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KINETIC STABILITY PROPERTIES OF AN INTENSE RELATIVISTIC ELECTRON RING IN A HIGH-CURRENT BETATRON ACCELERATOR

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The kinetic stability properties of an intense relativistic electron ring located at the midplane of an externally applied betatron field are investigated within the framework of the linearized Vlasov-Maxwell equations, including the important influence of electromagnetic effects and surface-wave perturbations. Stability properties are calculated for eigenfrequency ω near harmonics of the relativistic cyclotron frequency ω_{cz} in the applied betatron field. Making use of the large-aspect-ratio assumption ($R_0 >> a_c$), a closed algebraic dispersion relation is obtained for the longitudinal instability, assuming that the electron ring is located inside a perfectly conducting toroidal shell. Several points are noteworthy in this analysis. First, transverse electromagnetic effects can provide complete stabilization provided the ring current is sufficiently large. Second, for the case where the betatron focussing force exceeds the self-field defocussing force ($\mu \equiv \omega_{cz}^2/\omega_\beta^2 - 1/\gamma_b^2 > 0$), it is found that stabilization occurs at sufficiently low transverse temperature of the beam electrons. Third, for the case where $\mu < 0$ and the transverse temperature of the beam electrons is sufficiently low, it is found that surface perturbations on the electron beam drive a radial kink instability.

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1. INTRODUCTION

There is a growing literature[1-16] on the equilibrium and stability properties of intense relativistic electron rings with applications to high-current betatron accelerators, including the conventional high-current betatron as well as the modified betatron (Fig. 1). An intense relativistic electron ring is likely subject to various macro- and micro-instabilities.[10-16] The most deleterious instabilities appear to be associated with the class of longitudinal instabilities, including the negative-mass and resistive-wall instabilities.[13-16] The majority of previous analyses of the longitudinal stability properties of an intense electron ring have been carried out within the framework of a rigid-beam model.[15,16] In a recent calculation,[14] we developed a kinetic formalism describing longitudinal instabilities in a relativistic electron ring, including the important influence of finite beam temperature, wall resistivity, and self-field effects on stability behavior. Strictly speaking, a more accurate theoretical analysis is required to incorporate the full influence of electromagnetic effects and surface perturbations on longitudinal stability properties. In this regard, the present article extends the previous kinetic treatment[14] to a broader range of system parameters, eliminating various restrictive assumptions, and incorporating the important influence of electromagnetic effects and surface-wave perturbations. In particular, the present analysis allows for kink-like perturbations with $\partial/\partial\Phi \neq 0$ in Fig. 1.

The equilibrium properties and basic assumptions are briefly summarized in Sec. 2. The eigenvalue equation is derived in Sec. 3, making use of the linearized Vlasov-Maxwell equations. The analysis is fully electromagnetic. Moreover, in obtaining the perturbed distribution function, and subsequently the perturbed charge and current densities, we take toroidal effects into account. In particular, the radial variation of the azimuthal electron velocity produces surface-charge and surface-current perturbations corresponding to a kink-type perturbation with $\partial/\partial\Phi \neq 0$. In order to simplify the expression for the perturbed distribution function, we assume that the eigenfrequency ω is approximately equal to a harmonic of the electron cyclotron frequency ω_{cz} in the applied betatron field. In the limit of large aspect ratio, the eigenvalue equation [Eq. (45)] is solved analytically in Sec. 4, leading to a closed algebraic dispersion relation [Eq. (66)] for the complex eigenfrequency ω.

Detailed stability properties are investigated in Sec. 5, with particular emphasis on the influence of electromagnetic effects and surface-wave perturbations on stability behavior. In order to illustrate the strong stabilizing influence of electromagnetic effects, we consider the case where the conducting wall is located very close to the surface of the electron beam ($a_c \tilde{\approx} a$), and the surface-wave contributions in the dispersion relation (66) can be neglected (Sec. 5.A). In a parameter regime corresponding to the conventional negative-mass instability ($\mu = \omega_{cz}^2/\omega_\beta^2 - 1/\gamma_b^2 > 0$), it is found that electromagnetic effects can lead to complete stabilization of the instability provided the beam current is above some critical value [Eq. (75)]. The important stabilizing influence of electromagnetic effects is found for both conventional and modified betatron accelerators.

To clarify the terminology used in the preceding paragraph, it is clear that the terms proportional to ω/ck in Eq. (66) are related to electromagnetic effects. In this regard, it is customary in conventional treatments of longitudinal stability properties to approximate terms such as $1 - \beta_b\omega/ck$ by $1 - \beta_b^2$, where use is made of $\omega \simeq \ell\omega_{cz} = k\beta_b c$. Here, $\beta_b c = R_0\omega_{cz}$ is the mean azimuthal velocity, and $k = \ell/R_0$. To be more precise, we should express

$$1 - \beta_b \frac{\omega}{ck} = \left(1 - \beta_b^2\right) - \beta_b\left(\frac{\omega - \ell\omega_{cz}}{ck}\right),$$

which is the procedure followed in analyzing the dispersion relation in Sec. 5.

Indeed, retaining the contributions proportional to $\beta_b\chi = \beta_b(\omega-\ell\omega_{cz})/ck$, it is found that the inclusion of the concommitant "electromagnetic effects" can have a large influence on detailed stability properties as summarized in the previous paragraph.

The effects of surface-wave perturbations on stability behavior are also investigated (Sec. 5.B), including the surface-wave contribution in the dispersion relation (66), but (arbitrarily) turning off the (stabilizing) electromagnetic contribution. Two points are noteworthy from the stability analysis. First, in the case where the focussing force of the betatron field exceeds the self-field defocussing force, it is found that the negative-mass instability for the modified betatron is stabilized by reducing the effective transverse temperature of the beam electrons to sufficiently low values. For the conventional betatron, it is shown that there is a range of parameters for which the system is stable. The stability criterion is easily satisfied provided the electron density is sufficiently large [Eqs. (82) and (83)]. This stabilization results from the inclusion of surface-wave perturbations. Second, in the case where the betatron focussing force is less than the defocussing self-field force, conventional theory predicts that the system is stable. However, by reducing the effective transverse temperature to sufficiently low values, the present analysis (including surface-wave perturbations) predicts that the system is unstable. The instability, which originates from the surface-wave contribution in the eigenvalue equation, corresponds to a radial kink instability.[16] Although the radial kink instability can be derived from fluid or rigid-beam descriptions, only a kinetic model based on the Vlasov-Maxwell equations can accurately predict detailed stability behavior because of the sensitive dependence on transverse thermal effects.

2. EQUILIBRIUM PROPERTIES AND BASIC ASSUMPTIONS

As illustrated in Fig. 1, the equilibrium configuration consists of a relativistic electron ring located at the midplane of an externally applied betatron field $B_{0r}^{ext}(r,z)\hat{\underset{\sim}{e}}_r + B_{0z}^{ext}(r,z)\hat{\underset{\sim}{e}}_z$. In addition, the electron ring is located inside a toroidal conductor with minor radius a_c. An externally applied toroidal magnetic field $B_{0\theta}^{ext}\hat{\underset{\sim}{e}}_\theta$, together with the betatron field, act to confine the ring both axially and radially. Here, $\hat{\underset{\sim}{e}}_r$, $\hat{\underset{\sim}{e}}_\theta$, and $\hat{\underset{\sim}{e}}_z$ are unit vectors in the r-, θ-, and z-directions, respectively. For $B_{0\theta}^{ext} = 0$, we recover the conventional betatron configuration. The equilibrium radius of the electron ring is denoted by R_0 and the minor dimensions of the ring are denoted by 2a (radial dimension) and 2b (axial dimension), respectively. In addition to the cylindrical polar coordinates (r,θ,z), we also introduce the toroidal polar coordinate system (ρ,Φ,θ) defined by

$$r-R_0 = \rho\cos\Phi, \quad z = \rho\sin\Phi , \tag{1}$$

where ρ is measured from the equilibrium radius R_0. The characteristic mean azimuthal velocity of the ring electrons ($V_{\theta b}^0=\beta_b c$) is in the positive θ-direction, which produces a self-magnetic field $\underset{\sim}{B}_0^s(\underset{\sim}{x})$. The ring is also assumed to be partially charge neutralized by a positive ion background with fractional charge neutralization f. That is, $n_i^0(r,z) = fn_b^0(r,z)$, where $n_b^0(r,z)$ and $n_i^0(r,z)$ are the equilibrium electron and ion densities, respectively, and f = const. = fractional charge neutralization.

To make the theoretical analysis tractable, we make the following simplifying assumptions in describing the electron ring equilibrium by the steady-state $(\partial/\partial t=0)$ Vlasov-Maxwell equations.

