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Neoclassical Flux-friction Relations In Arbitrary Closed-end Plasmas

Y. T. Lau

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Plasma Fusion Center
Massachusetts Institue of Technology
Cambridge, MA 02139 USA

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Neoclassical Flux-friction Relations In Arbitrary Closed-end Plasmas

Y. T. Lau

Plasma Fusion Center
Massachusetts Institue of Technology
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By following the moment approach of neoclassical transport theory, flux-friction relations in arbitrary closed-end plasmas are derived. As a simple application, the Pfirsch-Schlüter fluxes are obtained for arbitrary closed-end plasmas. In particular, the Pfirsch-Schlüter fluxes for a nonaxisymmetric toroidal plasma are calculated. The resonance effect for a model toroidal magnetic field is reproduced straightforwardly. The ambipolar potential and the parallel flows in the Pfirsch-Schlüter regime are also determined for arbitrary closed-end plasmas, showing that the fluxes associated with them are generally negligible in this regime.

I. INTRODUCTION

The moment approach of neoclassical transport theory has originally been developed by Hirshman and Sigmar for axisymmetric tokamaks.[1] This approach was then generalized to nonaxisymmetric toroidal systems.[2,3] Based on these works, neoclassical transport coefficients for nonaxisymmetric toroidal systems in various collisionality regimes were calculated.[4,5] The advantage of the moment approach, i.e., the parallel momentum conservation and the ambipolarity condition being considered at the fluid moment level, was also discussed.[6] In all these studies, flux coordinates were employed to express the magnetic field in a contravariant fashion. Subsequent formulas were then closely tied to the flux coordinates. However, these coordinates may not be convenient, since the magnetic field may either be expressed in a covariant fashion (as the gradient of a scalar field) or by its magnitude and field lines. Besides, for general closed-end systems with complicated magnetic axes, such as helical solenoids[7] and DRAKON,[8,9] they are not applicable. Therefore, it is desirable to formulate the moment approach by keeping the vector form of the magnetic field and merely assuming the existence of flux surfaces. The formulas derived in this way are applicable to arbitrary closed-end plasmas and another advantage of the moment approach—separation of the geometric effects from the kinetic ones—is manifest from the derivation.

In Sec. II, the first-order flows are solved from four lowest-order moment equations, leaving some flux functions undetermined. Then in Sec. III, by introducing a vector field **D**, the higher-order fluxes are related to the friction forces, viscosities, inductive electric field, and external sources. After solving for the vector field **D** in Sec. IV, we follow the conventional way to split the fluxes into different pieces to obtain the flux-friction relations in Sec. V. Corresponding relations for toroidal plasmas are derived in Sec. VI. In Sec. VII, Pfirsch-Schlüter fluxes are calculated for arbitrary closed-end plasmas. In particular, the effective safety factor for a model toroidal field is obtained, reproducing the resonance effect. The ambipolar potential and the parallel flows in the Pfirsch-Schlüter regime are determined in Sec. VIII. Finally, conclusions are given in Sec. IX.

II. LOWEST-ORDER SOLUTIONS

Expanding the four moment equations[1] to lowest order in $\delta(=\rho/L$, where ρ is the Larmor radius and L the scale length). we have for each species[10] (the species index α is omitted),

$$\frac{e}{c} n_0 \mathbf{u}_1 \times \mathbf{B} = \nabla P_0 + n_0 e \nabla \Phi_0, \tag{1a}$$

$$\frac{e}{c} \mathbf{q}_1 \times \mathbf{B} = \frac{5}{2} P_0 \nabla T, \tag{1b}$$

$$\nabla \cdot n_0 \mathbf{u}_1 = 0, \tag{2a}$$

$$\nabla \cdot \mathbf{q}_1 = -n_0 \mathbf{u}_1 \cdot \left(\frac{5}{2} \nabla T + e \nabla \Phi_0 \right), \tag{2b}$$

where $\mathbf{u}_1$ and $\mathbf{q}_1$ are the first-order velocity and heat flows. Assuming the existence of flux surfaces labeled by Ψ and charge neutrality $\sum_\alpha n_\alpha e_\alpha = 0$, we readily obtain from Eqs. (1) the temperature, pressure, density, and electric potential as functions of the flux variable only,

$$T = T(\Psi),$$

$$P_0 = P_0(\Psi),$$

$$n_0 = n_0(\Psi),$$

$$\Phi_0 = \Phi_0(\Psi).$$

Among these flux functions, Φ_0 are to be determined by the higher-order equations while the others are assumed to be given. Now Eqs. (1) imply that the first-order velocity and heat flows are vector fields lying within flux surfaces and their purpendicular components are azimuthal, i.e.,

$$\mathbf{u}_{1\perp} = \frac{c\mathbf{b} \times \nabla P_0}{e n_0 B} + \frac{c\mathbf{b} \times \nabla \Phi_0}{B} = \frac{\mathbf{b} \times \nabla \Psi}{B} \frac{c}{e} \left(\frac{P_0'}{n_0} + e\Phi_0' \right), \tag{3a}$$

$$\mathbf{q}_{1\perp} = \frac{\mathbf{b}\times\nabla\Psi}{B}\frac{5c}{2e}P_0T', \tag{3b}$$

where $\mathbf{b} \equiv \mathbf{B}/B$ is the unit vector along field lines. Also, Eqs. (2) indicate that the first order flows are divergence-free, i.e., $\nabla\cdot\mathbf{u}_1 = \nabla\cdot\mathbf{q}_1 = 0$. Therefore, if we write

$$\mathbf{u}_1 = \mathbf{u}_{1\perp} + (\mathcal{U} + \hat{U}_{\|}(\Psi))\mathbf{B}, \tag{4a}$$

$$\mathbf{q}_1 = \mathbf{q}_{1\perp} + (\mathcal{Q} + \hat{Q}_{\|}(\Psi))\mathbf{B}, \tag{4b}$$

we get two magnetic differential equations for $\mathcal{U}$ and $\mathcal{Q}$,

$$\mathbf{B}\cdot\nabla\mathcal{U} = -\nabla\cdot\left(\frac{\mathbf{b}\times\nabla\Psi}{B}\right)\frac{c}{e}\left(\frac{P_0'}{n_0} + e\Phi_0'\right), \tag{5a}$$

$$\mathbf{B}\cdot\nabla\mathcal{Q} = -\nabla\cdot\left(\frac{\mathbf{b}\times\nabla\Psi}{B}\right)\frac{5c}{2e}P_0T'. \tag{5b}$$

Noting that $\mathbf{J}$ satisfies $\mathbf{J} = c\nabla\times\mathbf{B}/4\pi \perp \nabla\Psi$, we can solve Eqs. (5) to obtain

$$\mathcal{U} = -\left(\int d\boldsymbol{\ell}\cdot\nabla\Psi\times\nabla B^{-2}\right)\frac{c}{e}\left(\frac{P_0'}{n_0} + e\Phi_0'\right), \tag{6a}$$

$$\mathcal{Q} = -\left(\int d\boldsymbol{\ell}\cdot\nabla\Psi\times\nabla B^{-2}\right)\frac{5c}{2e}P_0T'. \tag{6b}$$

The constants from the line integrals in Eqs. (6) will be included in the flux functions $\hat{U}_{\|}(\Psi)$ and $\hat{Q}_{\|}(\Psi)$, which remain indefinite in this order and can be determined by the parallel balance equations of higher order.

