PFC/JA-87-10

MHD CLUMP INSTABILITY

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January 1987

Submitted to The Physics of Fluids

ABSTRACT

The theory of MHD clump instability in a plasma with magnetic shear is presented. MHD clump fluctuations are produced when the mean magnetic field shear is turbulently mixed. Nonlinear instability results when, as the clump magnetic island structures resonantly overlap, the mixing overcomes clump decay due to magnetic field line stochasticity. The instability growth time is on the order of the Lyapunov time. The renormalized dynamical equation describing the MHD clump instability is derived from one fluid MHD equations and conserves the dynamical invariants of the exact equations. The renormalized equation is a nonlinear, turbulent version of the Newcomb equation of linear MHD stability theory and can be cast into the form of a nonlinear MHD energy principle. The growth rate of the instability is calculated. The instability is a nonlinear analogue of the Rayleigh-Taylor interchange instability in a magnetized fluid and, in the

fully stochastic case, of the tearing mode instability. MHD clump instability is a dynamical route to the Taylor state.

I. INTRODUCTION

It has been suggested that nonlinear fluctuations called clumps will occur in magnetohydrodynamic (MHD) plasma with magnetic shear.[1] Clumps were first discussed in studies of Vlasov turbulence, where the term was used to describe resonant fluctuations produced by the turbulent mixing of phase space density gradients.[2,3] The clump fluctuations arise because the phase space density is conserved in a Vlasov plasma: an element of phase space density is turbulently transported to a new region of phase space of different density. In the corresponding MHD case with magnetic shear, clumps arise because of the conservation of energy.[1] Clumps are localized, non-wave-like-structures and, in the Vlasov plasma, were shown to be holes in the phase space density.[4] In isolation, the Vlasov hole is a Bernstein-Green-Kruskal (BGK) mode and is the analogue of the modon in fluids.[5] In an MHD plasma with shear, the corresponding role is played by a hole in the current density, i.e., by the magnetic island. The Vlasov hole fluctuation is fundamentally nonlinear. Its self-consistent structure contains closed (trapped) orbits and, therefore, cannot be obtained from linear perturbation theory. The linear perturbation expansions also fail in the MHD case where resonantly trapped magnetic field lines close to form magnetic islands. Hole stability properties are also nonlinear. For example, Vlasov hole turbulence can grow in amplitude in linearly stable, current driven plasma.[6-9] A turbulent Vlasov plasma is composed of a random collection of these colliding, growing holes called clumps.[10-12] The MHD analogue of the Vlasov clump instability is the subject of this paper. The MHD clump instability describes the turbulence that results from the strong resonant interaction of magnetic islands at high magnetic Reynolds numbers. The turbulence has many features of fluid and Vlasov plasma turbulence, and thereby provides an interesting bridge between the two types of turbulence.

MHD clumps are resonant fluctuations localized near mode rational surfaces in the plasma. The fluctuations are produced when the mean magnetic field shear is turbulently mixed. Instability results when the mixing rate exceeds the clump decay rate due to the magnetic field line stochasticity generated at island (resonance) overlap. The stability boundary is nonlinear and, above a critical but small fluctuation amplitude, is below the linear stability boundary (Kruskal-Shafranov limit). The growth rate is amplitude dependent, and is on the order of the Lyapunov time. For low amplitudes, the Lyapunov time is long and the growth rate is a nonlinear analogue of that for the Rayleigh-Taylor interchange (mixing) instability. For large amplitudes, the Lyapunov time is short and the region of stochasticity is large. Then, the growth rate resembles that of the tearing mode driven by an anomalous resistivity due to stochastic magnetic field line diffusion. The instability threshold is a nonlinear analogue of the Kruskal-Shafranov condition where the Lyapunov length replaces the longitudinal wave length of the fluctuation. The turbulent mixing during MHD clump instability minimizes the energy subject to magnetic helicity conservation. Steady state MHD clump turbulence is described by the Taylor state.

The results presented here are developed from intuitive and physical models, as well as derived from a renormalized perturbation theory of the MHD equations. Though most fully developed for the nonlinear description of turbulence in simplified one dimensional plasma,[3,13] renormalized plasma theories can nevertheless be arcane. Applying the theories to the coupled, nonlinear, three dimensional vector equations describing MHD fluids would appear to only enhance this reputation. We have, therefore, applied the theory in a way that stresses its physical and conceptual features. This is made possible by earlier work on one dimensional Vlasov plasma where an integrated theoretical and computer simulation effort led to a detailed, but intuitive and tractable model of Vlasov turbulence.[3,4,6-12] This model, the clump/hole model, describes the plasma from the complementary viewpoints of localized, coherent structures (the holes) and their integration into a fully turbulent state of incoherent fluctuations (the clumps). The detailed results of the model agree well with the computer simulations. The MHD clump model extends the conceptual and mathematical features of this antecedent work. In particular, we expand upon the preliminary work of Ref. 1 with calculations based on Ref. 8.

Much work has been done by other investigators on turbulent relaxation in MHD fluids.[14-26] These include studies on stochastic diffusion of magnetic field lines,[14,15] selective decay and constraints of MHD conservation laws,[16,17,1] quasilinear models of turbulent dynamo action,[18,19] nonlinear dynamics of coherent[20] and interacting magnetic fluctuations,[21,24] and final force-free (Taylor) states of minimum energy.[16,24-25] Except for Ref. 26, where the self-consistent generation of (time independent) stochastic fields was considered, these investigations have assumed a given spectrum of fluctuations (e.g., Alfven waves, tearing modes). However, in the clump model of MHD turbulence, the self-consistent generation of fluctuations is treated on an equal footing with the nonlinear conservation laws. One is, therefore, led down a path different from the previous investigations. The result is a model that unifies many of their features and reveals the MHD clump fluctuation as a new constituent of MHD turbulence in sheared magnetic fields. Rather than having to assume a given spectrum of magnetic islands, a priori, the theory determines the fluctuations self consistently from the turbulence itself via Ampere's law and the conservation laws. In addition, unlike a statistical mechanical or variational calculation, the theory self-consistently determines nonlinear final states and the rates at which they are attained.[27] One pays a price for this--unlike a quasilinear model, the renormalized theory is manifestly nonlinear and, therefore, necessarily more complicated and approximate. However, we believe that the MHD clump theory, besides providing a unified picture, has a strong intuitive appeal that more than compensates for its additional complexity and approximations.

This paper describes MHD clump fluctuations and their instability to growth. Steady state MHD clump turbulence and the conservation of magnetic helicity are treated in a subsequent paper,[27] but a brief preview of the results is presented in Sec. IC below. The present paper is organized as follows. The current hole and its magnetic island structure are derived in Sec. IA. A physical discussion of resonantly interacting hole growth and decay is presented in Sec. IB. We conclude the Introduction with the preview of magnetic helicity conservation and its consequences for clump dynamics. The clump fields and their conservation laws are derived from the one fluid MHD equations in Part II. One and two point renormalizations of the clump field equations are presented in Part III. In Part IV, the two point equation is cast into the form of a nonlinear Newcomb-like equation and solved for the instability growth rate. In Part V, we show that the growth rate is analogous to that for the Rayleigh-Taylor instability in a magnetized fluid. In Part VI, we cast the clump growth rate into the form of a nonlinear MHD energy principle and show in Part VII that the stability boundary is a non-linear analogue of that for the linear kink mode. Part VIII compares the growth rate to that for Vlasov phase space density holes.

A. Current Density Hole

The fundamental nonlinear structure in the theory is a hole in the current density, i.e., a fluctuation $\delta J_z<0$ where J_z is the longitudinal current density. The current hole produces self-consistent magnetic fields which become "trapped" about spacial resonances (so-called mode rational surfaces) in a sheared magnetic field. The hole is localized near the mode rational surface where the restoring force to field line bending is minimal. The trajectory of these perturbed magnetic field lines forms a two dimensional vortex or magnetic island structure in the plane perpendicular to the current. A saturated tearing mode[28-29] is an example of such a vortex. A Kadomstev bubble[30], formed by a vacuum region inside the plasma, is an extreme example, where the hole depth is maximum, i.e., the current density is zero in the hole. This self-consistent island/vortex structure is the analogue of the trapped particle phase space vortex of the Vlasov hole[4] and of the modon in fluids.[5]

Consider MHD force balance

$$\underline{J} \times \underline{B} = 0 \tag{1}$$

($\underline{J}$ and $\underline{B}$ are current density and magnetic field) and Ampere's law for the self-consistent field

$$\underline{\nabla} \times \underline{B} = \mu_o \underline{J} \tag{2}$$

The equilibrium island structure of the current hole follows from the resonant solution to (1) and (2) in a sheared magnetic field. For simplicity, we use slab geometry and write $\underline{B} = \underline{B}_o + \delta\underline{B}$, $\underline{B}_o = \underline{B}_{\perp o} + B_{oz}\hat{z}$, and $\underline{B}_{\perp o} = \underline{B}_{oy}(x)\hat{y}$, where B_{oz} is assumed constant and $\delta\underline{B}$ is only in the transverse direction, i.e., $\delta\underline{B} = \delta\underline{B}_{\perp}$. For $B_{oz} >> B_{o\perp}$, this is a field with so-called tokamak ordering. Taking the $\hat{z}$ component of the curl of (1) and using $\underline{\nabla}.\underline{B} = \underline{\nabla}.\underline{J} = 0$, (1) reduces to[20]

$$\underline{B}.\underline{\nabla}\, J_z = 0 \tag{3}$$

(3) states that the J_z is constant along magnetic field lines. It is the reduced current equivalent of $\underline{J}\times\underline{B} = 0$, i.e., that $\underline{J}$ flows along magnetic field lines ($\underline{J} = \mu\underline{B}$). With $\underline{B}_1 = \underline{\nabla}\times(\psi\hat{z})$ defining the poloidal flux function and setting μ_o=1 in (2) for convenience, J_z and $\underline{B}$ in (3) are coupled self-consistently through Ampere's law:

$$\nabla_{\perp}^2\, \psi = -\, J_z \tag{4}$$

Expressing (3) in terms of the model field gives

$$B_{oz}\,\frac{\partial}{\partial z}\,J_z + B_{oy}(x)\,\frac{\partial}{\partial y}\,J_z + \delta B_x\,\frac{\partial}{\partial x}\,J_z = 0 \tag{5}$$

where for simplicity we have retained only δB_x in $\delta\underline{B}$. The two dimensional island structure is determined by the $\underline{B}$ line trajectory in the plane z=constant. The solution, therefore, is $J_z = J_z(\psi)$ where $\psi=\psi_o + \delta\psi$. This can be made more explicit by assuming a weak shear so that $B_{oy}(x) = B_{oy}'x$ where $B_{oy}' = \partial B_{oy}/\partial x$. Then, the structure equation is

$$B_{oy}'\, x\,\frac{\partial}{\partial y}\,J_z + \frac{\partial\delta\psi}{\partial y}\,\frac{\partial}{\partial x}\,J_z = 0 \tag{6}$$

with a solution of $J_z = J_z(\delta\psi - B_{oy}'\, x^2/2)$. The island is formed as the unperturbed field line trajectory $\underline{B}_o$ is perturbed and trapped near the resonance by the nonlinear line bending term δB_x in (5). The trajectories are bound (closed) when $B_{oy}'\, x^2/2 - \delta\psi < 0$, i.e., the island width is given by $\Delta x=(2\delta\psi/B_{oy}')^{1/2}$.

An analogous trapped structure occurs in a Vlasov plasma described by the Vlasov equation

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} - \frac{e}{m}\,\frac{\partial\phi}{\partial x}\,\frac{\partial f}{\partial v} = 0 \tag{7}$$

In steady state, unperturbed particle trajectories x=vt become perturbed and resonantly "turned around" or trapped by the potential ϕ. The phase space island structure is determined by contours of constant $f=f(\phi+mv^2/2e)$. The trajectories are bound when the total energy is negative, $mv^2/2 + e\phi < 0$, i.e., the island (trapping) width is $\Delta v = (-2e\phi/m)^{1/2}$. Poisson's equation determines the self-consistent potential required to trap the island contours of width Δv,

$$\frac{\partial^2}{\partial x^2} \delta\phi = -4\pi e \int dv \delta f \tag{8}$$

One divides δf into two parts: $\tilde{f}$ describing the island structure and f^c describing the nonresonant particles passing outside the island.[4] Then, we write (8) as

$$\frac{\partial^2}{\partial x^2} \delta\phi + 4\pi e \int dv f^c = - \int dv \tilde{f} \tag{9}$$

We Fourier transform (7) and evaluate f_k^c at the phase speed of the hole. Then, in terms of the mean distribution gradient, f_k^c is given by

$$f_k^c = - \frac{e}{m} \delta\phi_k \frac{1}{v} \frac{\partial f_o}{\partial v} \tag{10}$$

This nonresonant ($v>\Delta v$) particle distribution tends to shield out the resonant island potential. This leads to the definition of a dielectric shielding function, ε_k, from the Fourier transform of the left hand side of (9), i.e.,

$$\varepsilon_k \doteq 1 - \frac{4\pi e^2}{m} P\int dv \frac{1}{v} \frac{\partial f_o}{\partial v} \tag{11}$$

where P means principal value. The Fourier transform of (9) can then be written as

$$\phi_k = \frac{\tilde{\phi}_k}{\varepsilon_k} \tag{12}$$

where $\tilde{\phi}_{\underline{k}} = (4\pi e/k^2)\int dv\tilde{f}_{\underline{k}}$ is the hole or resonant part of the potential. Since the hole width is Δv and $\Delta v^2 =- 2e\phi/m$, (12) then yields an approximate relation between the hole depth and resonance width

$$\tilde{f} \sim - \varepsilon_k \frac{\Delta v}{v_t^{\,2}} (k\lambda_D)^2 \tag{13}$$

where v_t is the thermal speed and λ_D is the Debye length. Equation (13) gives the hole depth required to trap the particle trajectories into a bound hole structure of width Δv and k^{-1}. Localized, hole solutions result for $\varepsilon_k > 0$. A more rigorous calculation using maximal entropy arguments yields essentially the same hole solution as this physical balance of forces arguments.[4]

Imposition of self consistency (4) on (3) similarly yields the bound current hole. This "modon" solution follows from (4) with J_z=$J_z(\psi)$. We divide J_z into currents flowing inside (resonant) and outside (nonresonant) the island resonance as in (9). For small island width, the nonresonant current can be obtained from the linearized version of (3):

$$\delta J_{\underline{k}}^{NR} = i \frac{\delta B_{\underline{k}}}{\underline{k}\cdot\underline{B}_o} \frac{\partial}{\partial x} J_{oz} \tag{14}$$

where $\underline{k}\cdot\underline{B}_o = k_z B_{oz} + k_y B_{oy}(x)$. Eq. (14) is the usual nonresonant current response of tearing mode theory.[28] The $\underline{k}\cdot\underline{B}_o$ singularity in (14) implies that the field lines are particularly susceptible to bending at the mode rational surface where $\underline{k}\cdot\underline{B}$=0 = 0. The island structure will, therefore, be localized about this resonant surface at $x_s = - k_z B_{oz}/k_y B_{oy}'$. Inserting (14) into (4) gives

$$\left[\frac{\partial^2}{\partial x^2} - k_y^2 + \frac{k_y J'_{oz}}{\underline{k}\cdot\underline{B}_o}\right] \delta\psi_{\underline{k}} = - \delta J^R_{\underline{k}} \tag{15}$$

The well known Newcomb equation of linear stability theory follows from (15) with $\delta J^R_{\underline{k}} = 0$ and describes the currents flowing outside the island resonance.[28] Integrating (15) over these currents gives

$$\Delta'_{\underline{k}} = -|2k_y| - \frac{k_y}{\delta\psi_{\underline{k}}(x_s)} P\int dx \frac{\delta\psi_{\underline{k}}(x)}{\underline{k}\cdot\underline{B}_o} \frac{\partial}{\partial x} J_{oz} \tag{16}$$

Δ'_k is the usual stability parameter of linear tearing mode theory,

$$\Delta'_{\underline{k}} = - \frac{1}{\delta\psi_{\underline{k}}(x_s)} P\int dx \frac{\partial^2}{\partial x^2} \delta\psi_{\underline{k}}(x) \tag{17}$$

or, more conventionally[28],

$$\Delta'_{\underline{k}} = \frac{\delta\psi'_{\underline{k}}(1)}{\delta\psi_{\underline{k}}(1)} - \frac{\delta\psi'_{\underline{k}}(2)}{\delta\psi_{\underline{k}}(2)} \tag{18}$$

where (1,2) refer to positions (infinitesimally) on either side of the singularity at $x=x_s$. The jump in the logarithmic derivative of $\delta\psi$ is due to the $\underline{k}\cdot\underline{B}_o$ singularity in (14). In linear tearing mode theory, the singularity is resolved by collsional resistivity. In the nonlinear clump model, an analogous role is played by the nonlinear diffusion of stochastic field lines, i.e., resistivity due to the turbulence.

