QUASILINEAR STABILIZATION OF THE FREE ELECTRON LASER INSTABILITY FOR A RELATIVISTIC ELECTRON BEAM PROPAGATING THROUGH A TRANSVERSE HELICAL WIGGLER MAGNETIC FIELD

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ABSTRACT

A quasilinear model is developed that describes the nonlinear evolution and stabilization of the free electron laser instability in circumstances where a broad spectrum of waves is excited. The relativistic electron beam propagates perpendicular to a helical wiggler magnetic field $\underset{\sim}{B}_0 = -\hat{B}\cos k_0 z\, \hat{\underset{\sim}{e}}_x - \hat{B}\sin k_0 z\, \hat{\underset{\sim}{e}}_y$, and the analysis is based on the Vlasov-Maxwell equations assuming $\partial/\partial x = 0 = \partial/\partial y$ and a sufficiently tenuous beam that the Compton-regime approximation is valid ($\delta\phi \simeq 0$). Coupled kinetic equations are derived that describe the evolution of the average distribution function $G_0(p_z,t)$ and spectral energy density $\mathcal{E}_k(t)$ in the amplifying electromagnetic field perturbations. A thorough exposition of the theoretical model and general quasilinear formalism is presented, and the stabilization process is examined in detail for weak resonant instability with small temporal growth rate γ_k satisfying $|\gamma_k/\omega_k| \ll 1$ and $|\gamma_k/k\Delta v_z| \ll 1$. Assuming that the beam electrons have small fractional momentum spread ($\Delta p_z/p_0 \ll 1$), the process of quasilinear stabilization by plateau formation in the resonant region of velocity space ($\omega_k - kv_z = 0$) is investigated, including estimates of the saturated field energy, efficiency of radiation generation, etc.

I. INTRODUCTION AND SUMMARY

There have been several theoretical[1-15] and experimental[16-24] studies of coherent radiation generation by free electron lasers that use an intense relativistic electron beam as an energy source. Both transverse[1-11] and longitudinal[12-15] wiggler magnetic field geometries have been considered. For beam propagation through a transverse wiggler field, there have been many theoretical estimates (e.g., Refs. 1-11) of the gain (growth rate) during the linear phase of instability. Few calculations,[25-32] however, have addressed the nonlinear development and saturation of the instability. Particularly important for free electron laser applications is the development of a self-consistent theoretical model that estimates the saturated amplitude of the radiation field (and hence the overall efficiency of radiation generation) in terms of properties of the electron beam and the wiggler field.

In the present article, we develop a quasilinear model describing the nonlinear evolution and stabilization of the free electron laser instability. It is assumed that beam propagation is perpendicular to a transverse helical wiggler field and that a broad spectrum of waves is excited. Following a thorough exposition of the theoretical model and general quasilinear formalism (Secs. II-IV), we examine the stabilization process for weak resonant instability with small temporal growth rate γ_k satisfying $|\gamma_k/\omega_k|<<1$ and $|\gamma_k/k\Delta v_z|<<1$ (Secs. V and VI). Assuming that the beam electrons have small fractional momentum spread ($\Delta p_z/p_0<<1$), we investigate the process of quasilinear stabilization by plateau formation in the resonant region of velocity space ($\omega_k - kv_z = 0$), including estimates of the saturated field energy, efficiency of radiation generation, etc.

In the present analysis, we investigate free electron laser radiation generation by a low-density relativistic electron beam propagating in the z-direction perpendicular to a helical wiggler magnetic field [Eq. (2)]

$$\underset{\sim}{B}_0 = -\hat{B}\cos k_0 z\,\hat{\underset{\sim}{e}}_x - \hat{B}\sin k_0 z\,\hat{\underset{\sim}{e}}_y,$$

where $\hat{B} = \text{const.}$ is the wiggler amplitude and $\lambda_0 = 2\pi/k_0$ is the wavelength. As summarized in Sec. II, the theoretical model is based on the nonlinear Vlasov-Maxwell equations for the class of beam distribution functions $f_b(z,\underset{\sim}{p},t)$ of the form [Eq. (5)][7,8]

$$f_b(z,\underset{\sim}{p},t) = n_0\delta(P_x)\delta(P_y)G(z,p_z,t) \quad .$$

Here, $\partial/\partial x = 0 = \partial/\partial y$ is assumed, and P_x and P_y are the exact canonical momenta [Eq. (6)] in the combined wiggler and transverse radiation fields. Moreover, the electron beam is assumed to be sufficiently tenuous that the Compton-regime approximation is valid with negligibly small perturbations in the longitudinal fields ($\delta\phi \simeq 0$).

In Sec. III, we give a detailed derivation of the quasilinear equations describing the nonlinear evolution of the system for perturbations about the (slowly varying) average distribution function [Eq. (10)]

$$G_0(p_z,t) = \frac{1}{2L}\int_{-L}^{L} dz G(z,p_z,t),$$

where 2L is the periodicity length in the z-direction. The conservation relations satisfied by the exact Vlasov-Maxwell equations and the approximate quasilinear equations are discussed in Sec. IV and Appendix B. To briefly summarize the general quasilinear results, in response to the amplifying field perturbations it is found that $G_0(p_z,t)$ evolves according to the diffusion equation [Eqs. (30) and (83)]

$$\frac{\partial G_0}{\partial t} = \frac{\partial}{\partial p_z}\left(D\,\frac{\partial G_0}{\partial p_z}\right)$$

and the spectral energy density $\mathcal{E}_k(t)$ in the electromagnetic field perturbations satisfies [Eq. (44)]

$$\frac{\partial \mathcal{E}_k}{\partial t} = 2\gamma_k \mathcal{E}_k$$

where $\gamma_k(t)$ is the slowly varying growth rate. Here, the diffusion coefficient $D(p_z,t)$ is defined by [Eq. (30) and the equation prior to Eq. (84)]

$$D(p_z,t) = 2\left(\frac{e\hat{B}}{2k_0}\right)^2\left(\frac{\bar{\gamma}}{\gamma}\right)^2 \sum_{k=0}^{\infty}\left|\delta A^{+}_{k+k_0} + \delta A^{-}_{k-k_0}\right|^2 \exp\left\{2\int_0^t dt'\gamma_k(t')\right\}$$
$$\times \frac{k^2\gamma_k}{(\omega_k - kv_z)^2 + \gamma_k^2}$$

for $\gamma_k \geq 0$. Moreover, $\bar{\gamma}mc^2 = \text{const.}$ is the characteristic electron energy, γ is the relativistic mass factor defined by $\gamma = (1 + p_z^2/m^2c^2 + e^2\hat{B}^2/m^2c^4k_0^2)^{1/2}$ [Eq. (9)], $v_z = p_z/\gamma m$ is the axial velocity, $\omega_k + i\gamma_k$ is the complex eigenfrequency, and $\delta A^{\pm}_{k\pm k_0}$ are the dimensionless amplitudes of the vector potential. To complete the quasilinear description, the complex eigenfrequency $\omega_k(t) + i\gamma_k(t)$ is determined adiabatically in time from the linear dispersion relation [Eq. (40)]

$$D_{k+k_0}D_{k-k_0} = -\frac{1}{2}\,\frac{\hat{\omega}_c^2}{c^2k_0^2}\left[D_{k+k_0} + D_{k-k_0}\right]$$
$$\times\left[\alpha_3\omega_p^2 + \bar{\gamma}mc^2\omega_p^2\bar{\gamma}^2\int\frac{dp_z}{\gamma^2}\,\frac{k\partial G_0/\partial p_z}{\omega_k - kv_z + i\gamma_k}\right],$$

where $G_0(p_z,t)$ evolves according to Eq. (83). Here, $\hat{\omega}_c = e\hat{B}/\bar{\gamma}mc$ is the relativistic cyclotron frequency, $\omega_p^2 = 4\pi n_0 e^2/\bar{\gamma}m$ is the relativistic plasma frequency-squared, $D_{k\pm k_0} = (\omega_k + i\gamma_k)^2 - c^2(k\pm k_0)^2 - \alpha_1\omega_p^2$ [Eqs. (34) and (35)] are the transverse dielectric functions, and $\alpha_j (j = 1,3)$ is defined by $\alpha_j = \bar{\gamma}^j \int (dp_z/\gamma^j)G_0$.

As an application of the general theory developed in Sec. III, in Secs. V and VI we examine the quasilinear stabilization process for the upshifted branch of the dispersion relation [$D_{k-k_0} \simeq 0$ in Eq. (40)] in circumstances corresponding to weak resonant instability satisfying

$$\left|\frac{\gamma_k}{\omega_k}\right| \ , \ \left|\frac{\gamma_k}{k\Delta v_z}\right| << 1 \ .$$

Here, Δv_z is the characteristic spread in axial electron velocity of G_0. For a tenuous electron beam, the real frequency is $\omega_k \simeq c(k-k_0)$, and the instantaneous growth rate $\gamma_k(t)$ is given by [Eq. (77)]

$$\gamma_k(t) = \frac{\pi}{4} \ \frac{\hat{\omega}_c^2}{c^2k_0^2}\,\omega_p^2 \ \frac{\bar{\gamma}^{-3}m^2c^2}{\left[1+\left(\frac{e\hat{B}/mc}{ck_0}\right)^2\right]} \ \frac{k}{\omega_k|k|}\left[\gamma\,\frac{\partial}{\partial p_z}\,G_0(p_z,t)\right]_{v_z=\frac{\omega_k}{k}} \ .$$

For waves excited with positive phase velocity $\omega_k/k > 0$, it follows from Eq. (77) that $\gamma_k \gtrless 0$ accordingly as $[\gamma\partial G_0/\partial p_z]_{v_z=\omega_k/k} \gtrless 0$. That is, waves with phase velocity in the region of positive momentum slope in $G_0(p_z,t)$ are amplified, corresponding to instability with $\gamma_k > 0$. Moreover, for $\omega_k = kv_z$ and $\omega_k = (k-k_0)c$, the amplified wavenumber $\hat{k}$ and resonant particle velocity v_z are related by the familiar relation [Eq. (90)]

$$\hat{k} = \frac{k_0}{(1 - v_z/c)} \ .$$

Detailed growth properties are studied in Sec. V.B for the case where G_0 is instantaneously a gaussian [Eq. (78)] centered at $p_z = p_0$ with small fractional momentum spread $\Delta p_z/p_0 << 1$. An estimate of the characteristic maximum growth rate from Eq. (77) shows that the condition $|\gamma_k/k\Delta v_z| << 1$ requires that the beam density and wiggler amplitude be sufficiently small that [Eq. (114)]

$$\frac{\pi}{4} \frac{\hat{\omega}_c^2 \omega_p^2}{c^4 k_0^4} << \left(1 + \frac{v_0}{c}\right)^2 \left(\frac{v_0}{c}\right)^4 \left(\frac{\Delta p_z}{p_0}\right)^3 ,$$

where $v_0 = p_0/\bar{\gamma}m$ is the characteristic velocity of the beam electrons.

In Sec. VI.A, for general $G_0(p_z,t)$, we investigate the quasilinear stabilization process in the resonant region $(\omega_k - kv_z = 0)$ of velocity space where the diffusion coefficient $D(p_z,t)$ can be approximated by [Eq. (91)]

$$D_r(p_z,t) = 2\pi^2 e^2 \left(\frac{\hat{\omega}_c}{ck_0}\right)^2 \left(\frac{\bar{\gamma}}{\gamma}\right)^2 \left(\frac{c}{v_z}\right)^2 \int_0^\infty dk\, \mathcal{E}_k(t)\, \delta(\omega_k - kv_z)$$

in the continuum limit with $\sum_k \to \int dk$. It is shown that the coupled kinetic equations for $G_0(p_z,t)$ and $\mathcal{E}_k(t)$ can be integrated with respect to t to give the exact conservation relation [Eq. (96)] relating $G_0(p_z,t)$ and $\mathcal{E}_k(t)$ to their initial values. The time asymptotic state corresponds to plateau formation

$$\frac{\partial}{\partial p_z} G_0(p_z, t\to\infty)\Bigg|_{v_z=\omega_k/k} = 0$$

in the resonant region, with $\gamma_k(t\to\infty) = 0$.