III. HIGHER-ORDER SOLUTIONS

The exact momentum equation can be written as[1]

$$\frac{e}{c}n\mathbf{u}\times\mathbf{B} = ne(\nabla\Phi - \mathbf{E}^A) + \nabla P + \nabla\cdot\boldsymbol{\pi} - \mathbf{F}_1 - \mathbf{K}_1 + \left[nm\left(\frac{\partial\mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u}\right)\right], \tag{7a}$$

where $\boldsymbol{\pi} = \mathsf{P} - P\mathsf{I}$ is the viscosity tensor and $\mathbf{E} = -\nabla\Phi + \mathbf{E}^A$ the electric field (static and inductive). $\mathbf{F}_1$ is the friction and $\mathbf{K}_1$ is an external force. The heat flux equation is

$$\frac{e\mathbf{q}}{cT}\times\mathbf{B} = \nabla\Theta + \nabla\cdot\boldsymbol{\Theta} - \mathbf{F}_2 - \mathbf{K}_2 + \left[\frac{m}{T}\frac{\partial\mathbf{q}}{\partial t} + \frac{5\mathbf{u}}{2}\frac{\partial P}{\partial t} - \frac{5}{2}nm\mathbf{u}\cdot\nabla\mathbf{u}\right.$$

$$+\frac{\partial}{\partial t}\left(\frac{nmu^2}{2}\mathbf{u}-\mathbf{u}\cdot\pi\right)-\frac{\epsilon}{T}\mathbf{E}\cdot(\pi+\mathbf{uu}mn)\Bigg], \tag{7b}$$

where $\Theta\mathsf{I}+\boldsymbol{\Theta}=m\mathsf{r}/T-5\mathsf{P}/2$ is the heat viscosity tensor, and $\mathbf{F}_2=m\mathbf{G}/T-5\mathbf{F}_1/2$ the heat friction. ($\mathsf{r},\mathsf{P},\mathbf{G}$ are the energy-weighted stress tensor, pressure tensor, collisional rate of heat flux, as defined in Ref. 1) The external heat power flux $\mathbf{P}^{ex}$ is combined with $\mathbf{K}_1$ to form $\mathbf{K}_2=m\mathbf{P}^{ex}/T-5\mathbf{K}_1/2$. Next we observe that the terms in the square brackets of Eqs. (7) are of higher order than the other terms, thus to first order in δ, we have

$$\frac{e}{c}n_0\mathbf{u}_2\times\mathbf{B}=n_0e\nabla\tilde{\Phi}+\nabla\tilde{P}-\nabla\cdot\pi-\mathbf{F}_1-n_0e\mathbf{E}^A-\mathbf{K}_1, \tag{8a}$$

$$\frac{e\mathbf{q}_2}{cT}\times\mathbf{B}=\nabla\Theta+\nabla\cdot\boldsymbol{\Theta}-\mathbf{F}_2-\mathbf{K}_2, \tag{8b}$$

where $\tilde{\Phi}=\Phi-\Phi_0, \tilde{P}=P-P_0$ are the quantities that describe variations within flux surfaces. Also note that we have treated the external sources as first-order quantities. Now we are in a position to solve for $\mathbf{u}_2$ and $\mathbf{q}_2$. In order to do this, we must know the various moments on the right hand sides of Eqs. (8). However, some of these moments (like $\tilde{\Phi}$) are not easy to obtain unless we go to even higher order equations. Thus we seek a way to eliminate these scalar functions at the cost of losing some unnecessary information. Recall that we have assumed the existence of flux surfaces, thus for closed-end systems, the only dangerous fluxes are the flux surface averages of the cross-surface fluxes, i.e.,

$$\Gamma_\Psi=\langle n_0\mathbf{u}_2\cdot\nabla\Psi\rangle, \tag{9a}$$

$$q_\Psi=\langle \mathbf{q}_2\cdot\nabla\Psi\rangle. \tag{9b}$$

The flux surface average $\langle\,\rangle$ is defined in Appendix B. Now if we construct a vector field $\mathbf{D}$ that satisfies

$$\nabla\cdot\mathbf{D}=0, \tag{10a}$$

$$\mathbf{B}\times\mathbf{D}=\nabla G(\Psi), \tag{10b}$$

then we immediately have (using Eq. (B5)) $\left\langle \mathbf{D}\cdot\nabla\tilde{\Phi}\right\rangle = \langle \mathbf{D}\cdot\nabla P\rangle = \langle \mathbf{D}\cdot\nabla\Theta\rangle = 0$, and the dot products of Eqs. (14) with $\mathbf{D}$ yield

$$\Gamma_\Psi = \frac{c}{eG'}\left\langle(\nabla\cdot\pi - \mathbf{F}_1 - en_0\mathbf{E}^A - \mathbf{K}_1)\cdot\mathbf{D}\right\rangle, \tag{11a}$$

$$\frac{q_\Psi}{T} = \frac{c}{eG'}\langle(\nabla\cdot\mathbf{\Theta} - \mathbf{F}_2 - \mathbf{K}_2)\cdot\mathbf{D}\rangle. \tag{11b}$$

Note that G is an arbitrary flux function. Since $\mathbf{D}$ is always divided by G', the cross-surface fluxes are actually independent of this free function.

Taking the dot products of Eqs. (8) and $\mathbf{B}$, we get the parallel balance equations,

$$\left\langle(\nabla\cdot\pi - \mathbf{F}_1 - en_0\mathbf{E}^A - \mathbf{K}_1)\cdot\mathbf{B}\right\rangle = 0, \tag{12a}$$

$$\langle(\nabla\cdot\mathbf{\Theta} - \mathbf{F}_2 - \mathbf{K}_2)\cdot\mathbf{B}\rangle = 0. \tag{12b}$$

Summing up Eq. (12a) over all N species and applying momentum conservation $\sum_\alpha \mathbf{F}_{1\alpha} = 0$ and charge neutrality $\sum_\alpha n_\alpha e_\alpha = 0$, we obtain the parallel momentum conservation equation

$$\sum_\alpha\langle\mathbf{B}\cdot\nabla\cdot\boldsymbol{\pi}_\alpha\rangle - \langle\mathbf{K}_1\cdot\mathbf{B}\rangle = 0. \tag{13}$$

Furthermore, the ambipolar condition $\sum_\alpha \Gamma_\Psi^\alpha = 0$ leads to

$$\sum_\alpha\langle\mathbf{D}\cdot\nabla\cdot\boldsymbol{\pi}_\alpha\rangle - \langle\mathbf{K}_1\cdot\mathbf{D}\rangle = 0. \tag{14}$$

These $2N+1$ independent equations are just enough to determine the ambipolar potential and the undetermined parts of the first-order flows ($\hat{U}_\parallel$ and $\hat{Q}_\parallel$). In some special cases, Eqs. (13,14) together can determine the ambiploar potential, as will be shown in Sec. VIII.