Nonlinearly, (4) and (5) give the self-consistent hole solution. We note that the nonlinear term on the left hand side of (5) will resolve the $\underline{k}\cdot\underline{B}_o$ singularity. For example,[26] in the presence of field line stochasticity, the nonlinear term gets replaced by a diffusion operator with the effect of replacing $\underline{k}\cdot\underline{B}_o$ in (14) and (15) with $\underline{k}\cdot\underline{B}_o + i\delta$ where $\delta>0$. We can then integrate (15) over all space and, using (17), obtain

$$(\Delta'_{\underline{k}} + 2k_y + i\lambda)\ \delta\psi_{\underline{k}}(x_s) = - \int dx\delta J^R_{\underline{k}} \tag{19}$$

where

$$\lambda = \frac{\pi k_y}{|k_y B'_{oy}|} \left(\frac{\partial J_{oz}}{\partial x}\right)_{x_s} \tag{20}$$

The factor λ is an effective imaginary part of the stability parameter Δ'. Since the integral in (19) only contributes in the resonant region of width Δx, (19) implies that Ampere's law can be schematically written as $\nabla_\perp^2 \delta\psi \sim -M\delta J^R$ where $M^{-1} \sim [\Delta'_{\underline{k}} + 2k_y + \lambda]\Delta x$. M is an effective permeability shielding the island and is the analog of the dielectric shielding (11). Here, the shielding comes from nonresonant currents flowing outside the island. For $\Delta'_k + 2|k_y|>0$, the nonresonant currents reinforce the resonant part of the island field. Using $\Delta x \delta J^R$ to approximate the right hand side of (19) and the resonance width $\Delta x = (\delta\psi/J_{oz})^{1/2}$ from the solution to (6), we can estimate δJ^R from (19) and obtain the depth $(-\delta J_z^R)$ of a force-free hole fluctuation ($\delta J_z = J_z - J_{oz}$) of small ($|\delta J_z|<J_{oz}$), but finite amplitude,

$$\delta J_z^R \sim - \Delta' \Delta x J_{oz} \tag{21}$$

(21) is the MHD analogue of the Vlasov hole relation (13). As with the Vlasov hole, it relates the hole depth required to self-consistently trap the field lines about the resonance.

As in the BGK solution of the Vlasov case, the current hole (21) is one of many possible solutions to the coupled equations (3) and (4). The full nature of these solutions would require a detailed pseudo potential analysis. However, we take the position where that in the turbulent state, many details of isolated, coherent hole structure will be "washed out" by the turbulence. Therefore, any bound structures tending to form will do so by the approximate balance of forces just described. The boundary conditions and detailed behavior of flux contours near and outside of the separatrix are approximated by the parameter $\Delta'_{\underline{k}}$. An analogous approximation (i.e., the use of ε_k in (13)) has proven to be very successful in the Vlasov case.[4]

B. Growth and Decay of Current Holes - Mixing and Stochasticity

We are concerned primarily with the inviscid dynamics of current holes at large Reynolds numbers. Neglecting the dissipation effects of resistivity and viscosity, a localized magnetized fluid element--such as an MHD hole fluctuation--can dissipate its energy in several ways.[31] In addition to convection, dissipation can occur by the radiation of sound waves away from the hole. This effect of pressure is considered in Sec. IIIC. Another channel--and the one of importance here--is through the radiation of shear-Alfven waves down magnetic field lines away from the hole. The rate at which the energy is dissipated is $V_A \underline{k}.\underline{B}/B$, where V_A is the Alfven speed and $\underline{k}$ is the wave number of the fluctuation. Since the holes are localized near mode rational surfaces where $\underline{k}.\underline{B} = 0$, the dissipation is minimal. Indeed, the trapping of the field lines near the resonance eliminates the dissipation even for a finite amplitude island. This equilibrium can be disrupted, however, if island resonances strongly overlap. Then, the island structure discussed in the previous section is dissipated or "torn-up" as Alfven waves propagate down the resulting stochastic magnetic field lines away from the hole. This decay is calculated in detail in Sec. III below, but can be understood here by the following physical considerations.

The dissipation rate (τ^{-1}) for a finite amplitude island is approximately $V_A k_y x_d B_{oy}'/B$, where $B_{oy}' = \partial B_{oy}/\partial x$ is the shear strength and x_d is the width of the stochastic region. (In the turbulent regime, x_d plays the role of resonance width that the island width plays in the coherent island regime. For strongly overlapping islands, $x_d \sim \Delta x$). Hole energy is dissipated as an Alfven wave propagates down a stochastic field line that random walks radially a distance x_d for each longitudinal distance of $z_o \sim V_A \tau \sim (k_y x_d B'_{oy}/B)^{-1}$. For diffusing field lines, $(\delta x)^2 = 2 D^m z$, where D^m is the field line diffusion coefficient. Therefore, $x^2_d \sim D^m z_o$. In the limit of infinitesimally small island width, D^m is given by the well known expression[14,15]

$$D^m = \int \frac{dk}{(2\pi)^2} \langle \delta B^2 \rangle_{\underline{k}} \; \pi \delta(\underline{k} \cdot \underline{B}_o / B) \tag{22}$$

For finite sized islands, the $\underline{k} \cdot \underline{B}_o$ resonance in (22) and (15) becomes broadened to, approximately, $(\underline{k} \cdot \underline{B}_o/B + i z_o^{-1})$ as in Ref. 26. Equation (22) then becomes an equation for D^m. Its solution shows that, at island overlap, D^m becomes nonzero and the field lines become stochastic.[26] Two neighboring field lines will diverge apart radially by x_d after a length $z \sim z_o$. The length z_o is referred to as the Lyapunov length or Kolmogoroph entropy[15]. The dissipation rate, $\tau^{-1} \sim V_A/z_o$, is the inverse Lyapunov time. In terms of the radial diffusion of the field lines, τ can also be written as $\tau^{-1} \sim D/x_d^2$ where $D = V_A D^m$. Here, D is a turbulent resistivity which, along with D^m, becomes nonzero at island overlap. The time τ is the time that two neighboring stochastic field lines will remain correlated, i.e., diffuse together. The nonlinear time τ is a turbulent skin or resistive time.

The dissipation can be opposed by the production of new fluctuations. This is the origin of the instability. The instability occurs as the resonant interactions of the finite amplitude islands create new fluctuations (clumps) by turbulently mixing the average magnetic shear. The clumps are produced because the energy is conserved in inviscid (ideal) MHD. The energy plays the role here that the phase space density plays in the production of Vlasov clumps.[2] The turbulence transports an element of magnetic fluid of given energy density to a new region which, because of the magnetic shear, has a different energy density. Since the magnetic shear results from a longitudinal mean current density in the plasma, the mean current density is the free energy source for the instability. Net growth, i.e., instability, is achieved by the creation ($\mathbf{R}$) of the new fluctuations by turbulent mixing at a rate faster than that of the stochastic decay, thus yielding a net growth rate of the form

$$\gamma = \frac{1}{\tau}(\mathbf{R} - 1) \tag{23}$$

The characteristic time scale for growth is the Lyapunov time τ. This creation of MHD fluctuations by turbulent mixing is the nonlinear, turbulent analogue of the Rayleigh-Taylor (interchange) instability in a magnetized fluid.[29,31] The "-1" stochastic decay term in (23) is the turbulent analogue of the line bending (restoring) force of the Rayleigh-Taylor model. The factor $\mathbf{R}/\tau$ is the clump analogue of the mixing rate of light and heavy fluid. The analogy is discussed in detail in Sec. IIIF below.

The instability is fundamentally nonlinear and three dimensional, even though an individual island structure is two dimensional. This is because the interaction of island resonances with incomensurate field line pitches (sometimes called field line "helicities") is three dimensional, i.e., shear Alfven waves propagate down magnetic field lines. For strongly overlapping islands, this resonant interaction is very inelastic, causing significant magnetic field line stochasticity and mode coupling of energy to high wave numbers. The strength of the interaction increases with fluctuation amplitude. Nonlinearity is also important for the production of the fluctuations. Constrained nonlinearly by energy conservation, stochastic (turbulent) transport creates new fluctuations (clumps) by the mixing of the magnetic shear. In addition, the nonlinearity tends to cause a clump fluctuation, once produced, to self-organize into a localized island (hole) structures. Instability results from the competition between these nonlinear effects of clump production and decay. These strong nonlinear features of the instability imply that the turbulence cannot be described by a perturbative nonlinear model in which linear theory provides the lowest order approximation. The stability analysis relies fundamentally on the existence of two-dimensional magnetic island equilibria of finite amplitude and their three dimensional interactions. These features are reminiscent of the transition to turbulence in plane Poisuille fluid flow[32] and in current driven Vlasov plasma[6-9]. As there, we find that the transition to turbulence is strongly nonlinear and subcritical. The onset and evolution of the MHD turbulence is described by a fully nonlinear set of equations, rather than the linearized MHD energy principle or the Newcomb equation for linear disturbances.[29] In the description of turbulence in fluids, the inadequacy of the Orr-Sommerfeld equation for linear fluctuations is well known.[32-33] The MHD clump theory extends the linear energy principle and Newcomb equation to the nonlinear, turbulent regime. Subcritical MHD turbulence has been apparently observed in computer simulations by Waltz.[34]

The depth of the current density hole plays a role similar to that of vorticity in fluid turbulence. "Magnetic vorticity" (J) self-consistently determines the magnetic field through Ampere's law (2). Because of the existence of the finite amplitude island equilibrium, the magnetic vorticity fluctuations tend to organize into the localized, resonant island structures of Sec. IA above. Though both positive and negative fluctuations in the current density can occur, it is the current density holes, i.e., the negative magnetic vorticities, that can grow. These correspond to $\Delta'>0$ in (21). To see this explicitly, consider an isolated concentration of magnetic vorticity, δJ. Faraday's law is

$$\frac{\partial}{\partial t}\,\delta\psi = -\,D\,\delta J \tag{24}$$

where D is a turbulent resistivity modeling the stochastic background fluctuations (i.e., other islands). Growth, a positive value to the right hand side of (24), occurs for current holes, $\delta J<0$. Using the island width, and (3) for δJ, (8) yields a growth rate, $\lambda_H = \partial \ln\delta\psi/\partial t$, resembling that for the tearing mode,[20]

$$\gamma_H \sim D\,\frac{\Delta'}{\Delta x} \tag{25}$$

Growth occurs for $\Delta'>0$ (free energy available) and $D\neq 0$ (stochasticity from overlapping resonances). For $\Delta'>0$, the nonresonant current δJ^{NR} flowing outside the island reinforces the resonant island current δJ^{R} (see (21)) and the island grows. Though the instability may be precipitated by the overlap of two coherent islands (e.g., two tearing modes), a truly turbulent state will quickly develop as interactions ("collisions") between the islands lead to mode coupling (hole decay) and mixing (generation of new holes). The resulting turbulence will be composed of incoherent, strongly interacting magnetic vorticity concentrations which we call MHD clumps. Rather than a coherent island structure, an MHD clump fluctuation is a flux tube or bundle of correlated magnetic field lines with finite lifetime on the order of τ. The instability or turbulence is modeled as the creation, interaction, and growth of these clump fluctuations. In this turbulent regime, (25) is replaced by (23).

The derivation of (23) is the main objective of this paper. We find that the clump regeneration factor **R** in (23) can be written approximately as (see (112) and (114))

$$\mathbf{R} \sim \frac{\Delta_k' x_d}{|\Delta_{\underline{k}}' x_d|^2}\,(1 + \gamma\tau)^{-1} \tag{26}$$

The $\Delta_{\underline{k}}' x_d$ factor in the numerator of (26) is the magnetic shear driving force for the instability. The instability derives its free energy as in the tearing instability. The $|\Delta_{\underline{k}}' x_d|^{-2}$ factor in (26) gives the shielding of the clumps. The clump fields that randomly mix the magnetic shear are shielded by the nonresonant currents flowing outside the islands. $\Delta_{\underline{k}}'$ has a real and imaginary part because of the stochastic broadening of the $\underline{k}.\underline{B}_0$ resonance. The $(1+\gamma\tau)^{-1}$ factor in (26) occurs because the clump fields causing the turbulent mixing are growing in time. For $\gamma\tau>1$, growth can occur before any appreciable stochastic decay, and (23) and (26) give

$$\gamma^2 \sim \hat{R}/\tau^2 \tag{27}$$

where $\hat{R} = R(1+\gamma\tau)^{-1}$. In this regime, the instability is a nonlinear analogue of the Rayleigh-Taylor interchange (mixing) instability. When $\gamma\tau<1$, the effect of stochasticity is significant, and the instability resembles that of the tearing mode, but driven by an anomalous resistivity due to stochastic magnetic (and flow) field diffusion. In this limit, (23) and (26) give

$$\gamma \sim D\frac{\Delta_{\underline{k}}'}{x_d}(\hat{R} - 1) \tag{28}$$

since $\tau^{-1} \sim D/x_d^2$. Except for the factor in parenthesis, (28) resembles the growth rate of the tearing mode in the Rutherford regime.[20] Here, D replaces the Spitzer resistivity of the Rutherford model, and the turbulent resonance width x_d replaces the coherent island width Δx. The $(\hat{R}-1)$ factor in (28) accounts for the net regeneration of new fluctuations by mixing even as existing island structures are torn up by magnetic field line stochasticity. The $(\hat{R}-1)$ factor corrects the growth rate (25) calculated in Sec. VI of Ref. 1.

C. Magnetic Helicity Conservation

We derive the clump regeneration factor (26) in Sec. III from renormalized MHD equations that are an ensemble averaged version of the one fluid MHD equations. In the absence of collisional dissipation (resistivity and viscosity), the one fluid MHD equations admit three dynamical invariants: total energy, cross helicity, and magnetic helicity (see Sec. II). (Except for the smallest spacial (dissipation) scales, these will be approximate invariants for large Reynolds number turbulence as well). As we have discussed above, it is the turbulent mixing of the energy invariant that is responsible for the generation of clump fluctuations. However, in an MHD plasma with self-consistent fields, the turbulent mixing of the energy is not done arbitrarily, but rather is globally constrained by magnetic helicity conservation. While we only deal explicitly with energy (and cross helicity) conservation in this paper, magnetic helicity conservation has important consequences. These are outlined briefly below, but are derived in detail in a subsequent paper on steady state MHD clump turbulence.[27]

Mean magnetic helicity is conserved because the fields generating the clumps by the mixing of the mean shear derive self-consistntly from the clump currents themselves via Ampere's law. As a result, the mean magnetic field and, from Faraday's law, the mean longitudinal electric field follow from a mean nonlinear Ohm's law of the form

$$-\qquad E_{oz} = DJ_{oz} - FB_{oz} \tag{29}$$

The D term in (29) describes the turbulent diffusion of magnetic field lines and would be present for an arbitrary spectrum of stochastic fields. The F term reflects the self consistency of the fields, i.e., the fact that the source of the stochastic mixing fields is in fact the individual clumps themselves. In the limit of zero island width, magnetic helicity conservation is ensured by the cancellation of the D and F terms in (29). This situation is analogous to the vanishing of the net transport between like-like particles in a one dimensional Vlasov plasma. There, the transport vanishes because of momentum conservation. Here, we have helicity conserving, resonant interactions ("collisions") between islands (current holes) rather than between momentum conserving, shielded test particles. To next order in the island width, E_{oz} becomes

$$E_{oz} = - \underline{\nabla} \cdot \underline{\underline{D}} \cdot \underline{\nabla} J_{oz} \tag{30}$$

where $\mathbf{D}$=$D(\Delta x_{\perp})^2 \sim D[k_y{}^2 + (\Delta x)^{-2}]^{-1}$. Equation (30) is reminiscent of the net $\underline{E}_o \times \underline{B}_o$ flux of guiding centers in a guiding center plasma where, because of ambipolarity constraints, net transport across the magnetic field occurs at second order in the gyro radius. At island overlap, the deeply resonant part of the island structures "collide" in an analogous fashion to the guiding centers, with magnetic helicity conservation playing the role of the ambipolarity constraint. Equation (30) conserves the magnetic helicity because $\int d\underline{x}\ \underline{E}_o \cdot \underline{B}_o = 0$, i.e.,

$$\int d\underline{x}\ E_{oz} = 0 \tag{31}$$

in a strong and constant longitudinal field B_{oz}. For a cylindrical plasma surrounded by a perfectly conducting wall, the integral constraint can be combined with Faraday's law to give the equivalent constraint on the mean field

$$\frac{\partial}{\partial t} \int d\underline{r}\ r\ B_{o\theta} = 0 \tag{32}$$

Equation (32) is the analogue of momentum conservation of the Vlasov case and the ambipolarity constraint of the guiding center case.