(a) The minor dimensions of the electron ring are much smaller than its major radius, i.e.,

$$a,b \ll R_0 \ . \tag{2}$$

To further simplify the analysis, it is also assumed that the minor cross section of the electron ring is circular with a=b, which is consistent provided the external field index n satisfies

$$n = 1/2, \tag{3}$$

where $n=-[r\partial \ell n B_{0z}^{ext}(r,z)/\partial r]_{(R_0,0)}$.

(b) Consistent with Eq. (2), it is also assumed that the transverse (r,z) kinetic energy of an electron is small in comparison with the characteristic azimuthal energy $\gamma_b mc^2$.

(c) The maximum spread in canonical angular momentum δP_θ is assumed to be small with $|\delta P_\theta| \ll \gamma_b m\beta_b cR_0$.

(d) Finally, it is assumed that

$$\frac{\nu}{\gamma_b} = \frac{N_e}{2\pi R_0}\frac{e^2}{mc^2}\frac{1}{\gamma_b} \ll 1 \ , \tag{4}$$

where $\nu=(N_e/2\pi R_0)(e^2/mc^2)$ is Budker's parameter, N_e is the total number of electrons in the ring and e^2/mc^2 is the classical electron radius. For further discussion of the basic assumptions used in the present analysis, the reader is referred to Ref. 14.

For azimuthally symmetric equilibria ($\partial/\partial\theta=0$) with both r- and z-dependence, there are two exact single-particle constants of motion in the equilibrium field configuration. These are the total energy H,

$$H = (m^2c^4+c^2\underset{\sim}{p}^2)^{1/2} - e\phi_0(r,z) \ , \tag{5}$$

and the canonical angular momentum P_θ,

$$P_\theta = r[p_\theta - (e/c)A_\theta^0(r,z)] \ , \tag{6}$$

where $\underset{\sim}{p}=\gamma m\underset{\sim}{v}$ is the mechanical momentum, $\phi_0(r,z)$ is the equilibrium electrostatic potential, $-e$ and m are the electron charge and rest mass, respectively, c is the speed of light in vacuo, and $A_\theta^0(r,z)$ is the θ-component of the equilibrium vector potential. Within the context of Eqs. (2) and (3), it can be shown[14] that the canonical angular momentum

$$P_\Phi = \rho p_\Phi - (e/c)\hat{B}_\theta\rho^2 \ , \tag{7}$$

in the plane perpendicular to the toroidal magnetic field $B_{0\theta}^{ext}\hat{\underset{\sim}{e}}_\theta$ is an approximate single-particle invariant for a thin, circular electron ring with $a=b$ and $a<<R_0$. In Eq. (7), $\hat{B}_\theta$ is defined by $\hat{B}_\theta=B_{0\theta}^{ext}(R_0,0)$.

For present purposes, we consider the electron distribution function specified by[14]

$$f_b^0(H,P_\Phi,P_\theta) = \frac{\hat{n}_bR_0\Delta}{2\pi^2\gamma_bm}\,\frac{\delta(H-\omega_bP_\Phi-\hat{\gamma}mc^2)}{(P_\theta-P_0)^2+\Delta^2} \ , \tag{8}$$

where $\hat{n}_b=n_b^0(R_0,0)$ is the electron density at the equilibrium orbit $(r,z) = (R_0,0)$, ω_b=const. is the angular velocity of mean rotation in the Φ-direction, Δ is the characteristic spread in the canonical angular momentum P_θ, and $\hat{\gamma}$ is a constant. After some straightforward algebra, it can be shown that the combination $H-\omega_bP_\Phi$ occurring in Eq. (8) can be expressed in the approximate form[14]

$$H - \omega_b P_\Phi = \gamma_b mc^2 + p_\perp^2/2\gamma_b m + \psi(r,z) \ . \tag{9}$$

Here, $p_\perp^2 = p_\theta^2 + (p_\Phi - \gamma_b m\omega_b\rho)^2$ is the transverse momentum-squared in a frame of reference rotating with angular velocity ω_b about the toroidal axis. The envelope function $\psi(r,z)$ in Eq. (9) is defined by[14]

$$\psi(r,z) = \frac{1}{2}\gamma_b m\Omega_\beta^2\rho^2 \ , \tag{10}$$

for a thin ring with circular cross section (a=b). Moreover, the effective focussing frequency Ω_β^2 is defined by[14]

$$\Omega_\beta^2 = \omega_b\omega_{c\theta} - \omega_b^2 + \omega_\beta^2 \ , \tag{11}$$

where

$$\omega_\beta^2 = \frac{1}{2}\omega_{cz}^2 + \frac{1}{2}\omega_{pb}^2[\beta_b^2 - (1-f)] \ . \tag{12}$$

Here, $\omega_{pb}^2 = 4\pi\hat{n}_b e^2/\gamma_b m$ is the relativistic plasma frequency-squared of the beam electrons at $(r,z) = (R_0,0)$, and $\omega_{cz} = eB_{0z}^{ext}(R_0,0)/\gamma_b mc$ and $\omega_{c\theta} = e\hat{B}_\theta/\gamma_b mc$ are the relativistic cyclotron frequencies in the betatron and toroidal magnetic fields at the equilibrium orbit $(r,z) = (R_0,0)$.

Substituting Eq. (9) into Eq. (8) and evaluating the electron density profile $n_b^0(r,z) = \int d^3p\ f_b^0$, we find

$$n_b^0(r,z) = \hat{n}_b U(a-\rho) \ , \tag{13}$$

where $a=[2(\hat{\gamma}-\gamma_b)c^2/\gamma_b\Omega_\beta^2]^{1/2}$, and U(x) is the Heavisides step function defined by U(x) = +1 for x>0, and U(x)=0 for x<0. For the equilibrium to exist, it is necessary that $\hat{\gamma}>\gamma_b$ and $\Omega_\beta^2>0$. The condition $\Omega_\beta^2>0$ can be expressed in the equivalent form

$$\omega_b^- < \omega_b < \omega_b^+ \tag{14}$$

where the laminar rotation frequencies $\omega_b^\pm$ are defined by

$$\omega_b^\pm = \frac{\omega_{c\theta}}{2}\left[1 \pm \left(1 + \frac{4\omega_\beta^2}{\omega_{c\theta}^2}\right)^{1/2}\right] . \tag{15}$$

Making use of the definitions of $\omega_b^\pm$ in Eq. (15), the effective focussing frequency Ω_β defined in Eq. (11) can be expressed in the equivalent form

$$\Omega_\beta^2 = (\omega_b^+ - \omega_b)(\omega_b - \omega_b^-) . \tag{16}$$

It can be shown that the equilibrium pressure tensor in the (ρ,Φ) plane perpendicular to the θ-direction is isotropic with the perpendicular pressure $P_\perp^0(\rho) = n_b^0(\rho)T_\perp^0(\rho)$ given by

$$n_b^0(\rho)T_\perp^0(\rho) = 2\pi \int_0^\infty dp_\perp p_\perp \int_{-\infty}^\infty dp_\theta \, \frac{p_\rho^2 + (p_\Phi - \gamma_b m\rho\omega_b)^2}{2\gamma_b m} f_b^0 , \tag{17}$$

where $T_\perp^0(p)$ is the effective transverse temperature profile. Defining

$$\hat{T}_\perp = \frac{1}{2}\gamma_b m\Omega_\beta^2 a^2 = \frac{1}{2}\gamma_b m\omega_{c\theta}^2 r_L^2 , \tag{18}$$

and substituting Eq. (8) into Eq. (17) gives

$$T_\perp^0(\rho) = \hat{T}_\perp(1-\rho^2/a^2) , \tag{19}$$

for $0 \leq \rho \leq a$. In Eq. (18), r_L is the characteristic thermal Larmor radius of the ring electrons in the azimuthal magnetic field $\hat{B}_\theta$. Making use of Eq. (18) to eliminate Ω_β in Eq. (11) in favor of r_L, we solve Eq. (11) for the rotation frequency ω_b and obtain

$$\omega_b = \hat{\omega}_b^{\pm} \equiv \frac{\omega_{c\theta}}{2}\left\{1 \pm \left[1 + \frac{4\omega_\beta^2}{\omega_{c\theta}^2} - \left(\frac{2r_L}{a}\right)^2\right]^{1/2}\right\}, \tag{20}$$

which relates ω_b to the thermal Larmor radius r_L. The two signs (±) in Eq. (20) represent <u>fast</u> (+) and <u>slow</u> (-) rotational equilibria. Whenever the Larmor radius r_L approaches zero ($r_L/a \to 0$), the rotation frequencies defined in Eq. (20) approach the laminar (cold-fluid) rotation frequencies defined in Eq. (15).

