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STEADY STATE MHD CLUMP TURBULENCE

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ABSTRACT

The turbulent steady state of the MHD clump instability is investigated. Magnetic helicity conservation plays a decisive role in the steady state. The helicity invariant constrains the turbulent mixing of the mean magnetic shear driving the instability and modifies the instability growth rate. The steady state is determined by the balance between this helicity conserving growth by turbulent mixing and clump decay by field line stochasticity. The dynamical balance occurs when the mean current and magnetic field satisfy $\underline{J}_0 = \mu \underline{B}_0$, where μ depends on the mean square fluctuation level. Above this critical point ($\underline{J}_o > \mu \underline{B}_o$), the plasma is MHD clump unstable. This self-consistent generation of fields during MHD clump instability is a turbulent dynamo action. MHD clump instability is a <u>dynamical</u> route to the force free, Taylor state. For the steady state to exist, μ must exceed a threshold on the order of that required for B_{oz} field reversal. Only these Taylor states correspond to steady state MHD clump turbulence. From the μ threshold condition, the steady state fluctuation spectrum ($\delta B_{rms}/B$) is calculated and shown to increase with mean driving current as μ^3. The onset of the steady state corresponds to a phase transition where $\mu_c = \underline{J}.\underline{B}/B^2$ is the critical point. Fluctuation intermittency is discussed.

I. INTRODUCTION

This is the second paper in a series of three papers on the MHD clump instability. In the first paper (Ref. 1), we described MHD clump fluctuations and their instability to growth. The fluctuations are produced in an MHD plasma when the mean magnetic field shear is turbulently mixed. The turbulence transports a magnetized fluid element to a new region in the plasma where the mean energy density differs from that of the element's point of origin. The fluctuations are localized at the shear resonances of the plasma where the decay effect of shear Alfven wave emission is minimal. In isolation, the fluctuation is a hole ($\delta J_z < 0$) in the longitudinal current density $\bar{J}_z$. As the holes resonantly interact, their magnetic island structures become disrupted by magnetic field line stochasticity. Energy in the localized magnetic structures become dissipated as shear Alfven waves propagate down the stochastic field lines. This decay can be overcome as new fluctuations are regenerated by the turbulent mixing. The net growth rate of the mean square fluctuation level is of the form

$$\gamma = \frac{1}{\tau}(\mathbf{R} - 1) \tag{1}$$

where **R** is the mixing rate and τ is the Lyapunov (decay) time of the stochastic fields. Net growth (instability) occurs when **R**>1. Equation (1) can be cast into a perhaps more familiar form by recalling from Ref. 1 that the mixing rate is nonlinear and, therefore, evolves with the growth of the fluctuation level. In particular,

$$\mathbf{R} = \hat{\mathbf{R}}\,(1 + \gamma\tau)^{-1} \tag{2}$$

where

$$\hat{\mathbf{R}} = \frac{1}{\Delta_c' x_d} \tag{3}$$

Here, x_d is the turbulent resonance (island) width which generalizes the resonance width of an isolated island. x_d scales as the cube root of the field line diffusion coefficient (see (92)) and reduces to the island width Δx for the case of a single resonance. Δ_c' is a nonlinear version of the stability parameter ($\Delta_{\underline{k}}'$) of linear tearing mode theory and[2,3] gives the free energy available for nonlinear clump growth. With (2) and (3), (1) can be rewritten as

$$(\gamma + \tau^{-1})^2 = \hat{\mathbf{R}}/\tau^2 \tag{4}$$

(4) is analogous to the growth rate for the Rayleigh-Taylor interchange ("mixing") instability in magnetized fluids.[3,4] The stochastic decay (inverse Lyapunov time, τ^{-1}) plays the role of the restoring force to field line bending (shear Alfven emission rate) and the magnetic shear driving term ($\hat{\mathbf{R}}$) plays the role of the density gradient of light and heavy fluid. For large amplitude, fully stochastic fields, the Lyapunov time is short, and (4) takes the form

$$\gamma \sim \frac{\Delta_c'}{x_d} D\,(\hat{R} - 1) \tag{5}$$

where $D = x_d{}^2/\tau$ is the spacial diffusion coefficient of the stochastic field lines. The factor $(\Delta_c' D/x_d)$ in (5) resembles the growth rate of a tearing mode in the so-called Rutherford regime[5] but driven by a turbulent resistivity, D. It is an anomalous field line reconnection rate due to the stochasticity. The factor $(\hat{R}-1)$ describes the net regeneration of fluctuations by mixing even as existing fluctuations stochastically decay ("-1").

The nonlinear theory of Ref. 1 describes the strong resonant interaction (anomalous reconnection) of magnetic islands at high Reynolds numbers R_m. While, in the presence of weak collisional dissipation (i.e., $R_m \to \infty$), the theory conserves the total energy and cross helicity, magnetic helicity conservation is neglected. This flaw is remedied in this paper. Global conservation of magnetic helicity constrains the dynamical mixing of the mean shear and, therefore, $\hat{R}$ in the growth rate (5). We calculate the effect of this constraint below and find that Δ_c' in (3) is given by (116). If we define Δ' as the average value of $\Delta_{\underline{k}}'$ for unstable clumps of wave number $\underline{k}$, then, for clumps with island widths $\Delta x < k^{-1}$, $\hat{R}$ is approximatley given by

$$\hat{R} \sim \frac{1}{\Delta' x_d} \left[-\Delta x^2 \frac{\nabla^2 J_{oz}}{J_{oz}}\right] \tag{6}$$

where J_{oz} is the mean current density. The expression in brackets here is the correction due to magnetic helicity conservation. It can be understood intuitively (see next section) as a helicity modification to the mean electric field (E_{oz}) driving the clumps. Rather than a nonlinear Ohm's law of the form $E_{oz} = DJ_{oz}$ used in Ref. 1, magnetic helicity conservation constrains E_{oz} to be of the form $E_{oz} \sim - D(\Delta x)^2\nabla^2 J_{oz}$. An important consequence of (6) is that, for driven steady state MHD clump turbulence ($\hat{\mathbf{R}} = 1$ in (5)), J_{oz} satisfies

$$\nabla^2 J_{oz} + \mu^2 J_{oz} = 0 \tag{7}$$

where $\mu^2 \sim \Delta' x_d/\Delta x^2$ for $k\Delta x<1$. In the general case, we show in Section V that the mean vector current density $\underline{J}_o$ also satisfies (7) so that, for a wide stochastic spectrum where μ would be independent of position, $\underline{J}_o = \mu\underline{B}_o$ in the steady state. This relation is known elswhere as the Taylor state[6], but here plays the role of the stability boundary for the MHD clump instability.

The derivation of the magnetic helicity conserving source term **R** and its consequences for steady state MHD clump turbulence are the main objectives of this paper. The detailed derivations from the MHD equations are presented in Parts II and III. However, we first continue this Introduction with a brief review (Sec. IA) of the dynamical equations developed in Ref. 1 and of the importance of the conservation laws, in particular that of magnetic helicity. The helicity conserving MHD clump theory has an enlightening relationship to turbulent dynamo models[7], and this is discussed in Sec. IB. A physical discussion of the Taylor state equation, as well as onset conditions, mixing length relations and amplitude scalings for the turbulent steady state is presented in Sec. IC. The transition to MHD clump turbulence and its similiarity to plane Poisuille fluid flow is discussed in Sec. ID. Part IV deals with the possibility of current hole intermittency in MHD clump turbulence. Similarities with modon and phase space density hole intermittency in (respectively) fluid and Vlasov plasma turbulence are discussed.[8-11] In the Appendix, we discuss an interesting analogy between phase transitions and the onset of MHD clump instability.

A. Statistical Dynamics and Conservation Laws

The dynamical equations describing MHD clump turbulence are, of necessity, statistical. The two point fluctuation correlation function plays an essential role in the dynamical model. This correlation function follows from the conservation of energy and satisfies an equation of the form[1]

$$\left[\frac{\partial}{\partial t} + T(1,2)\right] C(1,2) = S(1,2) \tag{8}$$

A detailed derivation of (8) was carried out in Ref. 1. (The results are briefly outlined in Sec. III below). T(1,2) is, in the simplest case, a diffusion operator describing the resonant interaction between islands. In particular, it describes the exponential divergence of neighboring stochastic magnetic field lines or, equivalently, the mode coupling (cascade) of energy to high wave numbers. T(1,2) is a turbulent dissipation rate of fluctuation energy and is on the order of τ^{-1}. Note that the exponential divergence of field line orbits--sometimes referred to as orbit stochastic orbit instability[12]--describes the decay or "falling-apart" of the fluctuations. The growth of the fluctuations arise from S(1,2). The quantity S(1,2) is the source of fluctuations resulting from the turbulent mixing of the mean magnetic shear. It converts ordered (mean) equilibrium energy into turbulent fluctuations. The growth rate (1) follows from the solution of (8)--the term **R** deriving from S(1,2), and the "-1" term from T(1,2). The use of the Direct Interaction Approximation (DIA)[12,13] in models of fluid turbulence also yields equations of the form (8), with T(1,2) describing mode coupling via a turbulent viscosity, but with S(1,2) taken as a prescribed forcing function. Here, S(1,2) is more akin to mixing length models of fluid turbulence,[14] i.e., the mean-square clump energy, C(1,2), is determined self-consistently by turbulent mixing of the mean field shear. The model stresses this self-consistent production of the clump fluctuations produced by S(1,2) rather than the mode coupling spectra determined by T(1,2) alone. We think of (8) conceptually as a marriage between the DIA and mixing length models.

The T(1,2) and S(1,2) terms in (8) are derived from a renormalized perturbation theory and, therfore, only approximate the nonlinear terms in the exact MHD equations. However, the T(1,2) and S(1,2) terms must conserve the dynamical invariants of the exact equations. This is necessary for the preservation of the essential physics and, in particular, for a proper treatment of the mode coupling. In the absence of resistivity and viscosity, the invariants are total energy, cross helicity, and magnetic helicity.[11] In Ref. 1, we showed that the nonlinear mode coupling is treated by T(1,2) in a way that energy and cross helicity are conserved. Here, we show that a proper treatment of the turbulent mixing of the magnetic shear described by S(1,2) maintains the conservation of magnetic helicity. The situation is analogous to that of Vlasov turbulence where T(1,2) maintains phase space density conservation of the exact Vlasov equation, while S(1,2) ensures the conservation of momentum.[15] A mixing length-mode coupling equation such as (8) which follows the nonlinear evolution of fluctuations subject to dynamical invariants is a general feature of clump models of turbulence.

Collisional resistivity and viscosity also contribute to T(1,2) in (8). Their presence in MHD allows for changes in magnetic field topology that the Ohm's law

$$\underline{E} + \underline{V} \times \underline{B} = 0 \tag{9}$$

of ideal MHD prohibits ($\underline{E}$ and $\underline{V}$ are electric field and fluid velocity). Because of (9), magnetic field lines are frozen into the fluid flow, so magnetic flux surfaces are preserved. With collisional dissipation, field lines can "slip" from the flow, break and reconnect.[16] In a course-grained statistical sense, this also can occur in the presence of stochasticity. The stochastic bending and twisting of a field line down to finer and finer scale lengths will, when taken below the scale of the course graining, appear as a dissipation. This is the meaning of the inviscid part of T(1,2) in (8). The situation is similar to that of Vlasov turbulence. There, the exact Vlasov equation is time reversible and prohibits the breaking of orbit trajectories (contours of constant phase space density do not break). However, statistical course graining introduces irreversibility and dissipation (e.g., as in the Quasilinear Theory).[17] This irreversible mixing of phase space contours carrying different density reduces the mean, course-grained phase space density. A clear example is given in Fig. 5 of Ref. 18 where a time sequence of phase space density contours is shown. In the figure, the finite spacial grid used to solve for the contours also causes a course graining. As the phase space islands interact, their contours break as they become mixed and twisted down to scales less than that of the grid size. Magnetic flux contours in MHD can be similarly dissipated.

Since the energy in MHD is dissipated at a faster rate than magnetic helicity[11], we view (8) as the dissipation of the energy "invariant" by turbulent mixing subject to the constancy of the more "rugged" invariant, magnetic helicity. Magnetic helicity is conserved only in the volume averaged sense. Recall that magnetic helicity is conserved in ideal MHD if

$$\int d\underline{x}\ \underline{E}\cdot\underline{B} = 0 \qquad (10)$$

(10) is trivially satisfied because, from (9), $\underline{E}\cdot\underline{B} = 0$ on each flux surface. However, we take the view that, because of the course-graining or

collisional dissipation, only the flux surface at the conducting shell surrounding the plasma is preserved. Therefore, only the magnetic helicity associated with the total plasma volume is invariant. The situation is analogous to that of Vlasov turbulence where momentum can be exchanged locally but total momentum must be preserved globally,

$$\frac{\partial}{\partial t} \int dx \int dv \; vf = 0 \tag{11}$$

where the integrals are taken over the total plasma volume. Consider a cylindrical MHD plasma with sheared poloidal field, $B_{o\theta}(r)$, and strong longitudinal field, $B_{oz} \gg B_{o\theta}(r)$. Then, the helicity constraint (10) for the mean field becomes

$$\int d\underline{x} \; E_{oz} = 0 \tag{12}$$

which, when used in Faraday's law for $B_{o\theta}(r)$, yields a global constraint on the turbulent mixing of $B_{o\theta}(r)$,

$$\frac{\partial}{\partial t} \int d\underline{r} \; r \; B_{o\theta}(r) = 0 \tag{13}$$

where again, the integrals are taken over the plasma volume. The constraint (13) is the MHD clump analogue of (11) for the Vlasov case. The use of global helicity invariance has been previously proposed by Taylor.[6]

The two point energy conservation equation (8) is a Poynting theorem for the fluctuation energy. The T(1,2) term is the Poynting flux of fluctuation energy in and out of the volume. The S(1,2) term is the rate at which fluctuation energy is produced inside the volume by the turbulent mixing of the mean field shear, i.e., S(1,2) is the rate at which the energy in the mean field is dissipated into turbulent fluctuations inside the volume. In the simplest case, S(1,2) is equal to $E_{oz} J_{oz}$, where E_{oz} and J_{oz} are the mean longitudinal electric field and current density. The conservation of magnetic helicity constraining S(1,2), therefore, constrains the electric field profile $E_{oz}(\underline{x})$. In this paper, we show that the portion of the mean field Ohm's law due to the turbulence is

$$E_{oz} = D J_{oz} - F B_{oz} \tag{14}$$

where D and F are diffusion (turbulent resistivity) and dynamical friction (drag) coefficients. Only the D term in (14) was considered in Ref. 1. Inserting (14) into Faraday's Law and using (4) yields a time evolution equation for the mean magnetic field B_{oy} that is of the Fokker-Planck type. The diffusion term of this Fokker-Planck equation, coming from the D term of (14), describes the random motion of the holes. The second term of the equation, coming from the F term in (14), describes their self-consistent (correlated) motion. Because of magnetic helicity conservation, the two terms are connected--in the limit of zero resonance width, the D and F terms in (14) cancel. The resonant interactions between the holes lead to no net transport of the mean field $\underline{B}_o$. In this regard it is useful to think of these interactions as "collisions" between holes, i.e., hole-hole (island-island) collisions. The situation is analogous to the vanishing of the Fokker-Planck collision operator for identical particles in a one dimensional Vlasov plasma: collisions between like particles lead, because of momentum conservation, to no net transport of the mean particle distribution.[15] We show below that net transport from random hole collisions occurs in (14) at second order in the resonance width:

$$E_{oz} = - D(\Delta x)^2 \nabla_\perp^2 J_{oz} \tag{15}$$

where $\overline{\Delta x^2}$ is the mean-square step size (on the order of the island width squared). Using Ampere's law, insertion of (15) into Ampere's law yields a fourth order diffusion equation for magnetic field line diffusion. A similar cancellation of lowest order particle fluxes occurs in the guiding center plasma where like-like collisions cause transport (fourth order diffusion) at second order in the gyro radius.[19] The constraint of helicity conservation on Ohm's law has also been considered by Boozer.[20]

D. Turbulent Dynamo

The D and F terms of the clump model Ohm's law (14) can be interpreted as the β and α coefficients of dynamo theory[7]. In dynamo theory, the mean electric field is calculated from the fluctuating flow velocity, $\delta \underline{V}$, and magnetic fields in Ohm's law,

$$\underline{E}_0 = -\langle \delta \underline{V} \times \delta \underline{B} \rangle \qquad (16)$$

where <> denotes an ensemble average. Frequently, the view taken in dynamo theory is kinematic rather than dynamic (i.e., self-consistent) in that the $\delta \underline{V}$'s are prescribed and the δB's are derived from these flows using Faraday's law. The result is that (16) can be written as

$$\underline{E}_0 = \beta \underline{J}_0 + \alpha \underline{B}_0 \qquad (17)$$

While β depends on the mean square flow, α depends on the flow field and its derivatives. Consequently, the so-called α-effect, due to a non-zero α in (17), only occurs if the statistical properties of this background flow field lack reflexional symmetry.[7] In the clump model, we interpret (17) dynamically, i.e., as the self-consistent, helicity conserving Fokker-Planck process (14). We consider a "test particle" picture where the flow, $\delta \underline{V}$, is the sum of two terms: $\delta \tilde{\underline{V}}$ due to the presence of, self-consistent island structures, and $\delta \underline{V}^c$, the response that is phase coherent with the fields of background islands. With force balance used to evaluate $\delta \underline{V}$ in (16), the first term of (14) comes from $\delta \underline{V}^c$ and the second term from $\delta \tilde{\underline{V}}$. The D term is proportional to the mean square $\delta \underline{B}$ and would, therefore, be present for any $\delta \underline{B}$. The F term, however, is nonzero because the fields are correlated self-consistently in the island structure. This self-consistency causes the lowest order cancellation between the D and F terms of (14), thus yielding the magnetic helicity conserving form (15), the "net" α-effect. We demonstrate this cancellation for steady state turbulence in Sec. II. The nonvanishing of the F term in (14) can also be traced to the statistical prevalence of the current holes (over the "anti-holes"). Thus, the breaking of reflextion symmetry, as required by a turbulent dynamo, is achieved in the clump model by the formation and preferential growth of the current holes (As discussed in Sec. IB of Ref. 1, "anti-hole" fluctuations, $\delta J>0$, decay).

