rewrite (53) in terms of the variables ξ , N_{\perp} and ϕ defined by (36), (37) and (43). ϕ is the normalised electric field given by

$$\phi = E_y / (E_y)_{\text{inc}} \quad (54)$$

The x-component of the Poynting flux is $S_x = |E_y|^2 / 2\mu_0 c_A$. Choosing unit incident Poynting flux we express $P_{abs.}$ in terms of ϕ from (53) giving

$$P_{abs.} = \frac{\omega}{c_A} |\phi|^2 \text{Im}(N_{\perp}^2) \quad (55)$$

so that the total power absorbed is given by

$$\int_{x_1}^{x_2} P_{abs.} dx = \int_{\xi_1}^{\xi_2} |\phi|^2 \text{Im}(N_{\perp}^2) d\xi \quad (56)$$

which is the form appearing in the conservation law (50).

For $N_{\parallel} = 0$, Eq.(46) becomes (44) with a back-to-back cutoff - resonance combination. In order to integrate through the resonance we add a small amount of damping to resolve the singularity and subsequently take the limit of vanishing dissipation. This limiting process is done under integral sign, so that we make use of

$$\text{Im} \left(\lim_{\epsilon \rightarrow 0} \frac{1}{x - i\epsilon} \right) = \pi \delta(x) \quad (57)$$

We thus obtain

$$\int_{\xi_1}^{\xi_2} \phi \phi^* \text{Im}(Q) d\xi = \pi R_A \left(\frac{\eta}{4} + \frac{1}{2} \frac{v_{T1}^2}{c_A^2} \right) |\phi(-\xi_r)|^2 \quad (58)$$

with $\xi = -\xi_r$ the resonance point where $Q(-\xi_r) \rightarrow \infty$. It is clear from the discussion in section IV, concerning the replacement of the ion-Bernstein wave coupling by a localised perturbation on the fast-wave, that the resulting description of the fast-wave (Eqs.(28),(43)) combines the effects of mode conversion and cyclotron damping. For $N_{\parallel} = 0$ there is no cyclotron damping and the only contribution to the integral term in Eq.(50) comes from mode conversion. In this case, Eqs.(50) and (58)

clarify Budden's observation that $R + T \neq 1$ even in the limit of zero dissipation (Budden's "paradox"). Clearly, the missing power is to be interpreted as being mode converted.

VII. NUMERICAL RESULTS

The numerical method we employ to determine the transmission properties of the fast wave is now described, and results are presented for PLT and JET conditions, previously calculated^(16, 17, 18) by means of much more elaborate methods (i.e. numerical integration of fourth and sixth-order differential equations).

Consider the second-order equation

$$y'' + Q(x)y = 0 \quad , \quad (59)$$

with a generally complex potential $Q(x)$, which satisfies the following asymptotic conditions. Far away from the coupling region (situated, say, around $x = 0$) $Q(x)$ satisfies conditions for the validity⁽¹⁹⁾ of Liouville-Green-type solutions (48), (49), which, as $x \rightarrow \pm \infty$, smoothly become plane waves $\exp_{\pm}(ikx)$. This characterizes the nature of the fast wave far away from regions of mode-conversion, cyclotron damping, and cut-off (generically termed coupling).

In contrast to its slowly-varying character in the asymptotic region, the potential $Q(x)$ can vary rapidly in the coupling region due to the presence of cut-offs, resonances, damping and mode conversion. Under very limited conditions, WKBJ or phase-integral solutions⁽¹⁸⁾ can be constructed, but for complex potentials described by transcendental functions, such as in (43), the only reliable solution of Eq.(59) would be numerical. The problem then is to construct the transmission and reflection coefficients, which consists in matching boundary conditions of the type (48) and (49) to the numerical solution. This is described in detail in the Appendix.

As a first example, shown in Figs. 2a and 2b, we have considered heating

at $2\omega_{ci}$ in pure hydrogen for PLT parameters taken from Colestock and Kashuba⁽¹⁶⁾ (their Figs. 13 and 14).

The variation of the transmitted and reflected powers for low field incidence as a function of k_{\parallel} agree very well with the results of Colestock and Kashuba as does the variation of the power absorbed. The variation of the transmitted power as a function of electron density also shows a similar variation although the value we obtain at the highest densities is somewhat lower than that of Colestock and Kashuba⁽¹⁶⁾.