Because of magnetic helicity conservation, the turbulent mixing expels poloidal flux during the instability. Inside a confined plasma (where $\nabla_\perp^2 J_{oz} < 0$), $E_{oz} > 0$ and the flux decreases (see (30)). The converse occurs at the plasma edge (where $\nabla_\perp^2 J_{oz} > 0$). Equivalently, from the helicity constraint (31), an increase in E_{oz} inside a cylindrical plasma (where r is small) must be accompanied by a simultaneous, small decrease in E_{oz} near the plasma edge (where r is large). For a significantly large increase in E_{oz} on axis (i.e., the development of large fluctuation levels), E_{oz} will become negative near the plasma edge. We note that such radial electric field profiles have been observed during plasma disruptions in tokamak fusion devices.

Since magnetic helicity conservation constrains the dynamics of the mean shear profile, the turbulent mixing process generating the clump fluctuations is similarly constrained. While (26) derives from the nonlinear Ohm's law $E_{oz} = DJ_{oz}$, (30) implies that we must multiply (26) by $(-\nabla_\perp^2 J_{oz}/k_\perp^2 J_{oz})$ in order to ensure that the mixing process conserves magnetic helicity ($k_\perp^2 = k_y^2 + (\Delta x)^{-2}$ here). Then, the clump source term $\hat{R}$ in (26) becomes

$$\hat{R} \sim \frac{\Delta'_k x_d}{|\Delta'_{\underline{k}} x_d|^2} \left(- \frac{\nabla_\perp^2 J_{oz}}{k_\perp^2 J_{oz}}\right) \tag{33}$$

This constrained form of the clump mixing rate directly couples the fluctuation level to the mean profiles being mixed. For example, in steady state MHD clump turbulence ($\gamma = 0$ in (28)), the turbulent mixing balances the stochastic decay and J_{oz} satisfies

$$\nabla_\perp^2 J_{oz} + \mu^2 J_{oz} = 0 \tag{34}$$

where

$$\mu^2 \sim \Delta'_{\underline{k}}\, x_d \left(k_y^2 + \frac{1}{\Delta x^2}\right) \quad (35)$$

We show in the subsequent paper[27] that, (34) becomes

$$\nabla^2 \underline{J}_o + \mu^2 \underline{J}_o = 0 \quad (36)$$

in the general case of poloidal and toroidal currents. With $\underline{\nabla}\cdot\underline{J}_o = 0$, (36) has the solution

$$\underline{J}_o = \mu \underline{B}_o \quad (37)$$

MHD clump instability drives the plasma toward a force free state and is, therefore, a self-consistent, dynamical route to the Taylor state.[25] This occurs in the MHD clump model because turbulent mixing minimizes the energy subject to the constraint of magnetic helicity conservation. The self-consistent generation of currents and fields during MHD clump instability is a turbulent dynamo action. The D and F coefficients in (29) correspond to the β and α coefficients of dynamo theory.[19]

In the case of large, overlapping island resonances (as in a tokamak fusion device, where the mode rational surfaces are widely separated), the mixing lengths are large. Instability starts far from the Taylor state and results in significant disruption of the current profile. For more closely-packed resonances (as in a reversed field pinch fusion device), the initial mixing lengths are smaller and a steady state turbulence level is possible. However, in order for a fully stochastic ($\Delta x \sim x_d$) steady state to exist (i.e., Δx real in (35)), μ must exceed a threshold given approximately by $\mu^2 > 2\Delta'_{\underline{k}}\, k_y \sim 6$ for typical parameters. Since the Bessel function solution to (37) changes sign when $\mu > 2.4$, steady state MHD clump turbulence corresponds to B_{oz} field reversed Taylor states. MHD clump instability appears to provide a basis for a unified description of turbulent relaxation in tokamaks and RFPs. The instability occurs in its growth phase as the tokamak disruption and, in its steady state turbulence phase, in the RFP. Detailed comparisons between experiments and predictions of the MHD clump theory are presented in a second subsequent paper.[35]

II. MHD EQUATIONS

A. Dissipative, One Fluid Equations

The one fluid MHD equations provide a fluid description of the self-consistent magnetic and velocity fields in a plasma.[29] Self-consistency is achieved by combining momentum balance with Ampere's law, and using Faraday's law in conjunction with Ohm's law,

$$\underline{E} + \underline{V} \times \underline{B} = \eta_{sp}\underline{J}, \tag{38}$$

to obtain

$$\rho_o \left(\frac{\partial}{\partial t} + \underline{V}\cdot\underline{\nabla}\right)\underline{V} = \frac{1}{\mu_o}\,\underline{B}\cdot\underline{\nabla}\underline{B}_{\perp} - \underline{\nabla}_{\perp}\left(p + \frac{B_{\perp}^2}{2\mu_o}\right) + \nu\nabla_{\perp}^2\underline{V} \tag{39}$$

$$\left(\frac{\partial}{\partial t} + \underline{V}\cdot\underline{\nabla}\right) \underline{B}_{\perp} = \underline{B}\cdot\underline{\nabla}\underline{V} + \left(\frac{\eta_{sp}}{\mu_o}\right)\nabla_{\perp}^2\underline{B}_{\perp} \tag{40}$$

where η_{sp} is the Spitzer resistivity, ν is a collisional viscosity and, as above, we've assumed that B_z is constant and $\underline{V}=\delta\underline{V}$ is only transverse.

Three decay laws can be derived from (39) and (40). They are for the total energy,

$$\frac{\partial}{\partial t}\int d\underline{x} \left(\frac{1}{2}\rho_o V^2 + B^2/2\mu_o\right) = -2\nu\int d\underline{x}\ \omega^2 - 2\eta_{sp}\int d\underline{x}\ J^2 \tag{41}$$

the cross helicity,

$$\frac{\partial}{\partial t}\int d\underline{x}\ \underline{V}\cdot\underline{B} = -\mu_o\left(\frac{\nu}{\rho_o} + \frac{\eta_{sp}}{\mu_o}\right)\int d\underline{x}\ \underline{\omega}\cdot\underline{J} \tag{42}$$

and the magnetic helicity,

$$\frac{\partial}{\partial t}\int d\underline{x}\ \underline{A}\cdot\underline{B} = -2\,\frac{\eta_{sp}}{\mu_o}\int d\underline{x}\ \underline{B}\cdot\underline{J} \tag{43}$$

where $\underline{A}$ is the vector potential ($\underline{B}=\underline{\nabla}\times\underline{A}$) and $\underline{\omega}$ is the fluid vorticity ($\underline{\omega} = \underline{\nabla} \times \underline{V}$). Because of the difference in the number of gradient operators in the dissipation terms of (41) - (43), the magnetic helicity decays at a much slower rate than energy and cross helicity. This feature of the dissipation has been observed in computer simulations and is sometimes referred to as selective decay.[16] Energy and cross helicity cascade to high wave numbers while magnetic helicity remains approximately constant.

In order to define the Reynolds' number subsequently, it is useful to normalize $\underline{B}$ to B_{oz}, lengths to the current channel radius a, time to the Spitzer resistive time $\tau_R = \mu_o a^2/\eta_{sp}$, and $\underline{V}$ to a/τ_R. Then, (39) and (40) become

$$S^{-2}\left(\frac{\partial}{\partial t} + \underline{V}\cdot\underline{\nabla} - S_v\nabla_1^2\right)\underline{V} = \underline{B}\cdot\underline{\nabla}\underline{B}_\perp - \underline{\nabla}_1 p^* \tag{44}$$

$$\left(\frac{\partial}{\partial t} + \underline{V}\cdot\underline{\nabla} - \nabla_1^2\right)\underline{B}_\perp = \underline{B}\cdot\underline{\nabla}\underline{V} \tag{45}$$

where all quantities are dimensionless. The two scaling parameters in (44) and (45) are the Lundquist number $S=\tau_R/\tau_H$ ($\tau_H = a/V_A$ is the Alfven time), and the "Prandtl" number $S_v = \tau_R/\tau_v$ ($\tau_v = \rho_o a^2/\nu$ is the viscous time). Note also that: (1) the generalized pressure $(p + B_\perp^2/2\mu_o)$ in (39) is normalized in (44) to $p^*=(p+B_1^2/2\mu_o)(\mu_o/B^2{}_{oz})$, (2) with $\underline{J}\rightarrow(\mu_o a/B_{oz})\underline{J}$ in Eq. (2) Ampere's law becomes $\nabla\times\underline{B} = \underline{J}$, and (3) with $\underline{E}\rightarrow(\mu_o a/\eta_{sp}B_{oz})\underline{E}$ in (38), Ohm's law becomes $\underline{E} + \underline{V}\times\underline{B} = \underline{J}$. Though we will frequently consider only near ideal MHD effects and, therefore, neglect the collisional dissipation terms in (39)-(40), the particular normalized form (44)-(45) will be useful in identifying the physical significance of the nonlinear terms (e.g., as anomalous resistivities) and estimating the effects of collisional dissipation on MHD clump dynamics. The particular normalization will also be useful in Ref. 35 where the theory is compared to laboratory plasmas with finite S.

We will use the equation of state $\underline{\nabla}\cdot\underline{V} = 0$ so that p* in (44) is determined from $\underline{B}$ and $\underline{V}$. If the turbulent mixing occurs on a faster time scale and shorter spacial scales than the mean pressure profile, we can assume that mean pressure balance will be satisfied in a quasilinear sense, i.e.,

$$\langle \underline{B} \rangle \cdot \underline{\nabla} \langle \underline{B}_{\perp} \rangle - \underline{\nabla}_{\perp} \langle p^* \rangle = 0 \tag{46}$$

The mean pressure $p_o = \langle p^* \rangle$ responds quasistatically to the mean magnetic fields. Of course, in a low β plasma, the pressure p in p^* is small, and the mean current flows mainly along $\langle \underline{B} \rangle$, i.e., (46) reduces to $\underline{J}_o \times \underline{B}_o \cong 0$. In the case of Vlasov clumps, the mean distribution is also assumed to relax quasilinearly. The mean distribution changes slowly compared to the mixing rate for clump production. This occurs if the mixing length (resonance width) is less than the scale characteristic of the mean distribution. In the MHD clump case, this means $\Delta x < a$, where a is the radius of the current channel. From (45), (46), and $\underline{\nabla} \cdot \underline{V} = 0$, $\delta p^* = p^* - \langle p^* \rangle$ is determined by the equation

$$\nabla_{\perp}^{2} \delta p^* = \underline{\nabla}_{\perp} \cdot \left[\underline{B} \cdot \underline{\nabla} \delta \underline{B}_{\perp} + \delta \underline{B}_{\perp} \cdot \underline{\nabla} \langle \underline{B}_{\perp} \rangle - S_v S^{-2} \delta \underline{V} \cdot \underline{\nabla} \delta \underline{V} \right] \tag{47}$$

where $\delta \underline{B} = \underline{B} - \langle \underline{B} \rangle$. The system (44), (45) and (47) form a closed set for the determination of $\underline{B}$ and $\underline{V}$.

B. Conservation Laws

We are interested in nonlinear MHD instability and, therefore, cases of sizable fluctuation levels. In such cases, the magnetic Reynolds' number $R_m \sim S \delta B \Delta x$ (see (44) and (45)) is larger than unity. For $R_m \gg 1$, collisional dissipation will be a relatively small effect in (44) and (45), except for the small scales $\Delta x < x_c$ of the dissipation range ($x_c \sim (S \delta B)^{-1}$ is the dissipation scale). Neglecting collisional dissipation, i.e., the third term in the parenthesis of (44) and (45), the MHD equations admit three invariants: total energy, cross helicity, and magnetic helicity. Neglecting surface terms, the three conservation laws can be written as

$$\frac{\partial}{\partial t} \int d\underline{x} \; (B^2 + S^{-2} V^2) = 0 \tag{48}$$

for mean energy,

$$\frac{\partial}{\partial t} \int d\underline{x} \;\; \underline{V} \cdot \underline{B} = 0 \tag{49}$$

for mean cross helicity, and

$$\frac{\partial}{\partial t} \int d\underline{x} \;\; \underline{A} \cdot \underline{B} = 0 \tag{50}$$

for mean magnetic helicity. As discussed in Ref. 1, the invariants (48) and (49) result from the particular structure and symmetry of the nonlinear terms in (44) and (45). For example, total energy is conserved because the $\underline{V} \cdot \underline{\nabla}$ terms in (44) and (45) vanish separately upon integration (i.e., a mode coupling effect) while the $\underline{B} \cdot \underline{\nabla}$ terms, when integrated, cancel between (44) and (45) (i.e., a dissipation effect where magnetic and flow energy are converted into each other). Equation (50) results from $\int d\underline{x} \underline{E} \cdot \underline{B} = 0$ when $\underline{E} + \underline{V} \times \underline{B} = 0$. This can be seen directly by considering (45) explicitly in terms of $\underline{E}$, i.e., Faraday's law gives

$$\frac{\partial}{\partial t} \int d\underline{x} \; \underline{A} \cdot \underline{B} = -2 \int d\underline{x} \; \underline{E} \cdot \underline{B} \tag{51}$$

Since B_z is assumed constant and $\underline{V} = \delta \underline{V}$ is transverse only, we can set $\underline{B} = \underline{B}_{\perp}$ in (49). If we further assume an ordering where $B_{oz} \gg B_{o\perp}$, then (50) is equivalent to (31) or, with Faraday's law, (32). The conservation laws (48), (49), and (50) constrain the dynamics of the $\underline{B}_{\perp}$ and $\underline{V}$ fields. For example, consider the simple case where the cross helicity is initially zero. By (49), cross helicity will remain zero. Separating (48) into mean and fluctuating parts gives

$$\frac{\partial}{\partial t} \int d\underline{x} \, \langle \delta B_{\perp}^2 + S^{-2} \delta V^2 \rangle = - \frac{\partial}{\partial t} \int d\underline{x} \, \langle B_{\perp} \rangle^2 \tag{52}$$

where <> denotes an ensemble average. The conservation law (52) states that any rearrangement (such as turbulent mixing) of the mean magnetic field profile will necessarily produce fluctuation energy, i.e., MHD clump fluctuations. Note that arbitrary rearrangements are not allowed. Only those self-consistent motions that satisfy (50), or equivalently (31) or (32), are allowed. Equation (32) constrains the source of fluctuations on the right hand side of (52). Equivalently, we note that Faraday's law is linear, so that the right hand side of (52) can be rewritten in terms of the mean electric field $E_{oz} = \langle E_z \rangle$ to give

$$\frac{\partial}{\partial t} \int d\underline{x} \, \langle \delta B_{\perp}^2 + S^{-2} \delta V^2 \rangle = 2 \int d\underline{x} \, E_{oz} \, J_{oz} \tag{53}$$

The equivalent mean magnetic helicity constraint on E_{oz} in (53) is then (31).

Note that, in a course grained sense, the energy is dissipated. The turbulent mixing converts the large scale (mean) shear profile into smaller scale clump fluctuations. Since the magnetic helicity is conserved during the mixing, the energy is dissipated subject to the conservation of the helicity. This is reminiscent of the selective decay that occurs in the presence of collisional dissipation (see Sec. IIA above). Indeed, we show in Sec. III that, during the turbulent mixing, the nonlinear terms in (44) and (45) have the effect of anomalous resistivity and viscosity. This can be seen approximately here by writing $R_m \sim S(\delta B/B_{oz})$ $(\Delta x/a)$ in dimensional units as

$$R_m \sim [V_A \, \Delta x(\delta B/B_{oz})]/\eta_{sp} \tag{54}$$

The bracket in (54) is an anomalous resistivity due to magnetic fluctuations δB with transverse correlation lengths Δx. In the renormalized turbulence equations derived in Sec. III, the bracket in (54) gets replaced by a diffusion coefficient, $D=V_A D^m$, for diffusing stochastic magnetic field lines. It is also enlightening to write (54) as

$$R_m \sim \left[\frac{(\Delta x)^2}{\eta_{sp}}\right] \left[\frac{V_A}{\Delta x} \left(\frac{\delta B}{B_{oz}}\right)\right] \tag{55}$$

The first factor in brackets is the collisional resistive time in the resonant layer. The second factor is the inverse of the perturbed Alfven time in the resonant layer. If we denote these <u>nonlinear</u> times as $\tilde{\tau}_R$ and $\tilde{\tau}_H$ respectively, then $R_m = \tilde{\tau}_R/\tilde{\tau}_H$, i.e., the Lundquist number (S) defined nonlinearly in the resonant layer. The dimensional nonlinear resistive (Lyapunov) time is $\tau = \tilde{\tau}_R(\eta_{sp}/D) = \tilde{\tau}_R/R_m = \tilde{\tau}_H$, i.e., the Alfven time defined for the perturbed field in the resonant layer.