Comparison with the homopolar disc dynamo is enlightening.[7] A solid copper disc rotates about its axis with angular velocity Ω, and a current path between its rim and axle is made possible by a wire twisted in a loop around the axle (see Fig. 1.1 of Ref. 7). Rotation of the disc causes an electromotive force $M\Omega I$ which drives a current I in the loop given by

$$L \frac{dI}{dt} + RI = M\Omega I \tag{18}$$

where M is the loop/rim mutual inductance, and L and R are the self-inductance and resistence of the complete circuit. The system can be unstable to growth of current and magnetic fluctuations (MI is the magnetic flux induced across the disc.). Growth occurs when the source of free energy exceeds the dissipation rate, i.e., when $\Omega > R/M$ in (18). Ultimately, the disc rotation slows to the critical value $\Omega_c = R/M$, and a steady state is achieved. The situation is similar to that of the MHD clump instability where the mixing rate of the mean shear plays the role of the driving frequency Ω, and the stochastic decay (turbulent resistivity) corresponds to the resistance R in the disc/loop circuit. Growth of magnetic clump fluctuations (see (1)) occurs when turbulent mixing overcomes stochastic decay, i.e., $R>1$. Quasilinear relaxation of the shear gradients lowers R to the critical, steady state value $R_c = 1$. Multiplying (20) by I/R gives

$$\left(\frac{d}{dt} + \frac{R}{L}\right) I^2 = 2 \left(\frac{M}{L}\right)\Omega I^2 \tag{19}$$

(19) is analogous to (8) for the clump magnetic energy ($C \sim \langle\delta B^2\rangle$ in (8)). The resistive decay rate R/L corresponds to the clump stochastic decay rate $T(1,2) \sim \tau^{-1}$. The driving term on the right hand side of (19) is analogous to the clump source term S of (8). Note that $S \sim E_{oz} \sim D \sim \langle\delta B^2\rangle \sim C$ from (15). As in the disc dynamo where we must have $\Omega > 0$ for growth, a preferred sign for forcing is provided by $E_{oz} \sim -\nabla_\perp^2 J_{oz} > 0$ inside the plasma.

The MHD clump instability is a turbulent dynamo, but not of the usual type. Typically, dynamo models sustain or increase the mean magnetic field at the expense of currents flowing across the field. Were it not for the D term in (14), the F term would cause an increase in the mean magnetic field. However, the mean field does not increase because, to lowest order, the α-effect is balanced by the stochastic diffusion of the field lines. This balance is the result of magnetic helicity conservation. To next order (in the island width), the mean magnetic field decays according to the turbulent mixing process (15). The energy lost from the mean field goes, because of energy conservation, into the creation of the clump fluctuations. This degrading of the mean field occurs in the interior of a confined plasma where $\nabla_\perp^2 J_{oz} < 0$ in (15). In the exterior region where $\nabla_\perp^2 J_{oz} > 0$, (11) tends to support the mean field and, therefore, acts in the spirit of a traditional α-effect. The net effect of (15) is to expel mean poloidal flux from the plasma.

C. Steady State Turbulence - The Taylor State

The constraint of magnetic helicity conservation has a crucial effect on steady state MHD clump turbulence. As the instability proceeds, the quasilinear relaxation of the mean magnetic field gradients will drive the instability source term **R** toward zero. The net α-effect (15) will, therefore, vanish. This will result in decaying turbulence where the energy decays by mode coupling (i.e., the "-1" stochastic decay term in (5)) down to finer and finer scale lengths. However, this decay can be overcome if the mean field gradients and, therefore, the α-effect are maintained. This is the critical value $R_c = 1$ noted above. The turbulence is then driven, and a steady state turbulence (dynamo action) is possible. Of interest then is the structure of the driven clump fluctuations rather than the detailed features of the broad spectra produced in the case of decaying turbulence. The steady state clump fluctuation level follows from (1) and (6), and occurs when $\hat{\mathbf{R}} = 1$, i.e.,

$$\nabla_{\perp}^2 J_{oz} + \mu^2 J_{oz} = 0 \tag{20}$$

where

$$\mu^2 \sim \Delta' x_d \left(k_o^2 + \frac{1}{\Delta x^2}\right) \tag{21}$$

For low mode number, small amplitude holes $k_o \Delta x < 1$ and (21) gives $\mu^2 \sim \Delta'/x_d$ in the fully stochastic case. Then, multiplying (20) by Dx_d^2, and using (15) gives

$$E_{oz} + E_z^c = 0 \tag{22}$$

where $E_z^c = D\,\delta J_z^R = -\,D\Delta'\, x_d\, J_{oz}$ is the force turbulently dissipating the holes and

$$-\delta J^R = \Delta' x_d \; J_{oz} \tag{23}$$

is the root mean square hole depth necessary to form a trapped island structure (see Ref. 1). Since E_{oz} is the mean field force creating the holes, (22) is a statement of mean force balance. In addition to being a steady state (dynamo) condition on the mean-square fluctuation level (μ), (20) can also be thought of as a global equilibrium condition on the mean current density J_{oz}. In cylindrical coordinates, (20) gives $J_{oz} \sim J_o(\mu r)$ where J_o is the zeroth order Bessel function. A more enlightening view of this is obtained by considering poloidal as well as toroidal currents. An approximate calculation for this case is carried out in Sec. V. The equilibrium relation between the mean current profile and the mean-square fluctuation level is found to be a vector generalization of (20):

$$\nabla^2 \underline{J}_o + \mu^2 \underline{J}_o = 0 \tag{24}$$

Because Δ' and x_d are spectral averaged quantities, μ is relatively insensitive to position inside a broad, fully stochastic spectrum. (See the end of Sec. V for further discussion of this point). Therefore, we set μ = constant, and with $\underline{\nabla}.\underline{J}=0$, (24) has for a solution,

$$\underline{J}_o = \mu \underline{B}_o, \tag{25}$$

yielding again a force-free state ($\underline{J}_o \times \underline{B}_o = 0$). The global force balance relation (25), the so-called Taylor state, has been known previously as the MHD state of minimum mean energy with a given constant mean magnetic helicity[6]. That the solution (25) should result from steady state MHD clump turbulence is not surprising, since the clump dynamics minimize the mean energy via turbulent mixing subject to the constraint of global magnetic helicity conservation. Note that, in the clump theory, the parameter μ in (25) is a prescribed function (21) of the turbulence level. The steady state is turbulent. Equation (25) prescribes the mean profiles in terms of the mean-square fluctuation level (μ). The fluctuations generating the dynamo action are created self-consistently by the balance between turbulent mixing ($\underline{J}_o$) and the decay ($\mu \underline{B}_o$) caused by field line stochasticity.

Because μ depends on the mixing length x_d, (20) relates the mixing length to the shear. This relation follows if we multiply (20) by x_d^2 and use (23):

$$\delta J_z^R \sim x_d^2 \nabla_\perp^2 J_{oz} \tag{26}$$

Eq. (26) is actually an equivalent form of the MHD clump steady state mixing length relation

$$\langle \delta B_x^2 \rangle \sim - D\tau x_d^2 J_{oz} \nabla_\perp^2 J_{oz} \tag{27}$$

(27) follows from the steady state integration of (8) with the use of (15) for $S(1,2) = E_{oz} J_{oz}$ and $T(1,2) \sim \tau^{-1} \sim D/x_d^2$. Using the hole width, $x_d \sim (\delta\psi/J_{oz})^{1/2}$, and $\delta B_x = \partial\psi/\partial y \sim \delta\psi/\Delta y$ and $\delta J_z = -\partial\delta B_x/\partial y \sim -\delta B_x/\Delta y$, (27) reduces to (26). Equations (26) and (27) differ from the usual form of mixing length relations because, here, the mixing process is constrained by helicity conservation. This can be seen by integrating (8) in steady state, with only the diffusive term in (13) retained in S. Then, we obtain a mixing length relation of the standard form: $\delta B_x^2 \sim D\tau J_{oz}{}^2 \sim x_d^2 J_{oz}{}^2$. However, because of helicity conservation, each term in (14) must be retained--leading to the use of (15) in S(1,2) and the result (27).

Not all μ values (21) are consistent with the steady state. Solving (21) for x_d gives

$$x_d = \frac{1}{2\Delta' k_o^2} \left[\mu^2 \pm (\mu^4 - 4\Delta'^2 k_o^2)^{1/2}\right] \tag{28}$$

There are no real solutions for x_d (and, therefore, D) unless $\mu^2 > 2\Delta' k_o$. For reasonable values of Δ' and k_o, this means μ will be an order one quantity. At the threshold $\mu^2=2\Delta' k_o$, and $k_o x_d = 1$. Above the threshold, $k_o x_d > 1$ and

$$x_d \sim \mu^2/\Delta' k_o^2 \tag{29}$$

Note that μ's on the order of one are comparable to the μ values required for a B_{oz} field-reversed state, i.e., the solution $B_{oz} = J_{oz}/\mu \sim J_o(\mu r)$ to (25) reverses sign when $\mu r>2.4$. Therefore, of all the possible Taylor states (μ), the ones with reversed B_{oz} field correspond to steady state MHD clump turbulence. Smaller μ's apparently correspond to MHD clump instability.

At the threshold of the clump steady state, the mixing length relation (27) reduces to

$$\delta B_x \sim x_d J_{oz}\, k_o\, x_d \tag{30}$$

(30) follows from (27) by using (23), (20) for $\nabla_{\perp}^2 J_{oz}$, and $\mu^2 \sim \Delta' k_o x_d{}^2 \sim \Delta'/x_d$ at the threshold ($k_o x_d=1$). Since (23) can also be written as $\delta B_x \sim x_d J_{oz} k_o x_d$, (30) is just the fluctuation level necessary for the formation of trapped island structures. Since $k_o x_d = 1$ at the threshold, (30) is, at threshold, just the standard mixing length relation

$$\delta B_x \sim x_d J_{oz} \tag{31}$$

A similar result occurs for Vlasov holes characterized by a velocity trapping width Δv.[21] There, the mixing length level $\delta f \sim \Delta v \partial f_o/\partial v$ is the same order as the fluctuation level necessary for trapped hole formation, i.e., $\delta f \sim (\Delta v/v_{th})\bar{f}_o$. Therefore, in steady state turbulence, the fluctuations can "just barely" self-organize before they become dissipated by the turbulence. The turbulence is thus composed of colliding, growing, "amorphous" hole structures: clumps.

In steady state, the diffusion coefficient $D \sim x_d^2/\tau$ can also be written as $D \sim \gamma_{n\ell}/k_\perp^2$, since $k_\perp \sim x_d^{-1}$ is the typical radial wave number and $\gamma_{n\ell} \sim R/\tau \sim \tau^{-1}$ is the characteristic mixing rate. Were it not for the constraint of helicity conservation, $\gamma_{n\ell}/k_\perp^2$ would be the cross field diffusion coefficient, $D_\perp$, for the mean field flux, ψ_o. However, $D_\perp < D$ as can be seen by combining (15) with Faraday's law:

$$\frac{\partial \psi_o}{\partial t} = \frac{\partial^2}{\partial x^2} D(\Delta x)^2 \frac{\partial^2}{\partial x^2} \psi_o \tag{32}$$

Therefore, $D_\perp \sim D(\Delta x/a)^2 \sim (\gamma_{n\ell}/k_\perp^2)(\Delta x/a)^2$ where a is the radius of the current channel. An analogous reduction in cross field transport occurs in a guiding center plasma of identical particles.[19] Equation (32) is derived from the MHD equations in Sec. II and Sec. III. A quasilinear equation of the form (32) has also been derived for an assumed spectrum of tearing modes by Strauss and Bhattacharjee.[22,23]

D. <u>Transition to Turbulence</u>

Of course, the steady state condition (24) also describes the threshold condition for the onset of the instability. Recalling (5), it is enlightening to write (24) for arbitrary $\hat{\mathbf{R}}$,

$$\nabla^2 \underline{J}_o + \hat{\mathbf{R}} \mu^2 \underline{J}_o = 0 \tag{33}$$

with a solution corresponding to (25) of

$$\underline{J}_o = \hat{\mathbf{R}}^{1/2} \mu \underline{B}_o \tag{34}$$

Therefore, for a given μ, (i.e., amplitude), instability occurs when the mean driving current density exceeds the critical value

$$\underline{J}_c = \mu \underline{B}_o \tag{35}$$

For analysis of this instability threshold, it is useful to include additional dissipation effects such as collisional resistivity (η_{sp}) and viscosity (ν). Assuming a unit magnetic Prandtl number, the additional dissipations change $T(1,2) \sim \tau^{-1} \sim D/x_d^2$ in (8) to the net dissipation rate of $(D + \eta_{sp})\, x_d^2 = (1 + R_m^{-1})\tau^{-1}$, where $R_m = D/\eta_{sp}$ is the Reynolds number. The growth rate (5) then becomes

$$\gamma \sim \eta_a \frac{\Delta'}{x_d} (\hat{\mathbb{R}} - 1 - \frac{1}{R_m}) \tag{36}$$

Instability occurs when the mixing overcomes both collisional and turbulent dissipation. While we think of the "-1" term in (36) here as a mode coupling rate due to "hole-hole collisions", it is also useful to think of it as a turbulent eddy viscosity as in fluid turbulence.[12,13] Then, the effective viscous damping rate is $(1 + R_m^{-1})\tau^{-1}$. With these modifications, the threshold condition becomes $\underline{J}_o > \underline{J}_c$ where

$$\underline{J}_c = \mu\left(1 + \frac{1}{R_m}\right)^{1/2} \underline{B}_o \tag{37}$$

Thus, the instability condition on the Reynolds number is

$$R_m > \left[\left(\frac{J_{oz}}{\mu B_z}\right)^2 - 1\right]^{-1} \tag{38}$$

where we have considered only the longitudinal part of the current. Instability (MHD clump regeneration) occurs for a given amplitude (μ) and current profile if the Reynolds number exceeds the critical value given by the right hand side of (38). Since, for $k_o \Delta x < 1$, μ decreases with increasing amplitude, smaller Reynolds numbers require larger amplitudes for the onset of turbulence. The threshold is evidently nonlinear.

Consider the instability threshold in the case of two large islands located at different mode rational surfaces. When the island resonances begin to overlap, the region of initial stochasticity will be small compared to the island widths. The effect of this intermittent region of stochasticity is to replace the "-1" in (36) and (38) with a factor p_s, where p_s denotes the percentage of stochasticity compared to the island width. A further modification is due to the fact that parallel currents--and, therefore, two current holes--attract each other. This tendency for island coalescence will tend to inhibit the mode coupling decay rate of the islands. In order to account for this tendency, we further multiply the "-1" in (36) and (38) by the factor r_c, where $r_c<1$. The phenomenological p_s and r_c factors also occur in Vlasov hole turbulence. There, hole intermittency reduces the decay rate due to hole-hole collisions, while the attraction between holes causes hole fragments (produced from hole-hole collisions) to recombine into new holes. The magnetic island coalescence instability is the analogue of the Jeans instability for Vlasov holes.[18] The net result here is that, for $p_s r_c<<1$, the $J_{oz}/\mu B_z$ term dominates in the denominator of (38). Since μ is given by (21), and a critical amplitude (island width) is required for island overlap, the stability boundary is of the form depicted in Fig. 1, a form reminscent of plane Poisuille flow.[25] As discussed in the Appendix, Fig. 1 can be viewed as a phase diagram for the "phase transition" to steady state MHD clump turbulence.