In Figs. 3a and 3b we compare our results for minority D(H) heating with Figs. 5 and 9 of Colestock and Kashuba⁽¹⁶⁾ for PLT parameters. Again the variation and magnitude of the reflected and transmitted powers as a function of k_{\parallel} are in very good agreement with Colestock and Kashuba. Similarly, the power absorbed also agrees well, our second order analysis yielding both the position of the peak absorption and its magnitude. The variation of the transmitted and reflected powers for low field incidence and the power absorbed (by the protons) as a function of the minority to majority density ratio is also very well predicted by our second order model.

In Figs. 4a and 4b we compare our results with the fourth order model of Romero and Scharer⁽¹⁸⁾ applied to the JET plasma, again for D(H). We find excellent agreement in the variation and magnitude of the transmitted, reflected and absorbed powers as functions of k_{\parallel} and the minority to majority density ratio. We note in particular, that in both calculations the power absorbed peaks around $\eta = 0.03$.

Thus, as a final example we present a table of results obtained from our second order theory for the PLT "benchmark" problem selected at the ICRF workshop held in Madison, Wisconsin, May 1985. The results are given in Table I and show the transmission T and reflection coefficients R as a function of k_{\parallel} for D(H). Included for comparison are the results obtained by Imre et al.⁽¹⁷⁾ from a fourth order analysis. There is good quantitative agreement although the second order theory gives a consistently higher value for the reflection coefficient particularly at

the smallest values of k_{\parallel} .

We have verified for a number of cases of practical interest that the simple second-order ordinary differential equation representation (43) for the fast-wave in a two-ion species plasma, produces absorption, transmission and reflection coefficients compatible with results previously obtained from numerical integration of fourth and sixth-order ordinary differential equations. (16-18) The coupling of the fast-wave to the ion-Bernstein wave was treated here as a localized perturbation to the fast-wave. The localized perturbation retained the resonance properties characteristic of the coupling of the fast wave to the ion-Bernstein wave as well as the effects of cyclotron damping of both minority and majority ion species. As a result, the power "absorbed" which appears in the conservation law (47), is clearly a combination of mode-conversion and ion cyclotron damping. The second order equation (43) which we have derived in this paper is a generalized Budden equation which includes the effect of localized dissipation.

We note that although we have neglected the effect of electron Landau damping the power "absorbed" obtained from our second order model should give an upper limit to the power lost by the fast-wave in crossing the coupling region. This is because electron Landau damping is only expected to be significant for the ion-Bernstein wave. Since we have shown that the effect of mode conversion is included in our second order model it is immaterial to the fast wave how the mode converted energy is dissipated. Within the second-order model discussed in this paper we are unable to separate the effects of mode conversion and dissipation except for the case of perpendicular propagation. We have also obtained a criterion for the disappearance of mode conversion due to the overlapping of the region of strong cyclotron damping with that of mode coupling. Under these conditions the energy lost by the fast wave is dissipated locally due to cyclotron damping. The effect of the poloidal magnetic field, which we have neglected, will be of significance for electron dissipation. (16,23) For small values of N_{\parallel} in the coupling region ion

cyclotron damping will be weak and mode conversion strong. However, away from the coupling region the mode converted ion-Bernstein wave may undergo electron Landau damping as N_{\parallel} increases due to the effect of the rotational transform.

Thus, the neglect of electron-Landau damping and rotational transform is not expected to alter the total power absorbed but only the spatial location of the power dissipated by the electrons which could be quite far from the mode conversion region. Clearly, having replaced the propagating ion-Bernstein wave by a localised response we have lost some information on the power deposition profile and this is evidently the most significant deficiency of our second order model.

ACKNOWLEDGEMENT

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APPENDIX

It is well known that the solution of a system of linear ordinary differential equations can be written as a linear combination of the boundary conditions. In matrix form

$$|y(x)| = [T(x)]|y_0| \quad , \quad (A.1)$$

where $|y_0|$ is the given solution vector at $x = x_0$, and the transfer matrix $[T]$ depends on the system. While for most cases of interest $[T]$ cannot be determined analytically, the transfer between two particular points $x = x_1$ and $x = x_2$ is easily determined numerically. For example, in the case of interest here, we first write Eq.(59) in the form

$$y' = p \quad , \quad p' = -Qy \quad , \quad (A.2)$$

then select two points x_1 and x_2 away from and on opposite sides of the coupling region where individual waves of the form (48) and (49) can be identified, and integrate (A.2) generally twice with independent boundary conditions in order to obtain the transfer matrix. The solution at $x = x_2$ is then related to that at $x = x_1$ by the equation

$$\begin{pmatrix} y \\ p \end{pmatrix}_2 = [T] \begin{pmatrix} y \\ p \end{pmatrix}_1 \quad (A.3)$$

into which we substitute for y the asymptotic wave forms (48) and (49), and solve for τ and ρ .