IV. THE VECTOR FIELD D

The vector field $\mathbf{D}$ defined by Eqs. (10) has exactly the same geometric structures as the first-order flows: it is divergence-free and lies within the flux surfaces. Thus we can proceed in exactly the same way. The azimuthal component is

$$\mathbf{D}_\perp = -G'\frac{\mathbf{b}\times\nabla\Psi}{B}. \tag{15}$$

If we write

$$\mathbf{D} = \mathbf{D}_\perp + (\mathcal{D} + \hat{D}_{||}(\Psi))\mathbf{B}, \tag{16}$$

we get a magnetic differential equation for $\mathcal{D}$,

$$\begin{aligned}\mathbf{B}\cdot\nabla\mathcal{D} &= G'\nabla\cdot\left(\frac{\mathbf{b}\times\nabla\Psi}{B}\right) \\ &= G'\mathbf{B}\cdot\nabla\Psi\times\nabla B^{-2}.\end{aligned} \tag{17}$$

Where again, $\mathbf{J}\perp\nabla\Psi$ has been used in the last step. The solution for $\mathcal{D}$ is

$$\mathcal{D} = G'\int d\boldsymbol{\ell}\cdot\nabla\Psi\times\nabla B^{-2}, \tag{18}$$

which is completely determined by the magnitude of $\mathbf{B}$ and its field lines. Note that in general $\oint d\boldsymbol{\ell}\cdot\nabla\Psi\times\nabla B^{-2} \neq 0$, hence this integral determines $\mathcal{D}$ up to an arbitrary flux function $\hat{D}_{||}$. However, because of the parallel balance equations (Eqs. (12)), it does not contribute to the net cross-surface fluxes. Therefore we can absorb it into $\mathcal{D}$ and write

$$\frac{\mathbf{D}}{G'} = -\frac{\mathbf{b}\times\nabla\Psi}{B} + \frac{\mathcal{D}\mathbf{B}}{G'}. \tag{19}$$

This equation is purely geometric and will be called "general geometric relation". It will be shown later that it reduces to the corresponding relations in toroidal systems. Taking the curl of Eq. (10b), we obtain another geometric relation,

$$\mathbf{B}\cdot\nabla\mathbf{D} = \mathbf{D}\cdot\nabla\mathbf{B}. \tag{20}$$

An alternative way of finding $\mathbf{D}$ is to construct it from different components of $\mathbf{B}$ while keeping Eqs. (10) satisfied. This method will be illustrated in Sec. VI.

Finally, we can relate $\mathcal{U}$ and $\mathcal{Q}$ in Eqs. (6) to their counterpart $\mathcal{D}$ by

$$\mathcal{U} = -\frac{\mathcal{D}}{G'}\frac{c}{e}\left(\frac{P_0'}{n_0} + e\Phi_0'\right), \tag{21a}$$

$$\mathcal{Q} = -\frac{\mathcal{D}}{G'}\frac{5c}{2e}P_0T'. \tag{21b}$$

And the first-order flows are related to $\mathbf{D}$ by

$$\mathbf{u}_1 = -\frac{c}{e}\left(\frac{P_0'}{n_0} + e\Phi_0'\right)\frac{\mathbf{D}}{G'} + \hat{U}_{\parallel}\mathbf{B}, \tag{22a}$$

$$\mathbf{q}_1 = -\frac{5c}{2e}P_0 T' \frac{\mathbf{D}}{G'} + \hat{Q}_{\parallel}\mathbf{B}. \tag{22b}$$

V. FLUX-FRICTION RELATIONS

As in Ref. 1 and 2, the fluxes Γ_Ψ and q_Ψ can be separated into different pieces according to their different physical origins. Except for those due to external sources and inductive electric field, all other fluxes are directly related to the friction forces and viscosity tensors. These relations are the flux-friction relations.

The particle flux is divided into

$$\Gamma_\Psi = \Gamma_{\rm cl} + \Gamma_{\rm ex} + \Gamma_{\rm na} + \Gamma_{\rm PS} + \Gamma_{\rm bp} + \Gamma_{\rm in},$$

i.e., classical, external, nonaxisymmetric, Pfirsch-Schlüter, banana-plateau, and inductive fluxes respectively. Using the general geometric relation (Eq. (19)), we can express them as

$$\Gamma_{\rm cl} = \frac{c}{e}\left\langle \frac{\mathbf{b}\times\nabla\Psi}{B}\cdot\mathbf{F}_{1\perp}\right\rangle, \tag{23}$$

$$\Gamma_{\rm ex} = -\frac{c}{eG'}\langle \mathbf{D}\cdot\mathbf{K}_1\rangle\,, \tag{24}$$

$$\Gamma_{\rm na} = \frac{c}{eG'}\langle \mathbf{D}\cdot\nabla\cdot\boldsymbol{\pi}\rangle\,, \tag{25}$$

$$\Gamma_{\rm PS} = \frac{c}{eG'}\left\langle F_{1\parallel}\left(\frac{B\langle \mathcal{D}B^2\rangle}{\langle B^2\rangle} - \mathcal{D}B\right)\right\rangle, \tag{26}$$

$$\Gamma_{\rm bp} = -\frac{c}{eG'}\frac{\langle \mathcal{D}B^2\rangle}{\langle B^2\rangle}\left\langle B(F_{1\parallel} + n_0 e E_{\parallel}^A)\right\rangle, \tag{27}$$

$$\Gamma_{\rm in} = \frac{c}{eG'}\left\langle n_0 e E_{\parallel}^A\left(\frac{B\langle \mathcal{D}B^2\rangle}{\langle B^2\rangle} - \mathcal{D}B\right)\right\rangle + \frac{c}{e}\left\langle n_0 e \mathbf{E}_{\perp}^A\cdot\frac{\mathbf{b}\times\nabla\Psi}{B}\right\rangle. \tag{28}$$

Note that due to momentum conservation and charge neutrality the classical, Pfirsch-Schlüter, banana-plateau, and inductive fluxes are intrinsically ambipolar. The ambipolar potential Φ_0 is therefore determined by balancing the other fluxes of different species.

Likewise, the heat flux is divided into

$$q_\Psi = q_{cl} + q_{ex} + q_{na} + q_{PS} + q_{bp},$$

and the various fluxes are,

$$\frac{q_{cl}}{T} = \frac{c}{e}\left\langle \frac{\mathbf{b}\times\nabla\Psi}{B}\cdot\mathbf{F}_{2\perp}\right\rangle, \tag{29}$$

$$\frac{q_{ex}}{T} = -\frac{c}{eG'}\langle \mathbf{D}\cdot\mathbf{K}_2\rangle, \tag{30}$$

$$\frac{q_{na}}{T} = \frac{c}{eG'}\langle \mathbf{D}\cdot\nabla\cdot\mathbf{\Theta}\rangle, \tag{31}$$

$$\frac{q_{PS}}{T} = \frac{c}{eG'}\left\langle F_{2\|}\left(\frac{B\langle \mathcal{D}B^2\rangle}{\langle B^2\rangle} - \mathcal{D}B\right)\right\rangle, \tag{32}$$

$$\frac{q_{bp}}{T} = -\frac{c}{eG'}\frac{\langle \mathcal{D}B^2\rangle}{\langle B^2\rangle}\left\langle BF_{2\|}\right\rangle. \tag{33}$$

Since Ω^{-1} is much smaller than any other characteristic time, we can use the CGL forms[11] for the viscosity tensors, i.e., $\pi = \delta P(\mathbf{bb} - \mathsf{I}/3)$, and $\mathbf{\Theta} = \delta\Theta(\mathbf{bb} - \mathsf{I}/3)$ with $\delta P = (P_\| - P_\perp)$, and $\delta\Theta = (\Theta_\| - \Theta_\perp)$. Applying Eq. (B4) with $\mathbf{A}$ replaced by $\delta P\mathbf{bb}$ and f by $\mathbf{D}$, we obtain

$$\langle \mathbf{D}\cdot\nabla\cdot\delta P\mathbf{bb}\rangle = -\left\langle \frac{\delta P}{B}\mathbf{bB}:\nabla\mathbf{D}\right\rangle.$$