In a toroidal plasma, the stability condition $J_o<J_c$ sets a lower bound on the safety factor $q(a) = aB_z(a)/RB_\theta(a)$, where (a,R) are the (minor, major) radii of the plasma and (B_z, B_θ) are the (toroidal, poloidal) magnetic fields. This can be seen by integrating (35) over the plasma (minor) cross section to obtain

$$q_c(a) = 2/R\bar{\mu} \tag{39}$$

where $\bar{\mu}$ is μ averaged over the plasma cross section. Stability occurs if $q(a) > q_c(a)$. An alternate view of the instability threshold follows from $\bar{\mu} < \mu_c = 2/Rq(a)$. Since $\mu_c \sim q(a)^{-1}$ increases with driving current, instability results when too much current is driven for a given Taylor state (μ). For example, in a tokamak fusion device, large amplitude, low mode number magnetic islands frequently develop.[3] Upon overlap, strong instability ("disruption") can be expected if this current threshold is exceeded. In the reversed field pinch fusion device, by contrast, the overlapping islands have smaller initial amplitudes so that $\bar{\mu}$ will be comparable to μ_c (i.e., $\hat{R}$ comparable to one). The plasma will be near the Taylor state and thus relatively quiescent. With $\bar{\mu} \sim \mu_c$, (28) implies that an increase in driving current will just push the clump turbulence to higher fluctuation levels.

II. FIELD SELF-CONSISTENCY

As discussed in Sec. I, the Fokker-Planck structure of the mean field (14) results from the self-consistent generation of the clump fluctuations. It is this self-consistent structure that ensures the global conservation of magnetic helicity. We show this here by considering the simplified but illuminating case of a spacially stochastic fluctuation spectrum that is time stationary. A complimentary derivation from the time dependent MHD equations is presented in Sec. III.

We begin by taking the $\hat{z}$ component of the curl of the equation for steady state MHD momentum balance. Using slab geometry and the model sheared field[26] of "tokamak ordering" ($\underline{B}=\underline{B}_o + \delta\underline{B}$, $\underline{B}_o = \underline{B}_{\perp o} + \hat{z}B_{oz}$, $\underline{B}_{\perp o} = \hat{y}B_{oy}(x)$, B_{oz} = constant), we obtain

$$\underline{B} \cdot \underline{\nabla} J_z = 0 \tag{40}$$

or, more explicitly,

$$\frac{\partial J_z}{\partial z} + \frac{B'_{oy} x}{B_{oz}} \frac{\partial J_z}{\partial y} + \frac{\delta B_x}{B_{oz}} \frac{\partial J_z}{\partial x} = 0 \tag{41}$$

where we have retained only the δB_x component of $\delta\underline{B}$ for simplicity. Since we are considering self-consistently generated fields, we must couple (41) to Ampere's law

$$\nabla_{\perp}^2 \psi = - J_z \tag{42}$$

where $\underline{B}_{\perp} = \underline{\nabla}x(\psi\hat{y})$ defines the poloidal flux function ψ. Because of the self-consistency, we write the current fluctuation as a sum of two parts, $\delta J_z = \delta J_z^c + \tilde{J}_z$. $\tilde{J}_z$ describes the source of fluctuations in ψ generated via (42), while δJ_z^c describes the response to these fields via (41). In particular, $\tilde{J}_z$ describes the resonant clump fluctuations generated self-consistently by turbulent mixing at the mode rational surfaces, while δJ_z^c

describes the nonresonant currents flowing in response to the clumps. The decomposition is similar to that for an isolated current hole discussed in Sec. IA of Ref. 1 where the coherent analogues of $\tilde{J}_z$ and δJ_z^c describe the currents flowing (respectively) inside and outside the magnetic island structure of an isolated hole. In the case of stochastic fields, it is useful to treat (41) as a Vlasov equation for field line trajectories where z plays the role of time.[27] For weak fields, the response δJ_z^c can then be calculated as in the Quasilinear theory. Neglecting $\tilde{J}_z$, one obtains from the fluctuating part of (41)

$$\delta J_{\underline{k}}^c = i \frac{\delta B_{\underline{k}}}{\underline{k}\cdot\underline{B}_o} \frac{\partial}{\partial x} J_{oz} \tag{43}$$

where $\delta J_{\underline{k}}^c$ is the Fourier transform of δJ_z^c. Equation (43) is the usual current response of linear tearing mode theory.[2,5] Substitution into the ensemble averaged version of (41),

$$\frac{\partial}{\partial z} J_{oz} = - \frac{\partial}{\partial x} \sum_{\underline{k}} k_y \, \mathrm{Im} \, \langle \delta\psi_{\underline{k}}^* \, \delta J_{\underline{k}} \rangle \, B_{oz}^{-1}, \tag{44}$$

gives the diffusion equation

$$\frac{\partial}{\partial z} J_{oz} = \frac{\partial}{\partial x} D^m \frac{\partial}{\partial x} J_{oz} \tag{45}$$

Here,

$$D^m = \sum_{\underline{k}} \frac{\langle \delta B_x^2 \rangle_{\underline{k}}}{B_{oz}^2} \pi\delta(\underline{k}\cdot\underline{B}_o) \tag{46}$$

is the usual diffusion coefficient of stochastic instability models of magnetic fields.[28,29] Inclusion of $\tilde{J}_z$ in (44) gives the Fokker-Planck equation

$$\frac{\partial}{\partial z} J_{oz} = \frac{\partial}{\partial x} D^m \frac{\partial}{\partial x} J_{oz} - \frac{\partial}{\partial x} F^m J_{oz} \tag{47}$$

where

$$F^{m} = - \sum_{\underline{k}} k_y \; \mathrm{Im} \langle \delta\psi^*_{\underline{k}} \tilde{J}_{\underline{k}} \rangle (B_{oz} J_{oz})^{-1} \tag{48}$$

Since $\tilde{J}_z$ and δJ_z^c are related self-consistently through (42), F^m and D^m in (47) are also related. This relationship and its consequences for global transport are best seen by considering the nonlinear expressions for D^m and F^m. For stochastic δB_x, the nonlinear term in (41) can be approximated by a diffusion operator in the same way that (41) yields (45). δJ_z^c, therefore, follows from

$$\left(\frac{\partial}{\partial z} + \frac{B'_{oy}}{B_{oz}} x \frac{\partial}{\partial y} - \frac{\partial}{\partial x} D^m \frac{\partial}{\partial x}\right) \delta J_z^c = - \frac{\delta B_x}{B_{oz}} \frac{\partial}{\partial x} J_{oz} \tag{49}$$

so that

$$\delta J_{\underline{k}}^c = - g_{\underline{k}}(x) \frac{\delta B_{\underline{k}}}{B_{oz}} \frac{\partial}{\partial x} J_{oz} \tag{50}$$

where

$$g_{\underline{k}}(x) = \int_0^\infty dz \exp\left[iz \, \underline{k}.\underline{B}_o/B_{oz} - (z/z_o)^3\right] \tag{51}$$

is a broadened resonance function and

$$z_o^{-1} = \left[\frac{1}{3} (k_y \, B'_{oy}/B_{oz})^2 \, D^m\right]^{1/3} \tag{52}$$

is the distance along z for a field line to diffuse a distance $\langle \delta y^2 \rangle^{1/2} = k_y^{-1}$. Insertion of (50) into (44) then gives the nonlinear diffusion coefficient

$$D^m = \sum_{\underline{k}} \frac{\langle \delta B_x^2 \rangle_k}{B_{oz}^2} g_{\underline{k}}(x) \tag{53}$$

that is now to be used in both (47) and (50). Note that, for weak fields, $g_{\underline{k}} \rightarrow \pi\delta\,(\underline{k}.\underline{B}_o/B_{oz})$ and (53) reduces to (46).

The physics of the nonlinear diffusion is easily seen. For finite amplitudes, field line diffusion occurs when

$$|k_z + k_y B'_{oy} x/B_{oz}| < \frac{1}{z_o} \tag{54}$$

In terms of the position $x_s = - k_z B_{oz}/k_y B'_{oy}$ of the mode rational surface of mode $\underline{k}$, (54) is

$$|x - x_s| < x_m \tag{55}$$

where $x_m = (k_y B'_{oy} z_o/B_{oz})^{-1}$, i.e.,

$$x_m = (B_{oz} D^m/3k_y B'_{oy})^{1/3} \tag{56}$$

In the stochastic case, x_m plays the role of the island width

$$\Delta x = (\delta\psi/J_{oz})^{1/2} \tag{57}$$

so (55) is the condition for island overlap. To see this, consider the overlap of two neighboring resonances. From (53) and (51), $D^m \sim \langle\delta B\rangle^2_{res} z_o/B^2_{oz}$ where δB_{res} is the resonant portion of δB_x contributing to the integral in (53). Using now the definition (52) for z_o, D^m in (56) can be expressed in terms of δB_{res} so that (56) becomes

$$x_m \sim (\delta B_{res}/k_y B'_{oy})^{1/2} \tag{58}$$

or, equivalently, (57). Therefore, at island overlap, D^m becomes nonzero and a field line random walks radially as one moves along z, i.e., $(\delta x)^2 = 2D^m z$.

The nonlinear field line diffusion destroys finite amplitude magnetic island structures and causes global transport of the mean fields. Localized island structures are disrupted because neighboring field lines diffuse at different rates. As in Sec. IIIB of Ref. 1 (see also Ref. 27), this can be seen by deriving from (51) the two point correlation equation analogous to (49), i.e.,

$$\left(\frac{\partial}{\partial z} + \frac{B'_{oy}}{B_{oz}} x_- \frac{\partial}{\partial y_-} - \frac{\partial}{\partial x_-} D^m_- \frac{\partial}{\partial x_-}\right) \langle \delta J_z(1) \delta J_z(2) \rangle = 0 \qquad (59)$$

in the simplified case of $\partial J_{oz}/\partial x = 0$. Here, $x_-=x_1-x_2$, $y_-=y_1-y_2$ are the separations between field line trajectories and $D^m_- = D^m_{11} + D^m_{22} - D^m_{12} - D^m_{21}$ is the relative diffusion coefficient where

$$D^m_{12} = \sum_{\underline{k}} \frac{\langle \delta B_x^2 \rangle_k}{B_{oz}^2} g_{\underline{k}}(x_1) \cos k_y y_- \qquad (60)$$

For small separations, two field lines feel approximately the same forces and tend to diffuse together. Then, we can write $D^m_- = D^m k_o^2 y_-^2$ where (as in (93) of Ref. 1), k_o defines the typical scale length of the stochastic fields. Using this D^m_-, the characteristics of (59) imply (for $z_- = 0$) that

$$\langle y_-^2(z) \rangle = (y_-^2 - 2x_- y_- L_c/L_s + 2x_-^2 L_c^2/L_s^2) e^{z/L_c} \qquad (61)$$

where $L_s^{-1} = B'_{oy}/B_{oz}$ is the inverse shear length and

$$L_c = (12)^{-1/3} z_o = (4k_o^2 L_s^{-2} D^m)^{-1/3} \qquad (62)$$

is the z-exponentiation length or Kolmogoroph entropy.[29] From (61), two field lines, initially separated by x_-, y_- will diverge by k_o^{-1} in y_- after a distance traversed in z of

$$z_c = L_c \ln \frac{3k_o^{-2}}{\bar{y}_-^2 - 2x_- \bar{y} L_c/L_s + 2x_-^2 L_c^2/L_s^2} \qquad (63)$$

where $\bar{y}_- = y_- - x_+ z_-/L_s$. It is this orbit exponentiation process, frequently referred to as stochastic orbit instability, that tears coherent island structures apart. This spacial destruction of the islands is connected to their destruction in time by the Alven speed. As described in Ref. 1, the islands are disrupted as Alfven waves propagate down the spacially stochastic field lines. The Lyapunov time of Ref. 1 is $\tau = L_c/V_A$ (see (62) above and (95) of Ref. 1), where the diffusion coefficient of Ref. 1 is $D = D^m V_A$ in dimensional units (see (53) above and (73) of Ref. 1). Note that, in the temporal case with flows, the $\delta \underline{B}$ fields in (53) get replaced by the clump magnetic fields $\delta \underline{L}$ and $\delta \underline{N}$.

The nonlinear version of (47) can now be obtained as follows. We first substitute (50) into (42) to obtain

$$\left[\frac{\partial^2}{\partial x^2} - k_y^2 + i\, g_{\underline{k}}(x)\, \frac{k_y\, J'_{oy}}{B_{oz}}\right] \delta\psi_{\underline{k}} = - \tilde{J}_{\underline{k}} \tag{64}$$

Equation (64) is a resonance broadened Newcomb equation[2,30] that is driven by the clump currents $\tilde{J}_{\underline{k}}$. Let us define the clump or resonant part of the flux function as

$$\tilde{\psi}_{\underline{k}} = x_m \int dx\, \tilde{J}_{\underline{k}} \tag{65}$$

then, integration of (64) gives

$$\delta\psi_{\underline{k}} = \frac{\tilde{\psi}_{\underline{k}}}{(\Delta'_{\underline{k}} + 2|k_x|)x_m} \tag{66}$$

where

$$\Delta'_{\underline{k}} = -2|k_y| - \frac{ik_y/B_{oz}}{\delta\psi_{\underline{k}}(x_s)}\, P\!\int dx\, g_{\underline{k}}(x)\delta\psi_{\underline{k}}(x)\, \frac{\partial}{\partial x} J_{oz} \tag{67}$$

is the resonance broadened tearing mode stability parameter.[2] The denominator, as discussed in Ref. 1, describes the shielding of the clump field by the nonresonant currents $\delta J^c_{\underline{k}}$. The self-consistency relation (66) relates F^m to D^m. Using (66) and (50), the $\delta J^c_{\underline{k}}$ contribution to the bracket in (44) becomes

$$-\frac{k_y^2}{B_{oz}}\, g_{\underline{k}}(x)\, J'_{oz}(x)\, \frac{\langle \tilde{\psi}^*_{\underline{k}} \int dx'\, \tilde{J}^*_{\underline{k}}(x') \rangle}{|\Delta'_{\underline{k}} + 2|k_y||^2 x_m} \tag{68}$$

while the $\tilde{J}_{\underline{k}}$ contribution is

$$\frac{k_y^{\ 2}}{B_{oz}} \frac{\langle \tilde{\psi}_k^* \tilde{J}_k(x) \rangle}{|\Delta_{\underline{k}}' + 2|k_y||^2 x_m} \tag{69}$$

Passing now to the Fourier integral limit and noting that

$$\langle \tilde{J}(1)\tilde{J}(2) \rangle_{\underline{k}} = 2\mathrm{Re}g_{\underline{k}}(1) \langle \tilde{J}(1)\tilde{J}(2) \rangle_{k_y} \tag{70}$$

(44) becomes

$$\frac{\partial}{\partial z} J_{oz} = \frac{\partial}{\partial x} 2x_m^{-1} \; B_{oz}^{-2} \int \frac{d\underline{k}}{(2\pi)^2} \, k_y^2 \; |\Delta_{\underline{k}}' + 2||^{-2} \int dx' \; \mathrm{Re}g_{\underline{k}}(x') \mathrm{Re}g_{\underline{k}}(x)$$

$$\left[\langle \tilde{\psi}\tilde{J}(x') \rangle_{k_y} \frac{\partial J_{oz}(x)}{\partial x} - \langle \tilde{\psi}\tilde{J}(x) \rangle_{k_y} \frac{\partial J_{oz}(x')}{\partial x'} \right] \tag{71}$$

This can be cast in the form of the Fokker-Planck equation (47) where, using (65) and (70),

$$D^m = \frac{1}{B_{oz}^{\ 2}} \int \frac{d\underline{k}}{(2\pi)^2} \frac{k_y^2 \langle \tilde{\psi}^2 \rangle_k}{|\Delta_{\underline{k}}' + 2|k_y||^2 x_m^{\ 2}} \, g_{\underline{k}}(x) \tag{72}$$

and

$$F^m = -\frac{1}{B_{oz}} \int \frac{d\underline{k}}{(2\pi)^2} \, k_y \frac{\langle \delta \tilde{\psi}_k^* \tilde{J}_k \rangle}{|\Delta_{\underline{k}}' + 2|k_y||^2 x_m} \, \mathrm{Im}\Delta_{\underline{k}}' \tag{73}$$

Recalling (66), we see that (78) is just (53). Rather than the diffusion equation (45) for J_{oz} in an arbitrary (non--self-consistent) field δB_x, (47) describes the evolution of J_{oz} due to self-consistent, shielded island fields (66). Note that $F^m \neq 0$ because the fields are correlated self-consistently by Ampere's law (42). This correlation connects F^m with D^m and causes the right hand side of (47) to vanish to lowest order in the island width. To see this, we note that as $x_m \sim z_o^{-1} \to 0$ in (51), $\mathrm{Re}g_{\underline{k}}(x) \to \pi\delta(\underline{k} \cdot \underline{B}_o / B_{oz})$, and the two terms in the bracket in (71) cancel. There is no net radial transport of the field lines.