In order to obtain $[T]$ one may integrate from either side. If we choose to integrate from the incidence side where there are two independent waves, we have to produce two independent solutions $y^{(I)}$ and $y^{(II)}$. These can be generated respectively from the simple linearly independent, boundary conditions

$$\begin{pmatrix} Y \\ p \end{pmatrix}_1^{(I)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} Y \\ p \end{pmatrix}_1^{(II)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} . \quad (\text{A.4})$$

The transfer matrix is then obviously given in terms of the solutions at $x = x_2$, in the form

$$[T] = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} = \begin{pmatrix} Y^{(II)} & Y^{(I)} \\ p^{(II)} & p^{(I)} \end{pmatrix}_2 . \quad (\text{A.5})$$

In Eq.(A.3) we now substitute

$$Y_1 = w_{+1} + \rho w_{-1} \quad , \quad Y_2 = \tau w_{+2} \quad (\text{A.6})$$

$$p_1 = ik_1 (w_{+1} - \rho w_{-1}) \quad , \quad p_2 = ik_2 \tau w_{+1} \quad ,$$

where

$$w_{\pm} = k^{-1/2} \exp(\pm i \int k dx) \quad , \quad k = Q^{1/2} \quad , \quad (\text{A.7})$$

and we have agreed that w_+ propagates to the right. Solving for τ and ρ immediately gives

$$\rho = \frac{(-i k_2 T_{11} + k_1 k_2 T_{12} + T_{21} + i k_1 T_{22}) w_{+1}}{(i k_2 T_{11} + k_1 k_2 T_{12} - T_{21} + i k_1 T_{22}) w_{-1}} \quad (\text{A.8})$$

$$\tau = \frac{2i k_1 (T_{11} T_{22} - T_{12} T_{21}) w_{+1}}{(i k_2 T_{11} + k_1 k_2 T_{12} - T_{21} + i k_1 T_{22}) w_{+2}} \quad ,$$

and, by definition, the power transmission and reflection coefficients are

$$T = \tau \tau^* \quad , \quad R = \rho \rho^* \quad . \quad (\text{A.9})$$

The general solution y satisfying the boundary conditions (A.6) can be

written in terms of the numerical solutions $y^{(I)}$ and $y^{(II)}$ as

$$y = A y^{(I)} + B y^{(II)} \quad , \quad (\text{A.10})$$

where the constants A and B satisfy

$$\begin{pmatrix} Y \\ P \end{pmatrix}_1 = A \begin{pmatrix} 0 \\ 1 \end{pmatrix} + B \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad . \quad (\text{A.11})$$

Much simpler expressions for τ , ρ , and y can be obtained if we integrate from the transmission side, where only one wave propagates and so only one integration is needed. This can be seen as follows. With x_1 on the transmission side we now have $y_1 = \tau w_{1-}$, so that from

$$\begin{pmatrix} Y \\ P \end{pmatrix}_2 = T \begin{pmatrix} Y \\ P \end{pmatrix}_1 = \begin{pmatrix} T_{11} Y_1 + T_{12} P_1 \\ T_{21} Y_1 + T_{22} P_1 \end{pmatrix} \quad (\text{A.12})$$

we get

$$w_{-2} + \rho w_{+2} = \tau w_{-1} (T_{11} - ik_1 T_{12}) \quad (\text{A.13})$$

$$- ik_2 (w_{-2} - \rho w_{+2}) = \tau w_{-1} (T_{21} - ik_1 T_{22})$$

The unknown coefficients

$$T_{11} - ik_1 T_{12} \equiv a_1 \quad (\text{A.14})$$

$$T_{21} - ik_1 T_{22} \equiv a_2$$

are numerically obtained by integrating (A.2) with the boundary condition

$$\begin{pmatrix} Y \\ P \end{pmatrix}_1 = \begin{pmatrix} 1 \\ -ik_1 \end{pmatrix} \quad , \quad (\text{A.15})$$

obviously giving

$$\begin{pmatrix} Y \\ P \end{pmatrix}_2 = [T] \begin{pmatrix} 1 \\ -ik_1 \end{pmatrix} = \begin{pmatrix} T_{11} - ik_1 T_{12} \\ T_{21} - ik_1 T_{22} \end{pmatrix} \quad . \quad (\text{A.16})$$

Then, from (A.13),

$$\tau = \frac{2ik_2}{(ik_2 a_1 - a_2)} \frac{w_{-2}}{w_{-1}} \quad (\text{A.17})$$

$$\rho = \frac{(ik_2 a_1 + a_2)}{(ik_2 a_1 - a_2)} \frac{w_{-2}}{w_{+2}} \quad (\text{A.18})$$

and the solution y satisfying the given boundary conditions is

$$y = \tau w_{-1} y^{(\text{III})} \quad (\text{A.19})$$

where $y^{(\text{III})}$ is the numerical solution of the boundary value problem (A.15).