Finally, in Sec. VI.B, we make use of Eq. (96) to obtain an estimate of the saturated field energy density $\mathcal{E}_F(\infty) = \int_0^\infty dk\mathcal{E}_k(t\to\infty)$ for specified initial distribution function $G_0(p_z,0)$. Expanding $G_0(p_z,0)$ about the point of maximum initial momentum slope ($p_z = p_{z0}$), and retaining leading-order terms, we obtain [Eq. (101)]

$$\mathcal{E}_F(\infty) = \frac{1}{12}\; n_0 v_0\; (\Delta p_z)^3 \left[\frac{\partial}{\partial p_z} G_0(p_z,0)\right]_{p_{z0}} ,$$

where Δp_z is the characteristic range of p_z over which plateau formation occurs, and $v_0 = p_0/\bar{\gamma}m$ is the beam velocity. If we estimate $[\partial G_0(p_z,0)/\partial p_z]_{p_{z0}} \approx 1/(\Delta p_z)^2$, this gives

$$\mathcal{E}_F(\infty) \approx \frac{1}{12}\, n_0 v_0 \Delta p_z \ .$$

Moreover, for highly relativistic electrons with $v_0 \simeq c$ and $(\bar{\gamma}-1)mc^2 \simeq cp_0$, we find for the efficiency η of radiation generation [Eq. (105)]

$$\eta = \frac{\mathcal{E}_F(\infty)}{(\bar{\gamma}-1)n_0 mc^2} \approx \frac{1}{12}\;\frac{\Delta p_z}{p_0} \quad .$$

Since $\Delta p_z/p_0 \ll 1$ is assumed throughout Secs. V and VI, it is clear from Eq. (105) that the efficiency η of radiation generation associated with plateau formation is relatively small. The time scale τ_{rel} for plateau formation is estimated in Eq. (109), and τ_{rel} is typically a few times the inverse maximum growth rate $([\gamma_k]_{MAX})^{-1}$ associated with the initial distribution $G_0(p_z,0)$. Therefore, at least within a quasilinear model, if there is to be efficiency enhancement above the level associated with plateau formation in Eq. (105), it is necessarily associated with a long-time quasilinear degradation of the beam distribution that occurs on a time scale $t > \tau_{rel}$, which is beyond the scope of the analysis presented in Sec. VI.

II. BASIC ASSUMPTIONS AND PHYSICAL MODEL

We consider a collisionless, relativistic electron beam with uniform cross section propagating in the z-direction. It is assumed that the beam is sufficiently tenuous that equilibrium space-charge effects can be neglected with

$$\mathbf{E}_0 = 0 \ . \tag{1}$$

The electron beam propagates perpendicular to a helical wiggler magnetic field specified by

$$\mathbf{B}_0 = -\hat{B}\cos k_0 z \hat{\mathbf{e}}_x - \hat{B}\sin k_0 z \hat{\mathbf{e}}_y \ , \tag{2}$$

where $\hat{B}$ = const. is the wiggler amplitude, $\lambda_0 = 2\pi/k_0$ is the axial wavelength, and $\hat{\mathbf{e}}_x$ and $\hat{\mathbf{e}}_y$ are unit Cartesian vectors in the x- and y-directions, respectively. The approximate form of the wiggler field in Eq. (2) with $\hat{B}$ = const. is valid near the magnetic axis where

$$k_0^2(x^2+y^2) << 1 \ .$$

We assume that this inequality is satisfied. Moreover, the beam density and current density are assumed to be sufficiently low that the equilibrium self-magnetic field can be neglected in comparison with the applied field $\mathbf{B}_0$.

Perturbations are considered in which the spatial variations are one-dimensional with $\partial/\partial x = 0 = \partial/\partial y$, and $\partial/\partial z$ generally nonzero. The electron beam is assumed to be sufficiently tenuous that the

Compton-regime approximation is valid with negligibly small perturbations in electrostatic potential and longitudinal fields, i.e.,

$$\delta\phi(z,t) = 0 \ , \quad \delta E_z(z,t) = 0 \ .$$

The perturbed vector potential is expressed as

$$\delta \mathbf{A}(\mathbf{x},t) = \delta A_x(z,t)\hat{\mathbf{e}}_x + \delta A_y(z,t)\hat{\mathbf{e}}_y \ ,$$

where the transverse electromagnetic fields, $\delta \mathbf{E}_T(\mathbf{x},t)$ and $\delta \mathbf{B}_T(\mathbf{x},t)$, are given by

$$\delta \mathbf{E}_T = -\frac{1}{c}\frac{\partial}{\partial t}\delta \mathbf{A} \ , \quad \delta \mathbf{B}_T = \nabla \times \delta \mathbf{A} \ . \tag{3}$$

The relativistic, nonlinear Vlasov equation for the beam distribution function $f_b(z,\mathbf{p},t)$ is given by

$$\left\{\frac{\partial}{\partial t} + v_z\frac{\partial}{\partial z} - e\left(\delta \mathbf{E}_T + \frac{\mathbf{v}\times(\mathbf{B}_0+\delta \mathbf{B}_T)}{c}\right)\cdot\frac{\partial}{\partial \mathbf{p}}\right\} f_b(z,\mathbf{p},t) = 0 \ , \tag{4}$$

where $-e$ is the electron charge, c is the speed of light in vacuo, and the particle velocity $\mathbf{v}$ and momentum $\mathbf{p}$ are related by

$$m\mathbf{v} = \frac{\mathbf{p}}{(1 + \mathbf{p}^2/m^2c^2)^{1/2}} \ .$$

In the present analysis, we investigate the class of exact solutions to Eq. (4) of the form[7,8]

$$f_b(z,\underset{\sim}{p},t) = n_0\delta(P_x)\delta(P_y)G(z,p_z,t) \ , \tag{5}$$

where n_0 = const., and P_x and P_y are the canonical momenta transverse to the beam propagation direction. The canonical momenta P_x and P_y are exact single-particle invariants in the combined wiggler and radiation fields, i.e.,

$$\begin{aligned} P_x &= p_x - \frac{e}{c} A_x^0(z) - \frac{e}{c}\delta A_x(z,t) = \text{const.}, \\ P_y &= p_y - \frac{e}{c} A_y^0(z) - \frac{e}{c}\delta A_y(z,t) = \text{const.} \end{aligned} \tag{6}$$

In Eq. (6),

$$A_x^0(z) = (\hat{B}/k_0)\cos k_0 z \ , \quad A_y^0(z) = (\hat{B}/k_0)\sin k_0 z \ ,$$

are the components of vector potential associated with the wiggler field in Eq. (2), and p_x and p_y are the transverse mechanical momenta. From Eq. (5), the effective transverse motion of the beam electrons is "cold". Substituting Eq. (5) into Eq. (4) and integrating the resulting equation over p_x and p_y, gives [8]

$$\left(\frac{\partial}{\partial t} + v_z\frac{\partial}{\partial z} - \frac{\partial}{\partial z}\hat{H}(z,p_z,t)\frac{\partial}{\partial p_z}\right) G(z,p_z,t) = 0 \tag{7}$$

for the evolution of $G(z,p_z,t)$. In Eq. (7), $\hat{H}$ is defined by

$$\hat{H}(z,p_z,t) = \gamma_T mc^2 \equiv [m^2c^4 + c^2p_z^2 + e^2(A_x^0+\delta A_x)^2 + e^2(A_y^0+\delta A_y)^2]^{1/2} , \tag{8}$$

which is the particle energy for $P_x = 0 = P_y$. In the absence of radiation field ($\delta A_x = 0 = \delta A_y$), the energy is given by

$$\gamma mc^2 = (m^2c^4 + c^2p_z^2 + e^2\hat{B}^2/k_0^2)^{1/2} , \tag{9}$$

where use has been made of $(A_x^0)^2 + (A_y^0)^2 = \hat{B}^2/k_0^2 = \text{const.}$.

It is assumed that the distribution function G, the vector potential $\underset{\sim}{A}$, and $\hat{H}$ are spatially periodic with periodicity length 2L. In this regard, it is convenient to introduce the spatially averaged distribution function $G_0(p_z,t)$ defined by

$$G_0(p_z,t) = (2L)^{-1} \int_{-L}^{L} dz\, G(z,p_z,t) . \tag{10}$$

Also, for small $\delta\underset{\sim}{A}$, the particle energy [Eq. (8)] can be expanded according to

$$\gamma_T mc^2 = \gamma mc^2 + \frac{e^2}{\gamma mc^2} (A_x^0 \delta A_x + A_y^0 \delta A_y) + \ldots , \tag{11}$$

where γmc^2 is defined in Eq. (9). Moreover, the quantities G and $\hat{H}$ can be expressed as an average value plus a perturbation,

$$G(z,p_z,t) = G_0(p_z,t) + \delta G(z,p_z,t) , \tag{12}$$

$$\hat{H}(z,p_z,t) = \hat{H}_0(p_z,t) + \delta \hat{H}(z,p_z,t) . \tag{13}$$

Here, G_0 is defined in Eq. (10),

$$\hat{H}_0 = (m^2c^4 + c^2p_z^2 + e^2\hat{B}^2/k_0^2)^{1/2} = \gamma mc^2 ,$$

and $\delta\hat{H}$ is given to lowest order by

$$\delta\hat{H} = \frac{e^2}{\gamma mc^2} (A_x^0 \delta A_x + A_y^0 \delta A_y)$$
$$= \frac{e^2 \hat{B}}{\gamma mc^2 k_0} (\cos k_0 z \delta A_x + \sin k_0 z \delta A_y) \ . \qquad (14)$$

III. THE QUASILINEAR EQUATIONS

A. Kinetic Equation for the Average Distribution Function

We proceed with the nonlinear, one-dimensional Vlasov equation (7) and derive lowest-order quasilinear equations for G_0 and δG. Making the substitution

$$v_z = \frac{p_z}{\gamma_T m} = \frac{p_z c^2}{\hat{H}} , \tag{15}$$

Eq. (7) can be expressed in the equivalent form

$$\frac{\partial}{\partial t} G + p_z c^2 \frac{\partial}{\partial z} \left(\frac{G}{\hat{H}}\right) - \frac{\partial}{\partial p_z} \left(G \frac{\partial}{\partial z} \hat{H}\right) = 0 . \tag{16}$$

Taking the spatial average [denote by < >] of Eq. (16) over the periodicity length 2L, we obtain

$$\frac{\partial}{\partial t} G_0 = \frac{\partial}{\partial t} \langle G \rangle = \frac{\partial}{\partial p_z} \langle \delta G \frac{\partial}{\partial z} \delta\hat{H} \rangle . \tag{17}$$

Subtracting Eq. (17) from Eq. (7) leads to

$$\begin{aligned} &\frac{\partial}{\partial t} \delta G + \frac{p_z}{\gamma m} \frac{\partial}{\partial z} \delta G - \left(\frac{\partial}{\partial z} \delta\hat{H}\right) \frac{\partial}{\partial p_z} G_0 \\ &= \left(\frac{\partial}{\partial z} \delta\hat{H}\right) \frac{\partial}{\partial p_z} \delta G - \frac{\partial}{\partial p_z} \langle \delta G \frac{\partial}{\partial z} \delta\hat{H} \rangle + \frac{p_z}{\gamma^2 m^2 c^2} \delta\hat{H} \frac{\partial}{\partial z} \delta G , \end{aligned} \tag{18}$$

where use has been made of Eq. (15), and the inverse relativistic mass factor γ_T^{-1} [Eq. (8)] has been approximated by

$$\frac{1}{\gamma_T} = \frac{1}{\gamma} - \frac{e^2}{\gamma^3 m^2 c^4} (A_x^0 \delta A_x + A_y^0 \delta A_y) = \frac{1}{\gamma} - \frac{1}{\gamma^2 mc^2} \delta\hat{H} . \tag{19}$$

Equation (18), together with Maxwell's equations, determine the evolution of the perturbations, and Eq. (17) describes the evolution of the average background distribution function $G_0(p_z,t)$. In the approximation where only linear wave-particle interactions are retained in the description, the right-hand side of Eq. (18) (which is quadratic in the perturbation amplitude) is approximated by zero. In this case, $\delta G(z,p_z,t)$ evolves according to

$$\frac{\partial}{\partial t}\,\delta G + \frac{p_z}{\gamma m}\frac{\partial}{\partial z}\,\delta G - \left(\frac{\partial}{\partial z}\,\delta \hat{H}\right)\frac{\partial}{\partial p_z}\,G_0 = 0\ . \tag{20}$$

Equation (20) will be recognized as the linearized Vlasov equation for perturbations about the spatially uniform distribution function $G_0(p_z,t)$ which varies slowly with time according to Eq. (17). Equation (20) is solved treating G_0 as slowly varying, and the resulting expression for δG is substituted into Eq. (17) to determine the reaction of G_0 to the initially unstable field fluctuations.