For finite island widths, the two resonance functions in (71) overlap and cancellation does not occur. To show this, we follow the resonance expansion of the Fokker-Planck collision operator in Vlasov turbulence.[31] We assume that the correlation function is factorable:

$$\langle\tilde{\psi}\ \tilde{J}(x)\rangle_{k_y} = a(x)\ b(k_y) \tag{74}$$

Then, (71) can be written as

$$\frac{\partial}{\partial z} J_{oz} = \frac{\partial}{\partial x}\int dx'\ K(x'-x,\ \frac{x'+x}{2}) H(x,x') \tag{75}$$

where

$$K(x'-x,\ \frac{x'+x}{2}) = \frac{2x_m^{-1}}{B_{oz}^2}\int \frac{dk}{(2\pi)^2}\ \frac{\mathrm{Reg}_{\underline{k}}(x)\mathrm{Reg}_{\underline{k}}(x')}{|\Delta_{\underline{k}}'+2|k_y||^2}\ b(k_y) \tag{76}$$

and

$$H(x,x') = a(x')\ \frac{\partial J_{oz}(x)}{\partial x} - a(x)\ \frac{\partial J_{oz}(x')}{\partial x'} \tag{77}$$

Recalling (51), we note that $K \approx 0$ unless $|x-x'|<x_m$. Moreover, K is an even function of $\Delta x = x' - x$ and has a weak (nonresonant) dependence on $(x'+x)/2$. We, therefore, expand K as

$$K(\Delta x,\ x = \frac{\Delta x}{2}) = K(\Delta x,x) + \frac{\Delta x}{2}\frac{\partial}{\partial x}\ K(\Delta x,x) \tag{78}$$

and H as

$$H(x,x') = \left[\Delta x + \frac{(\Delta x)^2}{2}\ \frac{\partial}{\partial x}\right]\left[\frac{\partial a(x)}{\partial x}\ \frac{\partial J_{oz}(x)}{\partial x} - a(x)\ \frac{\partial^2 J_{oz}(x)}{\partial x^2}\right] \tag{79}$$

Substituting (78) and (79) into (75) gives

$$\frac{\partial}{\partial z} J_{oz} = -\frac{\partial^2}{\partial x^2}\int d\Delta x\ \frac{(\Delta x^2)}{2}\ K(\Delta x,x)\left[a(x)\ \frac{\partial^2 J_{oz}(x)}{\partial x^2} - \frac{\partial a(x)}{\partial x}\ \frac{\partial J_{oz}(x)}{\partial x}\right] \tag{80}$$

Because of the resonance functions in K, the $d\Delta x$ integral in (80) effectively replaces $\Delta x^2/2$ with the mean-square island width. Then, recalling (72), (74) and (76), the first term in (80) is

$-(\partial^2/\partial x^2)D^m(\Delta x)^2(\partial^2/\partial x^2)J_{oz}$. For D^m relatively insensitive to x inside a broad, fully stochastic spectrum, the last term in (80) can be neglected. Therefore, (80) becomes:

$$\frac{\partial}{\partial z} J_{oz} = - \frac{\partial^2}{\partial x^2} D^m (\Delta x)^2 \frac{\partial^2}{\partial x^2} J_{oz} \tag{81}$$

The resonant interaction of finite amplitude islands, therefore, cause net diffusive transport of the mean field, albeit fourth order diffusion. Since Alfven waves will carry energy away along the stochastic magnetic field lines at the Alfven speed, the effective global transport rate in time due to the stochasticity follows by multiplying (81) by V_A. Then, (81) gives

$$\frac{\partial}{\partial t} J_{oz} = - \frac{\partial^2}{\partial x^2} D(\Delta x)^2 \frac{\partial^2}{\partial x^2} J_{oz} \tag{82}$$

where, as we have discussed above, $V_A D^m \to D$. Assuming that D^m is relatively insensitive to x, we use the mean field part of (42), and thus (82) yields the time evolution equation (32) for the mean flux, ψ_o. In Sec. III, an alternative derivation from the time dependent MHD equations yields the same result as (32). Generalizing (32) to cylindrical coordinates and noting that $E_{oz} = -\partial\psi_o/\partial t$ from Faraday's law, (32) gives

$$E_{oz} = - \underline{\nabla}_\perp \cdot \underline{\underline{D}}(\Delta x)^2 \cdot \underline{\nabla}_\perp J_{oz} \tag{83}$$

Magnetic helicity is conserved since (83) satisfies (12). Note that this is ensured by the Fokker-Planck coefficients D and $\bar{F}$. To zeroth order in the island width, the right hand sides of (14) and (47) vanish. There is no field line transport and (12) is identically satisfied. Transport occurs at second order in the island width, thus leading to the helicity conserving form (83).

III. MAGNETIC HELICITY CONSERVATION

Here we consider the effect of magnetic helicity conservation on growing MHD clump fluctuations. For this purpose, we need the explicit time dependent equations. Time dependent equations for MHD clump turbulence were derived in Ref. 1. The equations were obtained from the full vector MHD equations, but with magnetic helicity conservation neglected. We showed that subtraction of the Alfven wave field from the total field $\underline{B}$ yields the clump or resonant part of the field $\underline{L}=\underline{B}-S^{-1}\underline{V}$, $\underline{N}=\underline{B}+S^{-1}\underline{V}$. [We use dimensional units here (see Sec. IIA of Ref. 1), where S is the Lundquist number ($S = \tau_R/\tau_H$ where τ_R and τ_H are, respectively, the resistive and Alfven times for the current channel radius)]. For example, neglecting pressure (shown in Ref. 1 to be small for the clumps), the $\underline{N}$ equation is

$$\left(\frac{\partial}{\partial t} - S\langle\underline{B}\rangle.\underline{\nabla} - S\delta\underline{L}.\underline{\nabla} - \nabla_{\perp}^2\right) \underline{N} = 0 \tag{84}$$

The $\langle\underline{B}\rangle.\underline{\nabla}$ term describes Alfven wave emission and, therefore, localizes the clumps near mode rational surfaces. The nonlinear $\delta\underline{L}.\underline{\nabla}$ term, when renormalized, becomes a diffusion operator as in (49) of Sec. II above. The diffusion coefficient is a turbulent or anomalous resistivity to be added to the collisional resistivity (the $\nabla_{\perp}^2$ term in (84)). The mean field $\langle\underline{N}\rangle$ satisfies

$$\frac{\partial}{\partial t}\langle N_y\rangle = \frac{\partial}{\partial x}(D+1)\frac{\partial}{\partial x}\langle N_y\rangle \tag{85}$$

where D is the turbulent resistivity in terms of the fluctuations $\langle\delta L^2{}_x\rangle_{\underline{k}}$. Because of the conservation of energy, the turbulent mixing (85) of the mean shear generates clump fluctuations $\langle\delta\underline{N}(\underline{x}_1,t).\delta\underline{N}(\underline{x}_2,t)\rangle \equiv \langle\delta\underline{N}_1.\delta\underline{N}_2\rangle$ satisfying

$$\left[\frac{\partial}{\partial t} - S\langle \underline{B}_-\rangle.\underline{\nabla}_- - \frac{\partial}{\partial x_-}(D_-+2)\frac{\partial}{\partial x_-}\right]\langle \delta\underline{N}_1.\delta\underline{N}_2\rangle$$

$$= 2\ D_{12}\frac{\partial\langle \underline{N}_1\rangle}{\partial x_1}\cdot\frac{\partial\langle \underline{N}_2\rangle}{\partial x_2} \qquad (86)$$

where $D_-=D_{11}+D_{22}-D_{12}-D_{21}=2\ (D-D_{12})$ with

$$D_{12} = S^2\int\frac{dk}{(2\pi)^2}\langle\delta L_x^2\rangle_{\underline{k}}\ G_{\underline{k}}\ \exp\ ik_y y_- \qquad (87)$$

The broadened resonance function is

$$G_{\underline{k}} = \int_0^\infty dt\ \exp\left[isk.\underline{B}_o t - (t/\tau_o)^3-\gamma t\right] \qquad (88)$$

where γ is the clump growth rate and $\tau_o = (1/3\ k_y^2\ B_{oy}'^2\ S^2 D)^{-1/3}$. Inverting the two point operator on the left-hand-side of (92) gives the clump fluctuation level (see (97) of Ref. 1):

$$\langle\delta\underline{N}_1.\delta\underline{N}_2\rangle = 2\tau_-\ D_{12}(B_{oy}')^2 \qquad (89)$$

where $\tau_- = \tau_{c\ell}\ (1+\gamma\tau)^{-1}$, with

$$\tau_{c\ell} = \tau\ \ell n\ \frac{3k_o^{-2}}{\bar{y}_-^2-2\bar{y}_-x_-S\tau B_{oy}'+2S^2B_{oy}'^2x_-^2\tau^2} \qquad (90)$$

and

$$\tau = (12)^{1/3}\ \tau_o = (4k_o^2S^2B_{oy}'^2D)^{-1/3} \qquad (91)$$

as the Lyapunov time. In (97), $\bar{y}_- = y_- - xB_{oy}'z_-$. The resonance width in (91) is

$$x_d = (4D/Sk_oB_{oy}')^{1/3} \qquad (92)$$

so that two field lines are only correlated if $|x_-|<x_d$. As discussed earlier, these results of the time dependent evolution are related to the spacially stochastic case of Sec. II by the Alfven speed (i.e., by S in dimensionalized units). We define the clump "flux" function $\Psi=\psi + S^{-1}\phi$,

where $\underline{B}_{\perp}=\underline{\nabla}\times(\hat{z}\psi)$ and $\underline{V}=\underline{\nabla}\times(\hat{z}\phi)$ define the poloidal flux function, ψ, and the velocity stream function, ϕ. The Fourier transform of (89) can then be written as

$$\left(\frac{\partial^2}{\partial x_-^2} - k_y^2\right) \langle\delta\Psi(1)\delta\Psi(2)\rangle_{\underline{k}} = - 2D\bar{\tau}_{c\ell}(x_-,\underline{k})(B'_{oy})^2 \tag{93}$$

where $\bar{\tau}_{c\ell}\,(x_-,\underline{k})$ is the Fourier transform of (90). As shown in Sec. IVA of Ref. 1, (93) can be cast in a form reminiscent of the Newcomb equation of linear MHD stability theory.

We would like to include magnetic helicity conserving terms in the time dependent equation (86) as we did in the spacially stochastic case of Sec. II. As there, we need to distinguish between the phase coherent and phase incoherent parts of the field (the Alfven wave part of the field has already been distinguished by the use of the $\underline{N}$ and $\underline{L}$ field variables). We write $\delta\underline{N} = \delta\underline{N}^c + \tilde{\underline{N}}$, where $\delta\underline{N}^c$, the part of $\delta\underline{N}$ phase coherent with $\delta\underline{L}$, is given by (70) of Ref. 1 and produces the diffusion equation (85). Note that $\delta\underline{N}^c$ would be present for any fluctuations $\delta\underline{L}$. The field $\tilde{\underline{N}}$ represents the resonant clump fluctuation produced self-consistently as the $\delta\underline{L}$ fields turbulently mix the mean shear. Because the mixing occurs at the overlap of resonances, $\tilde{\underline{N}}$ will be a random, incoherent function of the field phases. However, we will only need its correlation function. With the additional contribution of $\tilde{\underline{N}}$, (85) becomes (neglecting collisional dissipation for simplicity)

$$\frac{\partial}{\partial t} \langle N_y \rangle = \frac{\partial}{\partial x} \left[D^L \frac{\partial}{\partial x} \langle N_y \rangle - F^L \langle N_y \rangle \right] \tag{94}$$

where $F^L = S\langle \tilde{L}_x \tilde{N}_y \rangle / \langle N_y \rangle$ is a Fokker-Planck "drag" coefficient. We introduce the superscript L on D here to distinguish the $\langle \delta L^2 \rangle$ and $\langle \delta N^2 \rangle$ driven coefficients that we will consider below. A corresponding equation to (94) holds for $\partial \langle L_y \rangle / \partial t$, but with $F^N = -S\langle \tilde{N}_x \tilde{L}_y \rangle / \langle L_y \rangle$. The additional term F^L will modify the right-hand-side of (86) so that, when integrated, (90) will be replaced by

$$\langle \delta\underline{N}_1 \cdot \delta\underline{N}_2 \rangle = 2\tau_-^L \left[D_{12}^L B_{oy}'^2 + F_{12}^L B_{oy}' \right] \tag{95}$$

Similarly,

$$\langle \delta\underline{L}_1 \cdot \delta\underline{L}_2 \rangle = 2\tau_-^N \left[D_{12}^N B_{oy}'^2 - F_{12}^N B_{oy}' \right] \tag{96}$$

Strictly speaking, the $\tilde{\underline{N}}$ and $\tilde{\underline{L}}$ terms also produce F terms on the left-hand-sides of the two point equations such as (86). However, the two point propagators are less sensitive to magnetic helicity constraints than the clump source term. While the global conservation of helicity directly effects the mixing of the mean field, it has much less effect on the local mixing of the fluctuating part of the field. We, therefore, take $\tau_{c\ell}$ in (95) and (96) to be given by (90) as before. For simplicity, we consider the strong N/L coupling limit where $\langle \underline{N}.\underline{L}\rangle = 0$ so that $\tau^N_{c\ell} = \tau^L_{c\ell} = \tau_{c\ell}$ (see Sec. IIIB of Ref. 1). Adding (95) and (96) then gives the equation for the total energy correlation,

$$\langle \delta\underline{B}_1 \cdot \delta\underline{B}_2 + S^{-2}\delta\underline{V}_1 \cdot \delta\underline{V}_2\rangle = 2\,\tau_{c\ell}\,E_{oz}\,J_{oz} \tag{97}$$

where

$$E_{oz} = D\;J_{oz} - \langle \tilde{\underline{V}} \times \tilde{\underline{B}}\rangle_z \tag{98}$$

With $F = \langle \tilde{\underline{V}} \times \tilde{\underline{B}}\rangle_z / B_{oz}$, (98) corresponds to the Fokker-Planck form (14). We can also write $F = S\langle \tilde{\underline{N}} \times \tilde{\underline{L}}\rangle_z / B_{oz}$. F gives the self-consistent, correlated motion of the clump part of the magnetic field.

Insertion of (98) into Faraday's law gives a Fokker-Planck equation for the mean field B_{oy} with $F = \langle \tilde{\underline{V}} \times \tilde{\underline{B}}\rangle_z / B_{oy}$. Lowest order cancellation between the D and F terms in this equation is demanded by global magnetic helicity conservation constraining the dynamics of B_{oy}. The situation is analogous to the dynamics of the mean distribution (f_o) of clumps or discrete particles in a Vlasov plasma.[15] There, f_o also satisfies a Fokker-Planck equation of the form $\partial f_o/\partial t = \partial Q/\partial v$, where Q is a (x,v) phase space current given by $Q = D(\partial f_o/\partial v) - Ff_o$. D and F are velocity space diffusion and dynamical friction coefficients, e.g., $F \sim \langle \tilde{E}\tilde{f}\rangle / f_o$ where $\tilde{E}$ is the electric field from Poisson's equation and $\tilde{f}$ is the clump or discrete particle part of f. Because of global momentum conservation, these D and F terms cancel (Q=0) to lowest order in the resonance width Δv. To next order, Q~-

$D(\Delta v)^2\partial^3 f_o/\partial v^3$, thus giving a fourth order diffusion process for f_o. This is the well known effect of collisions between like particles (or clumps). In the MHD clump model, (98) plays the role of Q, with magnetic helicity conservation playing the constraining role analogous to momentum conservation.