C. Clump Fields

From the above, we conclude that MHD clump fluctuations can only be investigated with a dynamical model that conserves energy and magnetic helicity. Since these constraints are due to the structure and symmetry properties of the nonlinear terms in (44) and (45), special care must be taken in treating those terms. In particular, any renormalization method used to approximate the stochastic or turbulent portion of these terms must preserve their symmetry properties so that the conservation laws are satisfied. One way to facilitate this is to rewrite (44) and (45) in terms of new field variables $\underline{L} = \underline{B} - S^{-1}\underline{V}$ and $\underline{N} = \underline{B} + S^{-1}\underline{V}$. Neglecting δp^* terms, (44) and (45) can be written for unit Prandtl number ($S_V = 1$) as

$$\left(\frac{\partial}{\partial t} - S\underline{L}.\underline{\nabla} - \nabla_{\perp}^2\right) \underline{N} = 0 \tag{56}$$

$$\left(\frac{\partial}{\partial t} + S\underline{N}.\underline{\nabla} - \nabla_{\perp}^2\right) \underline{L} = 0 \tag{57}$$

where $\langle \underline{L} \rangle . \underline{\nabla} \langle \underline{N} \rangle$ and $\langle \underline{N} \rangle . \underline{\nabla} \langle \underline{L} \rangle$ terms are absent because of mean pressure balance (46). We have suppressed the δp^* terms here because, as we show in Sec. III, the δp^* effects are small compared to the magnetic shear driving terms for clump production. Additionally, we have taken $S_V=1$ in (56) and (57) because the symmetry is best displayed for this case. While $S_V=1$ is not generally appropriate for all interesting plasmas, the resulting $\nabla_\perp^2$ terms in (56) and (57) do provide an approximate and effective measure of collisional dissipation for large R_m clump turbulence. Again, the reason for this is that the nonlinear terms dominate the collisional terms during the instability. Since $\underline{\nabla} . \underline{L} = \underline{\nabla} . \underline{N} = 0$, the conservative structure of the nonlinear terms in (56) and (57) ensure that $\langle N^2 \rangle$ and $\langle L^2 \rangle$ are invariants in the absence of collisional dissipation. In the case of homogeneous turbulence, where a spacial average can be identified with an ensemble average, the conservation of $\langle (N^2+L^2) \rangle$ and $\langle (N^2-L^2) \rangle$ is equivalent to (48) and (49) respectively. The relevant magnetic helicity constraint is still (31) or (32), since $\partial \langle \underline{N} \rangle / \partial t = \partial \langle \underline{L} \rangle / \partial t = \partial \langle \underline{B}_\perp \rangle / \partial t$.

The advantage of (56) and (57) in describing clump dynamics is their resemblence to the flow of phase space fluid in a Vlasov plasma, i.e., to the Vlasov equation. In the absence of collisional dissipation, the conservation of $\langle N^2 \rangle$ and $\langle L^2 \rangle$ is analogous to the conservation of (mean square) electron and ion phase space densities in a Vlasov plasma.[2,3] As there, the conserved quantities are mixed to finer spacial scales during turbulent decay. Here, this cascade to high wave numbers occurs in the energy. With the neglect of collisional dissipation (i.e., $R_m \to \infty$), the flow in (56) and (57) is "incompressible" and a magnetized fluid element retains its energy density for a finite time (the smaller the volume element, the longer the lifetime). During this time, the turbulence randomly transports the fluid element to a new region of space where, for nonzero magnetic shear, the $\langle N^2 \rangle$ and $\langle L^2 \rangle$ energy densities are different. As in the Vlasov plasma, this mixing process is the origin of the clump fluctuations.

While (56) and (57) follow directly from the MHD equations, it is interesting to deduce them from the physical effects governing MHD clump evolution. We recall from Sec. IB that, in addition to convection, a localized clump fluctuation can dissipate its energy through the emission of shear Alfven waves. We, therefore, have two fluctuations to consider: wave-like (Alfven waves) and non-wave-like (clump) fluctuations. The two fluctuations tend to exist in mutually exclusive regions of space. While the clump is resonant and localized near the mode rational surface, the shear Alfven wave propagates in the nonresonant region away from the mode rational surface. The two fluctuations represent two degrees of freedom or excitation in the plasma. The Alfven wave magnetic fluctuation is given, in dimensional units, by $\delta \underline{B} = \pm \sqrt{\rho}\, \delta \underline{V}$, where ρ is the mass density.[29] Note that this expression is valid even for finite amplitude Alfven waves. The localized clump fluctuation is obtained by subtracting out this wave component from the total fluctuation $\delta \underline{B}$. This subtraction and distinction between wave-like and non-wave-like portions of the field fluctuations is analogous to the one made in (8)-(9). There, we wrote the total fluctuation as $\delta f = \delta f^c + \tilde{f}$ where $\tilde{f}$ denotes the localized (resonant) non-wave-like part of the fluctuation and δf^c determines the wave-like (nonresonant) response through the dielectric constant (see (10), (11)). Since, in the MHD case, the wave has two polarizations, we define $\underline{N} = \underline{B} + \sqrt{\rho}\underline{V}$ and $\underline{L} = \underline{B} - \sqrt{\rho}\underline{V}$ in dimensional units. Consider the dynamical equation for $\underline{N}$. Since the backward Alfven wave has been subtracted out, the $\underline{N}$ field decays by convection, collisional dissipation, and the propagation of forward Alfven waves. For unit Prandtl number, we have, in dimensional units,

$$\left(\frac{\partial}{\partial t} + \underline{V}\cdot\underline{\nabla}\right) N = \frac{1}{\sqrt{\rho}}\, \underline{B}\cdot\underline{\nabla}\underline{N} + \eta\nabla_{\perp}^{2}\underline{N} \tag{58}$$

The equation for $\underline{L}$ is similar, except that Alfven wave emission is due to the backward wave:

$$\left(\frac{\partial}{\partial t} + \underline{V}\cdot\underline{\nabla}\right)\underline{L} = -\frac{1}{\sqrt{\rho}}\, \underline{B}\cdot\underline{\nabla}\underline{L} + \eta\nabla_{\perp}^{2}\underline{L} \tag{59}$$

A rewriting of these two equations expresses their symmetry and coupling features:

$$\left(\frac{\partial}{\partial t} + (\rho)^{-1/2}\, \underline{N}\cdot\underline{\nabla} - \eta\nabla_{\perp}^{2}\right)\underline{L} = 0 \tag{60}$$

$$\left(\frac{\partial}{\partial t} - (\rho)^{-1/2}\, \underline{L}\cdot\underline{\nabla} - \eta\nabla_{\perp}^{2}\right)\underline{N} = 0 \tag{61}$$

These equations, when nondimensionalized, are just (56) and (57). The fields $\underline{N}$ and $\underline{L}$ have historically been known as Elasser variables.[31] Their physical significance here lies in their identification with the clump or resonant portion of the field.

Besides providing a natural basis for clump analysis, the fields $\underline{L}$ and $\underline{N}$ also define the correct variables for analogy with the unmagnetized fluid case (see Sec. 4-3 of Ref. 31). Turbulent transport coefficients which depend on the mean-square flow fluctuation in the pure fluid case will depend on $\langle L^2 \rangle$ and $\langle N^2 \rangle$ in the corresponding MHD case. This equal footing of the $\underline{B}$ and $\underline{V}$ fields in the turbulent transport processes can be traced to the symmetry of the nonlinear terms in the ideal MHD equations. Physically, the fields are "frozen-in" so that $\underline{B}$ is transported with the flow. This symmetry can be "broken" by collisional dissipation. For example, for $\eta_{sp} \neq 0$ but $\nu = 0$ as in the classic case of the tearing mode, the addition and subtraction of (44) and (45) will not lead to the particularly symmetric equations (56) and (57) expressible in terms of $\underline{L}$ and $\underline{N}$ alone. While the $\underline{N}$

and $\underline{L}$ dynamics are approximately conservative for $S_v \sim 1$ and $R_m >> 1$, if the turbulence "cools" down to $R_m \sim 1$, the symmetry will be broken and the clump-like, conservative form of equations (56) and (57) will not be preserved.

It is interesting to note that the symmetry of the nonlinear terms in (56) and (57)--specifically the absence of $\delta\underline{N}.\underline{\nabla}\delta\underline{N}$ and $\delta\underline{L}.\underline{\nabla}\delta\underline{L}$ terms. A similar situation also occurs in the Vlasov clump case.[3] There, because of Poisson's equation, the self-consistent electron and ion fields conserve momentum. As a consequence, contributions from like-like fields (electron-electron, or ion-ion) cancel between themselves. Net transport arises only from interactions between like and unlike fields. The analogous situation occurs in (56) and (57), since they are self-consistent to start with (i.e., Ampere's law has already been imposed). As a result, the nonlinear terms in (56) and (57) take on a symmetrical form--thus conserving energy and helicity and leading only to like-unlike (nonlinear) interactions between the fields.

As in the Vlasov case, the two point correlation function is the appropriate quantity to describe clump dynamics.[2,3] The reason is that one point equations cannot describe the correlated motion of neighboring fluid elements and, therefore, the clump lifetime of localized, correlated magnetized fluid elements in an MHD clump. When renormalized, the one point equations would predict that field lines at all radial positions in the plasma would diffuse independently. Consequently, fluid elements of <u>any</u> spacial extent would decay at the same rate. However, a fluid element of infinitely small scale will have an infinite lifetime, since, in the limit of zero separation, neighboring field lines feel the same forces, diffuse together, and are thus correlated forever. Such an infinite lifetime is merely a statement that the total energy, i.e., $\langle N^2 \rangle$ and $\langle L^2 \rangle$, is conserved. For a clump of finite spacial extent, neighboring field lines diffuse at different rates and diverge apart exponentially. The clump decays as energy cascades to high wave numbers. As discussed in Sec. IB, the characteristic time for decay is the perturbed Alfven time--the time for Alfven waves to propagate down the stochastic field lines away from the clump.

It is perhaps appropriate here to comment on the reduced MHD (Strauss) equations[36] for the poloidal flux ψ and the vorticity $U=\hat{z}.(\underline{\nabla}\times\underline{V}) = -\nabla_{\perp}^2\phi$. These follow from the inverse curl of (40) and the $\hat{z}$ component of the curl of (4):

$$\left(\frac{\partial}{\partial t} + \underline{V}.\underline{\nabla}\right)\psi = -\eta_{sp}J_z + \frac{\partial\phi}{\partial z} \tag{63}$$

$$\rho\left(\frac{\partial}{\partial t} + \underline{V}.\underline{\nabla}\right) U = \frac{1}{\mu_o}\underline{B}.\underline{\nabla}J_z + \nu\nabla_{\perp}^2 U \tag{64}$$

Though the Strauss equations have proven extremely useful in the study of resistive MHD fluctuations such as tearing modes (see, for example, Ref. 20), it appears that they are not the most appropriate equations for the study of nonlinear clump fluctuations. First, the essence of clump production is the conservation of "phase space density" and its resonant cascade under turbulent decay to high wave numbers. While U does cascade to high $\underline{k}$, (63) and (64) for ψ and U cannot be combined into such an appropriate conservative form. The problem can be traced to the fact that ψ is not mixed during turbulent decay but, rather, flows to long scale lengths (inverse cascade). As we have shown, however, the full unreduced MHD equations for $\underline{B}$ and $\underline{V}$ display the conservative MHD clump dynamics in a natural way. In addition, the essential resonant localization and decay properties of nonlinear clump fluctuations are not directly evident in (63) and (64). Because the Alfven wave response has not been explicitly distinguished (subtracted out as in (58) and (59)), the fluid element propagators appear to be nonresonant. However, as we have seen, an MHD clump fluctuation can decay resonantly by the emission of shear Alfven waves. It is the localization of the clumps near mode rational surfaces that minimizes this decay. The resonant property of the propagators in (56) and (57) displays this clump localization directly and leads to the exponential increase in the separation between neighboring field lines i.e., to the resonant decay of the clumps. Of course, the Strauss equations have the advantage of being scalar equations that do not involve the pressure p*. However, we shall see that the two point correlation equations we will need to describe the clumps involve only scalar quantities such as $\langle \underline{L}(1).\underline{L}(2)\rangle$ and $\langle \underline{N}(1).\underline{N}(2)\rangle$. We will also show that, for the clump problem, the δp^* terms in (44) and (45) can be neglected.

III. RENORMALIZED EQUATIONS

The renormalization of the clump field equations (56) and (57) is carried out below. While a straightforward renormalized perturbation procedure leads, as in the case of Vlasov turbulence[3,13], to numerous terms, we focus only on those terms in the perturbation expansion that have an identifiable, physical meaning. These are a Markovian diffusion term and a Fokker-Planck "dynamical friction" term[27] that ensures the conservation of magnetic helicity in the model. This is not to say that the other contributions to the renormalization are necessarily smaller in magnitude or less important. Any terms relating to clump self-energy (island coalescing), for example, must play a relevant role in the turbulence, but, we have not been able to identify such terms. In our view, the simplest, self-consistent, energy and helicity conserving model of the turbulence can be constructed from only the diffusion and Fokker-Planck drag terms generated from the formal renormalization procedure. Such a view has also proven useful in the Vlasov case.[3] There, a statistical model retaining only a diffusion and a momentum conserving (Fokker-Planck friction) term from the renormalization provides the basic but essential features of one dimensional Vlasov clump dynamics. The model, when modified phenomenologically for clump self-energy, agrees well with the results of computer simulations.[7,8,10,12] The physical meaning of additional terms in the Vlasov renormalization have proven to be obscure or impenetrable. The clump self-energy effects have been particularly illusive. The corresponding terms generated in the renormalization of the three dimensional MHD case are even more complex and obscure and we, therefore, ignore them in this investigation. An emphasis on conceptual and physically motivated models over formal, mathematical approaches has also been useful in the understanding of fluid turbulence.[37] As there, the model developed below focusses on the concepts of turbulent eddy diffusion and mixing.

It turns out that the effects of magnetic helicity conservation can be treated separately from those of energy and cross helicity conservation. Therefore, in the balance of this paper, we neglect magnetic helicity conservation (and, therefore, the Fokker-Planck drag terms) and focus only on renormalized equations of the diffusive type. This leads to (28) where the "-1" term derives from the diffusive decay of the clumps and the clump source term $\hat{R}$ derives from the diffusive mixing of the mean magnetic shear. Corrections to $\hat{R}$ due to magnetic helicity conservation are derived in a subsequent paper (Ref. 27), but have been reviewed in Sec. IC above.

A. One Point Renormalization

Though our goal is a renormalized equation for the two point correlation function, the diffusion coefficients in the two point equation depend on one point fluid element propagators. We obtain these in the spirit of Refs. 38 and 39. The renormalization is done at the level of the fluid element trajectories in position space, rather than in full Fourier transform space. This allows for a more transparent connection with the renormalized field line trajectories of Sec. IA, as well as making the physical meaning of neglected non-Markovian effects more clear.

Working in slab geometry for simplicity and retaining only the $\hat{x}$ component of the fluctuating field, we suppress the $\nabla_{\perp}^{2}$ collision term for simplicity and write the perturbed version of (56),

$$(\frac{\partial}{\partial t} - S\langle \underline{B} \rangle . \underline{\nabla} - S\delta L \frac{\partial}{\partial x})\ \delta N = S\delta L \frac{\partial}{\partial x} \langle N \rangle \qquad (65)$$

Consider the fluctuation δN in a large group of turbulent background fluctuations $\delta L_{\underline{k}}(x)$ with wave numbers $\underline{k} = (k_y, k_z)$. We seek the ensemble averaged effect of the background fluctuations in the nonlinear term of (65). To do so, we calculate that part of δN that is proportional to each $\delta L_{\underline{k}}^*$ in the nonlinear term of (65), i.e.,

$$\delta N(\underline{x},t)=S\int_0^t dt'\ G(t,t')\delta L_{\underline{k}}^*\ (x,t')e^{-i\underline{k}\cdot\underline{x}}\ \frac{\partial}{\partial x}\ \delta N(x,t') \qquad (66)$$

Note that there is also a part of δN coming from the $\partial\langle N\rangle/\partial x$ term on the right-side of (65). However, that term (see (70)) contributes to the equation for $\langle N\rangle$ rather than to the equation for ΔN. The <u>operator</u> $G(t,t')$ in (66) satisfies

$$(\frac{\partial}{\partial t} - S\langle\underline{B}\rangle.\underline{\nabla} - S\delta L\frac{\partial}{\partial x})\ G(t,t') = 0 \qquad (67)$$

with $G(t,t)=1$. $G(t,t')$ is a single element propagator that converts $\underline{x}$ into the time-dependent magnetized fluid element trajectory $\underline{x}(t)$ with initial condition $\underline{x}(t')=\underline{x}$ (i.e., Eulerian into Lagrangian variables). For random phase, stochastic fields, we retain only the ensemble averaged orbits in $G(t,t')$ and, therefore, set $G(t,t') = \langle G(t,t')\rangle = \langle G(t-t')\rangle$ in (66). We further make the Markovian approximation by setting $\delta N(x,t') = \delta N(x,t)$ and pulling $\partial\delta N(x,t)/\partial x$ outside of the integral in (66). This approximation is strictly valid only if the scale lengths and correlation times of the background fluctuations are much shorter than those of $\delta N(x,t)$, but it yields the physically appealing result of a diffusion equation. With these approximations, we insert (66) into the nonlinear term of (65) to obtain

$$(\frac{\partial}{\partial t} - S\langle\underline{B}\rangle.\underline{\nabla} - \frac{\partial}{\partial x}\ D\ \frac{\partial}{\partial x})\ \delta N = S\delta L\ \frac{\partial}{\partial x}\ \langle N\rangle \qquad (68)$$

where

$$D = S^2\int_0^\infty dt\ \int \frac{d\underline{k}}{(2\pi)^2}\ \langle\delta L^2(x,t)\rangle_{\underline{k}} e^{-i\underline{k}\cdot\underline{x}}\langle G(t)\rangle e^{i\underline{k}\cdot\underline{x}} \qquad (69)$$

is the diffusion coefficient. According to (68), the ensemble averaged effect of the turbulent background fluctuations is to cause a magnetized fluid element to diffuse. The mean field also diffuses. Inverting (68) by the use of $\langle G(t,t')\rangle$ gives the portion of δN that is phase coherent with $\delta L^*_{\underline{k}}$

$$\delta N(\underline{x},t)^c = S\int_0^t dt' \langle G(t,t')\rangle e^{-i\underline{k}\cdot\underline{x}} \delta L^*_{\underline{k}}(t') \frac{\partial}{\partial x} \langle N(x,t')\rangle \tag{70}$$

We again make the Markovian assumption by setting $\langle N(x,t')\rangle = \langle N(x,t)\rangle$ and pulling $\partial\langle N(x,t)\rangle/\partial x$ outside of the integral in (70). The Markovian assumption here is on a sounder footing than in (66) since the mean field evolves more slowly and on larger scales than the fluctuation δN. Insertion of (70) into (56) and ensemble averaging gives

$$\frac{\partial}{\partial t} \langle N\rangle = \frac{\partial}{\partial x} (D+1) \frac{\partial}{\partial x} \langle N\rangle \tag{71}$$

The turbulent diffusion coefficient D appears as an anomalous resistivity. Note that the positive diffusion terms in the equations for $\underline{N}$ and $\underline{L}$ are consistent with the cascade of these quantities (energy and cross helicity) to small spacial scales.