We introduce the Fourier series representations,

$$\begin{aligned}
G(z,p_z,t) &= G_0(p_z,t) + \sum_k{}'\,\delta G_k(p_z,t)\exp(ikz)\ ,\\
\hat{H}(z,p_z,t) &= \hat{H}_0(p_z,t) + \sum_k{}'\,\delta \hat{H}_k(p_z,t)\exp(ikz)\ ,\\
\delta A_x(z,t) &= \sum_k \delta A_x(k,t)\exp(ikz)\ ,\\
\delta A_y(z,t) &= \sum_k \delta A_y(k,t)\exp(ikz)\ .
\end{aligned} \tag{21}$$

In Eq. (21), $k = n\pi/L$ where n is an integer, and the summations run from $k=-\infty$ to $k=+\infty$. The prime on the summations denotes that the $k=0$ term is omitted. From Eqs. (13), (14), and (21) it follows that (for $k \neq 0$)

$$\delta\hat{H}_k = \frac{e^2\hat{B}}{2\gamma mc^2 k_0} [\delta A_x(k+k_0)+i\delta A_y(k+k_0)+\delta A_x(k-k_0)-i\delta A_y(k-k_0)] \ . \tag{22}$$

We also note that in Eq. (14), for $\delta\hat{H}$ to average to zero, it is required that $k \neq \pm k_0$. The time dependence of perturbed quantities is assumed to be of the form

$$\exp\left(-i\int_0^t \Omega_k(t')dt'\right) \ , \tag{23}$$

in circumstances where the time variation of $G_0(p_z,t)$ is sufficiently slow. In Eq. (23),

$$\Omega_k = \omega_k + i\gamma_k (= -\omega_{-k} + i\gamma_{-k}) \tag{24}$$

is the (complex) oscillation frequency of the waves, and γ_k is positive by hypothesis, corresponding to temporal growth. It is then useful to expand the field perturbations according to

$$\begin{aligned} \frac{e}{\bar{\gamma}mc^2}\exp(ik_0z)(\delta A_x - i\delta A_y) &= \sum_k \delta A^-_{k-k_0} \exp\left(ikz - i\int_0^t \Omega_k(t')dt'\right) \ , \\ \frac{e}{\bar{\gamma}mc^2}\exp(-ik_0z)(\delta A_x + i\delta A_y) &= \sum_k \delta A^+_{k+k_0} \exp\left(ikz - i\int_0^t \Omega_k(t')dt'\right) \ , \end{aligned} \tag{25}$$

where $\bar{\gamma}mc^2$ = const. denotes the characteristic mean energy of the beam electrons, and $\delta A^{\pm}_{k\pm k_0}$ are the dimensionless amplitudes

$$\delta A^{\pm}_{k\pm k_0} = \frac{e}{\bar{\gamma} mc^2}\left[\delta A_x(k\pm k_0,t) \pm i\delta A_y(k\pm k_0,t)\right]\exp\left(i\int_0^t \Omega_k(t')dt'\right) .$$

From Eqs. (14), (21), and (25), the quantity $\delta\hat{H}_k$ can be expressed as

$$\delta\hat{H}_k(p_z,t) = \frac{1}{2}\frac{e\hat{B}}{k_0}\frac{\bar{\gamma}}{\gamma}\left[\delta A^{+}_{k+k_0} + \delta A^{-}_{k-k_0}\right]\exp\left(-i\int_0^t \Omega_k(t')dt'\right) . \tag{26}$$

In Fourier variables, Eq. (17) becomes

$$\frac{\partial}{\partial t} G_0 = -\frac{\partial}{\partial p_z}\sum_k ik\delta\hat{H}_{-k}\delta G_k . \tag{27}$$

Solving Eq. (20) and neglecting free-streaming contributions to $\delta G_k(p_z,t)$, we obtain for the perturbed distribution function

$$\delta G_k(p_z,t) = -\frac{1}{2}\frac{e\hat{B}}{k_0}\frac{\bar{\gamma}}{\gamma}\frac{k\partial G_0/\partial p_z}{\Omega_k - kv_z}\left[\delta A^{+}_{k+k_0} + \delta A^{-}_{k-k_0}\right] \times \exp\left(-i\int_0^t \Omega_k(t')dt'\right) . \tag{28}$$

Here, $v_z = p_z/\gamma m$ and use has been made of Eq. (26). Substituting Eq. (28) into Eq. (27) yields the quasilinear kinetic equation for $G_0(p_z,t)$:

$$\frac{\partial}{\partial t} G_0(p_z,t) = i\left(\frac{e\hat{B}}{2k_0}\right)^2 \sum_k k^2\left(\delta A^{+}_{k+k_0} + \delta A^{-}_{k-k_0}\right)\left(\delta A^{+}_{-k+k_0} + \delta A^{-}_{-k-k_0}\right) \times \exp\left(2\int_0^t \gamma_k(t')dt'\right)\frac{\partial}{\partial p_z}\left[\left(\frac{\bar{\gamma}}{\gamma}\right)^2 \frac{\partial G_0/\partial p_z}{\Omega_k - kv_z}\right] . \tag{29}$$

In view of the reality of $\delta\underset{\sim}{A}(z,t)$ and $\delta\hat{H}(z,p_z,t)$, it follows that $\delta\hat{H}_{-k} = \delta\hat{H}^{*}_k$. Thus, Eq. (29) can also be expressed as

$$\frac{\partial}{\partial t} G_0(p_z,t) = i \sum_k k^2 \frac{\partial}{\partial p_z} \left(\frac{|\delta \hat{H}_k|^2 \partial G_0/\partial p_z}{\Omega_k - kv_z} \right)$$

$$= i \left(\frac{e^2 \hat{B}}{2mc^2 k_0} \right)^2 \sum_k k^2 \left| \delta A_x(k+k_0) + i\delta A_y(k+k_0) + \delta A_x(k-k_0) - i\delta A_y(k-k_0) \right|^2$$

$$\times \frac{\partial}{\partial p_z} \left(\frac{1}{\gamma^2} \frac{\partial G_0/\partial p_z}{\Omega_k - kv_z} \right) , \qquad (30)$$

where use has been made of Eq. (22). Note that Eq. (30) has the form of a diffusion equation for $G_0(p_z,t)$ in momentum space.

B. Adiabatic Dispersion Relation and Kinetic Equation for the Waves

The complex oscillation frequency $\Omega_k(t)$ is obtained adiabatically in terms of $G_0(p_z,t)$ from the linear dispersion relation. We now briefly outline the derivation of the linear dispersion relation, which is described in considerable detail in Ref. 3 . For the present configuration, the nonlinear evolution of the perturbations $\delta A_x(z,t)$ and $\delta A_y(z,t)$ is determined from the Maxwell equations

$$\left(\frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial z^2} \right) \delta A_x = - \frac{4\pi e}{c} \int d^3p \; v_x (f_b - f_b^0) , \qquad (31)$$

$$\left(\frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial z^2} \right) \delta A_y = - \frac{4\pi e}{c} \int d^3p \; v_y (f_b - f_b^0) . \qquad (32)$$

In Eqs. (31) and (32), f_b^0 is the unperturbed distribution function in the absence of radiation fields ($\delta A_x = 0 = \delta A_y$), and $f_b(z,\underset{\sim}{p},t)$ solves the nonlinear Vlasov equation (4). When Eq. (5) is substituted into Eqs. (31) and (32), and the resulting field equations are linearized, making use of Eqs. (11), (19), and (28), we obtain the matrix dispersion equation [8]

$$\begin{pmatrix} D_{k+k_0} + \frac{1}{2}\frac{\hat{\omega}_c^2}{c^2k_0^2}(\alpha_3\omega_p^2+\chi_k), & \frac{1}{2}\frac{\hat{\omega}_c^2}{c^2k_0^2}(\alpha_3\omega_p^2+\chi_k) \\ \frac{1}{2}\frac{\hat{\omega}_c^2}{c^2k_0^2}(\alpha_3\omega_p^2+\chi_k), & D_{k-k_0} + \frac{1}{2}\frac{\hat{\omega}_c^2}{c^2k_0^2}(\alpha_3\omega_p^2+\chi_k) \end{pmatrix} \begin{pmatrix} \delta A^+_{k+k_0} \\ \delta A^-_{k-k_0} \end{pmatrix} = 0 , \quad (33)$$

which relates the amplitudes $\delta A^+_{k+k_0}$ and $\delta A^-_{k-k_0}$, and determines the complex oscillation frequency Ω_k. In Eq. (33), the dielectric functions D_{k+k_0}, D_{k-k_0} and the effective susceptibility χ_k are defined by[8]

$$D_{k+k_0}(\Omega_k) = \Omega_k^2 - c^2(k+k_0)^2 - \alpha_1\omega_p^2 , \quad (34)$$

$$D_{k-k_0}(\Omega_k) = \Omega_k^2 - c^2(k-k_0)^2 - \alpha_1\omega_p^2 , \quad (35)$$

$$\chi_k(\Omega_k) = \bar{\gamma}mc^2\omega_p^2\,\bar{\gamma}^{-2}\int \frac{dp_z}{\gamma^2}\frac{k\partial G_0/\partial p_z}{\Omega_k - kv_z} , \quad (36)$$

where α_1 and α_3 are defined by

$$\alpha_1 = \bar{\gamma}\int\frac{dp_z}{\gamma}G_0 \quad , \quad \alpha_3 = \bar{\gamma}^3\int\frac{dp_z}{\gamma^3}G_0 , \quad (37)$$

and

$$\omega_p^2 = 4\pi n_0e^2/\bar{\gamma}m \quad (38)$$

is the relativistic plasma frequency-squared. Also in Eq. (33),

$$\hat{\omega}_c = e\hat{B}/\bar{\gamma}mc \quad (39)$$

is the relativistic cyclotron frequency associated with the wiggler amplitude $\hat{B}$. Requiring that the determinant of the matrix in Eq. (33) vanish gives the dispersion relation

$$D_{k+k_0}(\Omega_k)D_{k-k_0}(\Omega_k) = -\frac{1}{2}\frac{\hat{\omega}_c^2}{c^2k_0^2}[D_{k+k_0}(\Omega_k) + D_{k-k_0}(\Omega_k)]$$

$$\times [\alpha_3\omega_p^2 + \chi_k(\Omega_k)] \;, \tag{40}$$

which determines $\Omega_k(t) = \omega_k + i\gamma_k$ adiabatically in terms of $G_0(p_z,t)$.