Unlike the stochastic, but static magnetic field case of Sec. II, we have not been able to evaluate the correlation $\langle \tilde{\underline{N}}\times\tilde{\underline{L}}\rangle$ with our renormalization techniques and show that it ensures magnetic helicity conservation in E_{oz}. The static case involves the rather simple scalar equation (41) and yields the helicity conserving results (71) in straightforward way. The dynamic case with the full vector MHD equations is much more difficult to treat. Preserving the vector properties of the correlation is particularly difficult. Two points about the calculation are worth mentioning, however. Unlike kinetic dynamo models,[7] the evaluation of F in the clump model requires self-consistent rather than arbitrarily given flow fields $\underline{V}$. From momentum balance one obtains

$$\underline{V}^{SL} = S^2 \int^t dt' \, \underline{J}\,[\underline{x}(t'),t'] \times \underline{B}[\underline{x}(t'),t'] \tag{99}$$

so that

$$\langle \underline{V}^{SC} \times \underline{B}\rangle_z = -S^2 \int^t dt' \langle (\underline{B}.\underline{B}) \; J_z\rangle_{t'}, \tag{100}$$

where the subscript t' means evaluation of time and orbits at t=t'. Along with the so-called β term of kinetic dynamo theory (the β term being obtained from the time integration of Faraday's law in a given $\underline{V}$ field), (100) yields a contribution to E_{oz} of

$$\langle \delta\underline{V}\times\delta\underline{B}\rangle_z = -\int^t dt' \; \langle (\delta V^2 + S^2 \; \delta B^2) J_z\rangle_{t'}, \tag{101}$$

The coherent part of this response, coming from $J_z = J_{oz}$, gives the result E_{oz}

$= DJ_{oz}$. The response due to $J_z = \tilde{J}_z$ contributes to the incoherent or clump part of E_{oz}, i.e., the $F \sim \langle \tilde{\underline{V}} \times \tilde{\underline{B}} \rangle$ term in (98). This contribution depends on the three point correlation $\langle \delta B^2 \delta J_z \rangle$ and is sensitive to the distribution of fluctuation amplitudes. If the turbulence is equally populated with $\delta J_z > 0$ and $\delta J_z < 0$ fluctuations, this contribution to the α-effect will vanish. If, however, holes ($\delta J_z < 0$) are more prevalent because of their growth, the reflextion symmetry of the spectrum will be broken.[7] F and α will then be nonzero, α being negative and $F = -\alpha$ being positive. Of course, this effect is countered by the diffusion of the field lines (i.e., the D term in (14) and the β term in (17)). However, at the edge of a confined plasma, J_{oz} will be small and E_{oz} can expected to become negative as holes (bubbles) intermittently develop. Though interesting, these implications of (101) are not satisfactory since (101) does not ensure conservation of magnetic helicity. Clearly, additional contributions to $\langle \delta \underline{V} \times \delta \underline{B} \rangle$ are warranted, but we have not been able to identify or calculate them from the vector MHD equations.

What we have been able to do is to calculate the <u>net</u>, helicity conserving E_{oz} from a renormalized version of the reduced MHD (Strauss) equations. The Strauss equations[26] are scalar equations for the poloidal flux function, ψ, and the stream function, ϕ, where $\underline{V} = \underline{\nabla} \times \phi \hat{z}$. They are, neglecting collisional dissipation,

$$\frac{\partial \psi}{\partial t} = \underline{B} \cdot \underline{\nabla} \phi \tag{102}$$

$$S^{-2} \frac{\partial U}{\partial t} = \underline{B} \cdot \underline{\nabla} J_z \tag{109}$$

where $U = -\nabla_\perp^2 \phi$ is the vorticity. A helicity conserving form for E_{oz} can be obtained from these equations since the ensemble average of (102) can be written as

$$\frac{\partial}{\partial t}\langle\psi\rangle = \underline{\nabla}.\langle\delta\underline{B}_{\perp}\delta\phi\rangle \tag{104}$$

and, with $E_{oz}=-\partial\langle\psi\rangle/\partial t$, will automatically satisfy the magnetic helicity constraint (12). The renormalization merely determines the structure of $\langle\delta\underline{B}_{\perp}\delta\phi\rangle$ to be the current of a fourth order diffusion process. Since the Strauss equations are only valid for "tokamak ordering", the result we obtain is only valid in this limit. However, this is the limit that we are mainly concerned with in this paper. The result agrees with the physical argument leading to (14) and the stochastic magnetic field transport calculation of Sec. II.

Since we are interested in the self-consistent evaluation of (104), we will need the response $\delta\phi^{c}$ that is coherent with $\delta\underline{B}_{\perp}$ as well as the part $\delta\underline{B}_{\perp}^{c}$ that is coherent with $\delta\phi$. (Kinetic or quasilinear dynamo models focus on only one of the responses, usually $\delta\underline{B}_{\perp}^{c}$). For this purpose, we rewrite (104) as

$$\frac{\partial}{\partial t}\langle\psi\rangle = \underline{\nabla}.\langle\delta\underline{B}_{\perp}\delta\phi^{c}\rangle - \underline{\nabla}.\langle\delta\underline{V}_{\perp}\delta\psi^{c}\rangle \tag{105}$$

Next, we again distinguish between the wave-like (Alfven) and non-wave-like (clump) parts of the field. The Alfven wave response ($\delta\underline{B}^{W} = \pm S^{-1}\,\delta\underline{V}$ in dimensionless units) is $\delta J_{z}^{W} = \pm S^{-1}\delta U$ in terms of the fields of (102) and (103). The fluctuating part of the right-hand-side of (103) can then be written as $\underline{B}.\underline{\nabla}\tilde{J}_{z} + \underline{B}.\underline{\nabla}\delta J_{z}^{W} + \delta\underline{B}.\underline{\nabla}J_{oz}$, where $\tilde{J}_{z}$ is the clump part of the current density. The last term here gives δU^{c} (and therefore $\delta\phi^{c}$) that we seek. The first term is small near the resonance and, therefore, can be neglected as a source for $\delta\phi^{c}$. The δJ_{z}^{W} term, when written in terms of δU, can be brought to the left-hand side of (103), and has the effect of subtracting out the forward or backward Alfven wave as in Sec. IIC of Ref. 1. For $\delta J_{z}^{W} = S^{-1}\,\delta U$, the equation for δU^{c} is

$$S^{-2}\left(\frac{\partial}{\partial t} - S\,\underline{L}.\underline{\nabla}\right)\delta U^{c} = \delta\underline{B}.\underline{\nabla}J_{oz} \tag{106}$$

The renormalization converts the $\delta\underline{L}.\underline{\nabla}$ term into a diffusion operator as in Sec. II above. Therefore, $\delta\phi^{c}$ follows from

$$\delta U^{c}_{\underline{k}} = \frac{S^{2}\tilde{\underline{B}}_{k}.\underline{\nabla}J_{oz}}{iS\underline{k}.\underline{B}_{o}+[\gamma+(\tau^{L}_{o})^{-1}]} \tag{107}$$

where we've approximated the inverted propagator as a Lorentzian and set $\delta\underline{B}_{\underline{k}} = \tilde{\underline{B}}_{k}$ since we are only interested in the transport due to the clump part of the field. The governing equation for $\delta\psi^{c}$ can be obtained in similar fashion. For the fully stochastic case ($\gamma\tau<<1$), δJ^{c}_{z} can be obtained from the static version of (103), i.e., (40). Therefore,

$$\delta J^{c}_{\underline{k}} = i\,\frac{\delta\hat{\underline{B}}_{k}.\underline{\nabla}J_{oz}}{\underline{k}.\underline{B}} \tag{108}$$

where $\delta\hat{\underline{B}}$ is that part of $\delta\underline{B}$ that is driven by the clump part of $\delta\underline{V}$, and $\delta J^{c}_{\underline{k}}$ is the Fourier transform of δJ^{c}_{z}. We obtain $\delta\hat{\underline{B}}$ from (102), or, equivalently, its curl

$$\left(\frac{\partial}{\partial t} + \underline{V}.\underline{\nabla}\right)\underline{B}_{1} = \underline{B}.\underline{\nabla}\underline{V} \tag{109}$$

where we write $\underline{B}.\underline{\nabla}\delta\underline{V}$ as $\underline{B}.\underline{\nabla}\delta\underline{V}^{w} + \underline{B}.\underline{\nabla}\tilde{\underline{V}}$. Taking the forward Alfven wave response, $\delta\underline{V}^{w} = S\delta\underline{B}_{\perp}$, (109) becomes

$$\left(\frac{\partial}{\partial t} - S\underline{L}.\underline{\nabla}\right)\delta\hat{\underline{B}} = \underline{B}.\underline{\nabla}\tilde{\underline{V}} \tag{110}$$

Renormalization and time inversion of (110) then give

$$-\,\delta\hat{\underline{B}}_{\underline{k}} = \frac{i\underline{k}.\underline{B}\;\tilde{\underline{V}}_{k}}{iS\underline{k}.\underline{B}_{o}+[\gamma+(\tau^{L}_{o})^{-1}]} \tag{111}$$

so that

$$\delta J^{c}_{\underline{k}} = -\,\frac{\tilde{\underline{V}}_{k}.\underline{\nabla}\,J_{oz}}{iS\underline{k}.\underline{B}_{o}+[\gamma+(\tau^{L}_{o})^{-1}]} \tag{112}$$

Since $\nabla_{\perp}^2(\psi, \phi) = -(J_z, U)$, the equations (107) and (112) give

$$\left(\frac{\partial^2}{\partial x^2} - k_y^2\right)(\delta\psi_{\underline{k}}^c, -\delta\phi_{\underline{k}}^c) = \frac{(\tilde{\underline{V}}_k, S^2\tilde{\underline{B}}_k)\cdot\underline{\nabla}J_{oz}}{iS\underline{k}\cdot\underline{B}_o + (\gamma + \tau_o^{-1})} \tag{113}$$

where we have set $\tau_o^L = \tau_o$ since, in the strong N/L coupling limit, $\tau_o^L = \tau_o^N = \tau_o$. Note that, if we had subtracted out the backward Alfven wave ($S\underline{L} \to -S\underline{N}$ in (111) and (112)), S would change sign in (113). However, this change would not matter since we will only need the real part of the inverse propagator. Of course, this occurs because Alfven wave emission of either polarization leads to clump decay. For simplicity, we approximate the Laplacian operator in (113) with $k_{\perp}^2 \sim (\Delta x_k)^{-2} + k_y^2$, where Δx_k is the resonance width for mode k. Then, inverting (113) and substituting $\delta\phi^c$ and $\delta\psi^c$ into (105) gives

$$\frac{\partial}{\partial t}\langle\psi\rangle = \underline{\nabla}\cdot\underline{\underline{D}}\cdot\underline{\nabla}\ J_{oz} \tag{114}$$

where

$$\underline{\underline{D}} = S^2 \int \frac{dk}{(2\pi)^2} \langle\tilde{\underline{B}}\tilde{\underline{B}} + S^{-2}\tilde{\underline{V}}\tilde{\underline{V}}\rangle_{\underline{k}}\ G_{\underline{k}}\ k_{\perp}^{-2} \tag{115}$$

This is just (32) since, for field line steps mainly in the x direction ($k_y\Delta x \ll 1$), $k_{\perp}^{-1} \sim \Delta x$ so that $\mathbf{D} = D(\Delta x)^2$. Note also that in the strong N/L limit, the total energy correlation function in (115) is the same as $\langle\delta\underline{L}\delta\underline{L}\rangle_{\underline{k}}$ or $\langle\delta\underline{N}\delta\underline{N}\rangle_{\underline{k}}$. The magnetic helicity is conserved since (114) ensures (12). The form of (114) has been shown by Boozer to follow from general transport properties of the energy and magnetic helicity invariants.[20]

Strauss has obtained an equation similar to (114) for an assumed (given) quasi-linear spectrum of tearing modes.[22] However, the fluctuations and their evolution are very different from the clumps considered here. Equation (114) differs from the nonresonant, unrenormalized model of Strauss because MHD clumps are nonlinear, self-consistently generated resonant fluctuations whose dynamics conserve the magnetic helicity in both the growing ($\gamma>0$) and steady ($\gamma=0$) states. These features have a profound effect on MHD clump evolution. For example, clump turbulence does not approach the Taylor State by the vanishing of $\nabla_{\perp}^2 J_{oz}$. In the case of clump fluctuations, the turbulent mixing rate (114) must be large enough to overcome the nonlinear decay rate due to the stochastic magnetic field lines, i.e., $\hat{R}>1$ in (5). If $\nabla_{\perp}^2 J_{oz}$ vanished, no new fluctuations would be produced and any existing fluctuations (including, apparently, those of Ref. 22) would decay away due to the field line stochasticity. The turbulence level would decrease to zero in a time on the order of the Lyapunov time. Of course, the mixing and the continuous generation of clump fluctuations is maintained in the clump model by the maintenance of the J_{oz} profile, e.g., with an applied E_{oz}.

The use of the helicity conserving form $E_{oz}=-D(\Delta x)^2\nabla_{\perp}^2 J_{oz}$ instead of $E_{oz}=DJ_{oz}$ in the clump source term $2E_{oz}J_{oz}$ still produces (3) and (5), but with Δ_c' now given by (see (112) of Ref. 1).

$$\frac{1}{\Delta_c'} = \int_{-\infty}^{\infty} \frac{dk_y}{2\pi k_o} \frac{\mathrm{Re}\Delta_{\underline{k}}'+2|k_y|}{(\mathrm{Re}\Delta_{\underline{k}}'+2|k_y|)^2+\lambda^2} k_y^2 \, A(k_y)\left[-\frac{1}{k_{\perp}^2}\frac{J_{oz}'}{J_{oz}}\right] \tag{116}$$

where $\Delta_{\underline{k}}'$ is to be evaluated at $\underline{k}\cdot\underline{B}_o = 0$. Equation (116) is the magnetic helicity conserving form of Δ_c' which, when used in (3) and (5), gives the helicity conserving growth rate for MHD clump fluctuations.

IV. STEADY STATE CLUMP SPECTRUM

Rather than being assumed or given, the nonlinear fluctuations in the clump model are determined self-consistently from the turbulent mixing of the mean shear. The mean-square fluctuation amplitude or spectrum can be calculated for the case of driven, steady state turbulence as follows. We modify the spectrum equation (108) of Ref. 1 for magnetic helicity conservation ($DB'_{oy} = DJ_{oz} \rightarrow -(D/k_\perp^2)J''_{oz}$) and assume the strong N/L coupling limit. Then, expressing D^L in terms of the spectrum, we have

$$\langle\delta\Psi^2(0)\rangle_{\underline{k}} = \frac{-2\pi|B'_{oy}|\delta(\underline{k}\cdot\underline{B}_o)A(k_y)}{\Delta'_{\underline{k}} + 2|k_y|} \frac{J''_{oz}}{J_{oz}} \int_{-\infty}^{\infty} \frac{dk'}{(2\pi)^2} \left(\frac{k'_y}{k'_\perp}\right)^2 \frac{\langle\delta\Psi^2(0)\rangle_{\underline{k}'}}{iS\underline{k}'\cdot\underline{B}_o + \tau_o^{-1}} \tag{117}$$

where $A(k_y)$ is given by (110) of Ref 1 and we have again approximated $\mathrm{Re}G_{\underline{k}}$ by a Lorentzian. Setting (3) equal to unity and using (116), the solution to the integral equation (117) is

$$\langle\delta\Psi^2(0)\rangle_{\underline{k}} = M(x)\delta(\underline{k}\cdot\underline{B}_o)\left(-\frac{J''_{oz}}{J_{oz}}\right)\left[A(k_y)\frac{\mathrm{Re}\Delta'_{\underline{k}}+2|k_y|}{|\Delta'_{\underline{k}}+2|k_y||^2}\right] \tag{118}$$

where $M(x)$ is arbitrary if $\hat{\mathbb{R}} = 1$ and is zero otherwise. (118) is the steady state clump spectrum.

The fluctuations are driven by $\Delta'_{\underline{k}}$ and J''_{oz}. The factor $\mathrm{Re}\Delta'_{\underline{k}}$ in square brackets in (118) is the nonresonant free energy source for the fluctuations (as for the tearing mode). The (J''_{oz}/J_{oz}) factor is due to the constraint of magnetic helicity conservation. The delta function in (118) localizes the fluctuations at the mode rational surface where the tendency for field line bending is maximum (decay by Alfven wave emission is minimum). In reality, the $\delta(\underline{k}\cdot\underline{B}_o)$ singularity should be replaced by a broadened resonance function, since the unperturbed field line orbit used to obtain $\bar{y}_-$ in (90), and producing the $\delta(\underline{k}\cdot\underline{B}_o)$ in (117) (see (109) of Ref. 1) should be the nonlinear trajectory. The $A(k_y)$, coming from an x_- integral of $\tau_{c\ell}$, is a measure of the clump energy.