3. LINEARIZED VLASOV-MAXWELL EQUATIONS

In this section, we make use of the linearized Vlasov-Maxwell equation to investigate electromagnetic stability properties of the equilibrium ring configuration discussed in Sec. II. In the stability analysis, a normal-mode approach is adopted in which all perturbed quantities are assumed to vary according to

$$\delta\psi(\mathbf{x},t) = \delta\hat{\psi}(r,z)\exp\{i(\ell\theta-\omega t)\} ,$$

where $\mathrm{Im}\,\omega > 0$. Here, ω is the complex oscillation frequency and ℓ is the toroidal harmonic number. Integrating the linearized Vlasov equation from $t'=-\infty$ to $t'=t$ and neglecting initial perturbations, we find that the perturbed distribution function can be expressed as

$$\delta f_b(\mathbf{x},\mathbf{p},t) = \delta\hat{f}_b(\mathbf{x},\mathbf{p})\exp(-i\omega t),$$

where

$$\delta\hat{f}_b(\mathbf{x},\mathbf{p}) = e\int_{-\infty}^{0} d\tau\,\exp(-i\omega\tau) \times\left\{\delta\hat{\mathbf{E}}(\mathbf{x}') + (1/c)\mathbf{v}' \times \delta\hat{\mathbf{B}}(\mathbf{x}')\right\}\cdot\frac{\partial}{\partial\mathbf{p}'} f_b^0(\mathbf{x}',\mathbf{p}') . \tag{21}$$

In Eq. (21), $\tau = t'-t$, $\delta\hat{\mathbf{E}}(\mathbf{x})$ and $\delta\hat{\mathbf{B}}(\mathbf{x})$ are the perturbed electromagnetic field amplitudes, and the particle trajectories $\mathbf{x}'(t')$ and $\mathbf{v}'(t')$ satisfy $d\mathbf{x}'/dt' = \mathbf{v}'$ and $d\mathbf{p}'/dt' = -e(\mathbf{E}^0+\mathbf{v}'\times\mathbf{B}^0/c)$ with initial conditions $\mathbf{x}'(t'=t) = \mathbf{x}$ and $\mathbf{v}'(t'=t) = \mathbf{v}$. The Maxwell equations for $\delta\hat{\mathbf{E}}(\mathbf{x})$ and $\delta\hat{\mathbf{B}}(\mathbf{x})$ are given by

$$\nabla\times\delta\hat{\mathbf{E}} = (i\omega/c)\delta\hat{\mathbf{B}} \ ,$$

$$\nabla\times\delta\hat{\mathbf{B}} = (4\pi/c)\delta\hat{\mathbf{J}} - (i\omega/c)\delta\hat{\mathbf{E}} \ , \tag{22}$$

where $\delta\hat{\mathbf{J}} = -e\int d^3p\mathbf{v}\delta\hat{f}_b$ is the perturbed current density. From Eq. (22), it is readily shown that

$$\left(\nabla^2 + \frac{\omega^2}{c^2}\right)\delta\hat{\mathbf{E}} = 4\pi[\nabla\delta\hat{\rho} - i(\omega/c^2)\delta\hat{\mathbf{J}}] \ , \tag{23}$$

which is the form of Maxwell's equations used in the present stability analysis. In Eq. (23), $\delta\hat{\rho} = -e\int d^3p\delta\hat{f}_b$ is the perturbed charge density.

For present purposes, we consider the case where the toroidal conductor (radius $\rho=a_c$) has large aspect ratio with

$$a_c << R_0 \ . \tag{24}$$

It is further assumed that $\mathrm{Re}\omega \simeq \ell\omega_{cz}$, and that the waves are far removed from resonance with the transverse (r,z) motion of the electrons. That is, it is assumed that[14]

$$\left|\left(\frac{\omega_b^{\pm}}{\omega-\ell\omega_{cz}}\right)^2 - 1\right| \ , \ \left|\frac{\omega_b^{\pm}}{\omega}\right| >> \frac{a}{R_0} \ , \tag{25}$$

where $\omega_b^{\pm}$ are the characteristic (r,z) orbital oscillation frequencies defined in Eq. (15). To lowest order, consistent with Eq. (25), the azimuthal and radial orbits can be approximated by[14]

$$\theta' = \theta + (\omega_{cz} - \mu\delta P_\theta/\gamma_b mR_0^2)\tau \ , \tag{26}$$

$$r' = R_0 + (\omega_{cz}/\gamma_b m\omega_\beta^2 R_0)\delta P_\theta \ ,$$

where $\delta P_\theta = P_\theta - P_0$, and the oscillatory contributions (at frequencies $\omega_b^\pm$) have been neglected in Eq. (26). Here, the negative-mass parameter μ is defined by

$$\mu = \omega_{cz}^2/\omega_\beta^2 - 1/\gamma_b^2 \ , \tag{27}$$

where $\omega_\beta^2 = \omega_{cz}^2/2 + (\omega_{pb}^2/2)[\beta_b^2-(1-f)]$. Within the context of Eq. (25), and the assumption of a thin electron ring, we approximate $\delta\hat{\underset{\sim}{E}}(\underset{\sim}{x}') = \delta\hat{E}_\theta(r',z')\exp(i\ell\theta')\hat{\underset{\sim}{e}}'_\theta$ and $\delta\hat{\underset{\sim}{B}}(\underset{\sim}{x}') = (-ic/\omega)\{-(\partial/\partial z')\delta\hat{E}_\theta(r',z')\hat{\underset{\sim}{e}}'_r + (r')^{-1}(\partial/\partial r')[r'\delta\hat{E}_\theta(r',z')]\hat{\underset{\sim}{e}}_z\} \times \exp(i\ell\theta')$ on the right-hand side of Eq. (21).[14] In this regard, the perturbed electromagnetic force in Eq. (21) is approximated by

$$\begin{aligned} -e\left[\delta\hat{\underset{\sim}{E}}(\underset{\sim}{x}') + \frac{\underset{\sim}{v}'\times\delta\hat{\underset{\sim}{B}}(\underset{\sim}{x}')}{c}\right] &= -e\,\exp(i\ell\theta') \\ &\times\left\{-\frac{iv'_\theta}{\omega}\frac{1}{r'}\frac{\partial}{\partial r'}(r'\delta\hat{E}_\theta)\hat{\underset{\sim}{e}}'_r + \left[\delta\hat{E}_\theta + \frac{i}{\omega}\left(v'_z\frac{\partial}{\partial z'}\delta\hat{E}_\theta\right.\right.\right. \\ &\left.\left.\left.+\frac{v'_r}{r'}\frac{\partial}{\partial r'}(r'\delta\hat{E}_\theta)\right)\right]\hat{\underset{\sim}{e}}'_\theta - \frac{iv'_\theta}{\omega}\frac{\partial}{\partial z'}\delta\hat{E}_\theta\hat{\underset{\sim}{e}}_z\right\} \ , \end{aligned} \tag{28}$$

where $\delta\hat{E}_\theta = \delta\hat{E}_\theta(r',z')$. Substituting Eq. (28) into Eq. (21) and making use of the identity $\partial f_b^0/\partial\underset{\sim}{p}' = \underset{\sim}{v}'(\partial f_b^0/\partial H) + r'\hat{\underset{\sim}{e}}'_\theta(\partial f_b^0/\partial P_\theta)$, it is straightforward to show that the perturbed distribution function for the ℓ'th harmonic component can be expressed as

$$\delta\hat{f}_{b\ell}(r,z,\underset{\sim}{p}) = e\frac{\partial f_b^0}{\partial P_\theta}\int_{-\infty}^{0} d\tau\ r'\left\{\delta\hat{E}_\theta\right.$$

$$+ \frac{i}{\omega}\left[v_z' \frac{\partial}{\partial z'} \delta\hat{E}_\theta + v_r' \frac{1}{r'}\frac{\partial}{\partial r'}(r'\delta\hat{E}_\theta)\right]\Bigg\} \exp\left[i\ell(\theta'-\theta)-i\omega\tau\right] \tag{29}$$

$$+ e\frac{\partial f_b^0}{\partial H}\int_{-\infty}^{0} d\tau v_\theta' \delta\hat{E}_\theta \exp[i\ell(\theta'-\theta)-i\omega\tau] ,$$

where θ' and r' are defined in Eq. (26), and use has been made of the fact that $\partial f_b^0/\partial H$ and $\partial f_b^0/\partial P_\theta$ are independent of t' (i.e., constant along a particle trajectory in the equilibrium field configuration). Making use of the identity

$$\frac{d}{d\tau}\left[r'\delta\hat{E}_\theta(r',z')\exp(i\ell\theta'-i\omega\tau)\right]$$

$$= \exp(i\ell\theta'-i\omega\tau)r'\left\{-i\omega\delta\hat{E}_\theta + \frac{i\ell v_\theta'}{r'}\delta\hat{E}_\theta + v_z'\frac{\partial}{\partial z'}\delta\hat{E}_\theta + \frac{v_r'}{r'}\frac{\partial}{\partial r'}(r'\delta\hat{E}_\theta)\right\},$$

the term in the integrand proportional to $\partial f_b^0/\partial P_\theta$ in Eq. (29) can be simplified to give

$$r'\left\{\delta\hat{E}_\theta + \frac{i}{\omega}\left[v_z'\frac{\partial}{\partial z'}\delta\hat{E}_\theta + \frac{v_r'}{r'}\frac{\partial}{\partial r'}(r'\delta\hat{E}_\theta)\right]\right\}\exp(i\ell\theta'-i\omega\tau)$$

$$= \frac{\ell v_\theta'}{\omega}\delta\hat{E}_\theta\exp(i\ell\theta'-i\omega\tau) + \frac{i}{\omega}\frac{d}{d\tau}\left[r'\delta\hat{E}_\theta\exp(i\ell\theta'-i\omega\tau)\right] . \tag{30}$$