In concluding this section, we obtain the wave kinetic equation consistent with Eqs. (25), (40) and the quasilinear equation (30) for $G_0(p_z,t)$. The average energy density in the radiation field is given by

$$(2L)^{-1}\int_{-L}^{L} dz\,\frac{1}{8\pi}[(\delta \underset{\sim}{E}_T)^2 + (\delta \underset{\sim}{B}_T)^2]$$

$$= \sum_k \{[\delta \underset{\sim}{E}_k(t)\cdot\delta \underset{\sim}{E}_{-k}(t)] + [\delta \underset{\sim}{B}_k(t)\cdot\delta \underset{\sim}{B}_{-k}(t)]\}/8\pi$$

$$= \sum_k \left(|\delta \underset{\sim}{E}_{k+k_0}(t)|^2 + |\delta \underset{\sim}{E}_{k-k_0}(t)|^2 + |\delta \underset{\sim}{B}_{k+k_0}(t)|^2 + |\delta \underset{\sim}{B}_{k-k_0}(t)|^2\right)/16\pi, \tag{41}$$

where $\delta \underset{\sim}{E}_T(z,t)$ and $\delta \underset{\sim}{B}_T(z,t)$ have been Fourier-transformed according to Eq. (21). Referring to Eq. (3), the Fourier components of the electromagnetic fields in Eq. (41) can be expressed in terms of the dimensionless quantities $\delta A^+_{k+k_0}$ and $\delta A^-_{k-k_0}$ by making use of Eqs. (25) and (21). After some straightforward algebra, we obtain

$$\frac{1}{8\pi}\sum_k\left(|\delta \underset{\sim}{E}_k(t)|^2 + |\delta \underset{\sim}{B}_k(t)|^2\right)$$

$$= \left(\frac{\bar{\gamma}mc^2}{2e}\right)^2 \frac{1}{4\pi c^2} \sum_k \left\{ |\delta A^+_{k+k_0}|^2 [|\Omega_k|^2 + c^2(k+k_0)^2]\right.$$

$$+ |\delta A^{-}_{k-k_0}|^2 [|\Omega_k|^2 + c^2(k-k_0)^2]\Big\} \exp\left(2\int_0^t \gamma_k(t')dt'\right) \tag{42}$$

$$= \sum_k \mathcal{E}_k(t) \ ,$$

where the spectral energy density $\mathcal{E}_k(t)$ is defined by

$$\mathcal{E}_k(t) \equiv \left(\frac{\bar{\gamma}mc^2}{2e}\right)^2 \frac{1}{4\pi c^2} \exp\left(2\int_0^t \gamma_k(t')dt'\right)$$
$$\times \quad \left\{|A^{+}_{k+k_0}|^2\left(|\Omega_k|^2 + c^2(k+k_0)^2\right) + |A^{-}_{k-k_0}|^2\left(|\Omega_k|^2 + c^2(k-k_0)^2\right)\right\} . \tag{43}$$

From Eq. (43), it follows that $\mathcal{E}_k(t)$ evolves according to

$$\frac{\partial}{\partial t}\mathcal{E}_k(t) = 2\gamma_k(t)\mathcal{E}_k(t) \ , \tag{44}$$

where the linear growth rate $\gamma_k(t)$ is determined adiabatically in terms of $G_0(p_z,t)$ from Eq. (40).

Equations (30), (40) and (44) then form a closed quasilinear description of the system including the effects of linear wave-particle interactions. Justification of these quasilinear equations for small-amplitude perturbations requires a sufficiently broad spectrum of unstable waves that the inequalities [33]

$$|\Delta(\omega_k - kv_z)|^{-1} \ll \gamma_k^{-1}, \quad \tau_{rel} \ , \tag{45}$$

$$|\Omega_k^{-1} \frac{d}{dt} \Omega_k| \ll \gamma_k \ , \tag{46}$$

are satisfied. Here, τ_{rel} is the characteristic quasilinear relaxation time of $G_0(p_z,t)$ from Eq. (30), γ_k is the characteristic growth rate of the unstable waves, and $\Delta(\omega_k - kv_z)$ is the characteristic spread in values of $(\omega_k - kv_z)$ over the unstable k-spectrum.

IV. CONSERVATION LAWS

In Sec. IV.A, using the nonlinear Vlasov-Maxwell equations, we outline the derivation of three exact conservation relations, corresponding to conservation of (average) particle density, total energy, and total axial momentum. In Sec. IV.B and Appendix B, we derive the analogs of these conservation relations within the framework of the quasilinear kinetic equations developed in Sec. III.

A. Conservation Relations from the Fully Nonlinear Vlasov-Maxwell Equations

The fully nonlinear Vlasov-Maxwell equations possess three exact conservation relations. These are: average density,

$$\int_{-L}^{L} \frac{dz}{2L} \int d^3p f_b(z,\underset{\sim}{p},t) = \text{const.}, \tag{47}$$

total average plasma kinetic energy density plus electromagnetic field energy density,

$$\int_{-L}^{L} \frac{dz}{2L} \left\{ \int d^3p (\gamma_T - 1) mc^2 f_b(z,\underset{\sim}{p},t) + \frac{1}{8\pi} \left((\delta \underset{\sim}{E}_T)^2 + (\underset{\sim}{B}_0 + \delta \underset{\sim}{B}_T)^2 \right) \right\} = \text{const.}, \tag{48}$$

and total average plasma momentum density plus electromagnetic field momentum density,

$$\int_{-L}^{L} \frac{dz}{2L} \left\{ \int d^3p \, p_z f_b(z,\underset{\sim}{p},t) + \frac{1}{4\pi c} \left(\delta \underset{\sim}{E}_T \times \underset{\sim}{B}_0 + \delta \underset{\sim}{E}_T \times \delta \underset{\sim}{B}_T \right)_z \right\} = \text{const.} \tag{49}$$

In Eqs. (47) - (49), γ_T and $\int d^3p$ are defined by

$$\gamma_T = [1+\underset{\sim}{p}^2/m^2c^2]^{1/2} \tag{50}$$

and

$$\int d^3p \equiv \int_{-\infty}^{\infty} dp_x \int_{-\infty}^{\infty} dp_y \int_{-\infty}^{\infty} dp_z \ .$$

From the Vlasov equation (4), the time derivative $(\partial/\partial t)f_b(z,\underset{\sim}{p},t)$ can be expressed as

$$\frac{\partial}{\partial t} f_b = -v_z \frac{\partial}{\partial z} f_b + e\delta\underset{\sim}{E}_T \cdot \frac{\partial}{\partial \underset{\sim}{p}} f_b + \frac{e}{c} \underset{\sim}{v} \times (\underset{\sim}{B}_0+\delta\underset{\sim}{B}_T) \cdot \frac{\partial}{\partial \underset{\sim}{p}} f_b \ . \tag{51}$$

Equation (47) is readily verified by integrating Eq. (51) over $\int d^3p$, and taking the spatial average. It is assumed that f_b, $\delta\underset{\sim}{E}_T$ and $\delta\underset{\sim}{B}_T$ are periodic in z direction with periodicity length 2L, and use is made of the relation

$$\frac{\partial}{\partial \underset{\sim}{p}} \left(\frac{1}{\gamma_T}\right) = - \frac{\underset{\sim}{p}}{m^2c^2} \ \frac{1}{\gamma_T^3} \ . \tag{52}$$

Equation (48) can be verified by showing that

$$\frac{\partial}{\partial t} \left\{\int_{-L}^{L} \frac{dz}{2L} \int d^3p(\gamma_T-1)mc^2 f_b(z,\underset{\sim}{p},t)\right\} = \int_{-L}^{L} \frac{dz}{2L} \left(-e\delta\underset{\sim}{E}_T \cdot \int d^3p\underset{\sim}{v} f_b(z,\underset{\sim}{p},t)\right) \ , \tag{53}$$

and also that

$$\frac{\partial}{\partial t} \left\{\int_{-L}^{L} \frac{dz}{2L} \frac{1}{8\pi} \left((\delta\underset{\sim}{E}_T)^2 + (\underset{\sim}{B}_0+\delta\underset{\sim}{B}_T)^2\right)\right\} = \int_{-L}^{L} \frac{dz}{2L} \left(e\delta\underset{\sim}{E}_T \cdot \int d^3p\underset{\sim}{v} f_b(z,\underset{\sim}{p},t)\right) \ . \tag{54}$$

Equation (53) follows by substituting Eq. (51) into the left-hand side of Eq. (53) and carrying out the integrations. The Maxwell equations for the electromagnetic fields may be utilized to obtain Eq. (54), i.e.,

$$\frac{1}{c}\frac{\partial}{\partial t}\,\delta E_x + \frac{\partial}{\partial z}\,B_y = \frac{4\pi e}{c}\int d^3p v_x f_b\ , \tag{55}$$

$$\frac{1}{c}\frac{\partial}{\partial t}\,\delta E_y - \frac{\partial}{\partial z}\,B_x = \frac{4\pi e}{c}\int d^3p v_y f_b\ , \tag{56}$$

where $\underset{\sim}{B} = \underset{\sim}{B}_0 + \delta \underset{\sim}{B}_T$. Equations (55) and (56) are multiplied by δE_x and δE_y respectively and added together. To obtain Eq. (54), the resulting equation is averaged, and use is made of the Maxwell equation

$$\nabla\times\delta\underset{\sim}{E}_T = -(1/c)(\partial/\partial t)\delta\underset{\sim}{B}_T\ . \tag{57}$$

In a similar manner, Eq. (49) can be proved by showing that

$$\frac{\partial}{\partial t}\left\{\int_{-L}^{L}\frac{dz}{2L}\int d^3p\ p_z f_b(z,\underset{\sim}{p},t)\right\} = \int_{-L}^{L}\frac{dz}{2L}\left(\frac{e}{c}\,(\underset{\sim}{B}_0+\delta\underset{\sim}{B}_T)\times\int d^3p\underset{\sim}{v}f_b(z,\underset{\sim}{p},t)\right)_z\ , \tag{58}$$

and also that

$$\frac{\partial}{\partial t}\left\{\int_{-L}^{L}\frac{dz}{2L}\,\frac{1}{4\pi c}\,(\delta\underset{\sim}{E}_T\times\underset{\sim}{B}_0+\delta\underset{\sim}{E}_T\times\delta\underset{\sim}{B}_T)_z\right\}$$
$$= \int_{-L}^{L}\frac{dz}{2L}\left(-\frac{e}{c}\,(\underset{\sim}{B}_0+\delta\underset{\sim}{B}_T)\times\int d^3p\underset{\sim}{v}f_b(z,\underset{\sim}{p},t)\right)_z\ . \tag{59}$$

Equation (58) follows by making use of Eqs. (51) and (52). In obtaining Eq. (59), the time derivatives $(\partial/\partial t)\delta\underset{\sim}{E}_T$ and $(\partial/\partial t)\delta\underset{\sim}{B}_T$ are eliminated by making use of Eqs. (55) - (57).

B. Conservation Relations from Quasilinear Theory

We now demonstrate that the conservation relations (47) - (49) are upheld by the quasilinear kinetic equations derived in Sec. III. The distribution function f_b is taken to be of the form $f_b(z,\underset{\sim}{p},t) =$

$n_0\delta(P_x)\delta(P_y)G(z,p_z,t)$ [Eq. (5)]. In Eqs. (47) - (49), we expand quantities such as $\gamma_T mc^2$ and retain up to second-order terms in perturbation amplitudes.

Number Conservation: Substituting Eq. (5) into Eq. (47), and making use of Eq. (10), we obtain

$$\frac{\partial}{\partial t}\int_{-\infty}^{\infty} dp_z n_0 G_0(p_z,t) = 0. \tag{60}$$

Clearly this is true for $\partial G_0/\partial t$ given by Eq. (30), since Eq. (30) is in the form of a diffusion equation in momentum space and the integrand is an exact differential.