The double dot product is defined by $\mathbf{AB}:\mathbf{CD} \triangleq \sum_{i,j} A_i B_j C_j D_i$. Then Eq. (20) readily gives us

$$\Gamma_{na} = -\frac{c}{eG'}\left\langle \delta P\frac{\mathbf{D}\cdot\nabla B}{B}\right\rangle. \tag{34}$$

Similarly, the nonaxisymmetric heat flux

$$\frac{q_{na}}{T} = -\frac{c}{eG'}\left\langle \delta\Theta\frac{\mathbf{D}\cdot\nabla B}{B}\right\rangle. \tag{35}$$

VI. FORMULAS FOR TOROIDAL PLASMAS

We now show that with appropriate choices of **D**, the general flux-friction relations reduce to the conventional expressions for toroidal plasmas. For nonaxisymmetric toroidal plasmas, the coordinates (Ψ, θ, ζ) are employed with θ and ζ as the poloidal and toroidal angles. The magnetic field can be expressed in a contravariant fashion (see Appendix A),

$$\begin{aligned} \mathbf{B} &= \chi'(\Psi)\nabla\zeta\times\nabla\Psi + \psi'(\Psi)\nabla\Psi\times\nabla\theta \\ &= \mathbf{B}_p + \mathbf{B}_t \\ &= B^2\mathbf{e}_2 + B^3\mathbf{e}_3. \end{aligned} \tag{36}$$

where $B^2 = \chi'/\sqrt{g}$, $B^3 = \psi'/\sqrt{g}$ are the contravariant components of **B** and the Jacobian is

$$\sqrt{g} = (\nabla\Psi\cdot\nabla\theta\times\nabla\zeta)^{-1}. \tag{37}$$

Now if we choose

$$\mathbf{D} = \alpha_1\mathbf{B}_t + \alpha_2\mathbf{B}_p, \tag{38}$$

where α_1 and α_2 are arbitrary flux functions, then **D** is evidently divergence-free and through Eq. (10b) we find that

$$G' = \frac{\alpha_{12}\psi'\chi'}{\sqrt{g}}, \tag{39}$$

where $\alpha_{12} = \alpha_1 - \alpha_2$. Since G is required to be a flux function, we must choose a coordinate system such that g is a flux function too. A convenient one is the Hamada coordinates[12] (V, θ, ζ), whose $g \equiv 1$. (V is the volume enclosed by each flux surface) Using these coordinates and defining I by

$$\mathcal{D} = \frac{\mathbf{D}\cdot\mathbf{B}}{B^2} = \frac{I}{B^2}, \tag{40}$$

we have the following form of the general geometric relation (Eq. (19)),

$$\frac{\mathbf{D}}{\alpha_{12}\psi'\chi'} = -\frac{\mathbf{b}\times\nabla\Psi}{B} + \frac{I\mathbf{b}}{\alpha_{12}\psi'\chi' B} \tag{41}$$

The neoclassical particle fluxes then take the forms,

$$\Gamma_{\text{ex}} = -\frac{c}{e\alpha_{12}\psi'\chi'}\langle \mathbf{D}\cdot\mathbf{K}_1\rangle\,, \tag{42}$$

$$\Gamma_{\text{na}} = -\frac{c}{e\alpha_{12}\psi'\chi'}\left\langle \delta P\frac{\mathbf{D}\cdot\nabla B}{B}\right\rangle, \tag{43}$$

$$\Gamma_{\text{PS}} = \frac{c}{e\alpha_{12}\psi'\chi'}\left\langle F_{1\|}\left(\frac{B\langle I\rangle}{\langle B^2\rangle} - \frac{I}{B}\right)\right\rangle, \tag{44}$$

$$\Gamma_{\text{bp}} = -\frac{c}{e\alpha_{12}\psi'\chi'}\frac{\langle I\rangle}{\langle B^2\rangle}\left\langle B(F_{1\|} + n_0 e E_{\|}^{A})\right\rangle, \tag{45}$$

$$\Gamma_{\text{in}} = \frac{c}{e\alpha_{12}\psi'\chi'}\left\langle n_0 e E_{\|}^{A}\left(\frac{B\langle I\rangle}{\langle B^2\rangle} - \frac{I}{B}\right)\right\rangle + \frac{c}{e}\left\langle n_0 e\mathbf{E}^{A}\cdot\frac{\mathbf{b}\times\nabla\Psi}{B}\right\rangle. \tag{46}$$

The neoclassical heat fluxes are,

$$\frac{q_{\text{ex}}}{T} = -\frac{c}{e\alpha_{12}\psi'\chi'}\langle \mathbf{D}\cdot\mathbf{K}_2\rangle\,, \tag{47}$$

$$\frac{q_{\text{na}}}{T} = -\frac{c}{e\alpha_{12}\psi'\chi'}\left\langle \delta\Theta\frac{\mathbf{D}\cdot\nabla B}{B}\right\rangle, \tag{48}$$

$$\frac{q_{\text{PS}}}{T} = \frac{c}{e\alpha_{12}\psi'\chi'}\left\langle F_{2\|}\left(\frac{B\langle I\rangle}{\langle B^2\rangle} - \frac{I}{B}\right)\right\rangle, \tag{49}$$

$$\frac{q_{\text{bp}}}{T} = -\frac{c}{e\alpha_{12}\psi'\chi'}\frac{\langle I\rangle}{\langle B^2\rangle}\left\langle BF_{2\|}\right\rangle. \tag{50}$$

Writing $\mathbf{D} = \alpha_{12}\mathbf{B}_t + \alpha_2\mathbf{B} = \alpha_1\mathbf{B} - \alpha_{12}\mathbf{B}_p$ and recalling the parallel balance equations, we can easily see that the net total fluxes are independent of the free functions α_1 and α_2. Different forms of the fluxes given in Ref. 2 can also be reproduced easily from the forms given above.

For axisymmetric toroidal plasmas, we use the coordinates (Ψ, θ, ζ) and express $\mathbf{B}$ as[1]

$$\begin{aligned}\mathbf{B} &= \chi'(\Psi)\nabla\zeta\times\nabla\Psi + I(\Psi,\theta)\nabla\zeta \\ &= \mathbf{B}_p + \mathbf{B}_t.\end{aligned} \tag{51}$$

Where $2\pi\chi$ is the poloidal flux and Ψ the toroidal flux. Choosing

$$\mathbf{D} = R^2\nabla\zeta, \tag{52}$$

with R being the major radius and noting that $\nabla\Psi\cdot\nabla\zeta = \nabla\theta\cdot\nabla\zeta = 0$ and $|\nabla\zeta| = 1/R$, we have from Eq. (10b),

$$G' = \chi' = 1/2\pi q(\Psi), \tag{53}$$

where q is the safety factor. Then since

$$\mathcal{D} = \frac{I(\Psi,\theta)}{B^2}, \tag{54}$$

the general geometric relation reads as follows,

$$\frac{R^2\nabla\zeta}{\chi'} = -\frac{\mathbf{b}\times\nabla\Psi}{B} + \frac{I\mathbf{b}}{\chi' B}. \tag{55}$$