The turbulent diffusion of the mean field in time is related by the Alfven speed to the diffusion in z of spacially stochastic field lines discussed in Sec. IB. To show this, we note that

$$\left(\frac{\partial}{\partial t} - S\langle\underline{B}\rangle\cdot\underline{\nabla} - \frac{\partial}{\partial x} D \frac{\partial}{\partial x}\right) \langle G(t,t')\rangle = 0, \tag{72}$$

a result that follows from (67) in the same manner in which (71) has just been derived from (65). Assuming the model sheared field of Sec. IB $\langle G(t,t')\rangle \exp\bar{i}(k_y y+k_z z)$ can be evaluated from (72) and inserted into (69) to give

$$D = \int \frac{d\underline{k}}{(2\pi)^2} S^2 \langle\delta L^2\rangle_{\underline{k}} G_{\underline{k}} \tag{73}$$

where

$$G_{\underline{k}} = \int_0^\infty dt \exp[iS\underline{k}.\underline{B}_o t - (t/\tau_o)^3 - \gamma t] \tag{74}$$

with

$$\tau_o = (\frac{1}{3} k_y^2 B_{oy}'^2 S^2 D)^{-1/3} \tag{75}$$

The quantity γ in (74) follows from the assumed slow growth of the mean-square fluctuation level, i.e., $\langle\delta L^2(t)\rangle = \langle\delta L^2\rangle \exp \int^t dt'\gamma(t')$ in (69). For weak static fields, $G_{\underline{k}} = \pi\delta(S\underline{k}.\underline{B}_o)$, so that in dimensional units, the anomalous resistivity D in (71) is $Da^2/\tau_R = SaD^m/\tau_R = V_A D^m$ where D^m is given by (22) with δL (the clump portion of the magnetic field) replacing δB. For finite amplitudes, it is useful to approximate (74) as a Lorentzian, $G_{\underline{k}} = i(S\underline{k}.\underline{B}_o + i\tau_o^{-1})^{-1}$. As with z_o (see Sec. IA), τ_o broadens the resonance and leads to a nonzero D at resonance overlap. From (74), D becomes nonzero when

$$|x-x_s| < (D/3Sk_y B_{oy}')^{1/3} \equiv x_o \tag{76}$$

where $x_s = -k_z B_{oz}/k_y B_{oy}'$ is the position of the rational surface. Equation (76) is just the island resonance overlap condition. To see this, we write $D \sim S^2 \delta B^2_{res} \tau_o$ where δB_{res} here denotes the clump part of the field that is within a resonance width of x_s. Equation (76) then gives $D \sim S\delta B_{res} x_o$ and x_o becomes

$$x_o \sim (\delta B_{res}/k_o B_{oy}')^{1/2} \tag{77}$$

Since (77) is on the order of the island width in the resonant modes, (76) becomes $|x-x_s| < \Delta x$ and, therefore, D becomes nonzero at island overlap. Note that $D \sim x_o^2/\tau_o \sim (\Delta x)^2/\tau_o$. The field lines diffuse with a step size of the island width and with a time step of the Lyapunov time. This suggests the model of colliding islands that we have alluded to in Sec. I. In dimensional units, $D \sim (\delta B_{res}/B_{oz}) V_A x_o$ where we interpret x_o as the transverse correlation length.

Similar calculations as these can be carried out on (57). For simplicity, we consider only the case of $\underline{L}$ and $\underline{N}$ coupling where $\langle \underline{V} \rangle = 0$ and $\langle \underline{V}.\underline{B} \rangle = 0$. This is a "strong" coupling where $\langle N^2 \rangle = \langle L^2 \rangle = \langle \underline{L}.\underline{N} \rangle$. As a consequence, $\langle \underline{L} \rangle$ also satisfies (71). Note that $D^L = D^N = D$. The change in sign of S (i.e., the direction of Alfven wave propagation) in going from (56) for $\underline{N}$ to (57) for $\underline{L}$ does not effect the diffusion coefficients since only the real part of the propagator $G_{\underline{k}}$ is required for D^L or D^N. Physically, Alfven wave emission causes clump decay--regardless of the direction of the wave propagation. The strong coupling limit means that the mean-square magnetic and flow fields are transported similarly in accordance with the (approximate) frozen-in property of the fields. The two fields respond nonlinearly on the same time scale, i.e., $\tau_0^L = \tau_0^N = \tau_0$. The situation is analogous to the case of electron and ion Vlasov clumps where the electron and ion masses are equal.[8] The strong coupling limit has an additional advantage: we need only consider the energy and magnetic helicity invariants (48) and (50). The third invariant of cross helicity is satisfied identically for all time (see (49)).

B. Two Point Renormalization

The renormalization of the two point equations for $\langle \underline{N}_1.\underline{N}_2 \rangle = \langle \underline{N}(x_1,t).\underline{N}(\underline{x}_2,t) \rangle$ and $\langle \underline{L}_1.\underline{L}_2 \rangle = \langle \underline{L}(\underline{x}_1,t).\underline{L}(\underline{x}_2,t) \rangle$ can be done in the spirit of the last section. We only sketch the procedure here, but the reader can find details in Sec. VII of Ref. 1.

The two point version of (56) is

$$\left(\frac{\partial}{\partial t} - S\underline{L}_1.\underline{\nabla}_1 - S\underline{L}_2.\underline{\nabla}_2\right) \underline{N}_2.\underline{N}_2 = 0 \tag{78}$$

where we have again suppressed collisional dissipation terms for simplicity. Separating mean and fluctuating quantities by $\underline{N}_1 \cdot \underline{N}_2 = \langle \underline{N}_1 \cdot \underline{N}_2 \rangle + \delta(\underline{N}_1 \cdot \underline{N}_2)$, (78) becomes

$$\left(\frac{\partial}{\partial t} - S\langle \underline{B}_1\rangle \cdot \underline{\nabla}_1 - S\langle \underline{B}_2\rangle \cdot \underline{\nabla}_2\right) \langle \underline{N}_1 \cdot \underline{N}_2\rangle - S \sum_{i=1,2} \underline{\nabla}_i \cdot \langle \delta \underline{L}_i \, \delta(\underline{N}_1 \cdot \underline{N}_2)\rangle = 0 \tag{79}$$

In order to evaluate the nonlinear term in (79), we use the two point version of (70), i.e.,

$$\delta(\underline{N}_1 \cdot \underline{N}_2)^c = S\int_0^t dt' \, \langle G_{12}(t,t')\rangle \sum_{i=1,2} \delta \underline{L}_{\underline{k}}^*(t') \, e^{-i\underline{k}\cdot\underline{x}_i} \cdot \underline{\nabla}_i \langle \underline{N}_1(t') \cdot \underline{N}_2(t')\rangle \tag{80}$$

where $G_{12}(t,t')$ satisfies

$$\left(\frac{\partial}{\partial t} - S\underline{L}_1 \cdot \underline{\nabla}_1 - S\underline{L}_2 \cdot \underline{\nabla}_2\right) G_{12}(t,t') = 0 \tag{81}$$

with $G_{12}(t,t) = 1$. Again we make the Markovian approximation by setting $\langle \underline{N}_1(t') \cdot \underline{N}_2(t')\rangle = \langle \underline{N}_1(t) \cdot \underline{N}_2(t)\rangle$ in (80). Substituting (80) into (79) and reintroducing the collisional dissipation term then gives the bivariate diffusion equation

$$\left[\frac{\partial}{\partial t} - S \sum_{i=1,2} \langle \underline{B}_i\rangle \cdot \underline{\nabla}_i - \sum_{i=1,2} \underline{\nabla}_i \cdot (\underline{\underline{D}}_{ij} + 2) \cdot \underline{\nabla}_j\right] \langle \underline{N}_1 \cdot \underline{N}_2\rangle = 0 \tag{82}$$

where

$$\underline{\underline{D}}_{ij} = S^2 \int_0^\infty dt \int \frac{d\underline{k}}{(2\pi)^2} \langle \delta \underline{L}(x_i,t) \delta \underline{L}(x_j,t)\rangle_{\underline{k}} e^{-i\underline{k}\cdot\underline{x}_i} \langle G_{12}(t)\rangle \, e^{i\underline{k}\cdot\underline{x}_j} \tag{83}$$

Similar calculations can be carried out on (81) and, again using the Markovian approximation, we find that $\langle G_{12}(t)\rangle$ satisfies the same equation as $\langle \underline{N}_1 \cdot \underline{N}_2\rangle$, i.e., (82). The orbit function $\langle G_{12}(t)\rangle \exp i\underline{k}\cdot\underline{x}_j$ in (83) can then be evaluated as in the last section. We again use the model sheared field of Sec. IA and find that

$$\underline{\underline{D}}_{ij} = S^2 \int \frac{d\underline{k}}{(2\pi)^2} \langle \delta\underline{L}\ \delta\underline{L}\rangle_{\underline{k}} G_{\underline{k}} \exp\ ik_y(y_i - y_i) \tag{84}$$

where $G_{\underline{k}}$ is given by (74) and, for the near resonance case, we have set $x_1 = x_2$ in the spectral function and set $\underline{k}.(\underline{x}_i - \underline{x}_j) = k_y(y_i - y_j)$ in the orbit function. Note that the xx component of (84), which we denote by D_{ij}, is just (73) as $y_1 \to y_2$, i.e., $D_{11} = D_{22} = D$.

Retaining only the x component of the field fluctuations for simplicity, we can straightforwardly obtain the equation for $\partial\langle \underline{N}_1\rangle.\langle \underline{N}_2\rangle/\partial t$ from (71) and subtract the result from (82) to obtain the equation for $\langle \delta\underline{N}_1 . \delta\underline{N}_2\rangle = \langle \underline{N}_1 . \underline{N}_2\rangle - \langle \underline{N}_1\rangle.\langle \underline{N}_2\rangle$. Since uncorrelated fluid elements diffuse independently, we can equivalently deduce that $\langle \underline{N}_1\rangle.\langle \underline{N}_2\rangle$ satisfies (37), but with $\underline{\underline{D}}_{12} = \underline{\underline{D}}_{21} = 0$. Assuming that $\langle \delta\underline{N}_1 . \delta\underline{N}_2\rangle$ depends only on the relative coordinate $\underline{x}_- = \underline{x}_1 - \underline{x}_2$, an approximation valid for scale sizes less than the mean shear length, the equation for $\langle \delta\underline{N}_1 . \delta\underline{N}_2\rangle$ can be written as

$$\left[\frac{\partial}{\partial t} - S\langle \underline{B}_-\rangle . \underline{\nabla}_- - \frac{\partial}{\partial x}(D_- + 2)\frac{\partial}{\partial x}\right] \langle \delta\underline{N}_1 . \delta\underline{N}_2\rangle$$

$$= 2\ D_{12} \frac{1}{\partial x_1} \cdot \frac{\partial\langle \underline{N}_2\rangle}{\partial x_2} \frac{\partial\langle \underline{N}_1\rangle}{\partial x_1} \tag{85}$$

where $D_- = D_{11} + D_{22} - D_{12} - D_{21}$ and, with δL as the x component of $\delta\underline{L}$,

$$D_{12} = S^2 \int \frac{dk}{(2\pi)^2} \langle \delta L^2(x_- = 0)\rangle_{\underline{k}}\ G_{\underline{k}} \exp\ ik_y\ y_- \tag{86}$$

The equation for $\langle \delta\underline{L}_1 . \delta\underline{L}_2)\rangle$ corresponding to (85) can be obtained in a similar way. Note that we have evaluated D_{12} on the right hand side of (85) at $x_- \sim 0$ since, for $|x_-| > x_d$, $D_{12} \to 0$ (see (84)) and the clump source term vanishes. Also, since the linear and nonlinear shear damping (Alfven wave emission) terms vanish on the left-hand side of (85) as $\underline{x}_- \to 0$, the clump source term on the right-hand-side of (85) makes is largest contribution when $\underline{x}_- = 0$.

The correlated and relative diffusion coefficients D_{12} and D_- have the following meaning. As $\underline{x}_1 \to \underline{x}_2$, two points in a magnetized fluid element feel the same stochastic forces and, thus, diffuse together, i.e., $D_- \to 0$. For the shearless case this means, in the absence of collisional dissipation, that $\langle N^2 \rangle$ is conserved in (82). Since an analogous result holds for $\langle L^2 \rangle$, the nonlinear diffusion terms generated in the renormalization conserve total energy and cross helicity. In the case of nonzero shear, energy and cross helicity are again conserved. The conservation of $\langle N^2 \rangle$ means that the diffusive mixing (rearrangement) of $\langle N \rangle^2$ must produce fluctuations $\langle \delta N^2 \rangle$, i.e., the clumps. This is the meaning of (85) and, in the strong $\underline{N}/\underline{L}$ coupling limit where $\langle \underline{V} . \underline{B} \rangle = 0$, is just the energy fluctuation production relation (52). In this limit, the equation for $\langle \delta \underline{L}_1 . \delta \underline{L}_2 \rangle$, being the same as (85) but with $S \to -S$, yields the same result. For large $|x_1 - x_2|$, two fluid elements feel different forces and thus diffuse independently, i.e., $D_{12} \to 0$ and $D_- \to 2D$. The net effect of D_- is to diffusively mix magnetic fluid elements of different spacial scales. In this way, energy cascades to small spacial scales (high wave numbers).

Along with the conservation of magnetic helicity imposed in Ref. 27, the renormalization preserves all of the dynamical invariants of the original one fluid MHD equations. This is of crucial importance. First, it ensures us that, while some terms have been neglected, the renormalized equations we have derived maintain the essential physics of the original equations. One could go so far as to say that the preservation of the dynamical invariants is the most important requirement of the renormalization procedure.

Since the two point correlation function is peaked about $\underline{x}_- = 0$, the turbulent and collisional diffusion terms on the left-hand-side of (85) cause the decay of localized fluctuations. This is in contrast to the corresponding diffusion terms in the one point equations, e.g., (68) and (24). There, $\nabla_{\perp}^2 \delta\psi = - \delta J_z > 0$ for a hole fluctuation so that the diffusion terms produce growth (as for the tearing mode). Physically, the diffusion operators cause decay in the two point equations because neighboring elements in a localized clump fluctuation undergo stochastic orbit instability, and thereby diffuse apart radially. Correlations are destroyed by this effect.