The MHD clump spectrum (118) is similar to that for Vlasov clumps. The steady state, mean-square electric potential spectrum for Vlasov clumps has been calculated without momentum constraints in Ref. 32. Modifying that result by the factor $(-\mathrm{Im}\varepsilon^i/\mathrm{Im}\varepsilon^e)$ to ensure a momentum conserving clump source term,[15] the electron Vlasov clump spectrum is (neglecting ion nonlinearity) proportional to

$$\left(-\frac{\mathrm{Im}\varepsilon_k^i}{\mathrm{Im}\varepsilon_k^e}\right)\left[\frac{(\mathrm{Im}\varepsilon_k^e)^2}{|\varepsilon_k|^2}\;A(k)\right] \tag{119}$$

where ε_k^e is the electron dielectric function for mode k. (118) and (119) can be cast into an even more similar form by using the model for $\Delta_{\underline{k}}'$ given in (26) of Ref. 2. With $\mathrm{Re}\Delta_{\underline{k}}' \sim (\underline{k}\cdot\underline{B}_o')^2 \sim J_{oz}^2$, the free energy driving term $(-J_{oz}''/J_{oz})\,\mathrm{Re}\Delta_{\underline{k}}'$ in (119) resembles the $\mathrm{Im}\varepsilon \sim \partial f_o/\partial v$ driving terms in (119). Note that momentum conservation constrains the clump source term and thus requires $\mathrm{Im}\varepsilon^i\ \mathrm{Im}\varepsilon^e < 0$ for Vlasov clump instability. Similarly, magnetic helicity conservation requires $J_{oz}''\ \mathrm{Re}\Delta_{\underline{k}}' < 0$ for MHD clump instability. The main difference between (118) and (119) is the delta function resonance factor in (118). A delta function localization factor does not appear in (119) because it gets integrated over by the velocity integral in Poisson's equation. The lack of a corresponding integral in Ampere's law leaves the delta function in (118). This difference also leads, unlike in the Vlasov case, to an amplitude dependent instability threshold, i.e., multiplying (118) by $G_{\underline{k}}$ and integrating over $\underline{k}$ to obtain the diffusion coefficient produces, because of $\delta(\underline{k}\cdot\underline{B}_o)$ in (118), the $(\gamma+\tau^{-1})^{-1}$ factor in (2). This Γ factor is just what is left in $G_{\underline{k}}$ when $\underline{k}\cdot\underline{B}_o=0$ and what gives $\hat{\mathbf{R}} \sim \tau_o \sim x_d^{-1}$ in (3). As a result, the MHD clump turbulence is strongly resonant. The

MHD energy spectrum $\langle \delta B^2 + S^{-2}\delta V^2 \rangle_{\underline{k}}$ due to clumps will peak only at the mode rational surfaces, i.e., $\underline{k}$'s where $\underline{k}.\underline{B}_o=0$. By contrast, the electric field spectrum $\langle E^2 \rangle_{k\omega}$ for one dimensional Vlasov clumps does not reveal such peaks. One must look to the phase space density correlation for the resonant structure of Vlasov clump fluctuations. The electric field and charge density spectra in Poisson's equation integrate over these resonances.[9]

Multiplying (118) by $G_{\underline{k}}$ and integrating over $\underline{k}$ gives the steady state diffusion coefficient

$$D = \tau_o S^2 M(x)\left(-\frac{J''_{oz}}{J_{oz}}\right) \int \frac{dk_y}{(2\pi)^2} k_y^2 \; A(k_y) \; \frac{\mathrm{Re}\Delta'_{\underline{k}}+2|k_y|}{|\Delta'_{\underline{k}}+2|k_y||^2} \qquad (120)$$

Though M(x) can be arbitrary, the physical parameters of the plasma set limits on M(x). D(x) must be a smooth, well behaved function of x so that the island overlap criterion is smoothly satisfied. Further, the diffusion (Markovian) approximation demands that the fluctuation auto-correlation time (τ_{ac}) be short compared to τ_o. This will occur for a wide spectrum of strongly overlapping resonances, i.e., a fully stochastic spectrum of roughly equal amplitude modes. D(x) will thus be relatively independent of x in the unstable region and zero outside. The amplitudes of M and D are also limited. Clearly, $|\delta J_z|<J_{oz}$ sets an upper limit on the amplitude. A current density hole cannot be deeper than a vacuum bubble. Also, since $\tau_o \sim D^{-1/3}$, the $\tau_{ac}<\tau_o$ condition limits D to even smaller values. Since $\tau_{ac} \sim (S\, \bar{k}_y\, B'_{oy}\, \Delta x_{sp})^{-1}$, where Δx_{sp} is the spacial width of the spectrum, the constraint $\tau_{ac} < \tau_o$ is equivalent to $\Delta x<\Delta x_{sp}$, where Δx is the island width. If we make the reasonable assumption that the unstable spectrum encompasses a sizeable fraction of the current channel radius, we must have $\Delta x < a$.

The k_y dependence of the clump spectrum is, in general, nontrivial.

For example, the dependence of $\Delta'_{\underline{k}}$ is sensitive to the mean current profile and $A(k_y)$. Moreover, the k_o factor necessary in the evaluation of $A(k_y)$ must be obtained from the selfconsistent solution of (114b) and (118) of Ref. (1). Because of these complexities, we have not determined a general k_y dependence of the steady state spectrum. However, if we use the hyperbolic tangent model current profile of Eq. (27) of Ref. 2, we obtain $\int dx \langle \delta\Psi^2(x) \rangle_{\underline{k}} \sim A(k_y)$ which, for $k_y > k_o$, scales as k_y^{-2} but, for $k_y < k_o$, scales as $(1-k_y^2/k_o^2)$. Again, the large k_y dependence reflects the localization of the clumps to y_- scales $k_o y_- \lesssim 1$.

V. CLUMP DERIVATION OF $J_o = \mu B_o$

In Sec. IC, we obtained the Taylor state $\underline{J}_o = \mu \underline{B}_o$ as a solution to (24). There, we suggested (24) as the vector current generalization of (20). While we have derived (20) in Sec. III, and (24) appears to be a reasonable generalization to (20), we would like to derive (24) also. Ideally one would derive (24) by including both $B_z(x)$ and $B_y(x)$ shear in (86) for the clump dynamics. We would then proceed to derive the generalization of (93), and, upon integration, set the generlized $\hat{\mathbf{R}}$ equal to one and obtain (24). However, we have not been able to derive (24) in this way. While the assumption of tokamak ordering greatly simplified the calculation of $\hat{\mathbf{R}}$ from (86), the general calculation is much more difficult. For example, the $B_z'(x) \neq o$ effects will contribute to both the mixing term for self-consistent clump generation and to the propagators for clump decay. These additional B_z' vector contributions must be evaluated in a way that the self-consistency and conservation properties of the two point equations are maintained. Alternatively, one might imagine trying a quasilinear calculation in the spirit of Ref. 22. However, such calculations are not fully nonlinear and consider only nonresonant diffusion effects. For example, we note that the propagators in the diffusion coefficients in Ref. 22 are of the form γ/ω_k^2. We recognize this as the well known γ/ω_k^2 structure of non-resonant diffusion in quasilinear theory (here, $\omega_k = \underline{k} \cdot \underline{B}_o$ is the linear Alfven frequency). For clump fluctuations, the resonant contribution to D dominates, since, at the resonance, the turbulent mixing makes its largest contribution and the decay by Alfven wave emission is minimal. The nonlinear terms neglected in the quasilinear approach are also crucial to clump dynamics. For example, the diffusion coefficients in the nonlinear mixing term on the right hand side of (86) would, without their resonance broadening factors, diverge in the (ideal) solution for the clump growth rate. The broadening provided by the renormalization resolves this $\underline{k}\cdot\underline{B}_0=0$ singularity and leads to the amplitude dependent threshold ($\hat{R}=1$) and, for $\gamma\tau>1$, the γ^2 hydrodynamic-like growth of the instability. It is the amplitude dependence of this threshold parameter $\hat{R}$ that determines the amplitude dependence of μ (see (20) and (21)). The nonlinear terms are also important for clump decay since, no matter how small in magnitude, they will dominate near the resonances of clump localization. Therefore, while either the assumption of tokamak ordering or small amplitudes is simplifying enough to allow tractable analytical calculation, a rigorous calculation appears to be prohibitively difficult in the general case where neither assumption is made. Unfortunately, such is the case for a rigorous derivation of (24).

Assuming some reasonable symmetry properties for the steady state clump spectrum and diffusion coefficients, a hybrid derivation based on direct calculation and, when prohibitive, physical argument and analogy with the derivation of (114) is possible, however. For simplicity, we work in rectilinear, slab geometry. However, our final result (137) will be a vector equation valid in any coordinate system. We also work with dimensionalized variables where magnetic fields are normalized to the spacially averaged mean magnetic field $\bar{B}_0$ and lengths to the current channel radius a (see Sec. IIA of Ref. 1 for details).

We begin with the generalization of the mean-square clump source term $E_{oz}\,J_{oz}$ to $\underline{E}_o\cdot\underline{J}_o = E_{oz}J_{oz} + E_{oy}J_{oy}$. In the presence of J_{oz} and J_{oy}, the magnetic field $\underline{B}=\underline{\nabla}\times\underline{A}$ is now given in terms of the poloidal ($A_z = \psi_p$) and toroidal ($A_y = \psi_T$) flux functions. As with E_{oz}, the mean electric field E_{oy} will have the Fokker-Planck form (14), i.e., $E_{oy} = DJ_{oy} - F\,B_{oy}$. As discussed in the Introduction, this structure is a consequence of the self-consistent nature of the clumps, the F term being due to the incoherent part of the clump field. Because of magnetic helicity conservation, the D and F terms cancel to lowest order in the island width and lead to a fourth order diffusion process of the form (114) or, more simply, (15). Recall that **D** in (114) for $E_{oz} = -\partial\langle\psi_p\rangle/\partial t$ is due to the poloidal component of clump flux perturbation $\delta\Psi$, i.e., $\delta\Psi_p$ so that $\mathbf{D}\rightarrow\mathbf{D}^p$ in (114). Therefore, in an analogous fashion, $E_{oy} = -\partial\langle\psi_T\rangle/\partial t$ will be of the form (114), but with **D** replaced by the $\delta\Psi_T$ driven coefficient $\mathbf{D}^T$. This assumes that the contribution from poloidal and toroidal flux fluctuations are independent, and produces a model where their diffusive contributions are additive. Note that this assumption does not preclude the possibility that the mean parts of the poloidal and toroidal flux functions are correlated (i.e., as in (25)). These E_{oy} and E_{oz} components combine to give an $\underline{E}_o$ of the form $\underline{B}_o/B_o^2$ $\underline{\nabla}\cdot\underline{H}$, where $\underline{H}$ is the current for the fourth order (toroidal and poloidal) diffusion process. The factor $\underline{B}_o/B_o^2$ here is required since, in the general case, (10) rather than (12) preserves the magnetic helicity. (Note that for tokamak ordering, the full vector form here reduces to $E_{oz} = \underline{\nabla}\cdot\underline{H}$ (see (83) or (114)) and the helicity is preserved via (12) as before). We also assume for simplicity that, in the steady state, the turbulence is isotropic so that $\hat{\mathbf{D}}$ is diagonal, with the $\hat{x}\hat{x}$ and $\hat{y}\hat{y}$ components equal and each denoted by $\hat{\mathbb{D}}$. Therefore, with $J_{\parallel} = \underline{J}_o\cdot\underline{B}_o/B_o$, the helicity conserving, two point source term for poloidally and toroidally shear driven clumps is

$$-2\,\frac{\bar{J}_{\parallel}}{B_o}\,(\hat{\mathbb{D}}^p_{12}\,\nabla_1^2\,J_{oz} + \hat{\mathbf{D}}^T_{12}\,\nabla_1^2\,J_{oy}) \qquad (121)$$

where $\hat{\mathbf{D}}^p$ is given by (115) but with $G_{\underline{k}}$ replaced by $B_o G_{\underline{k}}$ (similarly for $\hat{\mathbf{D}}^T$). This additional B_o factor in the **D**'s provides for the correct magnetic field normalization of $G_{\underline{k}}$ in the general case (see (29)). Note that in the tokamak ordered case, $\underline{B}_o = \hat{z}\, B_{oz} = \hat{z}$ in dimensionless units, so $\hat{\mathbb{D}}^p = D^p$ and the first term of (121) gives the previous result calculated from the Strauss equations.

Equation (121) replaces the right-hand-side of (97). Integration along the two point orbits gives

$$-2\,\frac{J_{\parallel}}{B_o}\,(\tau^p_{c\ell}\,\hat{\mathbf{D}}^p_{12}\,\nabla^2_1\,J_{oz} + \tau^T_{c\ell}\,\hat{\mathbf{D}}^T_{12}\,\nabla^2_1\,J_{oy}) \tag{122}$$

where $\tau^p_{c\ell}$ and $\tau^T_{c\ell}$ are the clump lifetimes in the poloidal and toroidal flux surfaces respectively. Since the essential correlation between poloidal flux surfaces comes from their y_- separation, we set $\exp i\underline{k}.\underline{r}_- = \exp ik_y y_-$ in $\hat{\mathbf{D}}^p{}_{12}$ and put $\langle\underline{B}_-\rangle.\underline{\nabla}_- = B'_{oy}\; x_-\; \partial/\partial y_-$ in the two point propagator used to evaluate $\langle y_-^2(t)\rangle$ and thus $\tau^p_{c\ell}$. The clump poloidal lifetime $\tau^p_{c\ell}$ is then given by (90). Its x_- integral is given by (109) of Ref. 1. Similarly the essential correlation between toroidal flux surfaces comes from their z_- dependence, so we, therefore, set $\exp i\underline{k}.\underline{r}_- = \exp ik_z z_-$ in $\hat{\mathbf{D}}^T{}_{12}$ and put $\langle\underline{B}_-\rangle.\underline{\nabla}_- = B'_{oz}\; x_-\; \partial/\partial z_-$ in the two point propagator used to evaluate $\langle z_-^2(t)\rangle$ and thus $\tau^T_{c\ell}$. Then, the x_- integral of the clump toroidal lifetime $\tau^T_{c\ell}$ is given by (109) of Ref. 1, but with the replacements $B'_{oy} \rightarrow B'_{oz}$, $\delta(k_z + k_y\; B'_{oy}\; x_+/B_{oz}) \rightarrow \delta(k_y + k_z\; B'_{oz}\; x_+/B_{oy})$, and $A(k_y) \rightarrow A(k_z)$.