The fluxes are then reduced to the familiar forms,[1]

$$\Gamma_{\text{ex}} = -\frac{c}{e\chi'}\left\langle R^2\nabla\zeta\cdot\mathbf{K}_1\right\rangle, \tag{56}$$

$$\Gamma_{\text{na}} = -\frac{c}{e\chi'}\left\langle \delta P\frac{R^2\nabla\zeta\cdot\nabla B}{B}\right\rangle \equiv 0, \tag{57}$$

$$\Gamma_{\text{PS}} = \frac{c}{e\chi'}\left\langle F_{1\|}\left(\frac{B\langle I\rangle}{\langle B^2\rangle} - \frac{I}{B}\right)\right\rangle, \tag{58}$$

$$\Gamma_{\text{bp}} = -\frac{c}{e\chi'}\frac{\langle I\rangle}{\langle B^2\rangle}\left\langle B(F_{1\|} + n_0 e E_\|^A)\right\rangle, \tag{59}$$

$$\Gamma_{\text{in}} = \frac{c}{e\chi'}\left\langle n_0 e E_\|^A\left(\frac{B\langle I\rangle}{\langle B^2\rangle} - \frac{I}{B}\right)\right\rangle + \frac{c}{e}\left\langle n_0 e \mathbf{E}^A\cdot\frac{\mathbf{b}\times\nabla\Psi}{B}\right\rangle, \tag{60}$$

$$\frac{q_{\text{ex}}}{T} = -\frac{c}{e\chi'}\left\langle R^2\nabla\zeta\cdot\mathbf{K}_2\right\rangle, \tag{61}$$

$$\frac{q_{\text{na}}}{T} = -\frac{c}{e\chi'}\left\langle \delta\Theta\frac{R^2\nabla\zeta\cdot\nabla B}{B}\right\rangle \equiv 0, \tag{62}$$

$$\frac{q_{\text{PS}}}{T} = \frac{c}{e\chi'}\left\langle F_{2\|}\left(\frac{B\langle I\rangle}{\langle B^2\rangle} - \frac{I}{B}\right)\right\rangle, \tag{63}$$

$$\frac{q_{\text{bp}}}{T} = -\frac{c}{e\chi'}\frac{\langle I\rangle}{\langle B^2\rangle}\left\langle BF_{2\|}\right\rangle. \tag{64}$$

Note that for this particular choice of $\mathbf{D}$, the nonaxisymmetric fluxes are identically zero because B has no ζ dependence for axisymmetric plasmas. In general, we can have

$$\mathbf{D} = \alpha_1 R^2 \nabla\zeta + \alpha_2 R^2 \mathbf{B}_p / I(\psi, \theta)$$

with $G' = \alpha_{12}\chi'$, then the nonaxisymmetric fluxes will no longer be zero.

VII. PFIRSCH-SCHLÜTER FLUXES

As a simple application of the flux-friction relations presented in Sec. V, we consider here the Pfirsch-Schlüter fluxes. For a simple electron-ion plasma with $T_e \sim T_i$ and $m_e \ll m_i$, the friction-flow relations derived in Ref. 1 are simplified to[2]

$$\mathbf{F}_{1e} = -\mathbf{F}_{1i} = l_{11}^e \left[(\mathbf{u}_i - \mathbf{u}_e) + \frac{3\mathbf{q}_e}{5P_e} \right], \tag{65a}$$

$$\mathbf{F}_{2e} = l_{11}^e \left[\frac{3}{2}(\mathbf{u}_e - \mathbf{u}_i) - 1.86\frac{\mathbf{q}_e}{P_e} \right], \tag{65b}$$

$$\mathbf{F}_{2i} = -\frac{2}{5} l_{22}^i \frac{\mathbf{q}_i}{P_i}. \tag{65c}$$

where $l_{11}^e = n_e m_e \nu_{ei}$, $l_{22}^i = \sqrt{2} n_i m_i \nu_{ii}$ and the collision frequency

$$\nu_{\alpha\beta} = \frac{4}{3\sqrt{\pi}} \frac{4\pi e_\alpha^2 e_\beta^2 n_\beta \ln\Lambda}{m_\alpha^2 v_{t\alpha}^3}. \tag{66}$$

Since we are calculating the second-order fluxes, we can use the first-order flows in Eqs. (4) for $\mathbf{u}$ and $\mathbf{q}$ in these equations. From the definitions of the Pfirsch-Schlüter fluxes Eqs. (26) and (32), we find that $\hat{U}_\parallel$ and $\hat{Q}_\parallel$ do not contribute to these fluxes, thus we can obtain the Pfirsch-Schlüter fluxes without resorting to the parallel balance equations. Plugging Eqs. (21) with $n_0 = n_0^e = n_0^i$ and $e = e_i = -e_e$ into Eqs. (65) and then using the definitions of the Pfirsch-Schlüter fluxes, we obtain

$$\Gamma_{\mathrm{PS}}^e = \Gamma_{\mathrm{PS}}^i = -\left(\frac{c}{eG'}\right)^2 \left(\left\langle \mathcal{D}^2 B^2 \right\rangle - \frac{\left\langle \mathcal{D} B^2 \right\rangle^2}{\langle B^2 \rangle} \right) m_e \nu_{ei} \left(P_{0e}' + P_{0i}' - \frac{3}{2} n_0 T_e' \right), \tag{67a}$$

$$\frac{q_{\mathrm{PS}}^e}{T} = -\left(\frac{c}{eG'}\right)^2 \left(\left\langle \mathcal{D}^2 B^2 \right\rangle - \frac{\left\langle \mathcal{D} B^2 \right\rangle^2}{\langle B^2 \rangle} \right) m_e \nu_{ei} \left(-\frac{3}{2}(P_{0e}' + P_{0i}') + 1.86\frac{5}{2} n_0 T_e' \right), \tag{67b}$$

$$\frac{q^i_{\rm PS}}{T} = -\left(\frac{c}{\epsilon G'}\right)^2 \left(\left\langle \mathcal{D}^2 B^2 \right\rangle - \frac{\langle \mathcal{D} B^2\rangle^2}{\langle B^2\rangle}\right) \sqrt{2} n_0 m_i \nu_{ii} T_i'. \qquad (67c)$$

Notice that the geometric effects are completely separate from the kinetic ones. Letting $\Psi = \pi r^2 B_0$, we can define an effective safety factor that contains all the geometric information,

$$q^2_{\rm eff} = \frac{1}{2(2\pi r G')^2}\left[\left\langle \mathcal{D}^2 B^2 \right\rangle - \frac{\langle \mathcal{D} B^2\rangle^2}{\langle B^2\rangle}\right]. \qquad (68)$$

Now if we set $T_e = T_i = T$, the Pfirsch-Schlüter fluxes become

$$\Gamma^e_{\rm PS} = -2 q^2_{\rm eff} D^e_{\rm cl}\left(\frac{\partial n_0}{\partial r} + \frac{1}{4}\frac{n_0 \partial T}{T \partial r}\right), \qquad (69a)$$

$$\begin{pmatrix} q^e_{\rm PS} \\ q^i_{\rm PS} \end{pmatrix} = -2 q^2_{\rm eff} \begin{pmatrix} 0.825\kappa^e_{\rm cl} & -3\kappa^e_{\rm cl}/2 \\ \kappa^i_{\rm cl}/\sqrt{2} & 0 \end{pmatrix} \begin{pmatrix} \partial T/\partial r \\ T \partial \ln n_0/\partial r \end{pmatrix}. \qquad (69b)$$

There $D^e_{\rm cl} = \nu_{ei}(2T/m_e\Omega_e^2)$ is the classical diffusion coefficient and $\kappa^e_{\rm cl} = n_0\nu_{ei}(2T/m_e\Omega_e^2)$, $\kappa^i_{\rm cl} = n_0\nu_{ii}(2T/m_i\Omega_i^2)$ are the classical thermal conductivities.