C. Effect of Pressure

Because the p* pressure terms conserve energy by themselves when $\underline{\nabla}.\underline{V}=0$, we have been able to treat the renormalization and conservation properties of the nonlinear $\underline{N}.\underline{\nabla}$ and $\underline{L}.\underline{\nabla}$ terms separately from the pressure terms. We now consider the pressure terms and show that their effect on (85) is negligible compared to the current (magnetic shear) driven clump effects we have already considered. The main reason for the neglect of $\langle p \rangle$ is that $\langle p \rangle$ does not effect the mean magnetic field profile. Indeed, because of mean pressure balance (46), it is $\langle \underline{B} \rangle$ that determines $\langle p \rangle$. Because of (46), $\langle p \rangle$ has no effect on the correlation function equations. While the remaining δp^* terms do have an effect, the effect is small since, from $\underline{\nabla}.\underline{V}=0$, δp^* does not contribute to the total energy balance. The vanishing of $\int d\underline{x} \langle \underline{V}.\underline{\nabla}\delta p^* \rangle$ means that the corresponding δp^* contributions to the two point correlation function equation for the energy will vanish as $x_- \to 0$. Since the shear driving (mixing) term on the right-hand-side of (85) is finite and indeed at its largest value at $x_-=0$, the effect of the δp^* terms are, by comparison, negligible. It is useful to see this in detail by considering the linear part of δp^* in (47) (consideration of the nonlinear leads to the same conclusion). Using $\underline{\nabla}.\langle\underline{B}_\perp\rangle=0$ and $\underline{\nabla}.\delta\underline{B}=0$, this part of δp^* satisfies

$$\nabla_\perp{}^2\delta p^* = 2\, B'_{oy} \frac{\partial}{\partial y}\, \delta B \tag{87}$$

where, as above, δB is the x component of $\delta\underline{B}$. If we again neglect the correlation between $\delta\underline{V}$ and $\delta\underline{B}$ (strong coupling limit), δp^* makes the following contribution to the $\delta\underline{N}$ correlation function equation (85):

$$-\frac{\partial}{\partial x_1}\langle\delta p_1^*\ \delta\underline{B}_2\rangle - \frac{\partial}{\partial x_2}\langle\delta p_2^*\delta\underline{B}_1\rangle \tag{88}$$

Since $\delta p^* \sim \delta B$ from (87) and $\langle\delta B\delta B\rangle$ is a function of $\underline{x}_-$, the term $\partial\langle\delta p_1^*\delta B_2\rangle/\partial x_-$ can be evaluated from (87) so that (88) gives a contribution to (85) of

$$-4\ B'_{oy} \frac{\partial}{\partial y_-} \int_0^{x_-} dx'_-\ \ \langle\delta B(1)\delta B(2)\rangle \tag{89}$$

In wave number space, this δp^* contribution is proportional to $\underline{k}.[\underline{B}_o(1)-\underline{B}_o(2)]$. The additional source term to (89) vanishes as $x_-\to 0$, thus making it negligible to the shear mixing term already present on the right-hand-side of (85). The terms are in ratio of $x_-/\dot{x}_d$ for small x_-. A term corresponding to (89) appears in the $\langle\delta\underline{L}_1.\delta\underline{L}_2\rangle$ equation, but with a plus sign.

IV. CALCULATION OF THE GROWTH RATE

A. Nonlinear Newcomb Equation

The dynamical equation (85) for the mean-square fluctuation level can be inverted in time to yield a nonlinear, turbulent version of the Newcomb equation. The nonlinear equation will dominate in the resonant region $|x_1-x_2|<\Delta x$ and will determine the effect of turbulent mixing and decay of fluctuations in the vicinity of a mode rational surface a $x_s=x_1$. The behavior of the fluctuations away from the resonance, $|x_s-x|>\Delta x$, will be determined by the resonance broadened version of the Newcomb equation. In analogy with linear tearing mode theory, a matching of the two solutions relates the growth rate γ in the inversion of (85) to Δ' of the Newcomb equation.

Consider first the case of time stationary turbulence where the mean-square fluctuation level is not growing in time. Assuming that the fluctuations have scale sizes less than the mean shear length, the time inversion of (85) along the stochastic two point orbits can be written as

$$\langle \delta \underline{N}_1 \cdot \delta \underline{N}_2 \rangle = 2D \quad \tau_-(x_-,y_-,z_-) \; B_{oy}'^{\,2} \tag{90}$$

where

$$\tau_-(\underline{x}_-) = \frac{S^2}{D} \int \frac{d\underline{k}}{(2\pi)^2} \langle \delta L^2 \rangle_{\underline{k}} G_{\underline{k}} \int^t dt' \, \langle \exp\, i k_y y_-(t') \rangle \tag{91}$$

and we have noted that $\partial \langle \underline{N} \rangle / \partial x = B_{oy}' \hat{y}$. Equation (90) means that the localized clump fluctuations are driven by the large scale magnetic shear (B_{oy}'), a result consistent with the cascade of energy from large to small scale lengths. The meaning of the τ_- factor in (90) can be understood from the mixing process generating the clumps. Since a clump fluctuatin arises from the diffusive (D) transport of a magnetized fluid element to a region of differing energy density, a larger fluctuation results as the element diffuses farther and farther away from its density of origin. If the clump is not susceptible to decay during the transport, the fluctuation would get inexorably larger with time, i.e., τ_- in (90) would be equal to the elapsed time t. However, because of stochastic decay processes, a fluctuation of finite size has a finite lifetime (τ_-), so that the magnitude of the fluctuation is limited to the value given by (90).

The clump lifetime is determined by the operators on the left-hand-side of (85). These operators approximate the energy cascade to high wave numbers by a relative diffusion (mode coupling) processes in the two point correlation. In principle, the cascade involves all scale lengths in the clump but, because of complexities in an analytical evaluation of τ_-, we will be forced to consider only the turbulent dissipation of scales smaller than the clump size. We will find that the decay occurs as neighboring field lines in the clump diverge apart exponentially with time at a rate $(\tau_{cl})^{-1}$. While the $\tau_- = \tau_{cl}$ expression we calculate below is only strictly valid for scales less than the clump size, it nevertheless gives a reasonable result for the characteristic clump decay time. Moreover, the precise analytical expression for τ_- is not crucial, since, as we shall see, we will only need an integral of τ_-. similar approximations for the Vlasov clump case have yielded a clump lifetime in good agreement with computer simulations.[6]

We expand the randomly fluctuating (stochastic) orbit function $\exp[iky_-(t)]$ of (90) in cumulants[40,2] and, for a normally distributed y_-, obtain $\langle \exp iky_- \rangle = \exp(-1/2\, k_y^2 \langle y_-^2 \rangle)$. The time evolution of $\langle y_-^2 \rangle$ can be calculated from the two point orbit characteristics of the left hand side of (85). Using the model sheared field of Sec. IA, $\langle \underline{B}_- \rangle . \underline{\nabla}_- = B'_{oy} x_- \, \partial/\partial y_-$ for $z_1 = z_2$, the characteristic equations imply that

$$\frac{\partial^3}{\partial t^3} \langle y_-^2 (t)\rangle = 2S^2B_{oy}'^2 \langle D_-\rangle \quad (92)$$

(92) can be solved approximately by expanding $D_- = 2(D-D_{12})$ for small y_- so that $\langle D_-\rangle = 2k_o^2 D\langle y_-^2\rangle$, where k_o^2 is defined by

$$k_o^2 = \frac{1}{2D} \left(\frac{\partial^2 D_-}{\partial y_-^2}\right)_{y_-=0} \quad (93)$$

Because of the cascade of energy to high $\underline{k}$, one might worry that this small $k_y y_-$ expansion will quickly become invalid during the cascade. However, the instability growth rate is on the order of τ^{-1}, so that the expansion is valid during most of the first e-folding growth time of the instability. Using this $\langle D_-\rangle$ in (92), the time asymptotic solution with initial condition $y_-(0) = y_-$ is

$$\langle y_-^2(t)\rangle = (y_-^2 - 2y_- SB_{oy}' x_- \tau + 2S^2 B_{oy}'^2 x_-^2 \tau^2) e^{t/\tau} \quad (94)$$

where

$$\tau = (4k_o^2 S^2 B_{oy}'^2 D)^{-\frac{1}{3}} = (12)^{-\frac{1}{3}} \tau_o \quad (95)$$

is the characteristic time for neighboring field line orbits to diverge apart exponentially. This divgerence causes the cascade of the energy to high wave numbers, not a surprising result, since (94) is determined by the diffusive dynamics of D_- in (85). In dimensional units, $\tau = z_o/V_A$ where z_o is the Lyapunov scale. While (94) has been calculated for $z_-=0$, the z_- dependence can be recovered by replacing y_- in (94) with $\bar{y}_- = y_- - x_+ z_- B_{oy}'$ where $x_+ = 1/2\ (x_1 + x_2)$. Since k_o is the typical k_y in the $\underline{k}$ integral of (91), the time integral in (91) will converge for $t > \tau_{cl}$, where $k_o^2\langle y_-^2(\tau_{cl})\rangle/2 \sim 1$. Using (93), τ_{cl} is given by

$$\tau_{cl}(x_-,y_-,z_-)=\tau \ \ln \frac{3}{k_o^2(\bar{y}_-^2-2\bar{y}_-SB'_{oy}x_-+2S^2B'^2_{oy}x_-^2\tau^2)} \tag{96}$$

for arg $\ln>1$ and is zero otherwise. Since the time integral in (91) is on the order of τ_{cl}, we conclude that $\tau_-\sim\tau_{cl}$ in (90). When the fluctuations are growing during the instability, the two point propagator on the left-hand size of (85) is effectively $(\gamma+\tau_{cl}^{-1})$, where γ is the growth rate of the correlation function. Therefore, the time inversion of (85) can be generally written as

$$\langle\delta\underline{N}_1\cdot\delta\underline{N}_2\rangle = 2D\tau_{cl}\ B'^2_{oy}\ (1+\gamma\ \tau_{cl})^{-1} \tag{97}$$

The equation for $\langle\delta\underline{L}_1\cdot\delta\underline{L}_2\rangle$ follows by making the replacements $S\to-S$ and $D=D^L\to D^N$ on the right hand side of (85) (see (56), (57) and (68)).

With $\gamma=0$, (97) is in the form of a standard mixing length relation.[37] The mean (large scale) shear (B'_{oy}) is mixed to generate the smaller scale clump fluctuations. As discussed above, τ_{cl} in (97) does not mean that the clumps are driven by the small scales. τ_{cl} describes the decay or lifetime of the clumps and, along with the factor D, determines the mixing length. The mean-square mixing length is $x_d^2 = D\tau$. Using (94),

$$x_d = (4D/Sk_oB'_{oy})^{1/3} \tag{98}$$

so that $\tau k_oSB'_{oy}x_d=1$. x_d is the two point generalization of the one point resonance width x_o (see (76)). Two field line trajectories are in resonance (feel the same stochastic forces) when $|x_-|<x_d$.

The resonant, nonlinear version of the Newcomb equation is derived from (51). We define the clump "flux" functions $\Psi_\pm \equiv \psi \pm S^{-1}\phi$ so that

$$\langle\underline{N}_1\cdot\underline{N}_2\rangle = \langle\frac{\partial\Psi_+(1)}{\partial x_1}\frac{\partial\Psi_+(2)}{\partial x_2} + \frac{\partial\Psi_+(1)}{\partial y_1}\frac{\partial\Psi_+(2)}{\partial y_2}\rangle \tag{99}$$

Therefore,

$$\langle \delta \underline{N}_1 \cdot \delta \underline{N}_2 \rangle_{\underline{k}} = \left(-\frac{\partial^2}{\partial x_-^2} + k_y^2\right) \langle \delta\Psi_+(1)\ \delta\Psi_+(2)\rangle_{\underline{k}} \tag{100}$$

and the Fourier transform of (97) can be written as

$$\left(\frac{\partial^2}{\partial x_-^2} - k_y^2\right) \langle \delta\Psi_+(1)\delta\Psi_+(2)\rangle_{\underline{k}} = -\,2DB_{oy}'^2\ \bar{\tau}_{cl}(x_-,\underline{k}) \tag{101}$$

where

$$\bar{\tau}_{cl}(x_-,\underline{k}) \equiv \int dz_-\, e^{-ik_z z_-} \int dy_-\, e^{-ik_y y_-} \left(\frac{\tau_{cl}}{1+\gamma\ \tau_{cl}}\right) \tag{102}$$

Recalling (73) and (74), we can write $D \sim S^2 \langle \delta L^2 \rangle_{\underline{k}} (\gamma+\tau_o^{-1})^{-1} \sim S^2 k_y^2 \langle \delta\Psi^2 \rangle_{\underline{k}} (\gamma+\tau^{-1})^{-1}$ near the mode rational surface. Substituting this into (100) and rearranging produces the Newcomb-like equation

$$\left[\frac{\partial^2}{\partial x_-^2} - k_y^2 + S^2 k_y^2\, \frac{\bar{\tau}_{cl}(B_{oy}')^2}{(\gamma+\bar{\tau}^{-1}\ )}\right] \langle \delta\Psi_+(1)\delta\Psi_+(2)\rangle_{\underline{k}} = 0 \tag{103}$$

Fluctuations are created as the mean magnetic shear (B_{oy}') is turbulently mixed (D) near the resonant surface. This mixing is opposed by stochastic line bending forces ($Sk.B \sim \tau^{-1}$), but for a large enough mixing rate, net growth ($\gamma > 0$) of the mean-square fluctuation level is possible. We show in Sec. IIIF that the instability is a nonlinear, turbulent version of the Rayleigh-Taylor instability in a magnetized fluid.

Near the mode rational surface ($|x_-| < x_d$), the instability is described by the nonlinear Newcomb equation (101). Close to the resonance, clump decay by stochastic line bending (shear Alfven wave emission) is small. In the limit, a clump of infinitesimally small scale will, because of energy conservation, have an infinite life time ($\tau_{cl} \to \infty$). The mean-square source of fluctuations on the right hand side of (101) will increase secularly with time and thus diverge. In the nonresonant region away from the mode rational surface, shear Alfven wave emission (propagation) dominates.

Energy is carried away from the clumps along stochastic field lines, thus yielding a short clump life-time ($\bar{\tau}_{cl}-\tau_{cl}\to 0$). The clump source term in (101) then goes to zero and, as we show in the next section, (101) goes over to the two point Newcomb equation with broadened resonance[26]

$$\left[\frac{\partial^2}{\partial x_-^{\,2}} - k_y^2 + \frac{k_y J_o'}{k_y B_{oy}' x_- + i z_o^{-1}}\right]\langle\delta\psi(1)\,\delta\psi(2)\rangle_{\underline{k}} = 0 \qquad (104)$$

Eq. (104) follows by evaluating the Newcomb equation ((15) with $\delta J^R = 0$) at position (1) and multiplying by $\delta\psi(2)$ and ensemble averaging, where (2) is at the mode rational surface, i.e., at $x_2 = - k_z B_{oz}/k_y B_{oy}'$ where $\underline{k}\cdot\underline{B}_o(2) = 0$. Note that we have also included the z_o broadening in (104) and set $\partial^2/\partial x_1^2 = \partial^2/\partial x_-^2$ since $x_1 - x_2 = x_-$. This z_o broadening resolves the singularity at the mode rational surface. However, because of energy conservation, the singularity in (101) exists to all orders in the field amplitude. The source term for turbulent mixing in (101), therefore, dominates in the resonance region.

B. Growth Rate

Since D in (101) depends on $\langle\delta\Psi^2(0)\rangle_{\underline{k}}$ (see (86)), we can solve (100) for $\langle\delta\Psi^2\rangle$. Let $x_+ = (x_1+x_2)/2$ be located at the mode rational surface. If ε denotes the region about this singular surface, then, using $x_1=x_++x_-$ and $x_2=x_+-x_-$, we have

$$\left[\frac{\partial}{\partial x_-}\langle\delta\Psi(1)\delta\Psi(2)\rangle\right]_{-\varepsilon}^{\varepsilon} = \langle[\delta\Psi'(x_++x_-)\delta\Psi(x_+-x_-)$$

$$- \delta\Psi(x_++x_-)\ \delta\Psi'(x_+-x_-)]\rangle\Big|_{-\varepsilon}^{\varepsilon}$$

$$= 2\ \langle[\delta\Psi'(x_++\varepsilon)\delta\Psi(x_+-\varepsilon) - \delta\Psi(x_++\varepsilon)\ \delta\Psi'(x_+-\varepsilon)]\rangle$$

$$= 2\ \langle\delta\Psi'(1)\delta\Psi(2) - \delta\Psi(1)\delta\Psi'(2)\rangle$$

$$= 2\ \left\langle\left[\frac{\delta\Psi'(1)}{\delta\Psi(1)} - \frac{\delta\Psi'(2)}{\delta\Psi(2)}\right]\delta\Psi(1)\ \delta\Psi(2)\right\rangle$$

$$= 2\ \delta'\ \langle\delta\Psi(1)\ \delta\Psi(2)\rangle \tag{105}$$

where δ' denotes the jump (discontinuity) in (101), and (1) and (2) in (105) denote $x_++\varepsilon$ and $x_+-\varepsilon$ respectively. Therefore,

$$P\int dx_-\ \frac{\partial^2}{\partial x_-^2}\langle\delta\Psi(1)\delta\Psi(2)\rangle_{\underline{k}} = -\ 2\delta'_{\underline{k}}\ \langle\delta\Psi^2(0)\rangle_{\underline{k}} \tag{106}$$

$\langle\delta\Psi(1)\delta\Psi(2)\rangle_{\underline{k}}$ reduces to the Newcomb solution $\langle\delta\psi(1)\delta\psi(2)\rangle_{\underline{k}}$ outside the singularity when $\gamma\tau<1$. To see this, we note from Faraday's law that $\gamma\delta\psi\sim\underline{k}\cdot\underline{B}\ \delta\phi\ \sim\ k_yB'_{oy}x\delta\phi\ \sim\ \tau^{-1}(\delta\phi/S)(x/x_d)$, so $\delta\psi\ \sim\ (\gamma\tau)^{-1}\ (x/x_d)(\delta\phi/S)$. Therefore, away from the singularity, $\delta\Psi\sim\delta\psi$ except when $\gamma\tau$ is very large. Since we are interested primarily in the stochastic regime where $\gamma\tau<1$, this approximation is a good one. The Newcomb equation for $\langle\delta\psi(1)\delta\psi(2)\rangle_{\underline{k}}$ is given by (57). Since

$$\frac{\partial}{\partial x_-} \langle \delta\psi(1)\delta\psi(2)\rangle = 2 \left\langle \left[\frac{\delta\psi'(1)}{\delta\psi(1)} - \frac{\delta\psi'(2)}{\partial\psi(2)}\right] \delta\psi(1)\delta\psi(2)\right\rangle$$

$$= 2\, \Delta'_{\underline{k}} \langle \delta\psi(1)\delta\psi(2)\rangle \,, \qquad (107)$$

the discontinuity in (104) is just $\Delta'_{\underline{k}}$, the usual tearing mode stability parameter. Therefore, as in the tearing mode theory[28], we match the inner and outer solutions by setting $\delta'_{\underline{k}} = \Delta'_{\underline{k}}$. Since (101) describes the growth of current hole turbulence, this procedure just matches (for each $\underline{k}$) the resonant hole solution to the Newcomb solution. Near the resonance, the current hole has the same radial dependence as the tearing mode (i.e., see Sec. IA, IB where $\delta J_z = -\partial^2\delta\psi/\partial x^2 < 0$) so that, for an isolated coherent hole, the matching is done as in standard tearing mode theory, i.e., one equates the logarithmic derivatives δ' and Δ'. The results (106) and (107) mean that, in the turbulent case, the matching of solutions is also obtained by setting $\delta' = \Delta'$.