It is convenient to introduce the effective perturbed potential $\delta\hat{\psi}_\theta(r,z)$ defined by

$$\delta\hat{\psi}_\theta(r,z) = r\delta\hat{E}_\theta(r,z) ,$$

and the orbit integral I defined by

$$I = \int_{-\infty}^{0} d\tau \;\dot{\theta}'\; \delta\hat{\psi}_\theta(r',z')\exp[i\ell(\theta'-\theta)-i\omega\tau]\;. \tag{31}$$

Here, use has been made of $v'_\theta = r'd\theta'/dt'$. Substituting Eq. (30) into Eq. (29), and integrating by parts with respect to τ then gives for the perturbed distribution function

$$\begin{aligned}\delta\hat{f}_{b\ell}(r,z,\underset{\sim}{p}) &= \frac{e}{\omega}\frac{\partial f_b^0}{\partial P_\theta}\left[\ell I + i\delta\hat{\psi}_\theta(r,z)\right] \\ &\quad + e\frac{\partial f_b^0}{\partial H} I\;.\end{aligned} \tag{32}$$

Within the context of Eq. (25), it is valid to approximate

$$\delta\hat{\psi}_\theta(r',z') = \delta\hat{\psi}_\theta(R_0,0) + \frac{\omega_{cz}\delta P_\theta}{\gamma_b m R_0 \omega_\beta^2}\left(\frac{\partial}{\partial r}\delta\hat{\psi}_\theta\right)_{(R_0,0)} \tag{33}$$

in Eqs. (31) and (32), where we have neglected the small-amplitude oscillatory modulations in the r- and z-orbits. Substituting Eqs. (26) and (33) into Eq. (31), we find that the orbit integral I can be approximated by

$$\begin{aligned} I = i&\left(\omega_{cz} - \frac{\mu}{\gamma_b m R_0^2}\delta P_\theta\right) \\ &\times \frac{[\delta\hat{\psi}_0 + \delta\hat{\psi}'_0(\omega_{cz}/\gamma_b m R_0^2\omega_\beta^2)\delta P_\theta]}{\omega - \ell\omega_{cz} + \ell\mu\delta P_\theta/\gamma_b m R_0^2}\;,\end{aligned} \tag{34}$$

where the abbreviated notation

$$\delta\hat{\psi}_0 \equiv \delta\hat{\psi}_\theta(R_0,0)\;, \qquad \delta\hat{\psi}'_0 \equiv R_0\left(\frac{\partial}{\partial r}\delta\hat{\psi}_\theta\right)_{(R_0,0)}$$

has been introduced. In Eq. (32), we approximate $\delta\hat{\psi}_\theta(r,z) \simeq \delta\hat{\psi}_\theta(R_0,0) = \delta\hat{\psi}_0$, and define

$$I_1 = \frac{1}{\omega}\,[\ell I + i\delta\hat{\psi}_0]\ . \tag{35}$$

Making use of Eqs. (34) and (35), it is straightforward to show that I_1 can be expressed as

$$I_1 = \frac{i[\delta\hat{\psi}_0 + \delta\hat{\psi}_0'(\omega_{cz}/\gamma_b m R_0^2\omega_\beta^2)\delta P_\theta]}{\omega - \ell\omega_{cz} + \ell\mu\delta P_\theta/\gamma_b m R_0^2}\ . \tag{36}$$

Therefore, the perturbed distribution function in Eq. (32) is given by

$$\delta\hat{f}_b(r,z,\underset{\sim}{p}) = e\,\frac{\partial f_b^0}{\partial P_\theta}\,I_1 + e\,\frac{\partial f_b^0}{\partial H}\,I\ , \tag{37}$$

where I and I_1 are defined in Eqs. (34) and (36), respectively.

The perturbed charge and current densities are obtained from the integration of Eq. (37) over momentum space. Whenever the momentum integration is carried out, it is convenient to integrate by parts with respect to p_θ, making use of the identity

$$\begin{aligned}\alpha\,\frac{\partial}{\partial P_\theta}\,f_b^0(H-\omega_b P_\Phi,\ P_\theta) &= \frac{1}{r}\,\frac{\partial}{\partial p_\theta}\,[\alpha f_b^0(H-\omega_b P_\Phi,\ P_\theta)]\\ &- \left\{f_b^0(H-\omega_b P_\Phi,\ P_\theta)\,\frac{\partial\alpha}{\partial P_\theta} + \frac{v_\theta}{r}\,\frac{\partial}{\partial H}\,[\alpha f_b^0(H-\omega_b P_\Phi,\ P_\theta)]\right\}\ ,\end{aligned} \tag{38}$$

where α is an arbitrary function of P_θ, and $v_\theta = \partial H/\partial p_\theta$ is the azimuthal velocity.

Taking the θ-component of Eq. (23), and making use of the large-aspect-ratio assumption in Eq. (24), we obtain the eigenvalue equation for $\delta\hat{E}_\theta(\rho,\Phi)$,

$$\left(\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho}+\frac{1}{\rho^2}\frac{\partial^2}{\partial\Phi^2}-q^2\right)\delta\hat{E}_\theta = 4\pi\left(ik\delta\hat{\rho}-\frac{i\omega}{c^2}\delta\hat{J}_\theta\right) , \tag{39}$$

where $k=\ell/R_0$, $q^2=k^2+1-\omega^2/c^2$, and the Laplacian operator ∇^2 has been approximated by $\nabla^2 \simeq \rho^{-1}(\partial/\partial\rho)(\rho\partial/\partial\rho)+\rho^{-2}(\partial^2/\partial\Phi^2)-k^2-1$. Here, (ρ,Φ) is the toroidal polar coordinate system introduced in Eq. (1) and Fig. 1. To complete the description, we evaluate $4\pi(ik\delta\hat{\rho}-i\omega\delta\hat{J}_\theta/c^2) = -4\pi eik\int d^3p\delta\hat{f}_b(1-\omega p_\theta/c^2k\gamma m)$ on the right-hand side of Eq. (39). To the required accuracy, $(1-\omega p_\theta/c^2k\gamma m)$ can be approximated in the integrand by $(1-\omega\beta_b/ck) - \omega\delta P_\theta/c^2k\gamma_b^3mR_0$, where $\omega \simeq \ell\omega_{cz} = k\beta_bc$ is assumed. Making use of Eqs. (37) and (39) then gives the eigenvalue equation for $\delta\hat{\Psi}_\theta(\rho,\Phi)$

$$\begin{aligned}&\left(\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho}+\frac{1}{\rho^2}\frac{\partial^2}{\partial\Phi^2}-q^2\right)\delta\hat{\psi}_\theta(\rho,\Phi)\\ &\quad = -4\pi e^2 i\ell\int d^3p\left[(1-\omega\beta_b/ck)-\frac{\omega\delta P_\theta}{c^2k\gamma_b^3mR_0}\right]\\ &\qquad\times\left(\frac{\partial f_b^0}{\partial P_\theta}I_1+\frac{\partial f_b^0}{\partial H}I\right) .\end{aligned} \tag{40}$$

Making use of the identity in Eq. (38), the eigenvalue equation (40) can also be expressed as

$$\begin{aligned}&\left(\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho}+\frac{1}{\rho^2}\frac{\partial^2}{\partial\Phi^2}-q^2\right)\delta\hat{\psi}_\theta(\rho,\Phi)\\ &\quad = -4\pi ie^2\ell\int d^3p\left\{(1-\omega\beta_b/ck)\left(I-\frac{v_\theta}{r}I_1\right)\frac{\partial f_b^0}{\partial H}\right.\\ &\qquad\left.-f_b^0\frac{\partial}{\partial P_\theta}\left[\left(1-\frac{\omega\beta_b}{ck}-\frac{\omega\delta P_\theta}{c^2k\gamma_b^3mR_0}\right)I_1\right]\right\} .\end{aligned} \tag{41}$$

In obtaining Eq. (41), use has been made of Assumption (c) in Sec. 2.