In order to test the outlined schemes numerical integrations were performed, using the IMSL library routine DREBS, based on the Bulirsch-Stoer⁽¹⁹⁾ extrapolation method. Accuracy was tested for a number of potentials, whose T and R are known analytically, such as, for example, special cases of the complex parabolic barrier⁽²⁰⁾. Typically, the numerical and analytic results agreed to within five significant digits. For complex potentials the conservation law (47), involving a domain integral of the solution itself, was equally well satisfied. This high degree of accuracy was verified to hold in cases of evanescent solutions, for which numerical difficulties were previously^(16,18,21) reported.

The integrations are, however, not always free of difficulties, and these occur when the coupling potential Q becomes singular on (or very close to) the real axis. Such a singular point arises, as illustrated in Fig. 1 and discussed in section IV, for negligible $\text{Im } Z(a_2\xi)$ and small k_{\parallel} and/or large enough minority concentration, and is identified with strong mode-conversion. The integration then fails as a result of exceedingly large $|dy/dx|$. To avoid this happening, we use the same device as

described at the end of Section VI for the integration of the Budden Eq.(44), which is to integrate along a contour bypassing the singularity under the real axis.

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FIGURE CAPTIONS

Fig. 1: Illustration of how the roots $\xi_{1,2}$ of Eq.(40) depend on η and N_{\parallel} .

Fig. 2: Hydrogen second harmonic heating. Power transport coefficients T , R and P for PLT parameters $R_0 = 1.32$ m, $B_0 = 2.9$ T, $n_e = 4 \times 10^{19} \text{ m}^{-3}$, $T = 2$ keV, $f = 42$ MHz. a) Parallel wave-number scaling. b) Density scaling.

Fig. 3: Minority D(H) heating for parameters of Fig. 3. a) Parallel wave-number scaling for minority (Hydrogen) concentration $\eta = 0.1$. b) Minority concentration scaling for $k_{\parallel} = 10 \text{ m}^{-1}$.

Fig. 4: Minority D(H) heating for JET parameters (c.f. Ref. 18) $R_0 = 3$ m, $B_0 = 3.45$ T, $n_e = 3.3 \times 10^9 \text{ m}^{-3}$, $T = 5$ keV, $f = 53$ MHz. a) Parallel wave-number scaling for minority concentration $\eta = 0.05$. b) Minority concentration scaling for $k_{\parallel} = 6 \text{ m}^{-1}$.