The clump correlation function now contains contributions from both $\delta\Psi_p$ and $\delta\Psi_T$. We again assume that the poloidal and toroidal clump fluctuations can be treated independently so that the fluctuation correlation $\langle\delta\Psi_p\delta\Psi_T\rangle$ is negligible. For example, in the strong N/L coupling limit, this means that $\langle\delta N_x(1)\;\; \delta N_x(2)\rangle_{\underline{k}} = - k_y^2\langle[\delta\Psi_p(1)\;\; \delta\Psi_p(2)\rangle_{\underline{k}} - k_z^2\;\langle\delta\Psi_T(1)\;\; \delta\Psi_T(2)]\rangle_{\underline{k}}$, where $\langle\delta\Psi_p(1)\;\; \delta\Psi_p(2)\rangle = \langle[\delta\psi_p(1)\;\; \delta\psi_p(2) + S^{-2}\;\; \delta\phi_p(1)\;\; \delta\phi_p(2)]\rangle$. The Fourier transformed nonlinear Newcomb equation (93) now becomes

$$(\frac{\partial}{\partial x_-^2} - k_y^2) \langle \delta\Psi_p(1)\delta\Psi_p(2)\rangle_{\underline{k}} + (\frac{\partial^2}{\partial x_-^2} - k_z^2)\langle\delta\Psi_T(1)\delta\Psi_T(2)\rangle_{\underline{k}}$$

$$= 2\,\frac{J_{\parallel}}{B_o}\,(\bar{\tau}^p_{c\ell}\,\hat{\mathbb{D}}^p\,\nabla_\perp^2\,J_{oz} + \bar{\tau}^T_{c\ell}\,\hat{\mathbb{D}}^T\,\nabla_\perp^2\,J_{oy}) \tag{123}$$

where $\bar{\tau}^p_{c\ell}$ is the Fourier transform of $\tau^p_{c\ell}$ and we have set $\mathbf{D}_{12} = \mathbf{D}$ since the dominant x_- dependence comes from the $\tau_{c\ell}$ factors. If we integrate (123) as in Sec. IVB of Ref. 1, we obtain

$$\left[\delta'_{\underline{k}}(p)+2|k_y|\right]\langle\delta\Psi_p^2\ (0)\rangle_{\underline{k}} - \left[\delta'_{\underline{k}}(T)+2|k_z|\right]\ \langle\delta\Psi_T^2\ (0)\rangle_{\underline{k}}$$

$$= \frac{J_{\parallel}}{B_o}\,\frac{\pi}{S}\,\Big[A(k_y)\hat{\mathbb{D}}^p\delta(k_z + k_y\ B'_{oy}\ x_+/B_{oz})\ \frac{\nabla_\perp^2 J_{oz}}{J_{oz}}$$

$$+ A(k_z)\ \hat{\mathbb{D}}^T\ \delta(k_y + k_z\ B'_{oz}\ x_+/B_{oy})\ \frac{\nabla_\perp^2 J_{oy}}{J_{oy}} \tag{124}$$

where $\delta'_{\underline{k}}(\bar{p})$ is the discontinuity in $\langle\delta\Psi_p(1)\ \delta\Psi_p(2)\rangle_{\underline{k}}$. As in Ref. 1, we equate the discontinuities δ' to that of the Newcomb solution. In the general case, the Newcomb equation can be written in terms of the radial component of the magnetic field $\delta B_x = \partial\delta\psi_p/\partial y - \partial\delta\psi_T/\partial z$. We again assume that poloidal and toroidal fluctuations are uncorrelated ($\langle\delta\psi_p\delta\psi_T\rangle = 0$) so that

$$\langle\delta B_x(1)\delta B_x(2)\rangle_{\underline{k}} = k_y^2\ \langle\delta\psi_p(1)\delta\psi_p(2)\rangle_{\underline{k}} + k_z^2\ \langle\delta\psi_T(1)\delta\psi_T(2)\rangle_{\underline{k}} \tag{125}$$

Then,

$$\frac{\partial}{\partial x_-}\ \langle\delta B_x(1)\delta B_x(2)\rangle_{\underline{k}} = k_y^2\ \delta'_{\underline{k}}(p)\langle\delta\psi_p(1)\delta\psi_p(2)\rangle_{\underline{k}}$$

$$+ k_z^2\ \delta'_{\underline{k}}(T)\langle\delta\psi_T(1)\delta\psi_T(2)\rangle_{\underline{k}} \tag{126}$$

But, it is also true that

$$\frac{\partial}{\partial x_-}\ \langle\delta B_x(1)\ \delta B_x(2)\rangle_{\underline{k}} = \langle\left[\frac{\delta B'_x(1)}{\delta B_x(1)} - \frac{\delta B'_x(2)}{\delta B_x(2)}\right]\delta B_x(1)\delta B(2)\rangle \tag{127}$$

$$= \Delta'_{\underline{k}} \langle \delta B_x(1) \delta B_x(2) \rangle_{\underline{k}} \tag{128}$$

Therefore, the discontinuity of the general Newcomb solution can be written in terms of the δ' factors as

$$\Delta'_{\underline{k}} = \delta'_{\underline{k}}(p) \frac{k_y^2 \langle \delta\psi_p^2(0) \rangle_k}{k_y^2 \langle \delta\psi_p^2(0) \rangle_{\underline{k}} + k_z^2 \langle \delta\psi_T^2(0) \rangle_{\underline{k}}}$$

$$+ \delta'_{\underline{k}}(T) \frac{k_z^2 \langle \delta\psi_T^2(0) \rangle_k}{k_y^2 \langle \delta\psi_p^2(0) \rangle_{\underline{k}} + k_z^2 \langle \delta\psi_T^2(0) \rangle_{\underline{k}}} \tag{129}$$

We now assume that the poloidal and toroidal wave numbers of a resonant clump fluctuation are comparable in the steady state. This seems to be a reasonable assumption for the fully developed, isotropic spectrum considered here. Equation (129) then simplifies to

$$\Delta'_{\underline{k}} = \frac{\delta'_k(p) \langle \delta\psi_p^2(0) \rangle_k + \delta'_k(T) \langle \delta\psi_T^2(0) \rangle_k}{\langle \delta\psi_p^2(0) \rangle_{\underline{k}} + \langle \delta\psi_T^2(0) \rangle_{\underline{k}}} \tag{130}$$

(130) and a matching of the solutions then allows the left-hand-side of (124) to be written as

$$-(\Delta'_{\underline{k}} + 2\,|k_y|)\, \langle \delta\Psi_p^2(0) \rangle_{\underline{k}} - (\Delta'_{\underline{k}} + 2|k_z|)\, \langle \delta\Psi_T^2(0) \rangle_{\underline{k}} \tag{131}$$

Since the toroidal and poloidal fluctuations are assumed uncorrelated, we break (124) into its toroidal and poloidal components,

$$\langle \delta\Psi_p^2(0) \rangle_{\underline{k}} = -\frac{\pi J_{\parallel}}{SB_o} \frac{\mathrm{Re}\Delta'_k + 2|k_y|}{(\mathrm{Re}\Delta'_{\underline{k}} + 2|k_y|)^2 + \lambda^2} \hat{\mathbf{D}}^P \delta(k_z + k_y B'_{oy} x_+ / B_{oz}) \frac{\nabla_{\perp}^2 J_{oz}}{J_{oz}} \tag{132}$$

and

$$\langle \delta\Psi_T^2(0) \rangle_{\underline{k}} = -\frac{\pi J_{\parallel}}{SB_o} \frac{\mathrm{Re}\Delta'_k + 2|k_z|}{(\mathrm{Re}\Delta'_{\underline{k}} + 2|k_z|)^2 + \lambda^2} \hat{\mathbf{D}}^T \delta(k_y + k_z B'_{oz} x_+ / B_{oy}) \frac{\nabla_{\perp}^2 J_{oy}}{J_{oy}} \tag{133}$$

where $\lambda = \mathrm{Im}\ \Delta'_{\underline{k}}$. We now form the diffusion coefficients by multiplying

(132) by $S^2 B_o G_{\underline{k}} k_y^2/k_d^2$ and (133) by $S^2 B_o G_{\underline{k}} k^2{}_z/k_d^2$ where $k_d^2 = (\Delta x_{\underline{k}})^{-2} + k_y^2 + k_z^2$ is the generalization of $k_{\perp}^2$ in Sec. III. Again, the delta functions in (132) and (133) will collapse the $G_{\underline{k}}$ function and give the Lyapunov time τ_o. Thus, similar to (111) of Ref. 1,

$$\hat{\mathbf{D}}^p = - \frac{J_{\parallel}}{B_o} (S\tau_o\ k_o\ B_o)\ \hat{\mathbf{D}}^p I\ \frac{\nabla_{\perp}^2 J_{oz}}{J_{oz}} \tag{134}$$

$$\hat{\mathbf{D}}^T = - \frac{J_{\parallel}}{B_o} (S\tau_o k_o B_o)\ \hat{\mathbf{D}}^T I\ \frac{\nabla_{\perp}^2 J_{oy}}{J_{oy}} \tag{135}$$

where

$$I = \int \frac{dk_y}{2\pi k_o} \frac{\mathrm{Re}\Delta_{\underline{k}}'+2|k_y|}{(\mathrm{Re}\Delta_{\underline{k}}'+2|k_y|)^2+\lambda^2} \frac{k_y^2}{k_d^2} A(k_y) \tag{136}$$

and, as in (116), $\Delta_{\underline{k}}'$ is evaluated at $\underline{k}.\underline{B}_o=0$. Of course, (134) is just $1=\hat{\mathbf{R}}$ for the poloidal driving term whereas (135) is the toroidal analogue.

Provided the **D**'s are not zero, (134) and (135) can be expressed in vector form as

$$\nabla_{\perp}^2\ \underline{J}_o + \mu^2\ \underline{J}_o = 0 \tag{137}$$

where

$$\mu^{-2} = \frac{J_{\parallel}}{B_o} (S\tau_o\ k_o\ B_o) I \tag{138}$$

In the case B_{oz} = constant >> B_{oy}, then $J_{\parallel}/B_o=J_{oz}$, $k_o B_o=k_{oy}$, and $\tau_o^{-1}=Sk_{oy}J_{oz}x_d$ so that $\mu^2 \sim \Delta_{\underline{k}}' k_{\perp}^2 x_d$ and (137) reduces to (21). In the general case, $\tau_o^{-1}= Sk_o J_{\parallel}\ x_d$ so that $\mu^2=x_d/I$. Then, for a smooth, broad and relatively flat spectrum, μ is constant and, with $\underline{\nabla}\cdot\underline{J}_o = 0$, the solution to (137) and Ampere's law is the force-free Taylor state

$$\underline{J}_o = \mu \underline{B}_o \tag{139}$$

where

$$\mu^2 = x_d/I \tag{140}$$

We can redefine I to make it dimensionless and of order unity by writing

$$I_d = \Delta' \left[k_o^2 + (\Delta x_o)^{-2}\right] I \tag{141}$$

where Δ' is a mean value of $\Delta'_{\underline{k}}$ for the unstable modes ($\mathrm{Re}\Delta'_{\underline{k}}$ evaluated for $k_y = k_o$) and Δx_o is the island width for mode k_o. If we assume that island overlap is more than satisfied in the steady state, then $\Delta x_o \sim x_d$ and (141) can be rearranged and written as

$$2k_o x_d \left(\frac{\mu^2}{2\Delta_o' k_o} I_d - 1\right) = (1-k_o x_d)^2 \tag{142}$$

A real and positive solution for x_d (and, therefore, D) only occurs when the threshold condition

$$\mu^2 > 2\Delta' k_o/I_d \tag{143}$$

is satisfied. Otherwise, the plasma is unstable. Therefore, of all possible Taylor states (i.e., μ values), the one with μ values given by (143) correspond to steady state clump turbulence. Notice that this threshold behavior can be traced to the presence of the inverse squared step size in brackets in (141), i.e., to the constraint of magnetic helicity conservation. Since $I_d \sim 1$, (143) implies that μ is an order one quantity for reasonable parameters. Such μ's also lead to B_{oz} field reversal, since $B_{oz} = \mu^{-1} J_{oz} \sim J_o(\mu r)$ from (141). Therefore, steady state clump turbulence occurs for B_{oz} field reversed Taylor states. At the threshold, $k_o x_d = 1$ and $\mu^2 = 2\Delta' k_o/I_d$. Much above threshold, $k_o x_d \gg 1$ and

$$\mu^2 = \Delta' k_o^2 x_d/I_d \tag{144}$$

or, in terms of the pinch parameter ($\theta = \mu a/2$ in dimensional units),

$$x_d = 4 I_d \, \theta^2 / \Delta' \, k_o^2 \tag{145}$$

Since $x_d \sim D^{1/3} \sim \langle \delta B^2 \rangle^{1/3}$, (139) implies that δB_{rms} increases with increasing current (θ) as θ^3.

In the Taylor model where (25) is calculated as a final state, μ enters as a Lagrange multiplier in a variational (energy minimization) principle and is, therefore, a constant. In the MHD clump model, the dynamical route to the final state is prescribed and μ only becomes constant in space when the clump spectrum becomes broad, flat and fully stochastic. Such spectral properties result from a large number of overlapping resonances. As we have noted in the discussion following (120), such conditions on the spectrum are also required for the strict application of the Markovian diffusion approximation used throughout the MHD clump theory. Inside the spectrum where these conditions are satisfied, D and μ will be relatively independent of position. Near the edges of the spectrum, the individual mode structures become important and will give a spacial dependence to D and μ.

At first throught, one might conclude that the Taylor model makes a more restrictive prediction about the final state than the clump model, namely, μ = constant. Actually, the constancy of μ is also built in by assumption in the Taylor model. There, if one assumes that magnetic helicity is invariant on each flux surface, the parameter μ in $\underline{J} = \mu \underline{B}$ is determined by each flux surface, i.e., μ depends on position. However, Taylor notes that during "violent", turbulent relaxation in an MHD plasma, flux surfaces will not be preserved. Only the flux surface at the conducting wall surrounding the plasma is invariant. Therefore, assuming that magnetic helicity is only conserved at the plasma boundary, there is only one Lagrange multiplier in the Taylor model--the μ associated with the boundary. The variational principle then leads to $\underline{J} = \mu \underline{B}$, where now, μ is a constant. The constancy of μ in the Taylor model can, therefore, be traced to the destruction of flux surfaces during turbulent relaxation of the plasma. Of course, flux surfaces can only be "violently" destroyed over a significant portion of the plasma crosssection if their resonances strongly overlap. Strong mode coupling and mixing will ensue and dynamically, a wide, fully stochastic spectrum will develop. The resulting stochastic field lines will transport current throughout the plasma crosssection, thus "homogenizing" the turbulence. Therefore, both the Taylor and the MHD clump models assume the constancy of μ. Moreover, the justification for the assumption is essentially the same for each model.

IV. INTERMITTENCY

An outstanding issue in the model concerns the complete role of finite amplitude island equilibria in the turbulence. In equation (8), the nonlinear interaction term T(1,2) describes, in its simplest form, the exponential divergence of neighboring field line trajectories. Memory of the initial state, i.e., initial correlations, will be lost exponentially with time. This lack of predictability, however, will be lessened if, because of the nonlinearity, coherent island structures tend to form. With this tendency toward fluctuation self-organization, the net mean-square fluctuation decay rate T(1,2) will be reduced, and fluctuation intermittency will tend to develop. Such a tendency toward intermittency would qualitatively alter the conceptual picture of the turbulence.

The intermittency issue is of importance in Vlasov hole turbulence, and we can look there for some guidance. Computer simulations of Vlasov plasma graphically show the development of hole intermittency in decaying[33] and driven[9] (i.e., unstable) turbulence. It has been proposed by Dupree[21] that these features can be partially understood conceptually as the tendency to develop fluctuations that maximize the Maxwell-Boltzman entropy ($f\ell nf$) of the plasma subject to dynamical constraints of energy and momentum conservation. The variational principle yields a fluctuation amplitude of

$$\delta f \sim -\varepsilon \frac{\Delta v}{v_{th}^2} \qquad (146)$$

where v_{th}, ε, and Δv are thermal velocity, dielectric function, and fluctuation velocity width. Rather than assuming a particular measure of the entropy, we can also obtain (146) by minimizing the mean-square phase space density, f^2, subject to momentum and energy conservation. While in a Vlasov plasma, f^2 is a dynamical invariant (it is constant along particle orbits), it will, in a coarse-grained sense, be dissipated by turbulent electric fields in the plasma and cascade to high wave numbers. Such a minimization principle is similar to the selective decay hypothesis[11] of fluid turbulence (see below).

Equation (146) describes a self-consistent, bound phase space density hole structure. Using Poisson's equation for the hole potential ϕ, (146) yields the trapping condition

$$\Delta v \sim (e\phi/m)^{1/2} \tag{147}$$

for a virialized equilibrium hole (e/m is the charge to mass ratio). Hole material tends to self-attract and coalesce into new holes in much the same way as a self-gravitating fluid.[18] This is a further impetus for hole formation. An analysis of collisions between holes shows that a dual cascade occurs with δf tending toward small scales and energy toward large scales.[21] Given that the Vlasov fluid is governed by a two dimensional (in x and v) phase space incompressible flow, this cascade is analogous to that of two dimensional Navier Stokes turbulence.[11] The Vlasov phase space density plays the role of fluid vorticity. Because (146) is the most probable state for the distribution of energy, momentum and phase space density, fluctuations can be expected to self-organize into such hole structures in a turbulent plasma. Another reason for the development of "Vlasov vorticity" concentrations is that an isolated hole can be unstable to growth for arbitarily small free energies.[34] These tendencies for hole self-organization and growth imply a turbulent state that is composed of an intermittent distribution of colliding, growing holes. The clump/hole theory deals with this turbulence, strictly speaking, only in its extremes. In the intermittent case of isolated coherent holes, one deals with the hole model and considers self-consistent hole structure, dynamics, and growth.[21,34] In the opposite extreme where the packing fraction of the holes in phase space is one half (i.e., the distribution of δf is Gaussian), one deals with the clump model and considers two point correlation functions, mode coupling, and growth of the mean-square fluctuation level.[35] The tendency for hole formation and attraction is modeled phenomenologically in the statistical equations. The T(1,2) term describing decay by exponentially diverging orbits is reduced, in accord with computer simulations[33], by a multiplicative factor on the order of 1/3. Interestingly, in the intermediate regime where the region of applicability of two models overlap, the clump and hole models agree. For example, neglecting mode coupling (hole-hole collisions), the fluctuation growth rate of the clump model agrees with that of the isolated hole model. The short coming of the clump/hole theory lies in its inability to predict quantitatively the development of hole formation and intermittency from the statistical correlation function equations describing the turbulence. However, the theory can predict, based on fluctuation self-organization and isolated hole growth arguments, that hole formation and intermittency will develop. Probability arguments have also been used to calculate approximately under what conditions intermittency will occur.[34]

Intermittency in fluid turbulence is well known. Concentrations of fluid vorticity have been observed in computer simulations of decaying fluid turbulence.[36] As in decaying Vlasov turbulence, the vorticity concentrations, called modons, develop spontaneously and, except for brief encounters with other modons, persist. The space occupied by the modons (i.e., the modon "packing fraction") decreases with time, leading to nonGaussian statistics. Modons also develop in apparently driven, geophysical flows.[37,38] Their formation is important to the predictability of such flows. One route taken for the integration of modons into the turbulence is similar to that for holes in Vlasov turbulence. A variational or minimization principle is invoked, based on the so-called selective decay hypothesis.[11] Of all the inviscid dynamical invariants, the "dissipated" one that viscously decays the fastest (to high wave numbers) is selected to be minimized subject to the constancy of the remaining, more slowly decaying ("rugged") invariants. The variational principle yields modon solutions.[39] The stability and interactions between modons are the subject of active research.