The angular velocity of an electron at radius r is given by $\dot{\theta} = v_\theta/r = (\partial H/\partial P_\theta)$. Assuming that the spread in canonical angular momentum is small, the angular velocity can be approximated by

$$\left(\frac{v_\theta}{r}\right)_{P_\theta=P_0} \approx \omega_{cz}\left(1 - \frac{r-R_0}{R_0}\right). \tag{42}$$

To evaluate the momentum integrals in Eq. (41), use is made of Eqs. (8), (9), and (10). After some straightforward algebra, we find

$$\int dp_r dp_z \delta(H-\omega_b P_\Phi-\hat{\gamma}mc^2) = 2\pi\gamma_b mU(a-\rho) \ , \tag{43}$$

and

$$\int dp_r dp_z \frac{\partial}{\partial H} \delta(H-\omega_b P_\Phi-\hat{\gamma}mc^2) = -\frac{2\pi}{\Omega_\beta^2}\frac{\delta(\rho-a)}{a} \ , \tag{44}$$

where U(x) is the Heaviside step function defined by $U(x>0) = +1$, and $U(x)=0$ for $x<0$, and the quantity $\Omega_\beta^2 = (\omega_b^+-\omega_b)(\omega_b-\omega_b^-)$ is defined in Eq. (16). Substituting Eq. (42) into Eq. (41), and making use of Eqs. (43) and (44), the eigenvalue equation (41) reduces to

$$\begin{aligned}&\left(\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\Phi^2} - q^2\right)\delta\hat{\psi}_\theta(\rho,\Phi) \\ &= -S\cos\Phi\,\delta\hat{\psi}_0\frac{\delta(\rho-a)}{a} + N_0\delta\hat{\psi}_0\frac{U(a-\rho)}{a^2} \\ &\quad - N_1\delta\hat{\psi}_0'\frac{U(a-\rho)}{a^2} \ ,\end{aligned} \tag{45}$$

where use has been made of Eq. (1), and we have introduced the abbreviated notation

$$S = \frac{\omega_{pb}^2 \omega ka(1-\beta_b\omega/ck)}{\Omega_\beta^2(\omega-\ell\omega_{cz}+i|\mu k\Delta|/\gamma_b mR_0)} , \qquad (46a)$$

$$N_0 = \frac{k^2a^2\omega_{pb}^2[\mu(1-\beta_b\omega/ck) + \omega(\omega-\ell\omega_{cz})/k^2c^2\gamma_b^2]}{(\omega-\ell\omega_{cz}+i|\mu k\Delta|/\gamma_b mR_0)^2} , \qquad (46b)$$

$$N_1 = \frac{ka(1-\beta_b\omega/kc)\omega_{cz}\omega_{pb}^2 a/R_0}{\omega_\beta^2(\omega-\ell\omega_{cz}+i|\mu k\Delta|/\gamma_b mR_0)} . \qquad (46c)$$

It should be noted that the term in Eq. (45) proportional to $\delta(\rho-a)$ corresponds to a surface-perturbation at the boundary ($\rho=a$) of the electron ring. This term, which is absent in standard treatments of the negative-mass instability, originates from the perturbed charge density contribution in Eq. (41) proportional to $\partial f_b^0/\partial H$. We further note that the final two terms on the right-hand side of Eq. (45) are proportional to $U(a-\rho)$ and correspond to a body-wave perturbation that extends throughout the electron ring ($0 < \rho < a$).

Finally, the terms proportional to ω/ck in Eqs. (46a) - (46c) are clearly related to electromagnetic effects. In this regard, it is customary in conventional treatments of longitudinal stability properties to approximate terms such as $1-\beta_b\omega/ck$ by

$$1 - \beta_b\omega/ck \simeq 1 - \beta_b^2 ,$$

where use is made of $\omega \simeq \ell\omega_{cz} = k\beta_b c$. To be more precise, we should express

$$1 - \beta_b\omega/ck = (1-\beta_b^2) - \beta_b\left(\frac{\omega-\ell\omega_{cz}}{ck}\right) ,$$

which is the procedure followed in analyzing the dispersion relation in Sec. 5. Indeed, retaining the contributions proportional to $\beta_b\chi = \beta_b(\omega-\ell\omega_{cz})/ck$, it is found that the inclusion of the concommitant "electromagnetic effects" can have a large influence on detailed stability behavior (Sec. 5), at least in certain parameter regimes of betatron operation.

4. DISPERSION RELATION FOR LONGITUDINAL PERTURBATIONS

We now make use of Eq. (45) to derive the dispersion relation for longitudinal perturbations about an intense relativistic electron beam circulating in a modified betatron (or a conventional betatron in the absence of the toroidal field). Making use of the assumption of large aspect ratio $a_c << R_0$ in Eq. (24), the eigenvalue equation (45) is solved in the straight-beam approximation. Since the perturbed density on the right-hand side of Eq. (45) is non-zero inside the electron beam, the eigenfunction $\delta\hat{\psi}_\theta(\rho,\Phi)$ can be determined in terms of the appropriate Green's function for the left-hand side of Eq. (45). Assuming that the Green's function $G(\rho,\rho',\Phi,\Phi')$ satisfies

$$\left(\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho}+\frac{1}{\rho^2}\frac{\partial^2}{\partial\Phi^2}-q^2\right)G(\rho,\rho',\Phi,\Phi') = \frac{1}{\rho}\,\delta(\rho-\rho')\delta(\Phi-\Phi') \ , \tag{47}$$

the eigenfunction $\delta\hat{\psi}_\theta(\rho,\Phi)$ can be expressed as

$$\delta\hat{\psi}_\theta(\rho,\Phi) = \int_0^{a_c} d\rho'\rho' \int_0^{2\pi} d\Phi' G(\rho,\rho',\Phi,\Phi')C(\rho',\Phi') \ , \tag{48}$$

where the source term $C(\rho',\Phi')$ is defined by

$$C(\rho,\Phi) = -S\cos\Phi\,\delta\hat{\psi}_0\,\frac{\delta(\rho-a)}{a} + N_0\delta\hat{\psi}_0\,\frac{U(a-\rho)}{a^2} - N_1\delta\hat{\psi}_0'\,\frac{U(a-\rho)}{a^2} \ . \tag{49}$$

Making use of the identity

$$\delta(\Phi-\Phi') = \frac{1}{2\pi}\sum_{m=-\infty}^{\infty} \exp\{im(\Phi-\Phi')\} , \tag{50}$$

the Green's function $G(\rho,\rho',\Phi,\Phi')$ can be expressed as

$$G(\rho,\rho',\Phi,\Phi') = \frac{1}{2\pi}\sum_{m=-\infty}^{\infty} g_m(\rho,\rho')\exp\{im(\Phi-\Phi')\} , \tag{51}$$

where the radial Green's function $g_m(\rho,\rho')$ is the solution to the differential equation

$$\left\{\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho} - \left(\frac{m^2}{\rho^2} + q^2\right)\right\} g_m(\rho,\rho') = \frac{1}{\rho}\,\delta(\rho-\rho') . \tag{52}$$

After some straightforward algebraic manipulation, it is found that the appropriate radial Green's function that solves Eq. (52) is given by

$$g_m(\rho,\rho') = \begin{cases} g_m^+(\rho,\rho') = \dfrac{K_m(qa_c)I_m(q\rho')}{I_m(qa_c)}\left[I_m(q\rho) - \dfrac{I_m(qa_c)}{K_m(qa_c)}K_m(q\rho)\right] \\ \qquad \text{for } \rho' < \rho < a_c , \\ \\ g_m^-(\rho,\rho') = \dfrac{K_m(qa_c)I_m(q\rho')}{I_m(qa_c)}\left[1 - \dfrac{I_m(qa_c)\,K_m(q\rho')}{I_m(q\rho')K_m(qa_c)}\right]I_m(q\rho) \\ \qquad \text{for } 0 < \rho < \rho' , \end{cases} \tag{53}$$

where use has been made of $g_m^+(a_c,\rho') = 0$, $g_m^+(\rho',\rho') = g_m^-(\rho',\rho')$ and $(\partial g_m^+/\partial\rho)_{\rho'} - (\partial g_m^-/\partial\rho)_{\rho'} = 1/\rho'$. In Eq. (53), I_m and K_m are the modified Bessel functions of the first and second kinds, respectively, of order m.