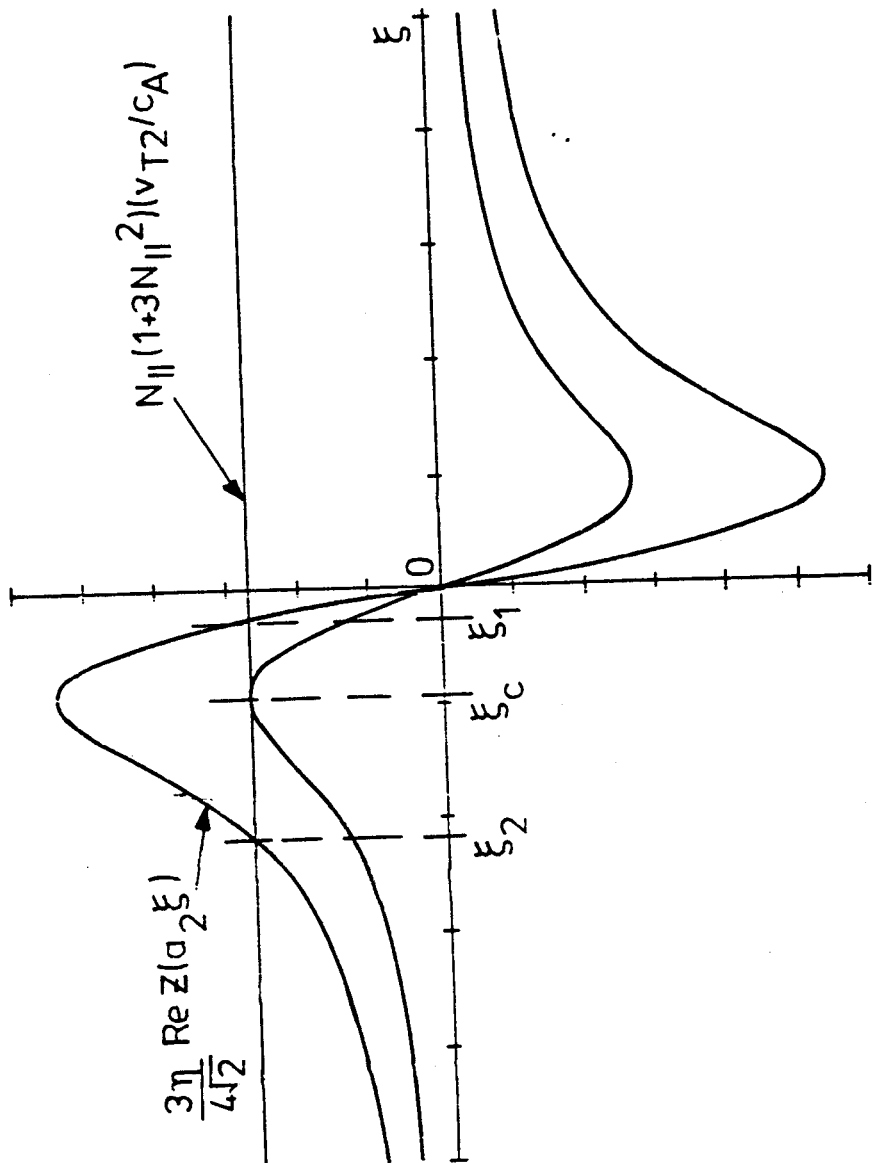
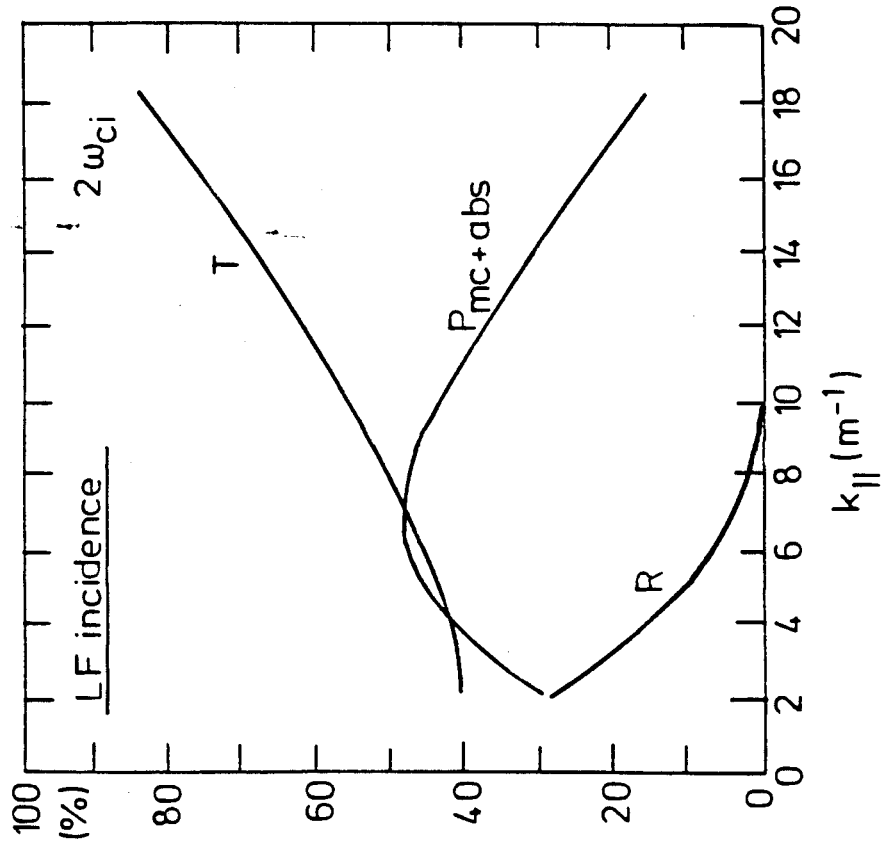
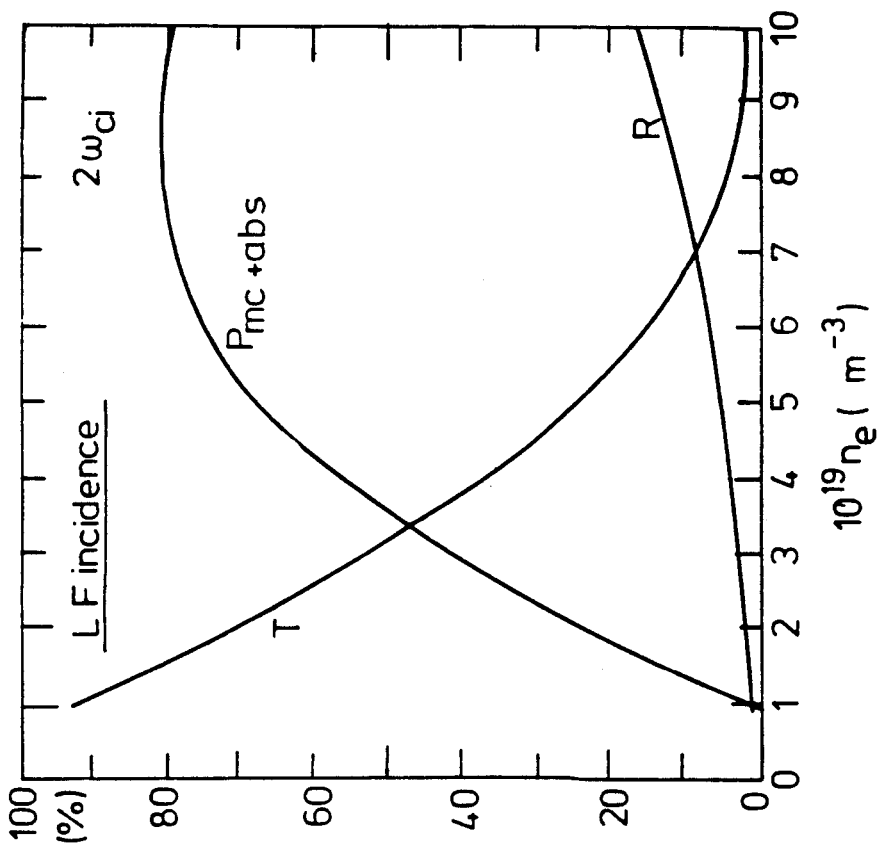