These results of the Pfirsch-Schlüter fluxes have different numerical factors compared with those obtained in Ref. 10 for toroidal systems. This is because we have used a simplified form of friction-flow relations (Eqs. (65)), where only the first two velocity moments $(\mathbf{u}, \mathbf{q})$ are included. If we take one more velocity moment and then find the linear relation between it and the previous two, as was done in Ref. 1, we will find that in the low-collisionality regimes, Eqs. (65) is still correct, while in the collisional regimes (Pfirsch-Schlüter and classical), the coefficients of their parallel parts are different. However, observing that the geometric effect is completely separated from the kinetic one, we can simply take the results from Ref. 10 and replace the geometric factor by the effective safety factor to obtain (with ion charge number $Z_i = Z_{\rm eff} = 1$)

$$\Gamma^e_{\rm PS} = -2 q^2_{\rm eff} D^e_{\rm cl}\left(0.656\frac{\partial n_0}{\partial r} + 0.385\frac{n_0 \partial T}{T \partial r}\right), \qquad (70a)$$

$$\begin{pmatrix} q^e_{\rm PS} \\ q^i_{\rm PS} \end{pmatrix} = -2 q^2_{\rm eff} \begin{pmatrix} 0.431\kappa^e_{\rm cl} & -0.543\kappa^e_{\rm cl} \\ 0.566\kappa^i_{\rm cl} & 0 \end{pmatrix} \begin{pmatrix} \partial T/\partial r \\ T \partial \ln n_0/\partial r \end{pmatrix}. \qquad (70b)$$

These results are valid in the collisional regimes for arbitrary closed-end plasmas.

As an example, let us consider a toroidal plasma with the model magnetic field

$$B = B_0\left(1 + {\sum_{l,m}}' b_{lm}\cos(l\theta - m\zeta + c_{lm})\right), \tag{71}$$

where the prime means l, m cannot be zero at the same time. $|b_{lm}| \ll 1$ are functions of r only, they describe the inhomogeneity of the field. The phase factors c_{lm} are also functions of r only. Besides the magnitude of the field, we use an approximate field line equation,

$$\zeta = q\theta + \zeta_0, \tag{72}$$

where q is the safety factor and ζ_0 a constant. Using Eq. (18) with $G = \Psi = \pi r^2 B_0$, we obtain to $O(b_{lm})$

$$\mathcal{D} = -2\pi r\frac{q}{B_0}{\sum_{l,m}}' 2b_{lm}\frac{l + mr^2/qR_0^2}{(l - mq)}\cos(l\theta - m\zeta + c_{lm}), \tag{73}$$

where R_0 is the average major radius. Keeping terms to $O(b_{lm}^2)$ in Eq. (68) and applying Eq. (B6), we get

$$q_{\rm eff}^2 = q^2{\sum_{l,m}}'\left(\frac{R_0 b_{lm}}{r}\right)^2\left(\frac{l + mr^2/qR_0^2}{l - mq}\right)^2. \tag{74}$$

This result exhibits resonances at rational surfaces $l = mq$. From its derivation, we know that the origin of these resonances is the following: when we travel along the field lines on these rational surfaces, we see a constant rather than an alternating field gradient. This spatial secularity adds up to a resonance of infinity. However, remember that our field line equation (Eq. (72)) is approximate, the spatial secularity should not be there had we use the exact field line equation. Therefore, we expect that in practice resonances of finite values rather than infinity will occur. Finally, without the term mr^2/qR_0^2, which is usually small except for large m, equation (74) is the same as the geometric factor obtained by Boozer in Ref. 13.

VIII. AMBIPOLAR POTENTIAL AND PARALLEL FLOWS

If there are no external sources, the ambipolar condition Eq. (14) and the parallel momentum conservation Eq. (13) are

$$\sum_\alpha \langle \mathbf{D}\cdot\nabla\cdot\pi_\alpha\rangle = -\sum_\alpha \left\langle \delta P_\alpha \frac{\mathbf{D}\cdot\nabla B}{B}\right\rangle = 0, \tag{75}$$

$$\sum_\alpha \langle \mathbf{B}\cdot\nabla\cdot\pi_\alpha\rangle = -\sum_\alpha \langle \delta P_\alpha \mathbf{b}\cdot\nabla B\rangle = 0, \tag{76}$$

where the same arguments for obtaining Eq. (34) have been used. Now suppose in a certain collisionality regime δP and $\delta\Theta$ take the following forms for species α, (we will omit the subscript zero in P_0, n_0 and P, n are understood to be flux functions)

$$\delta P_\alpha = -3\left(\hat{\mu}_1^\alpha \mathbf{u}_{1\alpha} + \hat{\mu}_2^\alpha \frac{2\mathbf{q}_{1\alpha}}{5P_\alpha}\right)\cdot\mathbf{H}(\mathbf{x}), \tag{77a}$$

$$\delta\Theta_\alpha = -3\left(\hat{\mu}_2^\alpha \mathbf{u}_{1\alpha} + \hat{\mu}_3^\alpha \frac{2\mathbf{q}_{1\alpha}}{5P_\alpha}\right)\cdot\mathbf{H}(\mathbf{x}), \tag{77b}$$

where $\hat{\mu}_j^\alpha$ are flux functions that contain kinetic effects and $\mathbf{H}(\mathbf{x})$ is a *species-independent* vector field determined by the magnetic field geometry. Note that both equations have the same $\hat{\mu}_2^\alpha$ is a consequence of the self-adjointness of the Coulomb collision operator.[1] Plugging Eq. (77a) into Eq. (75) and using Eqs. (22), we get

$$-3c\sum_\alpha\left[\hat{\mu}_1\left(\frac{P'}{en}+\Phi_0'\right)+\hat{\mu}_2\frac{T'}{e}\right]_\alpha \left\langle \mathbf{D}\cdot\mathbf{H}\frac{\mathbf{D}\cdot\nabla B}{G'B}\right\rangle + 3\sum_\alpha\left[\hat{\mu}_1\hat{U}_\parallel+\hat{\mu}_2\frac{2\hat{Q}_\parallel}{5P}\right]_\alpha \left\langle \mathbf{B}\cdot\mathbf{H}\frac{\mathbf{D}\cdot\nabla B}{G'B}\right\rangle = 0, \tag{78}$$

where $[\,]_\alpha$ means that relevant terms inside the bracket are those of species α. For Eq. (76) we get an equation of the same form except that $\mathbf{D}\cdot\nabla B/B$ is changed to $\mathbf{B}\cdot\nabla B/B$. So we arrive at two mutually exclusive equations and since $\mathbf{D}\neq\mathbf{B}$, the only solutions are

$$\sum_\alpha\left[\hat{\mu}_1\left(\frac{P'}{en}+\Phi_0'\right)+\hat{\mu}_2\frac{T'}{e}\right]_\alpha = 0, \tag{79}$$

$$\sum_\alpha\left[\hat{\mu}_1\hat{U}_\parallel+\hat{\mu}_2\frac{2\hat{Q}_\parallel}{5P}\right]_\alpha = 0. \tag{80}$$

Therefore an ambipolar potential that is *independent of geometry* is obtained as follows

$$\Phi_0' = -\sum_\alpha \left[\hat{\mu}_1 \frac{P'}{en} + \hat{\mu}_2 \frac{T'}{e}\right]_\alpha \Big/ \sum_\alpha \hat{\mu}_1^\alpha . \tag{81}$$

This is a direct consequence of the particular form of Eq. (77a).