Making use of Eq. (52), we obtain

$$g_m(\rho,\rho') = g_{-m}(\rho,\rho') \ . \tag{54}$$

In the subsequent discussion, the stability analysis is restricted to relatively low azimuthal mode numbers satisfying

$$ka_c = \frac{\ell a_c}{R_0} << 1 \ , \tag{55}$$

which is consistent with Eqs. (24) and (25). Within the context of Eq. (53), making use of the small-argument expansions of I_m and K_m, we approximate the radial Green's function to lowest order by

$$g_m^+(\rho,\rho') = \begin{cases} \ln(\rho/a_c) \ , & m=0 \ , \\ \frac{1}{2m}\left[\left(\frac{\rho\rho'}{a_c^2}\right)^m - \left(\frac{\rho'}{\rho}\right)^m\right] , & m \geq 1 \ , \end{cases} \tag{56}$$

and

$$g_m^-(\rho,\rho') = \begin{cases} \ln(\rho'/a_c) \ , & m=0 \ , \\ \frac{1}{2m}\left[\left(\frac{\rho\rho'}{a_c^2}\right)^m - \left(\frac{\rho}{\rho'}\right)^m\right] , & m \geq 1 \ . \end{cases} \tag{57}$$

Substituting Eqs. (51) and (53) into Eq. (48), the eigenfunction is given by

$$\delta\hat{\psi}_\theta(\rho,\Phi) = \sum_{m=-\infty}^{\infty} \delta\hat{\psi}_\theta^{(m)}(\rho)\exp\{im\Phi\} \ , \tag{58}$$

where

$$\delta\hat{\psi}_\theta^{(m)}(\rho) = \frac{1}{2\pi}\int_0^{2\pi} d\Phi' \exp\{-im\Phi'\}\left[\int_0^{\rho} d\rho'\rho' g_m^+(\rho,\rho')C(\rho',\Phi') + \int_\rho^{a_c} d\rho'\rho' g_m^-(\rho,\rho')C(\rho',\Phi')\right] . \tag{59}$$

The source term in Eq. (49) can also be expressed as

$$C(\rho,\Phi) = \sum_{m=-\infty}^{\infty} C_m(\rho)\exp(im\Phi) , \tag{60}$$

where the coefficients $C_m(\rho)$ are defined by

$$C_0(\rho) = (N_0\delta\hat{\psi}_0 - N_1\delta\hat{\psi}_0')\,\frac{U(a-\rho)}{a^2} \tag{61a}$$

$$C_{\pm1}(\rho) = -S\delta\hat{\psi}_0\,\frac{\delta(\rho-a)}{2a} , \tag{61b}$$

and

$$C_m(\rho) = 0 \text{ for } m = \pm2,\ \pm3,\ldots \quad . \tag{61c}$$

We now substitute Eq. (60) into Eq. (59) and integrate over Φ', using the identity

$$\int_0^{2\pi} d\Phi' \exp(-in\Phi') = \begin{cases} 2\pi , & n=0 \\ 0 , & n\neq0 . \end{cases}$$

Then, Eq. (59) can be simplified to give

$$\delta\hat{\psi}_\theta^{(m)}(\rho) = \int_0^{\rho} d\rho'\rho' g_m^+(\rho,\rho')C_m(\rho') + \int_0^{a_c} d\rho'\rho' g_m^-(\rho,\rho')C_m(\rho') . \tag{62}$$

It is evident from Eqs. (54) and (61B) that $\delta\hat{\psi}_\theta^{(1)}(\rho) = \delta\hat{\psi}_\theta^{(-1)}(\rho)$ in Eq. (62). Therefore, the eigenfunction in Eq. (58) can be expressed as

$$\delta\hat{\psi}_\theta(\rho,\Phi) = \delta\hat{\psi}_\theta^{(0)}(\rho) + 2\delta\hat{\psi}_\theta^{(1)}(\rho)\cos\Phi \ . \tag{63}$$

It is evident from Eqs. (59) - (62) that the eigenfunction $\delta\hat{\psi}_\theta(\rho,\Phi)$ in Eq. (63) is determined in terms of $\delta\hat{\psi}_0$ and $\delta\hat{\psi}_0'$. To derive the dispersion relation from Eq. (63), we evaluate $\delta\hat{\psi}_\theta(r,z)$ and $r(\partial/\partial r)\delta\hat{\psi}_\theta(r,z)$ at $(r,z) = (R_0,0)$. Thus, after some straightforward algebraic manipulation, we obtain

$$\begin{aligned}\delta\hat{\psi}_\theta(\rho,\Phi) = {} & \frac{S}{4}\,\delta\hat{\psi}_0\left(1 - \frac{a^2}{a_c^2}\right)\left(\frac{r-R_0}{a}\right) \\ & - \frac{1}{4}\,(N_0\delta\hat{\psi}_0 - N_1\delta\hat{\psi}_0')\left[2\ln\left(\frac{a_c}{a}\right) + 1 - \frac{\rho^2}{a^2}\right]\end{aligned} \tag{64}$$

for $0 \le \rho \le a$. Upon evaluating $\delta\hat{\psi}_\theta(r,z)$ and $r(\partial/\partial r)\delta\hat{\psi}_\theta(r,z)$ at $(r,z) = (R_0,0)$, we obtain two homogeneous equations relating the two amplitudes $\delta\hat{\psi}_0$ and $\delta\hat{\psi}_0'$. The condition for a nontrivial solution is that the determinant of the coefficients of $\delta\hat{\psi}_0$ and $\delta\hat{\psi}_0'$ be equal to zero. Setting the determinant equal to zero, we obtain the dispersion relation

$$1 + \frac{2\ln\left(\frac{a_c}{a}\right) + 1}{4}\left[N_0 - \frac{SN_1R_0}{2a}\left(1 - \frac{a^2}{a_c^2}\right)\right] = 0 \ , \tag{65}$$

where the quantities S, N_0, and N_1 are defined in Eq. (46). The term proportional to S in Eq. (65) originates from the surface perturbation in Eq. (45). It is evident from Eq. (65) that the contribution from the surface term vanishes as the conducting wall radius ($\rho=a_c$)

approaches the outer radius of the electron beam (ρ=a). Without presenting the details here, we find that the surface-driven instabilities obtained fro m Eq. (65) are easily stablized when the conducting wall radius is in sufficiently close proximity to the surface of the electron beam.

Equation (65) is one of the principal results of this paper and can be used to investigate detailed stability properties for a broad range of system parameters.

5. LONGITUDINAL STABILITY PROPERTIES

We now make use of the dispersion relation in Eq. (65) to determine stability properties in various parameter regimes of physical interest. Substituting the definitions in Eq. (46) into Eq. (65) gives the dispersion relation

$$1 + \frac{\nu}{\gamma_b}\left[2\ln\left(\frac{a_c}{a}\right) + 1\right] \frac{\ell^2 c^2/R_0^2}{(\omega - \ell\omega_{cz} + i|\mu k\Delta|/\gamma_b m R_0)^2}$$

$$\times \left\{\mu\left(1 - \beta\frac{\omega}{ck}\right) + \frac{\omega(\omega - \ell\omega_{cz})}{\gamma_b^2 k^2 c^2}\right.$$

$$\left. - \ell\frac{\omega_{pb}^2\omega_{cz}^2}{2\Omega_\beta^2\omega_\beta^2}\left(1 - \beta_b\frac{\omega}{ck}\right)^2\left(1 - \frac{a^2}{a_c^2}\right)\right\} = 0\ , \tag{66}$$

where use has been made of the definition of Budker's parameter ν, which is related to the plasma-frequency-squared by

$$4\frac{\nu}{\gamma_b}c^2 = \omega_{pb}^2 a^2\ . \tag{67}$$

Analyzing the full dispersion relation in Eq. (66), we can investigate stability properties for a broad range of system parameters, and determine the important influence of electromagnetic effects and surface-wave perturbations on stability behavior. We also emphasize that the spread in canonical angular momentum (Δ) has a strong influence[14] on stability properties.

A. Stabilizing Influence of Transverse Electromagnetic Effects

In order to illustrate the strong stabilizing influence of transverse electromagnetic effects, we first make use of the dispersion relation in Eq. (66) to investigate stability properties for the case where the conducting wall is very close to the surface of the electron beam ($a \simeq a_c$), or for the case of relatively high transverse beam temperature satisfying $|\ell\omega_{pb}^2/\gamma_b^2\Omega_\beta^2| << 1$. Neglecting the final term on the left-hand side of Eq. (66), the dispersion relation can be approximated by

$$1 + \frac{\nu}{\gamma_b} \frac{2\ell n(a_c/a) + 1}{(\omega - \ell\omega_{cz} + i|\mu k\Delta|/\gamma_b m R_0)^2} \times \left[\mu k^2 c^2 \left(1 - \beta_b \frac{\omega}{ck}\right) + \frac{\omega}{\gamma_b^2}\left(\omega - \ell\omega_{cz}\right)\right] = 0, \tag{68}$$

where $k=\ell/R_0$. For present purposes, we further assume zero spread in canonical angular momentum ($\Delta=0$). Defining the normalized Doppler-shifted frequency χ by

$$\chi = \frac{\omega - \ell\omega_{cz}}{kc} , \tag{69}$$

it is straightforward to show that Eq. (68) can be expressed (for $\Delta=0$) in the approximate form

$$\chi^2 + \frac{\nu}{\gamma_b^3}\left[2\ell n\left(\frac{a_c}{a}\right) + 1\right][\mu - (\mu\gamma_b^2 - 1)\beta_b\chi] = 0 , \tag{70}$$

where the term proportional to $\beta_b\chi$ represents the stabilizing influence of transverse electromagnetic effects. In obtaining Eq. (70), use has been made of Eq. (4).