Fig. 1.

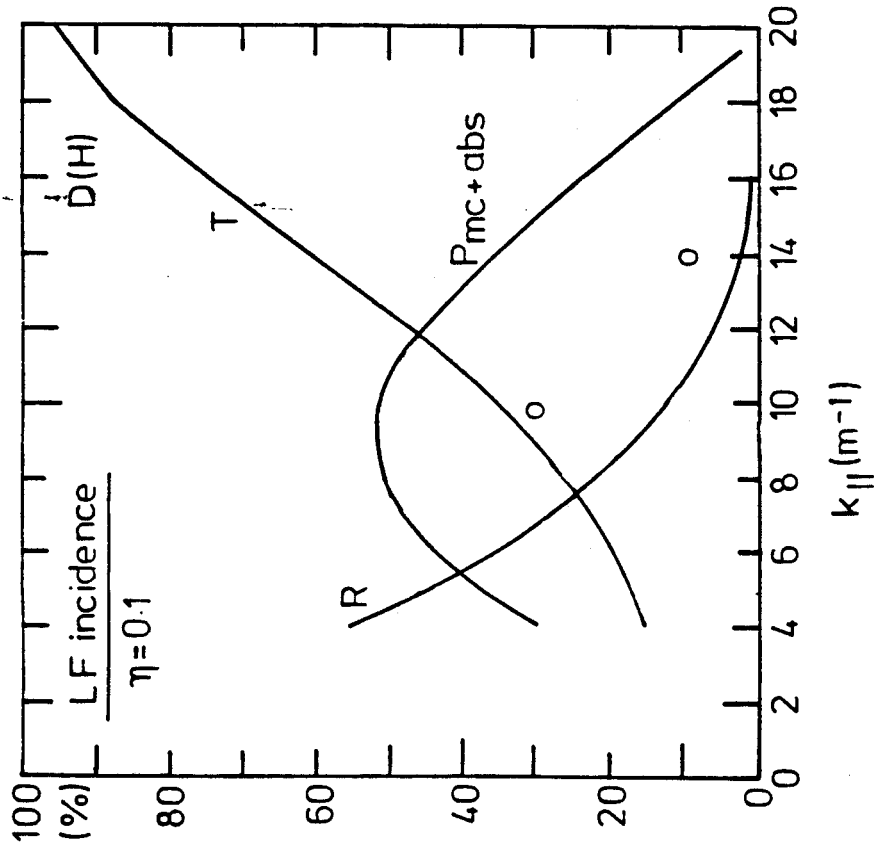


(2a)