Energy Conservation: To show energy conservation from quasilinear theory, the quantity $\gamma_T mc^2$ [Eq. (8)] is expanded according to

$$\gamma_T mc^2 = \gamma mc^2 + \frac{e^2}{\gamma mc^2}(A_x^0\delta A_x + A_y^0\delta A_y)$$
$$+ \frac{e^2}{2\gamma mc^2}(\delta A_x^2 + \delta A_y^2) - \frac{1}{2}\frac{e^4}{\gamma^3 m^3 c^6}(A_x^0\delta A_x + A_y^0\delta A_y)^2 + \ldots ,$$

where γmc^2 is defined in Eq. (9). Then the quasilinear analog of the exact energy conservation relation in Eq. (48) can be expressed as

$$\frac{\partial}{\partial t}\langle KED\rangle = -\frac{\partial}{\partial t}\left\langle \frac{1}{8\pi}\left[(\delta \underset{\sim}{E}_T)^2 + (\underset{\sim}{B}_0 + \delta\underset{\sim}{B}_T)^2\right]\right\rangle , \tag{61}$$

where $\langle KED\rangle$ is the average plasma kinetic energy density defined by

$$\langle KED\rangle = \int_{-L}^{L}\frac{dz}{2L}\int_{-\infty}^{\infty} dp_z n_0\Bigg((\gamma-1)mc^2 G_0 + \frac{e^2}{\gamma mc^2}(A_x^0\delta A_x + A_y^0\delta A_y)\delta G$$
$$+ \frac{e^2}{2\gamma mc^2}(\delta A_x^2 + \delta A_y^2)G_0 - \frac{1}{2}\frac{e^4}{\gamma^3 m^3 c^6}(A_x^0\delta A_x + A_y^0\delta A_y)^2\, G_0\Bigg) \tag{62}$$

correct to second order in the perturbation amplitude. In Appendix B, we make use of the quasilinear kinetic equations (29) and (44) and the dispersion relation (40) to verify that the energy conservation relation in Eq. (61) is satisfied to the level of accuracy of <KED> defined in Eq. (62).

Momentum Conservation: We now verify conservation of average total axial momentum. As noted in Appendix B regarding energy conservation, in order to obtain the quasilinear analog of Eq. (49), the approximate expression for the helical wiggler field given in Eq. (2) is substituted into the equation, and the term $(4\pi c)^{-1}\int_{-L}^{L} (dz/2L)(\delta \mathbf{E}_T \times \mathbf{B}_0)_z$ is set equal to zero. It can then be shown that consistent with this approximation, at the quasilinear level, is the neglect of second- and higher-order terms when expanding the current density $-e\int d^3p\,(v_x\hat{\mathbf{e}}_x + v_y\hat{\mathbf{e}}_y)f_b$ in powers of perturbed quantities. Thus, for the form of f_b given in Eq. (5),

$$
\begin{aligned}
-e\int d^3p(v_x\hat{\mathbf{e}}_x + v_y\hat{\mathbf{e}}_y)f_b
&= -\frac{en_0}{m}\left(\frac{e}{c}(A_x^0 + \delta A_x)\hat{\mathbf{e}}_x + \frac{e}{c}(A_y^0 + \delta A_y)\hat{\mathbf{e}}_y\right)\int_{-\infty}^{\infty}\frac{dp_z}{\gamma_T}\,G \\
= -\frac{n_0e^2}{mc}\int_{-\infty}^{\infty} dp_z &\left\{\left(\frac{1}{\gamma}(A_x^0 + \delta A_x)G_0 + \frac{1}{\gamma}A_x^0\delta G - \frac{e^2}{\gamma^3m^2c^4}(A_x^0\delta A_x + A_y^0\delta A_y)A_x^0G_0 + \dots\right)\hat{\mathbf{e}}_x\right. \\
&\left. + \left(\frac{1}{\gamma}(A_y^0 + \delta A_y)G_0 + \frac{1}{\gamma}A_y^0\delta G - \frac{e^2}{\gamma^3m^2c^4}(A_x^0\delta A_x + A_y^0\delta A_y)A_y^0G_0 + \dots\right)\hat{\mathbf{e}}_y\right\}.
\end{aligned}
\tag{63}
$$

In Eq. (63), G and $1/\gamma_T$ have been expressed as in Eqs. (12) and (19).

We obtain the quasilinear analog of Eq. (49) within the context of Eq. (63). To this end, Eq. (5) is substituted into Eq. (49), the resulting equation is differentiated with respect to time, use is made of Eq. (16), and we integrate by parts with respect to p_z. Utilizing Eq. (8), the time derivative of Eq. (49) can then be expressed as

$$- \frac{n_0 e}{mc} \int_{-L}^{L} \frac{dz}{2L} \left\{ \left(\frac{e}{c} (A_x^0 + \delta A_x) \frac{\partial}{\partial z} (A_x^0 + \delta A_x) + \frac{e}{c} (A_y^0 + \delta A_y) \frac{\partial}{\partial z} (A_y^0 + \delta A_y) \right) \right.$$

$$\left. \times \int_{-\infty}^{\infty} \frac{dp_z}{\gamma_T} G \right\} + \frac{\partial}{\partial t} \int_{-L}^{L} \frac{dz}{2L} \frac{1}{4\pi c} (\delta \underset{\sim}{E}_T \times \underset{\sim}{B}_0 + \delta \underset{\sim}{E}_T \times \delta \underset{\sim}{B}_T)_z = 0. \tag{64}$$

The term in curly brackets in Eq. (64) is proportional to the z-component of the cross-product of the current density appearing in Eq. (63) and the total magnetic field $[\nabla \times (\underset{\sim}{A}_0 + \delta \underset{\sim}{A})]$. The quasilinear analog of Eq. (49) is obtained from Eq. (64) by expanding $1/\gamma_T$, expressing $G = G_0 + \delta G$, and approximating the current density as indicated in Eq. (63). After some straightforward algebra, Eq. (64) can be expressed as

$$\frac{\partial}{\partial t} \langle PMD \rangle = - \frac{\partial}{\partial t} \left\langle \frac{1}{4\pi c} (\delta \underset{\sim}{E}_T \times \delta \underset{\sim}{B}_T)_z \right\rangle , \tag{65}$$

where the time rate of change of the average plasma momentum density is defined by

$$\frac{\partial}{\partial t} \langle PMD \rangle = \int_{-L}^{L} \frac{dz}{2L} \int_{-\infty}^{\infty} dp_z \left(- \frac{n_0 e^2}{mc^2} \right) \left(\frac{1}{\gamma} A_x^0 \delta G \frac{\partial}{\partial z} \delta A_x + \frac{1}{\gamma} A_y^0 \delta G \frac{\partial}{\partial z} \delta A_y \right.$$

$$\left. - \frac{e^2}{\gamma^3 m^2 c^4} G_0 (A_x^0 \delta A_x + A_y^0 \delta A_y)(A_x^0 \frac{\partial}{\partial z} \delta A_x + A_y^0 \frac{\partial}{\partial z} \delta A_y) \right) , \tag{66}$$

correct to second order in the perturbation amplitude. In Appendix B, we make use of the quasilinear kinetic equations (29) and (44) and the dispersion relation (40) to verify that the momentum conservation relation in Eq. (65) is satisfied to the quasilinear level of accuracy of $(\partial/\partial t)\langle PMD \rangle$ defined in Eq. (66).

V. QUASILINEAR STABILITY PROPERTIES

In circumstances where the beam density is sufficiently low and the wiggler amplitude is small, the dispersion relation (40) supports two solutions, near $D_{k+k_0}(\Omega_k) = 0$ and $D_{k-k_0}(\Omega_k) = 0$, respectively. In the subsequent analysis, we consider the upshifted branch with $D_{k-k_0} \simeq 0$, where $k_0 > 0$ is assumed. For $D_{k+k_0} \neq 0$ and $D_{k-k_0} \simeq 0$, the dispersion relation (40) can be approximated by

$$0 = \hat{D}(k,\Omega_k) = \Omega_k^2 - c^2(k-k_0)^2 - \alpha_1\omega_p^2 + \frac{1}{2}\frac{\hat{\omega}_c^2}{c^2k_0^2}\left[\alpha_3\omega_p^2 + \bar{\gamma}mc^2\omega_p^2\bar{\gamma}^{-2}\int\frac{dp_z}{\gamma^2}\frac{k\partial G_0/\partial p_z}{\Omega_k - kv_z}\right] , \tag{67}$$

where we have introduced the effective dielectric function $\hat{D}(k,\Omega_k)$ defined by

$$\hat{D}(k,\omega_k + i\gamma_k) \equiv (\omega_k + i\gamma_k)^2 - c^2(k-k_0)^2 - \alpha_1\omega_p^2 + \frac{1}{2}\frac{\hat{\omega}_c^2}{c^2k_0^2}\alpha_3\omega_p^2 + \frac{1}{2}\frac{\hat{\omega}_c^2}{c^2k_0^2}\bar{\gamma}mc^2\omega_p^2\bar{\gamma}^{-2}\int\frac{dp_z}{\gamma^2}\frac{k\partial G_0/\partial p_z}{\omega_k - kv_z + i\gamma_k} \tag{68}$$

for $\gamma_k = \mathrm{Im}\Omega_k > 0$.

A. Weak Resonant Instability

For present purposes, we consider weak resonant instability satisfying $|\gamma_k/\omega_k| \ll 1$ and $|\gamma_k/k| \ll \Delta v_z$, where Δv_z is the characteristic spread in axial electron velocity. Introducing $\hat{D}_r = \mathrm{Re}\hat{D}$ and $\hat{D}_i = \mathrm{Im}\hat{D}$, the dispersion relation $\hat{D} = \hat{D}_r + i\hat{D}_i = 0$ is expanded according to

$$0 = \hat{D}_r(k, \omega_k) + i \left[\gamma_k \frac{\partial}{\partial \omega_k} \hat{D}_r(k,\omega_k) + \hat{D}_i(k,\omega_k) \right] + \ldots, \tag{69}$$

and use is made of

$$\lim_{\gamma_k \to 0^+} \frac{1}{\omega_k - kv_z + i\gamma_k} = \frac{P}{\omega_k - kv_z} - i\pi\delta(\omega_k - kv_z), \tag{70}$$

where P denotes Cauchy principal value. Substituting Eqs. (69) and (70) into Eq. (67) and setting real and imaginary parts equal to zero, the real oscillation frequency ω_k is determined from

$$\begin{aligned} 0 = \hat{D}_r(k,\omega_k) = \omega_k^2 - c^2(k-k_0)^2 - \alpha_1\omega_p^2 + \frac{1}{2}\frac{\hat{\omega}_c^2}{c^2k_0^2}\alpha_3\omega_p^2 \\ + \frac{1}{2}\frac{\hat{\omega}_c^2}{c^2k_0^2}\bar{\gamma}mc^2\omega_p^2\bar{\gamma}^{-2} P\int \frac{dp_z}{\gamma^2}\frac{k\partial G_0/\partial p_z}{\omega_k - kv_z} \end{aligned}, \tag{71}$$

and the growth rate γ_k is given by

$$\gamma_k = -\frac{\hat{D}_i(k,\omega_k)}{\partial\hat{D}_r(k,\omega_k)/\partial\omega_k}, \tag{72}$$

where

$$\hat{D}_i(k,\omega_k) = -\frac{\pi}{2}\frac{\hat{\omega}_c^2}{c^2k_0^2}\bar{\gamma}mc^2\omega_p^2\bar{\gamma}^{-2}\int \frac{dp_z}{\gamma^2}\delta(\omega_k - kv_z)k\frac{\partial G_0}{\partial p_z}. \tag{73}$$

In Eqs. (71) and (73), the particle velocity v_z, momentum p_z, and energy γmc^2 are related by

$$v_z = p_z/\gamma m \quad , \quad \gamma mc^2 = (m^2c^4 + c^2p_z^2 + e^2\hat{B}^2/k_0^2)^{1/2}, \tag{74}$$

and the integrals in Eqs. (71) and (73) can be transformed from integrals over p_z to integrals over v_z by making use of

$$\frac{dv_z}{dp_z} = \frac{m^2c^4 + e^2\hat{B}^2/k_0^2}{\gamma^3m^3c^4}, \tag{75}$$

which follows from Eq. (74). Then $\hat{D}_i(k,\omega_k)$ can be expressed as

$$\hat{D}_i(k,\omega_k) = -\frac{\pi}{2}\frac{\hat{\omega}_c^2}{c^2k_0^2}\,\omega_p^2\gamma^{-3}mc^2\,\frac{m^3c^4}{(m^2c^4+e^2\hat{B}^2/k_0^2)}\,\frac{k}{|k|}\left[\gamma\frac{\partial G_0}{\partial p_z}\right]_{v_z=\frac{\omega_k}{k}} \tag{76}$$

In circumstances where the principal-value contribution in Eq. (71) is negligibly small, it follows that $\partial\hat{D}_r(k,\omega_k)/\partial\omega_k \simeq 2\omega_k$, and the growth rate γ_k in Eq. (72) reduces to

$$\gamma_k = \frac{\pi}{4}\frac{\hat{\omega}_c^2}{c^2k_0^2}\,\omega_p^2\gamma^{-3}m^2c^2\,\frac{1}{\left[1+\left(\frac{e\hat{B}/mc}{ck_0}\right)^2\right]}\,\frac{k}{\omega_k|k|}\left[\gamma\frac{\partial G_0}{\partial p_z}\right]_{v_z=\frac{\omega_k}{k}} \quad . \tag{77}$$

For waves excited with positive phase velocity $\omega_k/k > 0$, it follows from Eq. (77) that $\gamma_k \gtrless 0$ accordingly as $[\gamma\partial G_0/\partial p_z]_{v_z=\omega_k/k} \gtrless 0$. That is, waves with phase velocity in the region of positive momentum slope in $G_0(p_z,t)$ are amplified, corresponding to instability with $\gamma_k > 0$. (See Sec. V.B for discussion of a specific example.) For subsequent analysis of quasilinear stabilization in Sec. VI, it should be kept in mind that the linear growth rate $\gamma_k(t)$ in Eq. (77) varies slowly in time as the distribution function $G_0(p_z,t)$ evolves according to Eq. (30) in response to the amplifying field fluctuations.