If the term proportional to $\beta_b \chi$ is neglected in Eq. (70), we recover the familiar result,[14]

$$\chi^2 + \mu \frac{\nu}{\gamma_b^3} \left[2\ln \left(\frac{a_c}{a} \right) + 1 \right] = 0 \ , \tag{71}$$

which is the standard dispersion relation for the well known negative-mass instability. For future reference, the necessary and sufficient condition for instability ($\mathrm{Im}\omega > 0$) obtained from Eq. (71) is

$$\mu = \frac{\omega_{cz}^2}{\omega_\beta^2} - \frac{1}{\gamma_b^2} > 0 \ . \tag{72}$$

Equation (72) is the conventional instability criterion obtained in previous studies.[13,14] According to Eq. (72), the system is negative-mass unstable provided $\mu > 0$.

On the other hand, the necessary and sufficient condition for instability ($\mathrm{Im}\omega > 0$) obtained from the more accurate dispersion relation (70) is given by

$$\frac{\nu}{\gamma_b} \left[2\ln \left(\frac{a_c}{a} \right) + 1 \right] \left(\mu - \frac{1}{\gamma_b^2} \right)^2 < \frac{4}{\gamma_b^2 - 1} \mu \ , \tag{73}$$

which provides an upper bound on the parameter μ for instability to exist. Several points are noteworthy from Eq. (73). First, it is important to note that a sufficient condition for stability is

$$\mu = \frac{\omega_{cz}^2}{\omega_\beta^2} - \frac{1}{\gamma_b^2} < 0 \ , \tag{74}$$

where $\omega_\beta^2 = \omega_{cz}^2/2 + (\omega_{pb}^2/2)[\beta_b^2 - (1-f)]$. For f=0, the condition $\mu<0$ and the condition for existence of radially confined equilibria

[Eq. (15)] can be combined to give

$$1 < \omega_{pb}^2/\gamma_b^2\omega_{cz}^2 < 1 + \omega_{c\theta}^2/2\omega_{cz}^2 .$$

Evidently, for f=0, this inequality can be satisfied provided $\omega_{pb}^2/\omega_{cz}^2$ is sufficiently large. That is, the negative-mass instability can be completely stabilized provided equilibrium self-field effects are sufficiently strong. Second, for a given positive value of μ ($\mu>0$), we note from Eq. (73) that a sufficient condition for stability is

$$\frac{\nu}{\gamma_b} > \frac{4\mu/(\gamma_b^2-1)}{(\mu-1/\gamma_b^2)^2[2\ell n(a_c/a)+1]} . \tag{75}$$

For an ultrarelativistic electron beam with $\gamma_b >> 1$, the inequality in Eq. (75) can easily be satisfied provided the beam current is sufficiently large. The stabilization of the negative-mass instability for an intense relativistic electron beam originates from the inclusion of electromagnetic effects [the term proportional to $\beta_b\chi$ in Eq. (70)]. We emphasize that sufficient condition for stability in Eq. (75) is also applicable to an electron beam in a conventional betatron.

B. Influence of Surface-Wave Perturbations on Stability Behavior

In this section, we investigate the influence of surface-wave perturbations on stability behavior. In order to demonstrate the importance of surface-wave perturbations, we assume a fully nonneutral electron ring (f=0) with moderate energy. In this context, the approximations

$$1 - \beta_b \omega/ck \simeq 1/\gamma_b^2 \ ,$$

$$\omega_\beta^2 = (1/2)(\omega_{cz}^2 - \omega_{pb}^2/\gamma_b^2) \ , \tag{76}$$

are made in simplifying the dispersion relation in Eq. (66). After some straightforward algebra, it is found that the dispersion relation in Eq. (66) can be approximated by

$$\chi^2 + \frac{\nu}{\gamma_b^3}\left[2\ell n\left(\frac{a_c}{a}\right) + 1\right]\left\{\frac{\omega_{cz}^2}{\omega_\beta^2}\left[1 - \frac{\ell\omega_{pb}^2}{2\gamma_b^2\Omega_\beta^2}\left(1 - \frac{a^2}{a_c^2}\right)\right] - \frac{1}{\gamma_b^2}\right\} = 0 \ , \tag{77}$$

where use has been made of the definition of μ in Eq. (27), and zero spread in canonical angular momentum has been assumed (Δ=0). The term proportional to $(1-a^2/a_c^2)$ in Eq. (77) corresponds to the surface-wave contribution. The necessary and sufficient condition for instability ($\mathrm{Im}\omega>0$) obtained from Eq. (77) is given by

$$\frac{\omega_{cz}^2}{\omega_\beta^2}\left[1 - \frac{\ell\omega_{pb}^2}{2\gamma_b^2\Omega_\beta^2}\left(1 - \frac{a^2}{a_c^2}\right)\right] > \frac{1}{\gamma_b^2} \ , \tag{78}$$

for a moderate-energy electron beam with f=0 and Δ=0. In obtaining Eqs. (77) and (78), we emphasize that the stabilizing influence of transverse electromagnetic effects has been neglected arbitrarily.

In analyzing Eq. (78), we distinguish the two cases: (a) $\omega_\beta^2 > 0$ (betatron focussing force exceeds the defocussing self field force), and (b) $\omega_\beta^2 < 0$.

(a) $\underline{\omega_{cz}^2 > \omega_{pb}^2/\gamma_b^2}$: In this case, the inequality in Eq. (78) can be expressed in the equivalent form

$$\frac{r_L^2}{a^2} > \ell\left(1 - \frac{a^2}{a_c^2}\right)\frac{\omega_{cz}^2}{\omega_{c\theta}^2}\,\frac{\omega_{pb}^2/\omega_{cz}^2}{2\gamma_b^2-1+\omega_{pb}^2/\gamma_b^2\omega_{cz}^2}\ , \tag{79}$$

for instability to exist. In Eq. (79), the effective Larmor radius r_L is defined in Eq. (18). For a high-current electron beam with $\omega_{pb}^2/\gamma_b^2\omega_{cz}^2 \gtrsim 1$, it follows that perturbations with high azimuthal mode number are easily stabilized provided r_L^2/a^2 is sufficiently small. In other words, for the case where the betatron focussing force is larger than the defocussing self-field force, perturbations with high azimuthal mode number can be stabilized by reducing the effective transverse temperature of the beam electrons.

In a conventional betatron characterized typically by $\Omega_\beta^2 = \omega_\beta^2$ [see Eq. (11) for $\omega_{c\theta} = 0$ and $\omega_b = 0$], the necessary and sufficient condition for instability obtained from Eq. (78) is given by

$$\begin{aligned}\frac{\omega_{pb}^2}{\gamma_b^2\omega_{cz}^2} < &\left\{\left[\gamma_b^2+\gamma_b^2\ell\left(1-\frac{a^2}{a_c^2}\right)-1\right]^2 + 2\gamma_b^2 - 1\right\}^{1/2} \\ &- \gamma_b^2\left[1+\ell\left(1-\frac{a^2}{a_c^2}\right)\right] + 1\ .\end{aligned} \tag{80}$$

In the limiting case where the conductor is far removed from the electron ring $(a^2/a_c^2 \ll 1)$, Eq. (80) can be further simplified to give

$$\frac{\omega_{pb}^2}{\gamma_b^2\omega_{cz}^2} < \left\{[\gamma_b^2(\ell+1)-1]^2 + 2\gamma_b^2 - 1\right\}^{1/2} - \gamma_b^2(\ell+1) + 1\ , \tag{81}$$

for instability to exist. Therefore, in the nonrelativistic limit $(\gamma_b \simeq 1)$, the sufficient condition for stability can be expressed as

$$(\ell^2+1)^{1/2} - \ell < \frac{\omega_{pb}^2}{\omega_{cz}^2} < 1\ . \tag{82}$$

On the other hand, for an ultrarelativistic electron beam ($\gamma_b^2 >> 1$), the sufficient condition for stability can be expressed as

$$\frac{1}{\ell+1} < \frac{\omega_{pb}^2}{\gamma_b^2\omega_{cz}^2} < 1. \tag{83}$$

Equations (82) and (83) have been obtained from Eq. (81) combined with the condition for existence of radially confined beam equilibria, i.e., $\omega_{pb}^2 < \gamma_b^2\omega_{cz}^2$. Evidently, the inequality in Eq. (82), or in Eq. (83), can easily be satisfied for all $\ell \geq 1$ provided $\omega_{pb}^2/\omega_{cz}^2$ is sufficiently large that the inequality is satisfied for $\ell=1$. That is, the negative-mass instability in a conventional betatron can be completely stabilized for $\Delta=0$ provided the beam density is sufficiently large.