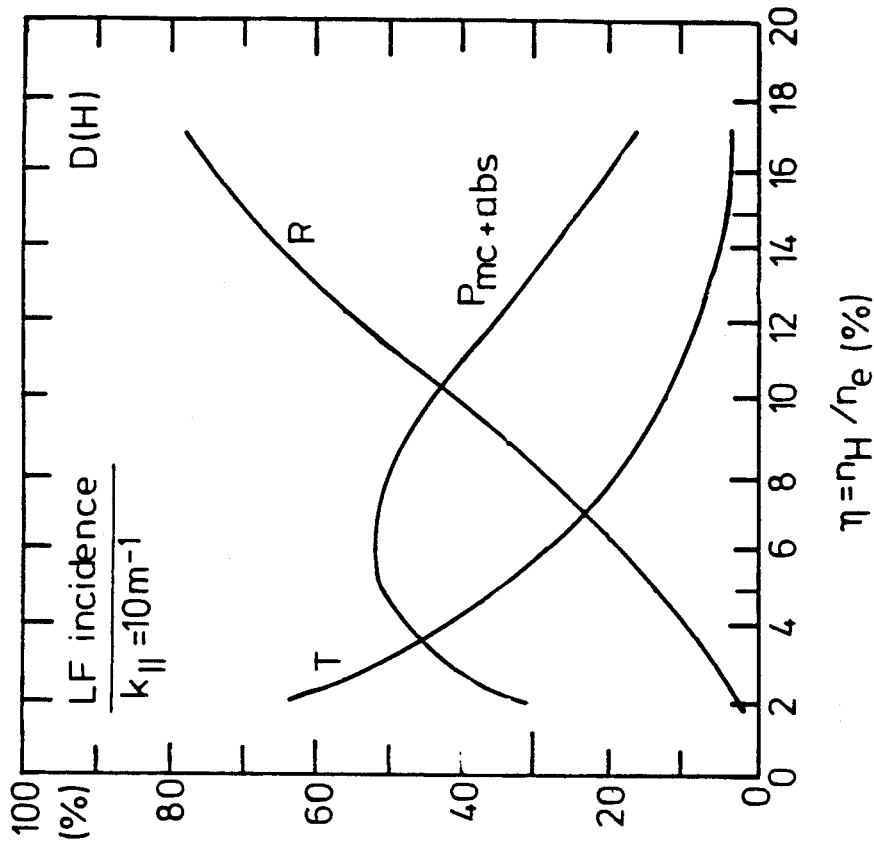


(2b)

Fig. 2



(3a)



(3b)