B. Stability Properties for Gaussian $G_0(p_z)$

As a specific example, and for purposes of estimating the relative size of the various terms on the right-hand sides of Eqs. (68) and (71), it is useful to consider the case where the instantaneous distribution function is gaussian in p_z with

$$G_0(p_z) = \left(\frac{2}{\pi}\right)^{1/2}\frac{1}{\Delta p_z}\,\exp\left[-\frac{2(p_z-p_0)^2}{\Delta p_z^2}\right] \quad . \tag{78}$$

It is further assumed that the characteristic axial momentum spread Δp_z of the beam electrons is sufficiently small that the quantities v_z and $1/\gamma^2$ can be expanded about the point $p_z = p_0$, i.e.,

$$v_z \simeq v_0 + (p_z - p_0)\left[\frac{dv_z}{dp_z}\right]_{p_z = p_0} \quad ; \quad \frac{1}{\gamma^2} \simeq \frac{1}{\bar{\gamma}^2} + (p_z - p_0)\left[\frac{d\gamma^{-2}}{dp_z}\right]_{p_z = p_0} . \tag{79}$$

Here, the derivatives are evaluated by means of Eqs. (74) and (75). Substituting Eqs. (78) and (79) into Eq. (68), the effective dielectric function $\hat{D}(k,\Omega_k)$ can be expressed as

$$\hat{D}(k,\omega_k + i\gamma_k) = (\omega_k + i\gamma_k)^2 - c^2(k - k_0)^2 - \alpha_1\omega_p^2 + \frac{1}{2}\frac{\hat{\omega}_c^2}{c^2k_0^2}\alpha_3\omega_p^2$$

$$- 2\frac{\hat{\omega}_c^2}{c^2k_0^2}\frac{\bar{\gamma}mc^2\omega_p^2}{\Delta p_z\Delta v_z}\left[1 + \xi Z(\xi)\right] \tag{80}$$

$$+ 2\sqrt{2}\frac{\hat{\omega}_c^2}{c^2k_0^2}\frac{v_0\omega_p^2}{\Delta v_z}\left[\xi + \xi^2 Z(\xi)\right] ,$$

where $Z(\xi)$ is the plasma dispersion function

$$Z(\xi) \equiv \pi^{-1/2}\int_{-\infty}^{\infty} dx\,\frac{\exp(-x^2)}{x - \xi} \quad ; \quad \xi = 2^{1/2}\frac{(\omega_k + i\gamma_k - kv_0)}{k\Delta v_z} .$$

Here, Δv_z is the characteristic spread in axial velocity defined by $\Delta v_z = \Delta p_z(m^2c^4 + e^2\hat{B}^2/k_0^2)/\bar{\gamma}^3m^3c^4$.

In the interesting case where $|\xi| \approx 1$, Eq. (80) cannot be further simplified by using the asymptotic expansions of $Z(\xi)$ for large or small values of the argument. Comparing the two terms involving square brackets in Eq. (80), the ratio of these terms for $\xi = -1/\sqrt{2}$ (corresponding to maximum growth rate) is of order $\bar{\gamma}mc^2/v_0\Delta p_z \gtrsim p_0/\Delta p_z$, which is typically much larger than unity. Further, referring to tabulated values of the function $Z(\xi)$, it is found in the range of interest that the magnitude of $[1+\xi Z(\xi)]$ is of order unity $(=0.285 - i\,0.76$ for $\xi = -0.7)$. Thus, the integral contribution in Eq. (68) is of order $2(\hat{\omega}_c^2/c^2k_0^2)\bar{\gamma}mc^2\omega_p^2/\Delta p_z\Delta v_z \gtrsim 2(\hat{\omega}_c^2/c^2k_0^2)\omega_p^2\ (p_0/\Delta p_z)(v_0/\Delta v_z)$. For sufficiently small fractional momentum spread $\Delta p_z/p_0$, this term is larger than the $\alpha_1\omega_p^2$ contribution in Eq. (68), which in turn is typically larger than $(\hat{\omega}_c^2/c^2k_0^2)\alpha_3\omega_p^2$. Moreover, for the interaction to take place in a parameter regime consistent with the Compton approximation <u>and</u> the condition $|\gamma_k/k| \ll \Delta v_z$ [Eq. (45)], it is required that

$$\frac{\hat{\omega}_c^2}{c^2k_0^2}\ \frac{\bar{\gamma}mc^2}{\Delta p_z\Delta v_z}\ \frac{\omega_p^2}{\omega_k^2} \ll 1 \quad . \tag{81}$$

For example, for the parameters in the Stanford experiment,[16] the quantities $(\hat{\omega}_c^2/c^2k_0^2)$, ω_p^2, $2(\hat{\omega}_c^2/c^2k_0^2)\bar{\gamma}mc^2\omega_p^2/\Delta p_z\Delta v_z$ and $(k-k_0)^2c^2(\simeq\omega_k^2)$ are calculated to be $\sim 2.24\times10^{-4}$, $1.3\ \times 10^{17}\text{sec}^{-2}$, 10^{22}sec^{-2} and $3.16\times10^{28}\text{sec}^{-2}$, respectively. We conclude that $\omega_k^2 - c^2(k-k_0)^2 \simeq 0$ and $\partial\hat{D}_r/\partial\omega_k \simeq 2\omega_k$ are excellent approximations for the parameter regime under consideration. [See discussion prior to Eq. (77).]

Returning to expression (77) for the growth rate, we find that γ_k can be approximated by

$$\gamma_k = (2\pi)^{1/2}\ \frac{\omega_p^2}{\omega_k}\,\frac{\hat{\omega}_c^2}{c^2k_0^2}\,\frac{k}{|k|}\,\frac{\left[1+\left(\frac{e\hat{B}/mc}{ck_0}\right)^2\right]}{\bar{\gamma}^2}\ \frac{c^2}{\Delta v_z^2} \times\frac{(v_0-\omega_k/k)}{\Delta v_z}\ \exp\left\{-\frac{2(\omega_k/k-v_0)^2}{\Delta v_z^2}\right\} \tag{82}$$

for the choice of gaussian distribution in Eq. (78). Here, $\bar{\gamma} \equiv [1+p_0^2/m^2c^2 + (e\hat{B}/mc^2k_0)^2]^{1/2}$, Δv_z is related to Δp_z by $\Delta v_z = \Delta p_z[1+(e\hat{B}/mc^2k_0)^2]/\bar{\gamma}^3m$, and $v_0 \equiv p_0/\bar{\gamma}m$. Maximum growth in Eq. (82) occurs for $\omega_k/k - v_0 \simeq -\Delta v_z/2$

where

$$[\gamma_k]_{MAX} = \left(\frac{\pi}{2}\right)^{1/2} \frac{\omega_p^2}{\omega_k} \frac{\hat{\omega}_c^2}{c^2 k_0^2} \frac{k}{|k|} \frac{\left[1 + \left(\frac{e\hat{B}/mc}{ck_0}\right)^2\right]}{\bar{\gamma}^2} \left(\frac{c}{\Delta v_z}\right)^2 \exp(-0.5) \ .$$

As a numerical example, the normalized growth rate $\gamma_k/k_0 c$ in Eq. (77) is plotted versus k/k_0 in Fig. 1 for $G_0(p_z)$ corresponding to the gaussian distribution in Eq. (78). The choice of parameters in Fig. 1 is: $\omega_p^2 = 1.25 \times 10^{17} \sec^{-2}$, $\bar{\gamma} = 47.1$, $p_0 = 1.286 \times 10^{-15}$ g-cm/sec, $\Delta p_z = 3.6 \times 10^{-18}$ g-cm/sec, wiggler field amplitude $\hat{B} = 2.4$ kG, and wiggler wavenumber $k_0 = 1.96\ \text{cm}^{-1}$, with corresponding dimensionless parameters $e\hat{B}/mc^2 k_0 = 0.718$, $\omega_p^2/c^2 k_0^2 = 3.62 \times 10^{-5}$ and $\Delta p_z/p_0 = 2.8 \times 10^{-3}$. Maintaining $\bar{\gamma} = 47.1$ and $e\hat{B}/mc^2 k_0 = 0.721$, in Fig. 2 the fractional momentum spread and normalized beam density are increased to $\Delta p_z/p_0 = 2 \times 10^{-2}$ and $\omega_p^2/c^2 k_0^2 = 1.28 \times 10^{-2}$, and there is a concomitant significant increase in instability bandwidth and growth rate (compare Figs. 1 and 2). For $k \gg k_0$, it can be shown from Eqs. (77) and (78) that the wavenumber k_m corresponding to marginal stability ($\gamma_{k_m} = 0$) is given by

$$k_m = \frac{2(p_0/mc)^2 k_0}{1 + (e\hat{B}/mc^2 k_0)^2} \ .$$

For example, $k_m/k_0 = 2930$ for the parameters chosen in Figs. 1 and 2. Shown in Fig. 3 are plots of $\gamma_k/k_0 c$ versus k/k_0 for two values of $e\hat{B}/mc^2 k_0$, $\omega_p^2/c^2 k_0^2 = 8.86 \times 10^{-3}$, and parameters otherwise identical to Fig. 2. For the moderately large values of wiggler amplitude assumed in Fig. 3, there is a notable downshift in the range of unstable k-values as $\hat{B}$ is increased. This is also true in Fig. 4 where $\gamma_k/k_0 c$ is plotted versus k/k_0 for $\bar{\gamma} = 10$, $\Delta p_z/p_0 = 3 \times 10^{-2}$, $\omega_p^2/c^2 k_0^2 = 1.60 \times 10^{-3}$, and the two values of normalized wiggler amplitude $e\hat{B}/mc^2 k_0 = 0.782$ and $e\hat{B}/mc^2 k_0 = 0.879$.

VI. QUASILINEAR STABILIZATION PROCESS

A. General Theory

In this section, we examine the quasilinear development of the system and describe the stabilization process within the context of Eqs. (30), (44) and (77).