Next we specialize to the Pfirsch-Schlüter regime, in which Eqs. (77) become

$$\delta P = -\frac{3P_\alpha}{\nu_\alpha}\left(\mu_1 \mathbf{u}_1 + \mu_2 \frac{2\mathbf{q}_1}{5P}\right)_\alpha \cdot \frac{\nabla B}{B}, \tag{82a}$$

$$\delta \Theta = -\frac{3P_\alpha}{\nu_\alpha}\left(\mu_2 \mathbf{u}_1 + \mu_3 \frac{2\mathbf{q}_1}{5P}\right)_\alpha \cdot \frac{\nabla B}{B}, \tag{82b}$$

where ν_α are self-collision frequencies and μ_j are constants. For an electron-ion plasma, $\mu_1^e = 0.733, \mu_2^e = 1.51, \mu_3^e = 6.06$, and $\mu_1^i = 1.365, \mu_2^i = 2.31, \mu_3^i = 8.78$. Equations (82) have been derived in Ref. 2 for nonaxisymmetric toroidal systems but they are generally true for arbitrary systems. Note that now $\hat{\mu}_j = P\mu_j/\nu$ and $\mathbf{H} = \nabla B/B$. Since $\nu_e/\nu_i \sim \sqrt{m_i/m_e}$, we can ignore the electron term in Eq. (79) and obtain

$$\Phi_0' = -\left(\frac{P'}{en} + \frac{\mu_2 T'}{\mu_1 e}\right)_i . \tag{83}$$

Thus to zeroth order in $\sqrt{m_e/m_i}$, the ambipolar potential is completely determined by the kinetics of ions.

Having obtained Φ_0', we can use Eq. (80) together with the parallel balance equations (Eqs. (12)) to calculate the flux functions $\hat{U}_\parallel, \hat{Q}_\parallel$ of the first-order flows. An iterative method involving two small parameters will be used.[2] With $E_\parallel^A = K_{1\parallel} = K_{2\parallel} = 0$, Eqs. (12) now read

$$\left\langle BF_{1\parallel}\right\rangle = \langle \mathbf{B}\cdot\nabla\cdot\boldsymbol{\pi}\rangle = -\langle \delta P \mathbf{b}\cdot\nabla B\rangle , \tag{84a}$$

$$\left\langle BF_{2\parallel}\right\rangle = \langle \mathbf{B}\cdot\nabla\cdot\boldsymbol{\Theta}\rangle = -\langle \delta \Theta \mathbf{b}\cdot\nabla B\rangle . \tag{84b}$$

In the Pfirsch-Schlüter regime, $v_{t\alpha}^2/\nu_\alpha^2 L^2 \ll 1$ ($v_t^2 = 2T/m$ is the thermal velocity and L the longitudinal scale length such as the connection length qR_0), the right-hand sides of Eqs. (84) are much smaller than the left-hand sides and therefore to zeroth order in $v_{t\alpha}^2/\nu_\alpha^2 L^2$, we have

$$\left\langle BF_{1\|}\right\rangle = \left\langle BF_{2\|}\right\rangle = 0. \tag{85}$$

Equations (65) then yield (the difference in the numerical factors due to the omission of the third velocity moment will not matter in this order)

$$\left\langle u_{1\|}^e B\right\rangle = \left\langle u_{1\|}^i B\right\rangle, \tag{86a}$$

$$\left\langle q_{1\|}^e B\right\rangle = \left\langle q_{1\|}^i B\right\rangle = 0. \tag{86b}$$

Putting the parallel parts of Eqs. (4) into these two equations, we get

$$\hat{U}_\|^i - \hat{U}_\|^e = \frac{\langle \mathcal{D}B^2\rangle}{G'\langle B^2\rangle}\frac{c}{|e|n}P'_{i+e}, \tag{87a}$$

$$\hat{Q}_\|^\alpha = \frac{\langle \mathcal{D}B^2\rangle}{G'\langle B^2\rangle}\left(\frac{5c}{2e}PT'\right)_\alpha, \tag{87b}$$

where $P_{i+e} \equiv P_i + P_e$ and $\alpha = e, i$. Furthermore, the zeroth-order approximation (ignoring the electron term) of Eq. (80) gives

$$\begin{aligned}\hat{U}_\|^i &= -\left(\frac{\mu_2 2\hat{Q}_\|}{\mu_1 5P}\right)_i \\ &= -\frac{\langle \mathcal{D}B^2\rangle}{G'\langle B^2\rangle}\left(\frac{\mu_2 c}{\mu_1 e}T'\right)_i,\end{aligned} \tag{88}$$

$$\hat{U}_\|^e = -\frac{\langle \mathcal{D}B^2\rangle}{G'\langle B^2\rangle}\left[\left(\frac{\mu_2 c}{\mu_1 e}T'\right)_i + \frac{c}{|e|n}P'_{i+e}\right]. \tag{89}$$

Therefore, by expanding the two small parameters $\sqrt{m_e/m_i}$ and $v_{t\alpha}^2/\nu_\alpha^2 L^2$, we have completely determined the first-order flows.

Substituting the above results into δP_e with $n = n_e = n_i$, $|e| = e_i = -e_e$ yields the first-order $\langle \mathbf{D}\cdot\nabla\cdot\pi_\alpha\rangle$, which are

$$\begin{aligned}\langle \mathbf{D}\cdot\nabla\cdot\pi_i\rangle &= -\langle \mathbf{D}\cdot\nabla\cdot\pi_e\rangle \\ &= -\frac{3P_e c}{\nu_e|e|}\left[\mu_1^e\left(\frac{P'_{i+e}}{n} + \frac{\mu_2^i}{\mu_1^i}T'_i\right) + \mu_2^e T'_e\right]G_{\mathrm{PS}},\end{aligned} \tag{90}$$

where the geometric factor is

$$G_{\text{PS}}(\Psi) = \left\langle \frac{\mathbf{D}\cdot\nabla B}{G'B}\mathbf{D}\cdot\nabla \ln B \right\rangle - \frac{\langle \mathcal{D}B^2\rangle}{G'\langle B^2\rangle}\langle \mathbf{b}\cdot\nabla B\mathbf{D}\cdot\nabla \ln B\rangle . \tag{91}$$

Similarly,

$$\langle \mathbf{D}\cdot\nabla\cdot\mathbf{\Theta}_e\rangle = \frac{3P_e c}{\nu_e|e|}\left[\mu_2^e\left(\frac{P'_{i+e}}{n} + \frac{\mu_2^i}{\mu_1^i}T_i'\right) + \mu_3^e T_e'\right] G_{\text{PS}}, \tag{92}$$

$$\langle \mathbf{D}\cdot\nabla\cdot\mathbf{\Theta}_i\rangle = -\frac{3P_i c}{\nu_i|e|}\left(\mu_2^i\frac{\mu_2^i}{\mu_1^i} - \mu_3^i\right) T_i' G_{\text{PS}}. \tag{93}$$

The expressions for $\langle \mathbf{B}\cdot\nabla\cdot\boldsymbol{\pi}\rangle$ and $\langle \mathbf{B}\cdot\nabla\cdot\mathbf{\Theta}\rangle$ are the same except that $\mathbf{D}\cdot\nabla \ln B$ is changed to $\mathbf{b}\cdot\nabla B$ inside G_{PS}.