Fig. 3

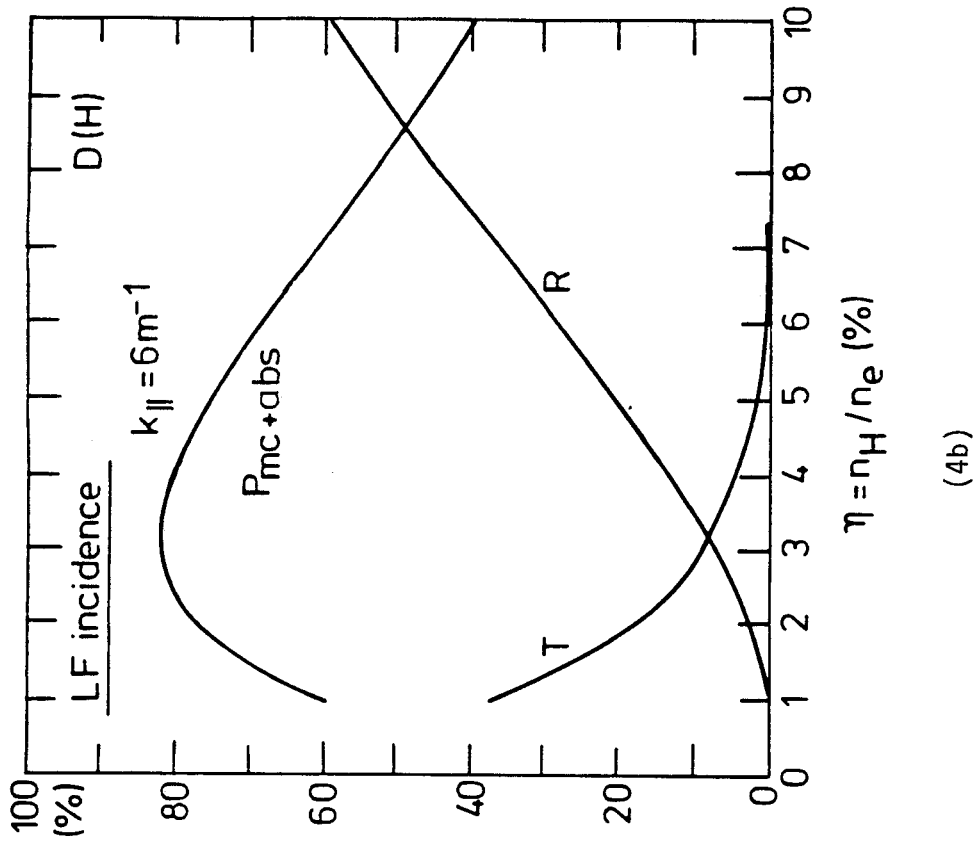
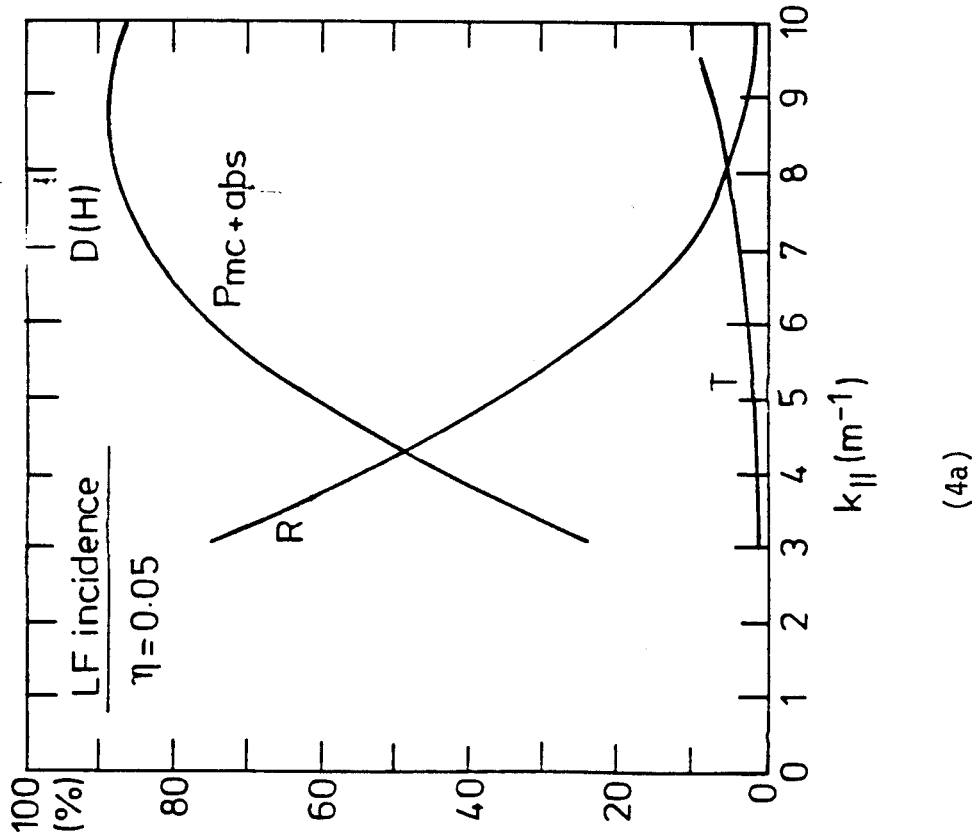


Fig. 4

Table I. PLT minority heating "benchmark" case [D(H), $\eta = 0.05$]

Comparison of our results with
those of Ref. 17 (last two rows)

$k_{\parallel} [m^{-1}]$	1	2	3	4	5	6	7	8	9	10
T	16.3	18.3	18.3	20.2	22.3	26.0	29.8	34.5	40.0	46.0
R	69.1	65.2	59.3	52.1	44.2	36.3	28.8	22.1	16.5	11.9
	(present results)									
T	15.4	16.2	17.5	19.5	22.1	25.4	29.5	34.4	40.0	46.4
R	61.2	57.9	52.9	46.7	39.8	32.9	26.4	20.4	15.3	11.1
	(Imre and Weitzner ⁽¹⁷⁾)									