Using (106) and $\delta'_{\underline{k}} = \Delta'_{\underline{k}}$, (107) gives

$$\langle \delta\Psi_+^2\ (0)\rangle_{\underline{k}} = \frac{2D^L B'^{\,2}_{oy}}{\Delta'_{\underline{k}} + 2|k_y|} \int_{-\infty}^{\infty} dx_-\ \bar{\tau}_{c\ell}(x_-,\underline{k}) \exp[-|k_y|x_-] \qquad (108)$$

where $\Delta'_{\underline{k}}$ is given by the resonance broadened version of (16) and the superscript L on D^L is a reminder that D^L depends on $\langle \delta L^2(0)\rangle_{\underline{k}}$, i.e., $\langle \delta\Psi_-^2\ (0)\rangle_{\underline{k}}$. An approximate way to evaluate the integral in (108) is to note that the integrand is nonzero in the range $ky_- \sim k_o y_- \sim 1/2$ and $x_-/x_d \sim 1/2$ so that we can replace $\gamma\tau_{c\ell}$ in the denominator of (102) with $\sim 2\gamma\tau$ (see Ref. 8), and $\exp(-k_y x_-) \sim 1$ in (108). Then, using (96) and (102), we do the x_- integral first by completing the square in the denominator of (96) and get

$$\int_{-\infty}^{\infty} dx_-\ \bar{\tau}_{c\ell}(x_-,\underline{k}) = \frac{\pi A(k_y)}{|SB'_{oy}|} (1+2\gamma\tau)^{-1} \delta(k_z + k_y x_+ B'_{oy}) \qquad (109)$$

where

$$A(k) = \frac{2\pi}{k^2} [1 - J_o (\sqrt{6}\, k/k_o)] \tag{110}$$

(J_o is the Bessel function). Equation (110) is proportional to the clump magnetic energy (see (90)). Recalling (73) and replacing the resonance function $G_{\underline{k}}$ with a Lorentzian (see (74)), we can construct D^N from the left-hand side of (108) to obtain

$$D^N = D^L \frac{|SB'_{oy}|}{1+2\gamma\tau^L} \frac{k_o}{\Delta'_c \Gamma^N} \tag{111}$$

where $\Gamma = \gamma + \tau_o^{-1}$ (with τ_o given by (75)), and

$$\frac{1}{\Delta'_c} = \int_{-\infty}^{\infty} \frac{dk_y}{2\pi k_o} \frac{\mathrm{Re}\Delta'_{\underline{k}}+2|k_y|}{(\mathrm{Re}\Delta'_{\underline{k}}+2|k_y|)^2+\lambda^2} k_y^2 \, A(k_y) \tag{112}$$

is an effective (inverse) $\Delta'_{\underline{k}}$ averaged over the clump spectrum. Here, $\underline{k}$ in $\Delta'_{\underline{k}}$ is $(k_y, -k_y B'_{oy} x_+)$ where λ is given by (20). An equation similar to (101) can be derived for $\langle \delta\Psi_-^2 \rangle$ in terms of D^N. The equation can be integrated and used to construct D^L:

$$D^L = D^N \frac{|SB'_{oy}|}{1+2\gamma\tau^N} \frac{k_o}{\Delta'_c \Gamma^L} \tag{113}$$

As in the Vlasov clump case (see Ref. 8), the coupled equations for the diffusion coefficients yield a quadratic equation for γ. We can simplify the situation here by taking the N/L strong coupling limit where, for zero cross helicity, we can set $\tau_o^N = \tau_o^L = \tau_o$ and $\tau^N = \tau^L = \tau$. Then, the two equations (111)-and (113) each yield the same equation for the total energy correlation function, i.e., for $D^L = D^N = D^E$ where D^E is given by (73), but with $\langle \delta\underline{L}\, \delta\underline{L} \rangle$ replaced by $\langle \delta\underline{B}\delta\underline{B} + S^{-2}\delta\underline{V}\delta\underline{V} \rangle$. Further, we note that $\tau_o = (12)^{1/3}\tau \sim 2\tau$, so (111) and (113) finally give

$$1 = \frac{\hat{R}}{(1+\gamma\tau_o)^2} \tag{114a}$$

where, since $|SB'_{oy}|\ k_o x_d \tau = 1$,

$$\hat{R} = \frac{2}{\Delta'_c x_d} \tag{114b}$$

Note that the B'_{oy} factors on the right-hand-side of (101)--ostensibly the clump driving terms--divide out in the derivation of (114) in favor of $\Delta'_{\underline{k}} \sim J'_{oz}$ as the driving term. One of the B'_{oy} factors divides out because $\int dx_- \tau_{c\ell} \sim (B'_{oy})^{-1}$ in (109). Because of (73) and (74), the evaluation of D at $\underline{k}\cdot\underline{B}_o = 0$ removes the other factor of B'_{oy}. The free energy source for instability is, therefore, the same as for the tearing mode.

In the limit $\gamma\tau > 1$, the Lyapunov time is long and growth can occur before field line stochasticity has any appreciable effect. In this hydrodynamic limit, (114a) becomes $\gamma^2 = \hat{R}/\tau^2$ which, with (114b), can be approximately written as

$$\gamma^2 \sim \frac{\mathrm{Re}\Delta'_{\underline{k}}}{|\Delta'_{\underline{k}} x_d|^2} \frac{x_d}{\tau^2} \tag{115}$$

If we now recall that $\tau^{-1} = k_o B'_{oy} S x_d$ and write (as in (77)) $x_d = (\delta B_{res}/k_o B'_{oy})^{1/2}$ for the island width in the resonant modes, (115) can be rewritten in dimensional units as

$$\gamma^2 \sim \frac{1}{\tau_H^2} \frac{\mathrm{Re}\Delta'_{\underline{k}}}{|\Delta'_{\underline{k}} x_d|^2} \frac{\langle\delta B^2\rangle_{res}}{x_d} \tag{116}$$

Except for the clump shielding factor $|\Delta'_{\underline{k}} x_d|^{-2}$, (116) is the same form as the growth rate calculated in Ref. 21. However, rather than the growth rate for a tearing mode in an assumed static background of stochastic fields, (70) describes the growth of resonant clump fluctuations. The clumps are self-consistently generated and shielded as other growing (background) clumps turbulently mix the mean shear.

In the limit where large amplitude islands overlap, the stochastic region is large and the Lyapunov time is short. This $\gamma\tau<1$ limit of (114) yields a growth rate of

$$\gamma \sim D \, \frac{\Delta'}{x_d} \, (\hat{R} - 1) \qquad (117)$$

where, for large amplitude islands, we have used $(2\hat{R})^{-1} \sim \Delta' x_d$ in front of the parenthesis of (117). Here, Δ' is an effective average value of $\Delta'_{\underline{k}}$ taken over the wave numbers of unstable clumps, i.e., the dominant Δ'_k in (112). Without the parenthesis, (117) resembles the growth rate of a finite amplitude, tearing mode island in the so-called Rutherford regime.[20] Replacing turbulent parameters with their corresponding coherent island quantities, i.e., Δ' with $\Delta'_{\underline{k}}$, x_d with Δx, and D with the collisional Spitzer value η_{sp}, we obtain the Rutherford result. The $(\hat{R}-1)$ factor in (117) brings the coherent island Rutherford result into the turbulent regime. Though an individual island will grow at the rate (25), the stochasticity resulting from the resonant interaction between islands will cause mode coupling and, hence, island decay. Net growth (117) of the mean-square fluctuation level results if the creation of new fluctuations by mixing $(\hat{R})$ occurs at a rate exceeding the decay rate due to stochasticity (-1), i.e., $\hat{R}>1$ in (117).

In the stochastic regime, the instability describes the nonlinear reconnection of field lines. The characteristic reconnection time is τ--the nonlinear or turbulent resistive time in the resonant layer ($\tau^{-1} \sim D/x_d^2$) or, equivalently, the Lyapunov time for stochastic field lines to diverge exponentially. This stochastic transport of field lines across the sheared-fields of the plasma randomly mixes elements of magnetized fluid at a rate τ^{-1}. Because of energy conservation, this random mixing process creates clump fluctuations at the same rate, and is the source of the instability. The speed at which field lines are randomly transported across the resonance--a speed referred to in the space plasma literature[41] as the "merging rate"--is x_d/τ. Since, in dimensional units, $\tau^{-1} \sim V_A k_y B'_{oy}\, x_d/B_{oz}$, and x_d is on the order of the island width, the merging rate is approximately given by $V_A(\delta B/B_{oz})$, where V_A is the Alfven speed in the longitudinal field B_{oz}. The merging rate is the Alfven speed in the local perturbed field. Again, the Alfven speed appears because the nonlinear dissipation of a clump is due to the propagation of Alfven waves down the stochastic field lines away from the clump.

Evaluation of (113) for γ requires k_o^2 which, from (86), (93), and $D_- = 2\,(D-D_{12})$ is given by

$$k_o^2 = \frac{1}{2D}\int \frac{dk_z dk_y}{(2\pi)^2}\, S^2\, k_y^4 \,\langle\delta\Psi^2(0)\rangle_{\underline{k}} = \frac{I_2}{I_o} \qquad (118)$$

where

$$I_n = \int_{-\infty}^{\infty} dk_y \,\frac{\mathrm{Re}\ \Delta'_{\underline{k}}+2|k_y|}{(\mathrm{Re}\Delta'_{\underline{k}}+2|k_y|)^2+\lambda^2}\, k_y^{n+2} A(k_y) \qquad (119)$$

Since $\hat{R}$ depends on k_o (see (114) and (112)), (114) and (118) have to be solved simultaneously in order to obtain the growth rate. Instability requires a value of $(\mathrm{Re}\Delta'_{\underline{k}} + 2|k_y|)$ that is positive and sufficiently large to ensure that $\hat{R}>1$. Moreover, the only scale length in (119) to determine k_o is $\mathrm{Re}\Delta'_{\underline{k}}$. While $\mathrm{Re}\Delta'_{\underline{k}}$ is rather sensitive to current profile, we expect less sensitivity in k_o and $\hat{R}$ since they are integrals over $\mathrm{Re}\Delta'_{\underline{k}}$. Still, the coupled integral equations are non-trivial and require numerical evaluation. For a rough estimate of these quantities, we consider a toroidal, confined laboratory plasma typical of a tokamak fusion device. There, $\mathrm{Re}\Delta'_{\underline{k}}>0$ for low mode number modes.[28] In particular, $\mathrm{Re}\Delta'_{\underline{k}} \sim a^{-1}$ in dimensional units where a is the minor radius of the plasma column. Equations (118), (119), and (114), therefore, imply that unstable MHD clump fluctuations will have low poloidal mode numbers and growth rates on the order of the inverse Lyapunov time τ^{-1}. Note that instability ($\hat{\mathbb{R}}>1$) requires $(\mathrm{Re}\Delta'_{\underline{k}} + 2|k_y|)>0$ rather than $\mathrm{Re}\Delta'_{\underline{k}}>0$, since the linear line bending restoring force term $2|k_y|$ in (16) is already taken into account in the nonlinear τ_o term in (114), i.e., the (-1) term in (117). Only the J'_{oz} part of $\mathrm{Re}\Delta'_{\underline{k}}$ appears in the clump growth factor $\hat{\mathbb{R}}$ of (117).

V. RAYLEIGH-TAYLOR ANALOGY

The net growth of fluctuations (113) by mixing can be obtained by analogy with the Rayleigh-Taylor interchange instability for a fluid with inverted density gradient.[29,31] In a magnetized plasma with shear, the instability is known as the Kruskal-Schwarzschild instability.[31] The MHD clump instability is a nonlinear, turbulent analogue of the Kruskal-Schwazschild instability.

Consider the Rayleigh-Taylor instability for an incompressible, unmagnetized fluid. In dimensional units, the fluid evolves according to mass conservation

$$\frac{\partial}{\partial t}\rho + \underline{V}\cdot\underline{\nabla}\rho = 0 \qquad (120)$$

and momentum balance

$$\rho\left(\frac{\partial}{\partial t} + \underline{V}\cdot\underline{\nabla}\right)\underline{V} = -\rho g\hat{x} - \underline{\nabla}p \qquad (121)$$

where p is the pressure and we have assumed that acceleration due to gravity points in the negative $\hat{x}$ direction. Linearizing (120) and (121), one obtains

$$\frac{\partial}{\partial x}(\rho_o\gamma^2)\frac{\partial}{\partial x}\delta V_{\underline{k}} = k_y^2\left(\rho_o\gamma^2 - g\frac{\partial\rho_o}{\partial x}\right)\delta V_{\underline{k}} \qquad (122)$$

where $\delta V_{\underline{k}}$ is the Fourier transform of the $\hat{x}$ component of $\delta\underline{V}$, γ is the growth rate, and ρ_o is the mean density. In the case of magnetized fluid with shear, one couples (120) and (121) to the Faraday/Ohm law

$$\frac{\partial}{\partial t}\underline{B} = \underline{\nabla}\times(\underline{V}\times\underline{B}) \qquad (123)$$

and (122) becomes

$$\frac{\partial}{\partial x}\left[\rho_o\gamma^2 + (\underline{k}\cdot\underline{B}_o)^2\right]\frac{\partial}{\partial x}\delta V_{\underline{k}}$$

$$= k_y^2 \left[\rho_o \gamma^2 + (\underline{k}\cdot\underline{B}_o)^2 - g\,\frac{\partial \rho_o}{\partial x}\right] \delta V_{\underline{k}} \tag{124}$$

Instability occurs when the forcing by the density gradient term overcomes the stabilizing effect of line bending. When $\partial\rho_o/\partial x > 0$, i.e., heavy fluid on top of light fluid, potential energy is decreased upon interchange or mixing of fluid elements. If the wave number of the frozen-in fluid elements is perpendicular to $\underline{B}_o$, the interchange will not alter the magnetic energy. For $\underline{k}\cdot\underline{B}_o \neq 0$, the mixing bends field lines and increases magnetic tension, and thus tends to stabilize the instability. Stabilization occurs as energy is carried away from the fluctuation in the form of linear shear-Alfven waves.