With these results, it is straightforward to calculate the nonaxisymmetric fluxes. The banana-plateau fluxes can be obtained from

$$\Gamma_{\text{bp}} = -\frac{c}{eG'}\frac{\langle \mathcal{D}B^2\rangle}{\langle B^2\rangle}\langle \mathbf{B}\cdot\nabla\cdot\boldsymbol{\pi}\rangle ,$$

$$\frac{q_{\text{bp}}}{T} = -\frac{c}{eG'}\frac{\langle \mathcal{D}B^2\rangle}{\langle B^2\rangle}\langle \mathbf{B}\cdot\nabla\cdot\mathbf{\Theta}\rangle .$$

If Eq. (18) is used to calculate $\mathcal{D}$, then in general the first term of G_{PS} will dominate, since $\mathcal{D}$ itself contains the gradient of B. Thus we can estimate the relative sizes of different fluxes as follows

$$\Gamma_{\text{PS}} : \Gamma_{\text{na}} : \Gamma_{\text{bp}} \sim 1 : (v_{te}/\nu_e L)^2 : (v_{te}/\nu_e L)^2(r/L).$$

Note that this remains true for axisymmetric toroidal plasmas if $\mathcal{D}$ is calculated from Eq. (18). But if we choose $\mathbf{D} = R^2\nabla\zeta$, then $\Gamma_{\text{na}} = 0$ and $\mathcal{D}$ is no longer proportional to ∇B. We have in this case $\Gamma_{\text{PS}} : \Gamma_{\text{bp}} \sim 1 : (v_{te}/\nu_e qR_0)^2$. The sum $\Gamma_{\text{na}} + \Gamma_{\text{bp}}$ is of course independent of the choice of $\mathbf{D}$. Therefore, in the Pfirsch-Schlüter regime, the Pfirsch-Schlüter fluxes calculated in Sec. VII always dominate, no matter what shapes the closed-end plasmas take.

IX. CONCLUSION

Flux-friction relations have been derived by following the moment approach of neoclassical transport theory. The introduction of the vector field **D** enables us to extract the geometric effects from the kinetic ones and the formulas obtained thereby are applicable to arbitrary closed-end plasmas. This advantage of the moment approach—separation of geometric effects from the kinetic one—is most evident in the Pfirsch-Schlüter fluxes, for which we can define an effective safety factor that contains all the geometric information. For a model toroidal magnetic field (Eq. (71)), this effective safety factor exhibits resonances at certain rational surfaces. These resonances are due to spatial secularities of the field gradient and will actually be finite in practice.

For regimes in which the geometric and kinetic effects are separable, the ambipolar condition and the parallel momentum conservaton together determine a general ambipolar potential that is independent of the magnetic field geometry. First-order parallel flows can be obtained by solving the parallel balance equations iteratively, as have been illustrated in Sec. VIII. In the Pfirsch-Schlüter regime, the nonaxisymmetric and banana-plateau fluxes associated with these flows are in general negligible.

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Appendix A. Gerneral Coordinate System

The general coordinates (x^1, x^2, x^3) are employed to describe closed-end plasmas. x^1 labels the flux surfaces, x^2 measures the poloidal angle, and x^3 specifies the longitudinal position. Note that any physical quantities should be periodic in x^2 and x^3. The contravariant bases are $\mathbf{e}^i = \nabla x^i$, from which we generate the covariant bases $\mathbf{e}_i$, and

$$\mathbf{e}_i = \epsilon_{ijk}\mathbf{e}^j\times\mathbf{e}^k\sqrt{g}, \tag{A1a}$$

$$\mathbf{e}^i = \epsilon^{ijk}\mathbf{e}_j\times\mathbf{e}_k/\sqrt{g}, \tag{A1b}$$

where the Jacobian is

$$\sqrt{g} = (\nabla x^1\cdot\nabla x^2\times\nabla x^3)^{-1}. \tag{A2}$$

The volume element is

$$d\mathbf{x} = \sqrt{g}dx^1dx^2dx^3. \tag{A3}$$

Suppose we have a divergence-free vector field $\mathbf{A}$ that lies within flux surfaces, i.e.,

$$\nabla\cdot\mathbf{A} = 0, \tag{A4a}$$

$$\mathbf{A}\cdot\nabla x^1 = 0, \tag{A4b}$$

then we can construct a contravariant representation for $\mathbf{A}$[14]

$$\begin{aligned}\mathbf{A} &= \chi'(x^1)\nabla x^3\times\nabla x^1 + \psi'(x^1)\nabla x^1\times\nabla x^2 \\ &= A^2\mathbf{e}_2 + A^3\mathbf{e}_3.\end{aligned} \tag{A5}$$

where $A^2 = \chi'/\sqrt{g}, A^3 = \psi'/\sqrt{g}$. A^1 is manifestly zero due to Eq. (A4b). The flux functions χ and ψ are related to the poloidal and longitudinal fluxes of $\mathbf{A}$ by

$$\chi = \frac{\Psi_p}{L} = \frac{\int\mathbf{A}\cdot d\mathbf{S}_p}{L}, \tag{A6}$$

$$\psi = \frac{\Psi_L}{2\pi} = \frac{\int\mathbf{A}\cdot d\mathbf{S}_L}{2\pi}. \tag{A7}$$

The field line equations are

$$\frac{d\ell}{A} = \frac{dx^i}{A^i}. \tag{A8}$$

Vector fields that have these geometric structures are $\mathbf{B}, \mathbf{D}, \mathbf{u}_1, \mathbf{q}_1$, and $\mathbf{J}$.

Appendix B. Flux Surface Average

The flux surface average $\langle\,\rangle$ is defined by[10]

$$\langle f \rangle \triangleq \int_{\Delta V} f d\mathbf{x} \bigg/ \int_{\Delta V} d\mathbf{x}\,, \tag{B1}$$

where ΔV is the volume between two neighboring flux surfaces. Labeling the flux surfaces by Ψ, we can derive from the above definition

$$\langle f \rangle = \frac{d\Psi}{dV} \int \frac{f dS}{|\nabla\Psi|}, \tag{B2}$$

If $\mathbf{A}$ lies within flux surfaces, i.e., $\mathbf{A}\cdot\nabla\Psi = 0$, then Gauss' law readily yields

$$\langle \nabla\cdot\mathbf{A} \rangle = 0, \tag{B3}$$

$$\langle f \nabla\cdot\mathbf{A} \rangle = -\langle \mathbf{A}\cdot\nabla f \rangle\,. \tag{B4}$$

Furthermore, if $\nabla\cdot\mathbf{A} = 0$, it follows that

$$\langle \mathbf{A}\cdot\nabla f \rangle = 0. \tag{B5}$$

Based on a divergence-free and on-surface vector field $\mathbf{A}$, another representation of $\langle\,\rangle$ can be established. Setting $x^1 = \Psi$ makes $\mathbf{A}$ satisfies Eqs. (A4). Then the use of Eqs. (A3,A5,A8) yields

$$\langle f \rangle = \frac{d\Psi_L}{dV} \int_0^{2\pi} \frac{dx^2}{2\pi} \int_0^L \frac{d\ell}{A} f, \tag{B6}$$

where $\Psi_L = 2\pi\psi$ is the longitudinal flux. For ergodic field lines, we can do the poloidal average by following the field line N circuits around the system. This yields

$$\langle f \rangle = \frac{d\Psi_L}{dV} \lim_{N\to\infty} \frac{1}{N} \int_0^{NL} \frac{d\ell}{A} f. \tag{B7}$$

A special case of this formula is

$$\lim_{N\to\infty} \frac{1}{N} \int_0^{NL} \frac{d\ell}{A} = \frac{dV}{d\Psi_L}, \tag{B8}$$

which relates the line integral to the specific volume $dV/d\Psi_L$.

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