Equation (30) can be rewritten in the general form of a diffusion equation in momentum space, i.e.,

$$\frac{\partial}{\partial t} G_0(p_z,t) = \frac{\partial}{\partial p_z}\left[D(p_z,t)\frac{\partial}{\partial p_z} G_0(p_z,t)\right] , \qquad (83)$$

where the diffusion coefficient $D(p_z,t)$ is defined by

$$D(p_z,t) = \left(\frac{e\hat{B}}{2k_0}\right)^2\left(\frac{\bar{\gamma}}{\gamma}\right)^2 \sum_{k=-\infty}^{\infty}\left|\delta A^+_{k+k_0} + \delta A^-_{k-k_0}\right|^2 \exp\left(2\int_0^t \gamma_k(t')dt'\right)$$

$$\times \frac{k^2}{-i\omega_k + ikv_z + \gamma_k} .$$

Making use of the symmetries in Eqs. (24) and (A4), the diffusion coefficient $D(p_z,t)$ can also be expressed as

$$D(p_z,t) = 2\left(\frac{e\hat{B}}{2k_0}\right)^2\left(\frac{\bar{\gamma}}{\gamma}\right)^2 \sum_{k=0}^{\infty}\left|\delta A^+_{k+k_0} + \delta A^-_{k-k_0}\right|^2 \exp\left(2\int_0^t \gamma_k(t')dt'\right)$$

$$\times \frac{k^2\gamma_k}{(\omega_k - kv_z)^2 + \gamma_k^2} .$$

We note that in obtaining Eq. (67) from Eq. (40), it has been assumed that $|D_{k-k_0}| \ll |D_{k+k_0}|$. In view of Eq. (A3), it is therefore consistent to approximate $D(p_z,t)$ by

$$D(p_z,t) = 2\left(\frac{e\hat{B}}{2k_0}\right)^2\left(\frac{\bar{\gamma}}{\gamma}\right)^2\sum_{k=0}^{\infty}\left|\delta A^-_{k-k_0}\right|^2 \exp\left(2\int_0^t \gamma_k(t')dt'\right)\frac{k^2\gamma_k}{(\omega_k-kv_z)^2+\gamma_k^2} \ . \quad (84)$$

Similarly, the average energy density in the radiation fields [Eqs. (41) and (42)] can be approximated by

$$\left(\frac{\bar{\gamma}mc^2}{2e}\right)^2\frac{1}{2\pi c^2}\sum_{k=0}^{\infty}\left|\delta A^-_{k-k_0}\right|^2\left[|\Omega_k|^2 + c^2(k-k_0)^2\right]\exp\left(2\int_0^t\gamma_k(t')dt'\right) = \sum_{k=0}^{\infty}\mathcal{E}_k(t),$$

where the spectral energy density $\mathcal{E}_k(t)$ is defined by

$$\mathcal{E}_k(t) = \left(\frac{\bar{\gamma}mc^2}{2e}\right)^2\frac{1}{2\pi c^2}\left|\delta A^-_{k-k_0}\right|^2\left[|\Omega_k|^2 + c^2(k-k_0)^2\right]\exp\left(2\int_0^t\gamma_k(t')dt'\right) . \quad (85)$$

Approximating $|\Omega_k|^2 \simeq \omega_k^2$ in Eq. (85), and combining Eqs. (84) and (85) gives for the diffusion coefficient

$$D(p_z,t) = 4\pi e^2\left(\frac{\hat{\omega}_c}{ck_0}\right)^2\left(\frac{\bar{\gamma}}{\gamma}\right)^2\int_0^{\infty}dk\,\frac{k^2c^2\,\mathcal{E}_k(t)}{[\omega_k^2 + c^2(k-k_0)^2]}\ \frac{\gamma_k}{(\omega_k - kv_z)^2 + \gamma_k^2} \ . \quad (86)$$

Here, we have taken the continuum limit $\sum_{k=0}^{\infty} \to \int_0^{\infty} dk$, and introduced the cyclotron frequency $\hat{\omega}_c = e\hat{B}/\bar{\gamma}mc$ associated with the wiggler amplitude.

In relation to the diffusion coefficient $D(p_z,t)$ defined in Eq. (86), we distinguish two regions of momentum or velocity space, namely the <u>resonant</u> region corresponding to particle velocities that satisfy

$$v_z = \omega_k/k \equiv v_{res} \quad (87)$$

for those k and ω_k making up the spectrum of unstable waves ($\gamma_k > 0$), and the <u>nonresonant</u> region corresponding to particle velocities that satisfy

$$(kv_z - \omega_k)^2 \gg \gamma_k^2 \ . \quad (88)$$

In momentum space, the resonant region corresponds to the region of positive momentum slope of $G_0(p_z,t)$ [Eq. (77)].

Only waves interacting with particles in this range can be excited and amplify. The momentum p_z of a resonant particle is related to the phase velocity ω_k/k of the wave with which it interacts by

$$p_z = p_{res} \equiv \left[1 + \frac{e^2\hat{B}^2/m^2c^2}{c^2k_0^2}\right]^{1/2} \left(1 - \frac{\omega_k^2}{c^2k^2}\right)^{-1/2} m\,\frac{\omega_k}{k} \tag{89}$$

where use has been made of Eq. (74). Moreover, combining the simultaneous resonance conditions $kv_z = \omega_k$ and $\omega_k = (k - k_0)c$, the wavenumber k and the resonant particle velocity v_z are connected through the relation

$$k = k_0c/(c-v_z) \equiv \hat{k}(v_z). \tag{90}$$

For small γ_k, we write $D(p_z,t) \simeq D_r(p_z,t)$ in the <u>resonant</u> region of velocity space ($\omega_k - kv_z = 0$), where

$$\begin{aligned} D_r(p_z,t) &= 4\pi^2e^2\left(\frac{\hat{\omega}_c}{ck_0}\right)^2\left(\frac{\bar{\gamma}}{\gamma}\right)^2\int_0^\infty dk\,\frac{k^2c^2\,\mathcal{E}_k(t)}{[\omega_k^2 + c^2(k-k_0)^2]}\,\delta(\omega_k - kv_z) \\ &= 2\pi^2e^2\left(\frac{\hat{\omega}_c}{ck_0}\right)^2\left(\frac{\bar{\gamma}}{\gamma}\right)^2\left(\frac{c}{v_z}\right)^2\frac{1}{(c-v_z)}\,\mathcal{E}_{\hat{k}}(t). \end{aligned} \tag{91}$$

In Eq. (91), we have made use of the identity (for $\omega_k = kv_z$)

$$\begin{aligned} \lim_{\gamma_k \to 0^+} \frac{\gamma_k}{(\omega_k - kv_z)^2 + \gamma_k^2} &= \pi\delta(\omega_k - kv_z) \\ &= \frac{\pi}{(c-v_z)}\,\delta[k - \hat{k}(v_z)], \end{aligned}$$

and expressed $[\omega_k^2 + c^2(k-k_0)^2] = 2k^2v_z^2$ for $\omega_k = kv_z$ and $\omega_k = (k - k_0)c$. Moreover, $\hat{k}(v_z)$ is defined in Eq. (90).

In the nonresonant region of velocity space, where $(\omega_k - kv_z)^2 >> \gamma_k^2$, we approximate $D(p_z,t) \simeq D_{nr}(p_z,t)$, where

$$D_{nr}(p_z,t) = 4\pi e^2 \left(\frac{\hat{\omega}_c}{ck_0}\right)^2 \left(\frac{\bar{\gamma}}{\gamma}\right)^2 \int_0^\infty dk \frac{k^2c^2\,\mathcal{E}_k(t)}{[\omega_k^2+c^2(k-k_0)^2]} \frac{\gamma_k}{(\omega_k-kv_z)^2} . \tag{92}$$

The nonresonant region includes the range of p_z where $\partial G_0/\partial p_z < 0$ and the wave spectrum is damped in Eq. (77). Within the context of the same approximation that the principal-value contribution in the dispersion relation (71) is negligibly small, it also follows from Eqs. (91) and (92) that the effects of nonresonant diffusion can be neglected in comparison with resonant diffusion. That is, for $k^2\Delta v_z^2 >> \gamma_k^2$, we approximate $D_{nr} \simeq 0$ in analyzing the quasilinear development of the system.

The quasilinear growth rate $\gamma_k(t)$ given in Eq. (77) involves the electron distribution function in the resonant region. Moreover, the system of equations (44), (77), (83), and (91) reduces to the following two coupled equations for the evolution of the wave spectral energy density and the distribution function in the resonant region:

$$\frac{\partial}{\partial t} G_0(p_z,t) = \frac{\partial}{\partial p_z} \left[2\pi^2 e^2 \left(\frac{\hat{\omega}_c}{ck_0}\right)^2 \left(\frac{\bar{\gamma}}{\gamma}\right)^2 \left(\frac{c}{v_z}\right)^2 \frac{1}{(c-v_z)} \mathcal{E}_{\hat{k}}(t) \frac{\partial}{\partial p_z} G_0(p_z,t)\right] , \tag{93}$$

and

$$\frac{\partial}{\partial t} \mathcal{E}_{\hat{k}}(t) = \frac{\pi}{2}\left(\frac{\hat{\omega}_c}{ck_0}\right)^2 \omega_p^2\, \bar{\gamma}^{-3} m^2 c^2 \frac{\mathcal{E}_{\hat{k}}(t)}{\left[1+\left(\frac{e\hat{B}/mc}{ck_0}\right)^2\right]} \frac{(c-v_z)}{k_0 v_z c} \gamma \frac{\partial}{\partial p_z} G_0(p_z,t) . \tag{94}$$

Here, $\hat{k}(v_z) = k_0 c/(c-v_z)$, and p_z is related to ω_k/k by Eq. (89). From Eq. (94), for positive $\partial G_0/\partial p_z$ at the point p_z, $\mathcal{E}_{\hat{k}}$ increases in time. However, the diffusion coefficient appearing in Eq. (93) is proportional to $\mathcal{E}_{\hat{k}}$ and hence as $\mathcal{E}_{\hat{k}}(t)$ increases, $G_0(p_z,t)$ diffuses to decrease the gradient $\partial G_0/\partial p_z$, thereby reducing adiabatically the rate of increase of $\mathcal{E}_{\hat{k}}$. Consequently, the behavior of this pair of equations is such as to limit the amplitude of $\mathcal{E}_{\hat{k}}$ and reduce $\partial G_0/\partial p_z$ and the growth rate. The time-asymptotic state is one for which $(\partial/\partial p_z)G_0(p_z,\infty) = 0$ in the resonant region, corresponding to the formation of a plateau with $\gamma_k(\infty) = 0$, while $\mathcal{E}_k(t)$ increases from its initial value to some steady asymptotic level $\mathcal{E}_k(\infty)$. Combining Eqs. (93) and (94) yields a conservation relation between the electron distribution function and the spectral energy density generated by the instability, i.e.,

$$\frac{\partial}{\partial t}\left\{G_0(p_z,t) - \frac{\partial}{\partial p_z}\left(\frac{4\pi e^2 k_0}{\omega_p^2 \bar{\gamma}^4 m^2}\left[1+\left(\frac{e\hat{B}/mc}{ck_0}\right)^2\right] \times\left(\frac{\bar{\gamma}}{\gamma}\right)^3 \frac{c}{v_z}\frac{1}{(c-v_z)^2}\,\mathcal{E}_{\hat{k}}(t)\right)\right\} = 0 , \tag{95}$$

which may then be integrated to give

$$G_0(p_z,t) = G_0(p_z,0) + \frac{\partial}{\partial p_z}\left\{\frac{4\pi e^2 k_0}{\omega_p^2 \bar{\gamma}^4 m^2}\left[1+\left(\frac{e\hat{B}/mc}{ck_0}\right)^2\right] \times\left(\frac{\bar{\gamma}}{\gamma}\right)^3 \frac{c}{v_z}\frac{1}{(c-v_z)^2}\left[\mathcal{E}_{\hat{k}}(t) - \mathcal{E}_{\hat{k}}(0)\right]\right\} , \tag{96}$$

relating $G_0(p_z,t)$ and $\mathcal{E}_{\hat{k}}(t)$.

Equations (93) and (94), which can be combined and integrated to give Eq. (96), constitute the final quasilinear equations that describe the nonlinear evolution of $G_0(p_z,t)$ in the resonant region and the concomitant evolution and saturation of the spectral energy density $\mathcal{E}_{\hat{k}}(t)$.