Extrapolating (124) to the nonlinear, turbulent regime yields the governing equation for the MHD clump instability. First, we must deal with the nonlinear mean-square fluctuation, rather than the linear, one point fluctuation. The conservation of mass density (120) is replaced with the conservation of energy density. Therefore, instead of an equation for $\delta V_{\underline{k}}$, one has an equation for the correlation functiôn $\langle\delta V(1)\delta V(2) + \rho_o^{-1}\delta B(1)\delta B(2)\rangle_{\underline{k}}$. Recalling that $\underline{B}_1 = \underline{\nabla}\times\hat{z}\psi$, and defining the stream function ϕ through $\underline{V}_1 = \underline{\nabla}\times\hat{z}\phi$, we will have a nonlinear dynamical equation for $k_y^2\langle\delta\Psi(1)\delta\Psi(2)\rangle_{\underline{k}} = k_y^2\langle[\delta\psi(1)\delta\psi(2) + \rho_o\delta\phi(1)\delta\phi(2)]\rangle_{\underline{k}}$. Second, the linear line bending term $\underline{k}\cdot\underline{B}_o = -\,i\underline{B}_o\cdot\underline{\nabla}$ in (124) must be generalized to the nonlinear, stochastic regime. This means that in (124), $[\rho_o\gamma^2+(\underline{k}\cdot\underline{B}_o)^2] = |\sqrt{\rho_o}\,\gamma+i\underline{k}\cdot\underline{B}_o|^2 \rightarrow \rho_o(\gamma+\tau^{-1})^2$, since $V_A\underline{k}\cdot\underline{B}(x)/B \sim V_A k_y B_{oy}' x_d/B \sim \tau^{-1}$ near the resonance. The finite amplitude and stochastic bending of the field lines is approximated by field line diffusion. Neighboring field lines (and frozen-in-fluid elements) diverge exponentially at the rate τ^{-1} and carry energy away stochastically in the form of nonlinear shear-Alfven waves. This stabilizing effect is countered by the creation of new fluctuations by turbulent mixing. Nonlinear, turbulent mixing of the magnetic shear replaces the gravitational forcing term for coherent interchange of the density in the Rayleigh-Taylor model. Therefore, the coherent mixing term $-g\, \partial\rho_0/\partial x \partial^2 W/\partial x^2$, where $W = -\rho_0 g x$ is the mean potential energy, must be generalized to the turbulent, current driven case. In the equation for the correlation function, this means $\partial^2 W(1,2)/\partial x_-^2$, where $W(1,2)$ is the two point, nonlinear magnetic potential energy of the clumps. Neglecting helicity conservation, we can use the turbulent mixing length relation $\langle \delta B^2 \rangle \sim \langle \delta x^2 \rangle (B_{oy}')^2 \sim D\tau_{c\ell}\, (B_{oy}')^2$ and estimate $\partial^2 W(1,2)/\partial x_-^2 \sim W/x_d^2 \sim D x_d^2\, \tau_{c\ell}\, (B_{oy}')^2 \sim (\tau_{c\ell}/\tau_o)(B_{oy}')^2$. Therefore, the turbulent, current driven analogue of (124) then becomes

$$\frac{\partial}{\partial x_-} \rho_o (\gamma+\tau_o^{-1})^2 \frac{\partial}{\partial x_-} \langle \delta\Psi^2(x_-) \rangle_{\underline{k}}$$

$$= k_y^2 \left[\rho_o (\gamma+\tau_o^{-1})^2 - \frac{\tau_{c\ell}}{\tau_o} (B'_{oy})^2\right] \langle \delta\Psi^2(x_-) \rangle_{\underline{k}} \qquad (125)$$

where we have divided out the k_y^2 factor from both sides and set $\langle \delta\Psi^2(x_-) \rangle = \langle \delta\Psi(1)\delta\Psi(2) \rangle$. Rearranging (125) gives

$$\left[\frac{\partial^2}{\partial x_-^2} - k_y^2 + \frac{\tau_{c\ell}}{\tau_o} \frac{(V_A k_y B'_{oy}/B_o)^2}{(\gamma+\tau_o^{-1})^2}\right] \langle \delta\Psi^2(x_-) \rangle_{\underline{k}} = 0, \qquad (126)$$

Since $\bar{\tau}_{c\ell} \sim \tau_{c\ell}\, (1 + \gamma\tau_o)^{-1}$, (126) is just (102) in dimensional units. The quadratic dependence of (113) on γ is thus due to the second order time derivative (i.e., acceleration) in the momentum balance. In the stochastic regime the acceleration goes over to diffusion ($\delta x^2 \sim t$) and the quadratic dependence on γ becomes a linear dependence as in (117).

VI. NONLINEAR ENERGY PRINCIPLE

The equations for the MHD clump instability can be recast into the form of an "energy principle" similar to that of linear MHD stability theory.[29] The linearized energy principle of MHD derives from the conservation of energy

$$\frac{\partial}{\partial t}\int d\underline{x}\,\frac{1}{2}\,\rho_o\,\langle\delta V^2\rangle + \frac{\partial}{\partial t}\int d\underline{x}\,\delta W = 0 \qquad (127)$$

where δW is the potential energy density driving the instability. For instance, in the case of the Rayleigh-Taylor instability of Sec. IIIF, a quadratic form can be constructed from (124) to give

$$\gamma^2 = -\frac{\int dx\ \delta W}{\int dx\frac{1}{2}\,\rho_o\langle\delta\xi_{\underline{k}}^2\rangle} \qquad (128)$$

where $\delta\xi_{\underline{k}}$ is the fluid displacement and, in the simplified case of $\partial\delta V_{\underline{k}}/\partial x = \gamma\partial\xi_{\underline{k}}/\partial x = 0$, δW is

$$\delta W = \left[(\underline{k}.\underline{B}_o)^2 - g\,\frac{\partial\rho_o}{\partial x}\right]\langle\delta\xi_{\underline{k}}^2\rangle \qquad (129)$$

Instability results when $\delta W<0$, i.e., when mixing of the density gradient overcomes the stabilizing effect of line bending.

For the MHD clump case, we deal with the total fluctuation energy and, thus remove $\langle\delta B^2\rangle$ from δW in (127). The two-point, clump analogue of (128) is

$$\gamma = -\frac{\int dx\_\,\delta W^c}{\int dx\_\langle\delta\underline{V}_1\cdot\delta\underline{V}_2 + S^2\,\delta\underline{B}_1\cdot\delta\underline{B}_2\rangle_{\underline{k}}} \qquad (129)$$

where

$$\delta W^c = \frac{1}{\tau_o}\langle\delta\underline{V}_1\cdot\delta\underline{V}_2 + S^2\delta\underline{B}_1\cdot\delta\underline{B}_2\rangle_{\underline{k}} - 2\,\frac{\bar{\tau}_{c\ell}}{\tau_o}\,DB_{oy}'^2 \qquad (130)$$

Instability ($\delta W^c<0$) results when turbulent mixing of the magnetic shear (the second term in (130)) overcomes stochastic line bending (the first term in

(130)). Insertion of (130) into (129) and integration over x_- gives

$$\gamma = \frac{-1}{\tau_o \langle \delta\Psi^2(0) \rangle_{\underline{k}}} \left[\langle \delta\Psi^2(0) \rangle_{\underline{k}} - \frac{DB'^2_{oy}}{\Delta'_{\underline{k}} + 2|k_y|} \int_{-\infty}^{\infty} dx_- \bar{\tau}_{c\ell} \exp[-|k_y| x_-] \right] \quad (131)$$

or, upon rearrangement and use of (108) and (73),

$$\gamma = -\frac{1}{\tau_o} \left[1 - \frac{\hat{\mathbb{R}}}{1+\gamma\tau_o} \right] = -\frac{1}{\tau_o} (1-\mathbf{R}) \quad (132)$$

This is just (113) or (23). Clearly, $\delta W^c < 0$ means that $\mathbf{R} > 1$, or, when $\gamma\tau_o \ll 1$, $\hat{\mathbf{R}} > 1$.

VII. NONLINEAR KRUSKAL-SHAFRANOV CONDITION

Though the unconstrained mixing calculation of the MHD clump instability presented here neglects magnetic helicity conservation, it does provide an interesting physical interpretation of the instability threshold. For this purpose we use the Re $\Delta'_{\underline{k}}$ model of Eq. (26) of Ref. 28 for long wavelength (low mode number) modes in a general current profile , i.e., Re $\Delta'_{\underline{k}} \sim (\underline{k}\cdot\underline{B}_o'/\underline{k}\cdot\underline{B}_o)^2/k_y \sim (B_{oy}'/B_{oy})^2/k_y$. Since the relevant k_y's are of order k_o, (112) gives $\Delta'_c \sim \mathrm{Re}\Delta_{\underline{k}}' \sim k_o$. The instability condition $\hat{\mathbb{R}} > 1$ is approximately then $x_d \mathrm{Re}\Delta_{\underline{k}}' < 1$, i.e., the "constant ψ approximation" in the stochastic layer.[28] Defining $L_c^{-1} \sim k_o x_d \; B_{oy}'/B_{oz}$, the instability threshold condition becomes in dimensional units

$$J_{oz} > \frac{1}{L_c} B_{oz} \tag{133}$$

where again we have set μ_o=1. As discussed in Sec. IB, $L_c \sim z_o$ is the z stochasticity length--the length one must move in z for the stochastic field lines to diffuse radially by x_d. For instability, (133) states that sufficient production of fluctuations by mixing ($J_{oz} \sim B_{oy}'$) must occur to overcome the spacial destruction of the localized clumps as neighboring field lines diverge apart stochastically. Equation (133) is reminiscent of the instability condition $J_{oz} > k_z B_{oz}$ for the linear kink mode (Kruskal-Shafranov condition).[29] In a conventional picture, the kink instability arises if the "pressure" due to bunching of the poloidal field lines generated by J_{oz} can overcome the resistance to this bending of the plasma column provided by the B_{oz}-field line tension. In the MHD clump instability, random localized bending and bunching of the field lines occurs nonlinearly as the mean poloidal shear profile is turbulently mixed. This process is opposed by the random restoring force of the stochastic magnetic field lines. Since this restoring force (and the Alfven wave emission it causes) is minimal near the mode rational surfaces, $k_z B_{oz}$ of the linear stability condition is replaced by $\underline{k}.\underline{B} \sim k_o B_{oy}' x_d \sim B_{oz} L_c^{-1}$ in the nonlinear (clump) stability condition.

A further connection with the linear kink instability can be obtained by integrating (133) over the minor cross section of a toroidal plasma. The instability threshold condition then is

$$q(a) < 2 \frac{L_c}{R} \qquad (134)$$

where, q is the "safety factor" and R is the major radius of the plasma. Equation (134) is the turbulent, nonlinear clump analogue of the Kruskal-Shafranov condition, $q(a) < 1$.[29] Instability results when the connection length is less than the sochasticity length. Expressed this way in terms of characteristic spacial scales of the stochastic fields, it is the spacial version of the temporal instability condition $\hat{\mathbf{R}}/\tau > 1/\tau$. Physically, after island overlap, clumps cannot regenerate if, as one moves along a magnetic field line, neighboring field lines inside a clump diverge radially by x_d (the clump scale size) before the distance $z \sim Rq$ characterizing the shear strength is reached. Since the field lines diverge by x_d after a distance traversed in z of L_c, the stability condition is $Rq > L_c$. For $L_c > Rq$, a connection length can be traversed before L_c is reached, the clumps can regenerate and instability results. With the dynamical constraint of magnetic helicity conservation, (133) gets replaced with $J_{oz} > \mu B_{oz}$.

At island overlap, the initial region of stochasticity will be small and confined to the island seperatrices. For such a small x_d at instability onset, $2L_c$ can easily be larger than R. Then, (134) implies a nonlinear stability boundary <u>below</u> the Kruskal-Schafranov limit, i.e., subcritical to the stability boundary of the linear kink mode. We note that the threshold for the Vlasov clump instability is also subcritical to the corresponding linear instability boundary.[6-9]

VIII. COMPARISON WITH VLASOV HOLE GROWTH

It is interesting to compare the growth rate of MHD current holes to that of phase space density holes in a Vlasov plasma. Consider the growth of a current hole in a stochastic bath of background holes. In the fully stochastic case ($\gamma\tau < 1$), the background stochasticity, as we've seen, can be modeled as an anomalous resistivity D. Inserting D into Faraday's law, we obtain (24) for the evolution of the poloidal flux function. The resulting growth rate (25) can be rewritten as $\gamma_H \sim (\Delta'_{\underline{k}} x_d)/\tau$. Expressing τ in terms of x_d, this becomes (in dimensional units)

$$\gamma_H \sim V_A (x_d/\Delta y)\ |B'_{oy}/B_{oz}|\ (\Delta'_{\underline{k}}\ x_d) \tag{135}$$

where $k_o^{-1} = \Delta y$ is the poloidal scale length of the hole. The first two factors in (135), coming from the reconnection rate $\tau^{-1} \sim D/x_d^2$, determine the characteristic growth time. Reconnection occurs as field lines bend stochastically at island overlap. This course-grained reconnection is driven by the stochastic $\underline{J}_{oz} \times \delta\underline{B}$, or field line tension force. The instability is current driven. The rate at which the reconnection occurs is given by the shear Alfven wave frequency $V_A \underline{k} \cdot \underline{B}_o / B_{oz} \sim V_A k_o B'_{oy} x_d / B_{oz} \sim \tau^{-1}$ in a resonant stochastic layer of width x_d about the island. Larger island fluctuations grow faster because the width of the stochastic or reconnection region x_d increases with amplitude as in (80). The free energy source for growth comes from the shielding function $\Delta'_{\underline{k}} x_d$, i.e., the current density gradient in the region outside the island. For $\Delta'_{\underline{k}} > 0$, there is a positive discontinuity or jump in the perturbed magnetic field across the island, and the island grows.

The growth rate of an isolated ion hole in the Vlasov plasma[7] is

$$\gamma_i \sim - (\Delta v/\Delta x)\ (v_i^2\, f'_{oi})\ (v_e^2 f'_{oe}) \qquad (136)$$

where v_i, v_e are ion and electron thermal velocities and $f'_{oi} = \partial f_{oi}/\partial v$, $f'_{oe} = \partial f_{oe}/\partial v$ are the velocity gradients of the ion and electron mean distributions. Equation (136) describes the growth of an ion hole of velocity width Δv and spatial width Δx. The hole grows in depth as it decelerates to regions of larger ion phase space density--hence, the second factor in (136). The free energy for the deceleration is provided by the electron gradient (the third factor in (136)). Electrons are resonantly reflected by the ion hole potential and exchange momentum with the hole. For $f'_{oe} > 0$, i.e., more electrons going faster than slower than the hole, the hole is accelerated by an electric field created by the reflecting electrons. The electric field or potential drop across the hole is proportional to $v_e^2\, f'_{oe} \sim \mathrm{Im}\varepsilon_e$, where $\mathrm{Im}\varepsilon_e$ is the imaginary or resonant part of the dielectric function due to the electrons. The potential drop structure is frequently referred to as a double layer.[10,11] The growth rate is amplitude dependent through the hole trapping time $\Delta x/\Delta v$. As the hole potential increases, the hole growth rate increases since more resonant electrons are reflected and each electron transfers more momentum to the hole.

Despite the similarities in (135) and (136), there are several important differences. While the MHD hole grows in amplitude at a fixed resonant position (i.e., mode rational surface) the Vlasov hole grows by decelerating to different resonant velocities where f_{oi} is larger. The free energy in the Vlasov case is resonant--coming from electrons within a velocity trapping width ($\sim\Delta v$) of the hole. In the MHD case, the free energy for growth derives from the nonresonant region. Consequently, free energy comes from the imaginary part of the shielding function ($\mathrm{Im}\varepsilon$) in the Vlasov case, but from the real part of the shielding function ($Re\Delta'_{\underline{k}}$) in the MHD case. However, in each case, growth is due to a jump or discontinuity in the field. [Note that the Kadomstev bubble also grows by a discontinuity in the field (see Ref. 30). The opposite signs of $\delta\psi'$ at the bubble and plasma boundaries generate opposing currents which force the bubble boundary into the plasma]. While the Vlasov hole can grow in isolation, the MHD hole requires the stochasticity (i.e., the turbulent resistivity D) produced by the overlap with other hole resonances. Since an additional effect of the stochasticity is the destruction of coherent islands by the exponential divergence of neighboring field lines, net growth must be achieved by the creation of new fuctuations by mixing. The MHD instability is, therefore, a clump instability rather than an isolated hole instability, i.e., the relevant growth rate is (28) rather than (25).

ACKNOWLEDGEMENT

With the premature start of Ref. 1 and a necessary diversion to develop and understand the instability of one dimensional Vlasov clumps (the "hydrogen atom" of clump fluctuations), this work has been many years in the making. Several people have influenced the outcome presented here. First, I would like again to thank B. Carreras and the ORNL group for sharing their computer simulation results early on in this work. Nothing has been more compelling to show the importance of mixing, stochastic decay, and the resonant nature of the fluctuations. I. Hutchinson has been a constant source of information on laboratory experiments as well as providing useful criticism of ideas. Along with A. Boozer, his championing of magnetic helicity conservation has been particularly stimulating. Acknowledgement for useful conversations also goes to J.B. Taylor, J.P. Freidberg, H. Strauss, R. Granetz, and to T.H. Dupree for constant encouragement. Critical reading of the manuscript by I. Hutchinson and T. Dupree is gratefully acknowledged. I would also like to thank The Department of Energy for supporting the interim study of the basic one dimensional Vlasov case, without which this work could not have been possible.

This work was supported by The Department of Energy, Office of Naval Research, and The National Science Foundation.

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