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**Theory of Global Interchange Modes in
Shaped Tokamaks with Small Central Shear**

J. J. Ramos

Plasma Fusion Center
Massachusetts Institute of Technology
Cambridge, MA 02139

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Plasma Fusion Center
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Abstract

The ideal MHD linear stability theory of arbitrary- n (including $n = 1$) interchange modes in shaped tokamaks with flat central rotational transform is developed. The unstable modes have very long parallel wavelength everywhere, but their perpendicular wavelength is assumed to be comparable to the plasma minor radius. It is shown that the stability condition against these radially extended modes is independent of their toroidal wavenumber n , and identical to the shearless limit of the $n \gg 1$ Mercier criterion.

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Sawtooth oscillations are one of the crucial phenomena that dominate the confinement dynamics of tokamak plasmas. The sawtooth phenomena have recently sparked much interest in the ideal MHD stability of tokamak configurations having a rather flat rotational transform profile over a sizable central portion of the plasma domain. The motivation for this is the experimental evidence that the sawtooth crash processes in the JET tokamak are consistent with the onset of an ideal MHD internal instability.^{1,2} This unstable mode has a toroidal wavenumber $n = 1$ and its poloidal structure is dominated by the $m = 1$ harmonic. Since the values of the poloidal beta at JET are very low, much theoretical work²⁻⁶ is being devoted to proving that such an ideal MHD instability can be excited at arbitrarily low values of β_p . An internally flat rotational transform profile, such that q is approximately equal to 1 over a sizable central core of the plasma, has been proposed^{2,3} in order to explain the sawtooth crash features observed at JET. In this case an interchange instability is possible, and this is the subject of the present work. Previous work on interchange instabilities in tokamak plasmas (e.g. Refs. 7 and 8) has relied on the assumption of short perpendicular wavelengths, by considering modes with large toroidal and poloidal wavenumbers, and/or a sharp radial localization about a magnetic surface. This leads to a fine scale instability, likely to be subject to finite-Larmor-radius stabilization, and with little chance of having macroscopic consequences. In a new development we prove here that, for configurations with small central shear, the short perpendicular wavelength constraint can be relaxed, and we work out a theory of large scale interchange instabilities which are capable of accounting for the sawtooth crashes in an elongated tokamak like JET. To carry out our analysis we make the fundamental assumption that, within the central core of interest whose inverse aspect ratio r_o/R_o is ordered as ϵ , q deviates from 1 (or more generally from a rational number m/n) by only an amount of order ϵ^2 . For our purposes it does not matter whether q stays above m/n or crosses this rational value one or several times: our result is independent of the details of the function $q - m/n \sim \epsilon^2$. We consider a macroscopic plasma perturbation, radially confined to the flat- q region, and having an interchange character, namely very long parallel wavelength ($k_{\parallel}R_o \sim \epsilon$) everywhere. However, we do not assume short perpendicular wavelengths, and $k_{\perp}r_o$ is taken to be of order unity. Because of the generally low toroidal and poloidal wavenumbers and broad radial

extent of the mode, we refer to it as a “global interchange”. We prove that the stability condition against these arbitrary- n global interchanges is independent of their toroidal wavenumber n , and identical to the near magnetic axis limit⁹ of the Mercier criterion⁷ against large- n , radially localized interchange modes:

$$\beta_{po} \left[1 - \frac{1}{q_o^2} - \frac{3(\kappa_o^2 - 1)(\kappa_o^2 - 2\tau_o)}{(\kappa_o^2 + 1)(3\kappa_o^2 + 1)} - \frac{4\beta_{po}(\kappa_o - 1)^2}{\kappa_o(\kappa_o + 1)(3\kappa_o^2 + 1)} \right] \geq 0, \quad (1)$$