Though current density concentrations (intermittency) have been observed in simulations of decaying, homogeneous MHD turbulence[11], the issue of isolated island formation and intermittency is less clear in shear driven MHD turbulence. However, there are suggestive parallels with Vlasov hole intermittency. One is the existence of most probable states. In parallel with the Vlasov case, one might calculate such a state by minimizing the energy subject to the constancy of magnetic helicity. Recall that it is the energy--being the MHD analogue of the Vlasov phase space density--that mixes and cascades to high wave numbers during turbulent mixing. In a coarse grained sense, the energy is dissipated. This is another variant of the selective decay hypothesis (see Montgomery's review). Performing the variational principle, one obtains a result known as Woltjer's theorem,[40]

$$\underline{\nabla} \times \underline{B} = \mu \underline{B} \tag{148}$$

where μ is the Lagrange multiplier for the magnetic helicity. Coupled with Ampere's law, (148) is just the force-free ($\underline{J}\times\underline{B} = 0$), Taylor state ($\underline{J} = \mu\underline{B}$). The global or mean field part of (148) is (25). As we showed explicitly in Ref. 1, one localized or resonant <u>fluctuation</u> solution to the self-consistent force-free state equations, $\underline{J} = \underline{\nabla}\times\underline{B}$ and $\underline{J}\times\underline{B} = 0$, is the magnetic island or current hole. The suggestion here is that, in a turbulent MHD plasma with magnetic shear, fluctuations will tend to self-organize into current density holes. These localized fluctuations described by (23) and (57) are the analogues of the Vlasov holes described by (146) and (147). The magnetic island coalescence instability[41]--in analogy with the self-gravitating, Jean's instability of Vlasov holes--would also tend to promote this self-organization into island structures. The decay effect of the T(1,2) term in (8) will be reduced by this tendency for fluctuation self-organization. Further impetus for magnetic vorticity concentrations comes from the growth of the current holes. We, therefore, expect that coherent, current (magnetic island) structures will form intermittently out of MHD clump turbulence.

Without the intermittent formation and growth of current holes, the turbulence would have a symmetric distribution of δJ_z fluctuations, i.e., $\delta J_z<0$ and $\delta J_z>0$ values would be equally likely. Preferential formation and growth of the holes ($\delta J_z<0$) breaks this symmetry. Such symmetry breaking is required of a turbulent dynamo, though it is the statistics of the flow (δV) field that are broken in conventional kinetic dynamo models.[7] Rather than assumed, as in a kinetic model, MHD clump turbulence is self-consistently generated via Ampere's law. Ampere's law determines both the hole structure and the free energy source for hole growth. The $\delta J_z \to -\delta J_z$ symmetry is thus self-consistently broken. The breaking occurs spontaneously as parallel current fluctuations $\delta J_z<0$ intermittently coalesce into self-consistent hole structures, and dynamically as the holes (once formed) preferentially grow via MHD clump instability (recall that the $\delta J_z>0$ fluctuations decay). An analogous situation occurs in the Vlasov plasma where the $\delta f \to -\delta f$ symmetry of weak turbulence is self-consistently broken via Poisson's equation. Phase space density holes ($\delta f<0$) spontaneously coalesce (Jean's instability) and grow (hole instability) while $\delta f>0$ fluctuations decay. Symmetry breaking by the spontaneous generation of nonperturbative, self-consistent structures also occurs in models of supeconductivity (Cooper pairs) and quantum field theory (Higgs field).[42-44] There, the symmetry breaking is interpreted as a phase transition. The condensation of holes out of Gaussian background turbulence in a Vlasov plasma can also be thought of as a phase transition. An interpretation of the MHD clump instability as a phase transition to the Taylor state is discussed in the Appendix.

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APPENDIX

In this Appendix, we suggest that the relaxation to the Taylor state via MHD clump instability can be viewed as a phase transition where $\mu_c = \underline{J}_0 \cdot \underline{B}_0 / B_0^2$ is the critical point. For analogy, we consider the Meissner effect in a superconducting, current carrying wire and the spontaneous magnetization in a ferromagnet. At least, the discussion below presents an alternative view of the instability. At best, the results suggest the MHD clump instability as an example of a dynamical model for phase transitions.

The salient features of superconductivity are well known.[42-44] The self-consistent, collective interaction of the electrons with the lattice ions in a superconductor produces an attractive force between the electrons. For small electron energies (i.e., low temperatures, T), this attractive force overcomes the normal Coulomb repulsion and binds the electrons together into so-called Cooper pairs. When an imposed magnetic field tries to penetrate into the superconductor, it induces a macroscopic flow of Cooper pair current given by

$$\underline{J} = - K^2 \underline{A} \tag{A1}$$

where $\underline{A}$ is the vector potential and K^2 is a positive constant. This diamagnetic current in turn self-consistently generates (via Ampere's law) a magnetic field which tends to cancel the imposed field. The magnetic field satisfies

$$\nabla^2 \underline{B} = K^2 \underline{B} \tag{A2}$$

so that the field only penetrates a characteristic distance K^{-1} into the superconductor. This expulsion of flux from a superconductor is known as the Meissner effect.

As implied above, these superconductivity effects only operate below a specific critical temperature (T_c). The metal is said to undergo a phase transition at $T=T_c$. Above T_c, the microscopic diamagnetic currents fluctuate with no preferred orientation. However, for $T<T_c$, this orientation symmetry is broken as the currents tend to line up to yield the macroscopic, nonzero mean field (A1). Such "long range ordering" is strongest when the energy of the collective electron motion is minimum, i.e. for the ground state. When T=0, all the current fluctuations line up in the same direction. In such a state of vanishing thermal motion, resistance to current flow is zero--a well known property of superconductors. A similar symmetry breaking and long range ordering also occurs in a ferromagnet where spin alignment leads to a net magnetization ($\underline{M}$) of

$$\underline{M}(T) = \underline{M}(0)\ [1-(\frac{T}{T_c})^2] \tag{A3}$$

for $T \leq T_c$ and $\underline{M} = 0$ for $T > T_c$. The bracketed expression in (A3) measures the amount of symmetry breaking and long range ordering of the spins below the critical point $T=T_c$. Such phase transitions can be described theoretically by the Ginzberg-Landau mean field model in which the nonzero magnetization of the lowest energy state is obtained by minimizing the Gibbs free energy subject to constant M. The conclusions of the mean field model are supported by the so-called BCS theory where the condensation of the ordered field into the ground state has been calculated quantum mechanically. These are equilibrium thermodynamic or statistical models. They do not address the dynamical route of the phase transition.

Consider now MHD clump instability in a current carrying plasma. As in Section IV, we imagine that the instability develops from background noise where current fluctuations are Gaussianly distributed. As we've seen, for $\Delta'>0$ and $\hat{R}>1$, current hole ($\tilde{J}_z<0$) fluctuations will preferentially grow--

thus breaking the $\delta J_z \to -\delta J_z$ symmetry of the pre-instability phase. For $\Delta'>0$, the currents (43) induced in response to the magnetic fields (δB in (43)) of the hole fluctuations, $\tilde{J}_z$, reinforce $\tilde{J}_z$ further. From Ampere's law, this is a paramagnetic effect (see (64) - (67)). In isolation, such a $\tilde{J}_z$ fluctuation would develop into a magnetic island. As discussed in Sec. IA of Ref. 1, the island structure forms as the self-consistent trapping (self binding) of the magnetic field lines balance (near the shear resonances) the variation in the field line pitch. However, in an environment of many overlapping resonances, an island will be dissipated by field line stochasticity. In the MHD clump instability, net growth of hole fluctuations occurs as the regeneration of new holes overcomes the stochastic decay. This occurs when $\hat{R}>1$, where $\hat{R}$ is given by (33). Here, the stochastic decay plays the role of decay by thermal agitation (T) of the superconductivity case. In the steady state, the fluctuation level is nonvanishing and, with $\hat{R} = 1$, (33) reduces to (137), or with Ampere's law,

$$\nabla^2 \underline{B}_o = - \mu^2 \underline{B}_o \tag{A4}$$

Unlike (A2), (A4) has a minus sign--a sign that reflects a paramagnetic (rather than diamagnetic) effect where, in the steady state, clump fluctuations sustain the field by dynamo action. In analogy with the superconductivity case, the self-organization of the fields occurs as the system relaxes to its lowest energy state. As we've seen, the solution to (A4) is the Taylor state (25), i.e., the state of lowest energy subject to constant magnetic helicity. It is a turbulent steady state where the symmetry has been broken to form a macroscopic but turbulent fluctuation level of current holes.

A measure of the strength of the phase transition to the steady state is the growth rate of the fluctuations. In the fully stochastic case

$(\gamma\tau<1)$, (4) gives

$$\gamma = \frac{1}{2\tau}(1 - \hat{R}^{-1/2}) \tag{A5}$$

which, when combined with (34), can also be written as

$$\gamma = \frac{1}{2\tau}[1 - (\frac{\mu}{\mu_c})^2] \tag{A6}$$

where $\mu_c = \underline{J}_o \cdot \underline{B}_o / B_o^2$. The critical point for the phase transition is $\mu=\mu_c$. For $\mu>\mu_c$, the plasma is stable--no current holes grow. For $\mu<\mu_c$, the symmetry is broken and current holes preferentially self-organize and grow. This is analogous to the development of long range order in the superconducting state.

An analogy with the "order parameter" M (see (A3)) can be made by relating (A6) to the steady-state hole fluctuation level. If $\langle\delta B_b^2\rangle$ is the background magnetic noise level, then the fluctuation amplitude resulting from MHD clump instability will be given at time t by

$$\langle\delta B^2\rangle = \langle\delta B_b^2\rangle \exp \int_o^t dt' \, \gamma \tag{A7}$$

where γ is given by (A6). If the plasma is driven, as we have discussed in Sec. I(C,D), the fluctuation level will grow and then saturate at a finite level. Let the time for the steady state to develop be denoted by t_R. Then, the current hole fluctuation level, ΔB^2, will be given approximately by

$$\Delta B^2 = \langle\delta B_b^2\rangle \gamma t_R \tag{A8}$$

where $\Delta B^2 = \langle\delta B^2(t_R)\rangle - \langle\delta B_b^2\rangle$. Then, (A6) and (A8) give

$$\Delta B^2(\mu) = \Delta B^2(0)\,[1 - (\frac{\mu}{\mu_c})^2] \tag{A9}$$

for $\mu \lesssim \mu_c$ and $\Delta B^2 = 0$ otherwise. Here, $\Delta B^2(0) \equiv \langle\delta B_b^2\rangle\, t_R/\tau$. The parameters μ and μ_c play the roles of T and T_c respectively in the superconductivity case. For the transition (instability) to occur, we need $\mu < \mu_c$ which means that $\hat{\mathbb{R}} > 1$, i.e., hole self-organization and growth overcomes stochastic decay caused by "collisions" with other holes. This is analogous to the condition $T_c > T$ where the self-ordering of the magnetization currents overcomes the competing effect of random thermal motion (interparticle collisions).

Note that an increase in driving current (i.e., in μ_c) leads to larger fluctuation levels. Because (A6) is only strictly valid for $\gamma\tau<1$ (fully stochastic limit), (A9) is only valid near the critical point $\mu=\mu_c$. For $\gamma\tau>1$, the instability is more of the interchange type, and the μ dependence of ΔB^2 changes. Of course, when new physics comes into play, similar changes in parametric dependence also occur in superconductivity transitions. If, as in Sec. ID, we consider the instability onset at the overlap of two magnetic islands, the stability boundary is $R_m = (\mu/\mu_c)^2$. With (21), this gives Fig. 1. Figure 1 is a phase diagram for the phase transition caused by MHD clump instability for finite R_m.

As we have discussed here and in Ref. 1, the clump fluctuations arise, because of energy and magnetic helicity conservation, from the mixing of the mean sheared fields. As the fluctuations grow during MHD clump instability, the mean (course grained) fields are dissipated as the mean flux is expelled from the plasma. This spontaneous expulsion of mean flux during MHD clump instability is analogous to the Meissner effect. The disruptive instability[3] in a tokamak fusion device is a "Meissner effect" in which the plasma confinement is lost before the Taylor state of lowest energy is reached. The phase transition does not reach completion. However, in a driven reversed field pinch device, the mean flux is maintained (replenished) and a steady state turbulence is possible. At the attainment of the steady state, the phase transition reaches the Taylor state of minimum energy and the mean fields are supported by the paramagnetic, dynamo action (A4), i.e., (25).

Equilibrium thermodynamic or statistical mechanical models of phase transitions are analogous to the Taylor model[6] of relaxation in an MHD plasma. Such energy minimization principles predict the final state of the transition, not the dynamical route taken. However, as we have seen, the MHD clump instability is a dynamical route through the transition to the Taylor state in an MHD plasma.

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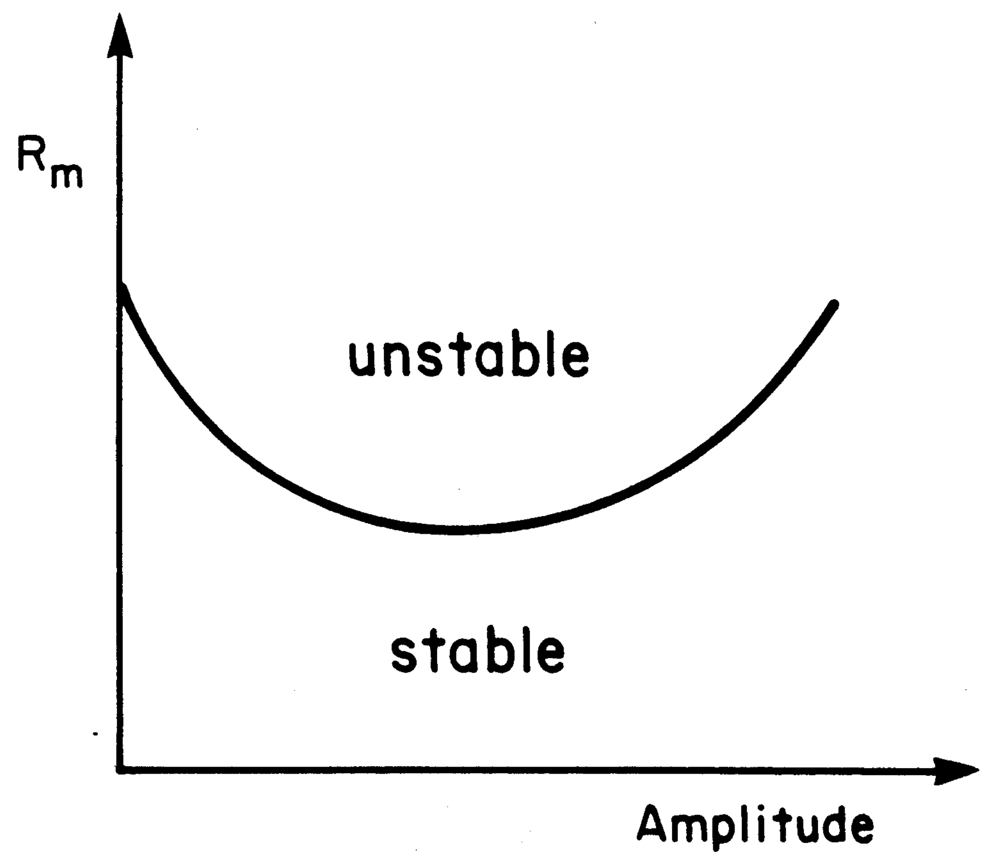
FIGURE CAPTION

1. Reynolds number R_m vs. amplitude for transition to MHD clump instability (schematic).


R_m
unstable
stable
Amplitude