(b) $\underline{\omega_{cz}^2 < \omega_{pb}^2/\gamma_b^2}$: In this case, the necessary and sufficient condition for instability in Eq. (78) can be expressed as

$$\frac{r_L^2}{a^2} < \ell\left(1-\frac{a^2}{a_c^2}\right)\frac{\omega_{cz}^2}{\omega_{c\theta}^2}\,\frac{\omega_{pb}^2/\omega_{cz}^2}{2\gamma_b^2-1+\omega_{pb}^2/\gamma_b^2\omega_{cz}^2}\,. \tag{84}$$

It is evident from Eq. (74) (obtained for $a \simeq a_c$) that the negative-mass instability can be stabilized by a sufficiently strong self-electric field. However, when $a<a_c$ and surface-wave contributions are included in the stability analysis, it is found that the inequality in Eq. (84) can be satisfied provided r_L^2/a^2 is sufficiently small. Thus, for sufficiently low effective transverse temperature, we conclude (for the case $\omega_{cz}^2 < \omega_{pb}^2/\gamma_b^2$) that the electron ring in a modified betatron exhibits instability. The toroidal variation of the azimuthal electron velocity [Eqs. (41) and (42)] produces a perturbed surface charge and current in the eigenvalue equation [Eqs. (45) and (46)], thereby resulting in a kink-type perturbation. Since the instability mechanism originates with the surface-wave perturbation in the radial direction, we refer to this instability as a radial kink instability. Evidently, the instability can be stabilized by increasing the effective transverse temperature $\hat{T}_{\perp} = (1/2)\gamma_b m r_L^2 \omega_{c\theta}^2$. We therefore conclude that the transverse temperature of the beam electrons plays a major role in stabilizing the radial kink instability.

To summarize, it has been shown in various parameter regimes that the longitudinal instability can be completely stabilized for $\Delta=0$ provided either (a) that the beam current is sufficiently large, or (b) that the transverse temperature is suitably adjusted. Otherwise, the instability can also be stabilized by a spread in the canonical angular momentum ($\Delta\neq 0$). For a detailed discussion of the stabilizing influence of a spread in canonical angular momentum, the reader is referred to Ref. 14.

Finally, the detailed stability properties of an intense electron ring in a modified betatron can be calculated numerically from the dispersion relation in Eq. (66) for a broad range of system parameters. Moreover, the numerical results are in good agreement with the analytical estimates in Secs. 5.A and 5.B.

6. CONCLUSIONS

In this paper, we have investigated the stability properties of an intense relativistic electron ring within the framework of the linearized Vlasov-Maxwell equations. The analysis was carried out for perturbations about a ring equilibrium located at the midplane of an applied betatron magnetic field combined with an applied toroidal magnetic field. The stability analysis was performed including the important influence of transverse electromagnetic effects and surface-wave perturbations. Stability properties were calculated for eigenfrequency ω near harmonics of ω_{cz}. The equilibrium properties and basic assumptions were summarized in Sec. 2, and the eigenvalue equation was derived in Sec. 3. Making use of the large-aspect-ratio assumption ($R_0 >> a_c$), the eigenvalue equation (45) was solved in Sec. 4, resulting in the dispersion relation (66) for the complex oscillation frequency. Detailed stability properties were investigated in Sec. 5, including a delineation of the important influence of transverse electromagnetic effects and surface-wave perturbations. In a regime where the surface contributions are negligibly small, it was shown (Sec. 5.A) that transverse electromagnetic effects can have a strong stabilizing influence on the negative-mass instability. One of the most important features of the analysis in this regime is that stabilization occurs (even for $\mu>0$) by increasing the beam current to a sufficiently high value. In the limit where electromagnetic effects are neglected, the influence of surface-wave perturbations on stability behavior was investigated (Sec. 5.B) for a wide range of system parameters. For the case where the betatron focussing force exceeds the defocussing self-field force ($\omega_{cz}^2 < \omega_{pb}^2/\gamma_b^2$), it was found that stabilization occurs when the transverse temperature of the beam electrons is reduced to a sufficiently low value. On the other hand, for the case where $\omega_{cz}^2 < \omega_{pb}^2/\gamma_b^2$, and the transverse temperature is sufficiently low, it was shown that the radial kink instability resulted as a consequence of the surface-wave perturbations. Finally, for the conventional betatron accelerator, it was shown that both electromagnetic effects and surface-wave perturbations have a strong stabilizing influence on the negative-mass instability.

ACKNOWLEDGMENTS

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FIGURE CAPTIONS

Fig. 1 Equilibrium configuration and coordinate system.

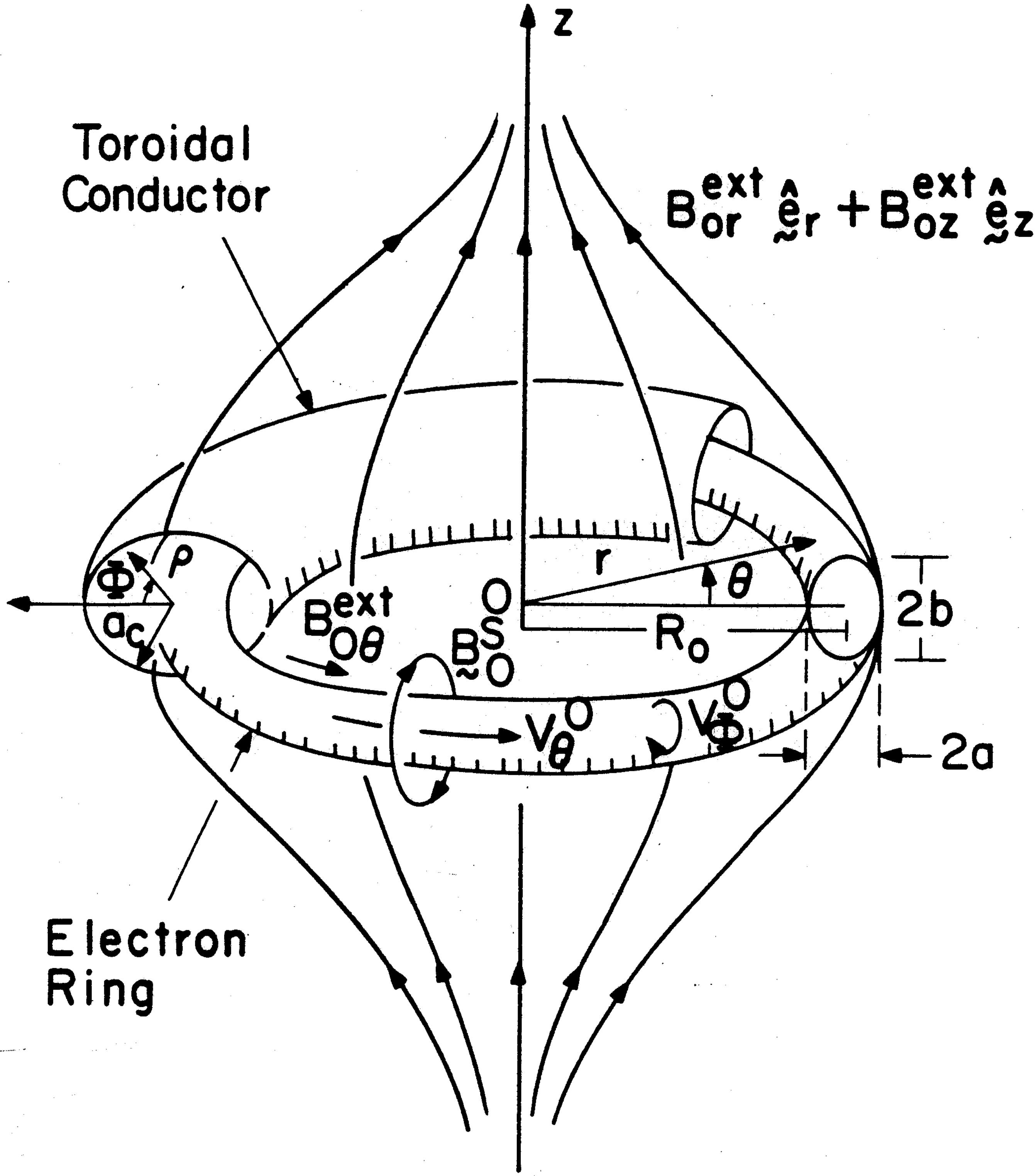
z
Toroidal
Conductor
$B_{or}^{ext}\,\hat{\underset{\sim}{e}}_r + B_{oz}^{ext}\,\hat{\underset{\sim}{e}}_z$
Φ
ρ
a_c
r
θ
O
$B_{O\theta}^{ext}$
$\underset{\sim}{B}_O^S$
R_o
2b
V_θ^O
V_Φ^O
2a
Electron
Ring

Fig. 1