where κ_o represents the elongation of the flux surfaces and τ_o is a measure of their triangularity. Thus, for $q_o = m/n = 1$, $\kappa_o^2 > 1$, $\kappa_o^2 > 2\tau_o$ and $\beta_{po} > 0$, we show the existence of a robust instability, independent of the details of $q - 1$, and with a significant growth rate (proportional to $\beta_{po}^{1/2}$) in the low- β_{po} regime. Only in the circular case ($\kappa_o = 1$) where our interchange modes become marginally stable, more elaborate analyses are needed to prove the existence of an instability. These must take into account either the contribution of a nonvanishing fluid displacement in the finite-shear region^{2,3} or (numerically) higher order effects sensitive to the details of the $q - 1$ function^{4,5}.

We begin by studying the plasma equilibrium in the flat- q region. We work in a flux coordinate system r, φ, θ defined as follows: $r(\psi)$ is the magnetic surface function defined by $rdr = R_o q(\psi) T(\psi)^{-1} d\psi$, where R_o is the radius of the magnetic axis, $2\pi\psi$ is the poloidal magnetic flux and $2\pi T$ is the poloidal current ($T = RB_t$); φ is the geometrical toroidal angle, i.e. $|\nabla\varphi| = R^{-1}$; θ is the poloidal coordinate that makes the magnetic field lines appear straight in the $\varphi - \theta$ space, and is defined by $\vec{B} \cdot \nabla\varphi = q(\psi)\vec{B} \cdot \nabla\theta$. Then we formulate the fundamental hypothesis that within a central core ($r < r_o$) of large aspect ratio [$r_o/R_o = O(\epsilon)$], the inverse rotational number q is approximately constant and rational [$q(r) = m/n + q_2(r)$, $q_2(r) = O(\epsilon^2)$]. This implies that, up until first non-trivial order, $O(\epsilon)$, all the equilibrium functions can be represented by polynomials of r/R_o for $r < r_o$. Since for our stability analysis we only need equilibrium information accurate to first non-trivial order, we can use the following general representation of the magnetic surface geometry in the flat- q region of interest:

$$R(r, \theta) = R_o \{1 + \rho \cos\theta + \rho^2 [-(2\sigma_o + \tau_o/2 + 1/2) + (\sigma_o + \tau_o/2 + 1/2)\cos 2\theta] + O(\epsilon^3)\},$$

$$Z(r, \theta) = R_o \kappa_o \{\rho \sin\theta + \rho^2 (\sigma_o + 1/2) \sin 2\theta + O(\epsilon^3)\},$$

where $\rho(r) = rR_o^{-1}\kappa_o^{-1/2}$ and κ_o, σ_o and τ_o are constants of order unity. The constants κ_o, σ_o and τ_o can be easily recognized to measure the elongation, shift and triangularity of the magnetic surfaces. The elongation and triangularity parameters κ_o and τ_o are free integration constants that are taken to be of order unity, as is the difference $\kappa_o - 1$. The shift parameter σ_o is determined by equilibrium conditions. By balancing the equilibrium equation to the required accuracy of $O(\epsilon)$, we obtain the following set of equilibrium relations that are needed in the stability analysis:

$$\frac{R_o^2 r}{T^2} \frac{dp}{dr} = \frac{r^2}{R_o^2} \left[-\frac{\beta_{po}(\kappa_o + \kappa_o^{-1})}{q_o^2} + O(\epsilon^2) \right],$$

$$\frac{R_o R j_t}{qT} = \frac{R_o^2}{r^2} \left(\frac{R^2 r}{T^2} \frac{dp}{dr} + \frac{r}{T} \frac{dT}{dr} \right) = -\frac{(\kappa_o + \kappa_o^{-1})}{q_o^2} \left(1 + \frac{2\beta_{po} r \cos\theta}{\kappa_o^{1/2} R_o} \right) + O(\epsilon^2),$$

$$2\sigma_o = \frac{2(\kappa_o^2 + 1)\beta_{po} + \kappa_o^2 - 2\tau_o}{3\kappa_o^2 + 1}.$$

Here $p(r)$ is the plasma pressure, j_t is the toroidal current density, $q_o = m/n$ and these relations serve to define the central poloidal beta parameter β_{po} . The latter is assumed to be of order unity so that our theory applies to the conventional low-beta regime, $\beta = O(\epsilon^2)$. Notice the essentially flat j_t profile that reflects our flat- q assumption.

We base our stability analysis on the ideal MHD energy principle and derive a stability criterion by minimizing the potential energy functional¹⁰ $W[\vec{\xi}]$. The fluid displacement vector $\vec{\xi}$ is assumed to vanish for $r > r_o$, i.e. outside the flat- q region. It can

also be shown that a minimizing parallel component of $\vec{\xi}$ that makes the mode incompressible, $\nabla \cdot \vec{\xi} = 0$, can be found. The minimization with respect to its components perpendicular to the magnetic field is carried out perturbatively in powers of ϵ^2 . Normalizing W to $B_o^2 R_o r_o^2 |\vec{\xi}/R_o|^2$, its leading order term is $O(\epsilon^{-2})$ and involves only the stable fast magnetosonic wave. Due to our low- β and flat- q assumptions, the instability driving terms associated with the pressure gradient and the parallel current gradient do not appear until $O(\epsilon^2)$, and the next order term, $O(1)$, again involves only the stable fast magnetosonic and Alfvén waves. The fast magnetosonic wave is suppressed by introducing a stream function Φ such that $\vec{\xi}_\perp \cdot \nabla r = \partial\Phi/\partial\theta$, and ordering the remainder $R_o T^{-1} \vec{\xi}_\perp \cdot (\vec{B} \times \nabla r) - \nabla r \cdot \nabla(r\Phi) \equiv \chi$ as $\chi/\Phi = O(\epsilon^2)$. To suppress the Alfvén wave to leading order, we must assume a long parallel wavelength, interchange-like mode. This is achieved by taking $\Phi = [\hat{\Phi}_o(r) + \hat{\Phi}_1(r, \theta)] \exp(im\theta - in\varphi)$ with $\hat{\Phi}_1/\hat{\Phi}_o = O(\epsilon)$. The $O(\epsilon^2)$ piece involves the instability drive for the first time. In this order a minimization of W with respect to χ can be carried out algebraically, thus resulting in a functional of Φ alone. All the terms involving the function $q_2(r) = O(\epsilon^2)$ can be combined into a perfect differential and eliminated by partial integration, so that the stability condition does not depend on the details of $q(r) - m/n$. After substituting for the equilibrium functions given above and carrying out some partial integrations, the potential energy functional reduces to:

$$W = \frac{\pi B_o^2 n^2}{2R_o} \int_0^{r_o} r dr \int_{-\pi}^{\pi} d\theta \left\{ D^2 \left| \frac{\partial \hat{\Phi}_1}{\partial \theta} + \frac{1}{im} \frac{\partial^2 \hat{\Phi}_1}{\partial \theta^2} \right|^2 + \frac{1}{D^2} \left| \frac{1}{im} \frac{\partial^2 (r \hat{\Phi}_1)}{\partial r \partial \theta} + u \left(\frac{\partial \hat{\Phi}_1}{\partial \theta} + \frac{1}{im} \frac{\partial^2 \hat{\Phi}_1}{\partial \theta^2} \right) \right|^2 \right. \\ \left. + \frac{(\kappa_o + \kappa_o^{-1})^2 \beta_{po} r^2}{R_o^2} K_o |\hat{\Phi}_o|^2 - \frac{2(\kappa_o + \kappa_o^{-1}) \beta_{po} r}{\kappa_o^{1/2} R_o} \left[\left(\cos\theta \hat{\Phi}_o + \frac{\sin\theta}{im} \frac{\partial (r \hat{\Phi}_o)}{\partial r} \right) \hat{\Phi}_1^* + c.c. \right] \right\},$$

where $K_o \equiv 1 - q_o^{-2} - 3(\kappa_o^2 - 1)(\kappa_o^2 - 2\tau_o)(\kappa_o^2 + 1)^{-1}(3\kappa_o^2 + 1)^{-1} + 8\beta_{po}(3\kappa_o^2 + 1)^{-1}$,

$D^2 \equiv (\kappa_o \cos^2\theta + \kappa_o^{-1} \sin^2\theta)^{-1}$ and $u \equiv -D^2(\kappa_o - \kappa_o^{-1}) \sin\theta \cos\theta$.

In the short perpendicular wavelength limit, namely $m \gg 1$ and/or $\partial\Phi/\partial r \simeq ik_r\Phi$ with $k_r r \gg 1$, W reduces to the familiar form encountered in the theory of large- n interchange and ballooning modes. Its minimization is then performed independently on each magnetic surface, where an ordinary differential equation in the poloidal variable θ results. The solution of this local Euler equation yields the Mercier criterion (1). However we are here interested in global, finite perpendicular wavelength modes for which W retains its full two-dimensional character. We shall be able to carry out the complete minimization of W in this general case. To this end we make first the change of variables:¹¹ $f_o(r) = R_o^{-1}r^2\hat{\Phi}_o$, $f_1(r, \theta) = 2^{-1}\kappa_o^{1/2}(\kappa_o + \kappa_o^{-1})^{-1}\beta_{po}^{-1}r\partial\hat{\Phi}_1/\partial\theta$. Then, after some partial integrations, we can cast W in the following form:

$$W = \frac{\pi B_o^2 n^2 (\kappa_o + \kappa_o^{-1})^2}{2R_o} \beta_{po} \int_0^{r_o} \frac{dr}{r} \int_{-\pi}^{\pi} d\theta \left\{ K_o |f_o|^2 + \frac{4\beta_{po} D^2}{\kappa_o} \left| f_1 + \frac{1}{im} \frac{\partial f_1}{\partial \theta} \right|^2 \right. \\ \left. + \frac{4\beta_{po}}{\kappa_o D^2} \left| \frac{r}{im} \frac{\partial f_1}{\partial r} + u(f_1 + \frac{1}{im} \frac{\partial f_1}{\partial \theta}) \right|^2 + \frac{4\beta_{po}}{\kappa_o} \left[f_o (\sin\theta f_1^* + \frac{\cos\theta}{im} f_1^* + \frac{r \cos\theta}{im} \frac{\partial f_1^*}{\partial r}) + c.c. \right] \right\}.$$

Now all the coefficient functions in W can be made independent of the radial variable by changing to $x(r) = \ln(r/r_o)$. Moreover, as functions of x , $f_o(x)$ and $f_1(x, \theta)$ are fast decaying when $x \rightarrow \pm\infty$. Therefore, we can express W in terms of their Fourier transforms $f_o(k_x)$ and $f_1(k_x, \theta)$. Then, minimization of W with respect to $f_1(k_x, \theta)$ yields the following ordinary differential equation in θ , with k_x as a parameter:

$$\left(1 + \frac{1}{im} \frac{\partial}{\partial \theta} \right) \left\{ \left[\left(\frac{\kappa_o + \kappa_o^{-1}}{2} \right) - \left(\frac{\kappa_o - \kappa_o^{-1}}{2} \right) \cos 2\theta \right] \left(1 + \frac{1}{im} \frac{\partial}{\partial \theta} \right) f_1 \right\} \\ - \frac{k_x}{m} \left(\frac{\kappa_o - \kappa_o^{-1}}{2} \right) \left[\left(1 + \frac{1}{im} \frac{\partial}{\partial \theta} \right) (\sin 2\theta f_1) + \sin 2\theta \left(1 + \frac{1}{im} \frac{\partial}{\partial \theta} \right) f_1 \right] \quad (2) \\ + \frac{k_x^2}{m^2} \left[\left(\frac{\kappa_o + \kappa_o^{-1}}{2} \right) + \left(\frac{\kappa_o - \kappa_o^{-1}}{2} \right) \cos 2\theta \right] f_1 + \left(\sin\theta + \frac{\cos\theta}{im} - \frac{k_x \cos\theta}{m} \right) f_o = 0.$$

Taking Eq. (2) back to W and integrating by parts we get:

$$W = \frac{\pi^2 B_o^2 n^2 (\kappa_o + \kappa_o^{-1})^2}{R_o} \beta_{po} \int_{-\infty}^{\infty} dk_x |f_o(k_x)|^2 \left[K_o + \frac{2\beta_{po}}{\pi \kappa_o} \int_{-\pi}^{\pi} d\theta \left(\sin\theta - \frac{\cos\theta}{im} - \frac{k_x \cos\theta}{m} \right) \frac{f_1(k_x, \theta)}{f_o(k_x)} \right]. \quad (3)$$

The periodic solution to the Euler equation (2) is given by the following double series:

$$f_1(k_x, \theta) = \frac{2m}{\kappa_o + \kappa_o^{-1}} f_o(k_x) \sum_{j=0}^{\infty} \left(\frac{\kappa_o - \kappa_o^{-1}}{\kappa_o + \kappa_o^{-1}} \right)^j \sum_{\substack{\ell=-(2j+1) \\ \ell \text{ odd}}}^{2j+1} c_{j,\ell}(k_x) e^{i\ell\theta},$$

with

$$c_{j,\ell} = -\frac{1}{2} \left[c_{j-1,\ell+2} \frac{k_x - i(m + \ell + 2)}{k_x + i(m + \ell)} + c_{j-1,\ell-2} \frac{k_x + i(m + \ell - 2)}{k_x - i(m + \ell)} \right],$$

and the first two coefficients $2 c_{0,-1} = [k_x + i(m - 1)]^{-1}$ and $2 c_{0,1} = [k_x - i(m + 1)]^{-1}$. This solution is taken to Eq. (3) with the remarkable result that the value of the θ -integral is equal to $-2\pi/(\kappa_o + 1)$ independent of both m and k_x . Hence the stability condition is $\beta_{po} [K_o - 4\beta_{po}\kappa_o^{-1}(\kappa_o + 1)^{-1}] \geq 0$, i.e. the Mercier criterion (1), for any $m = nq_o$ and any $f_o(r)$.

The specific radial structure of the mode given by the function $f_o(r)$ is determined by the solution of the MHD equations over the whole plasma domain, including the finite-shear region. Since the low- β interchange modes under consideration are strongly stabilized by the magnetic shear, the finite shear acts as a potential barrier that confines the mode to the flat- q region. This potential barrier quantizes the permissible $f_o(r)$'s, normally resulting in an infinite set of discrete radial eigenfunctions, and determines their specific form. In any case, only the flat- q region contributes to the incremental MHD potential energy W , due to the mode localization. Therefore the stability condition is given by the Mercier criterion (1) as evaluated before, for any toroidal number n and any radial eigenmode. This means that the marginal stability point $\omega^2 = 0$ is an accumulation point of the MHD frequency spectrum. These features have already been verified numerically.⁶

Finally we stress that, because of their finite perpendicular wavelength, our global interchange modes are not affected by finite-Larmor-radius stabilization provided the ion Larmor radius is much smaller than r_o . This is in marked contrast with the original, large- n Mercier interchanges,⁷ and provides the first opportunity for the Mercier stability criterion to have an observable, macroscopic effect on the behavior of tokamak plasmas. In summary, under rather general conditions we have obtained a robust, large scale instability which is likely to play an important role in elongated tokamaks if the prevalent transport processes produce flat central q -profiles. A possible scenario has this instability triggering the sawtooth internal disruption which in turn results in a flattened q -profile, thus a self-sustaining mechanism is established.

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