B. Time-Asymptotic Saturation Level of Radiation Fields

We now make use of Eqs. (95) and (91) to obtain estimates of the time-asymptotic energy density in the electromagnetic wave spectrum, the efficiency η of conversion of beam energy into radiation energy, and the characteristic time τ_{rel} for plateau formation in momentum space. The model we adopt is one in which plateau formation occurs over the interval $p_{z1} < p_z < p_{z2}$, where $p_z = p_{z0} \equiv p_{z1} + (1/2)(p_{z2} - p_{z1})$ is the point of maximum momentum slope of $G_0(p_z,0)$. Moreover, the wave spectrum $\mathcal{E}_k(t)$ is excited substantially over the interval $\hat{k}_1 < k < \hat{k}_2$, where the phase velocity ω_k/k falls in the region of maximum slope. Here, $\hat{k}=k_0 c/(c-v_z)$, and $\hat{k}_1$ and $\hat{k}_2$ are defined by $\hat{k}_1 \simeq k_0 c/(c - v_{z1})$ and $\hat{k}_2 \simeq k_0 c/(c - v_{z2})$.

(a) Saturated Wave Energy: Assuming that the initial energy density $\mathcal{E}_k(0)$ in the wave spectrum is negligible in comparison with the saturated level $\mathcal{E}_k(\infty)$, the conservation relation (96) can be integrated over p_z to give

$$\mathcal{E}_{\hat{k}}(t \to \infty) = \frac{\omega_p^2 \gamma^{-4} m^2}{4\pi e^2 k_0} \left[1 + \left(\frac{e\hat{B}/mc}{ck_0}\right)^2\right]^{-1} \left(\frac{\gamma}{\bar{\gamma}}\right)^3 \frac{v_z}{c} \times (c - v_z)^2 \int_{p_{z1}}^{p_z} \left[G_0(p_z', t \to \infty) - G_0(p_z', t = 0)\right] dp_z' , \tag{97}$$

where $\hat{k}=k_0 c/(c - v_z)$, and the time-asymptotic distribution function $G_0(p_z', t \to \infty) =$ const. is assumed to be flat, corresponding to plateau formation over the interval $p_{z1} < p_z < p_{z2}$. To estimate the right-hand side of Eq. (97), we Taylor expand the initial distribution function $G_0(p_z', t = 0)$ about p_{z0}, the point of maximum momentum slope. Since $[(\partial^2/\partial p_z'^2) G_0(p_z',0)]_{p_{z0}} = 0$, we express, correct to second order,

$$G_0(p_z',t=0) = G_0(p_{z0},t=0) + \left[\frac{\partial}{\partial p_z'} G_0(p_z',0)\right]_{p_{z0}} (p_z' - p_{z0}) + \ldots \quad . \tag{98}$$

Moreover, we take $G_0(p_z',t\to\infty) = G_0(p_{z0},t=0)$ in Eq. (97). Substituting Eq. (98) into Eq. (97), we find for the asymptotic spectral energy density

$$\mathcal{E}_{\hat{k}}(\infty) = \frac{\omega_p^2 \gamma^{-4} m^2}{8\pi e^2 k_0} \left[1 + \left(\frac{e\hat{B}/mc}{ck_0}\right)^2\right]^{-1} \left(\frac{\gamma}{\bar{\gamma}}\right)^3 \frac{v_z}{c}$$
$$\times (c - v_z)^2 (-p_z^2 + 2p_z p_{z0} + p_{z1}^2 - 2p_{z0}p_{z1}) \left[\frac{\partial G_0(p_z,0)}{\partial p_z}\right]_{p_{z0}} . \tag{99}$$

Making use of Eq. (99), the total saturated wave energy density $\mathcal{E}_F(\infty)$ is given by

$$\mathcal{E}_F(\infty) = \int_0^\infty d\hat{k}\mathcal{E}_{\hat{k}}(\infty) = \int_{p_{z1}}^{p_{z2}} dp_z \frac{d\hat{k}}{dp_z} \mathcal{E}_{\hat{k}}(\infty)$$
$$= \int_{p_{z1}}^{p_{z2}} dp_z \, \mathcal{E}_{\hat{k}}(\infty) \frac{k_0 c}{(c - v_z)^2} \frac{m^2c^4 + e^2\hat{B}^2/k_0^2}{\gamma^3 m^3 c^4}$$
$$= \frac{1}{2} n_0 v_0 \left[\frac{\partial}{\partial p_z} G_0(p_z,0)\right]_{p_{z0}} \tag{100}$$
$$\times \left[(p_{z2} - p_{z1})(p_{z0} - p_{z1})^2 - \frac{1}{3}(p_{z2} - p_{z0})^3 - \frac{1}{3}(p_{z0} - p_{z1})^3\right] .$$

In obtaining Eq. (100), use has been made of Eqs. (75) and (90), and we have approximated the factor v_z/c by $v_0/c = p_0/\bar{\gamma}mc$ in Eq. (99), corresponding to a narrow fractional velocity spread. Denoting $p_{z2}-p_{z1}=\Delta p_z$, and assuming that p_{z0} is centered between p_{z1} and p_{z2} with

$p_{z0} = p_{z1} + \Delta p_z/2 = p_{z2} - \Delta p_z/2$, then the saturated wave energy density in Eq. (100) can be expressed in the convenient form

$$\mathcal{E}_F(\infty) = \frac{1}{12} n_0 v_0 (\Delta p_z)^3 \left[\frac{\partial}{\partial p_z} G_0(p_z,0)\right]_{p_{z0}} \quad . \tag{101}$$

Equation (101) can be used to estimate $\mathcal{E}_F(\infty)$ for a wide range of initial distributions $G_0(p_z,0)$ corresponding to weak resonant instability. Here, keep in mind that p_{z0} corresponds to the point of maximum slope of $G_0(p_z,0)$. As an example, consider the gaussian distribution in Eq. (78) where $p_{z0} = p_0 - (1/2)\Delta p_z$. In this case $(\partial G_0/\partial p_z)_{p_{z0}} = (8/\pi)^{1/2}(1/\Delta p_z)^2 \exp(-0.5)$ and Eq. (101) reduces to

$$\mathcal{E}_F(\infty) = \frac{1}{12}\left(\frac{8}{\pi}\right)^{1/2} \exp(-0.5) n_0 v_0 (\Delta p_z). \tag{102}$$

Note from Eq. (102) that $\mathcal{E}_F(\infty)$ can be substantial, depending on the momentum spread Δp_z.

(b) <u>Efficiency of Radiation Generation</u>: We define the efficiency of radiation generation η as the ratio of saturated wave energy density $\mathcal{E}_F(\infty)$ to the beam kinetic energy density $(\bar{\gamma}-1)n_0 mc^2$. Making use of Eq. (101), this gives

$$\begin{aligned}\eta &= \frac{\mathcal{E}_F(\infty)}{(\bar{\gamma}-1)n_0 mc^2} \\ &= \frac{1}{12}\ \frac{v_0(\Delta p_z)^3}{(\bar{\gamma}-1)mc^2}\left[\frac{\partial}{\partial p_z} G_0(p_z,0)\right]_{p_{z0}} ,\end{aligned} \tag{103}$$

which can be used to calculate η for specified $G_0(p_z,0)$. For highly relativistic electrons, $v_0 \simeq c$ and $(\bar{\gamma}-1)mc^2 \simeq cp_0$, and Eq. (103) can be approximated by

$$\eta = \frac{1}{12}\,\frac{(\Delta p_z)^3}{p_0}\left[\frac{\partial}{\partial p_z}\,G_0(p_z,0)\right]_{p_{z0}} \quad . \tag{104}$$

If further we estimate $(\partial G_0/\partial p_z)_{p_{z0}} \approx 1/(\Delta p_z)^2$, then Eq. (104) gives for the efficiency

$$\eta \approx \frac{1}{12}\,\frac{\Delta p_z}{p_0} \quad . \tag{105}$$

(c) Time Scale for Plateau Formation: From $(\partial G_0/\partial t)=(\partial/\partial p_z)(D_r \partial G_0/\partial p_z)$, the characteristic time τ_{rel} for plateau formation in the resonant region is approximated by

$$\tau_{rel} \approx \frac{(\Delta p_z)^2}{D_r(p_{z0},\infty)} \quad , \tag{106}$$

where

$$D_r(p_z,\infty) = 2\pi^2 e^2 \left(\frac{\hat{\omega}_c}{ck_0}\right)^2 \left(\frac{\bar{\gamma}}{\gamma}\right)^2 \left(\frac{c}{v_z}\right)^2 \int_0^\infty dk\, \mathcal{E}_k(\infty)\,\delta(\omega_k - kv_z)$$

$$\simeq \frac{\pi}{2}\,\frac{\omega_p^2}{ck_0}\left(\frac{\hat{\omega}_c}{ck_0}\right)^2 \left(\frac{\bar{\gamma}}{\gamma}\right)^2 \left(\frac{c}{v_z}\right)^2 \frac{m\bar{\gamma}}{n_0}\,\frac{\hat{k}}{\Delta\hat{k}}\,\mathcal{E}_F(\infty) \quad . \tag{107}$$

In Eq. (107), $\hat{k}=k_0 c/(c-v_z)$, use has been made of Eq. (91), and we have expressed $\mathcal{E}_F(\infty)= \mathcal{E}_{\hat{k}}(\infty)\Delta\hat{k}$, where $\Delta\hat{k}$ is the characteristic width of the unstable k-spectrum. In Eq. (107), we estimate $\Delta\hat{k} \simeq k_0(\Delta v_z/c)(1-v_z/c)^{-2}$ and $(\Delta\hat{k})/\hat{k} \simeq (1+v_z/c)(\Delta v_z/c)(1-v_z^2/c^2)^{-1}$, and evaluate $D_r(p_{z0},\infty)$ with $\gamma \simeq \bar{\gamma}$, $v_z \simeq v_0$ and $\Delta v_z \simeq \Delta p_z[1+(e\hat{B}/mc^2k_0)^2]/\bar{\gamma}^3 m$, where $v_0 = p_0/\bar{\gamma}m$ and $\bar{\gamma} = [1+p_0^2/m^2c^2+(e\hat{B}/mc^2k_0)^2]^{1/2}$. After some straightforward algebra, this gives

$$D_r(p_{z0},\infty) \simeq \frac{\pi}{2}\,\frac{\omega_p^2\hat{\omega}_c^2}{c^3k_0^3}\left(\frac{c}{v_0}\right)^2 \frac{(\bar{\gamma}m)^2 c}{(1+v_0/c)n_0}\,\frac{\mathcal{E}_F(\infty)}{\Delta p_z} \quad . \tag{108}$$

Estimating $(\partial G_0/\partial p_z)_{p_{z0}} \approx 1/(\Delta p_z)^2$ in Eq. (101) gives $\mathcal{E}_F(\infty) \approx n_0 v_0(\Delta p_z)/12$, and substituting Eq. (108) into Eq. (106) then gives

$$\tau_{rel} \approx \frac{24}{\pi} \frac{c^3 k_0^3}{\omega_p^2 \hat{\omega}_c^2} \left(1 + \frac{v_0}{c}\right)\left(\frac{v_0}{c}\right)^3 \left(\frac{\Delta p_z}{p_0}\right)^2 . \tag{109}$$

Equation (109) gives a convenient order-of-magnitude estimate of the characteristic time scale for plateau formation in the resonant region in terms of ck_0, ω_p, $\hat{\omega}_c$, and the fractional momentum spread $\Delta p_z/p_0$. In the limit of highly relativistic electrons, $v_0 \simeq c$ and Eq. (109) can be approximated by

$$\tau_{rel} \approx \frac{48}{\pi} \frac{c^3 k_0^3}{\omega_p^2 \hat{\omega}_c^2} \left(\frac{\Delta p_z}{p_0}\right)^2 . \tag{110}$$

(d) Range of Validity: It is important to quantify the range of validity of the quasilinear analysis in Secs. V and VI which assumes weak resonant instability. First, the fractional momentum spread of the beam electrons is assumed small with

$$\frac{\Delta p_z}{p_0} << 1 . \tag{111}$$

In terms of the width of the wave spectrum, we estimate $\Delta\hat{k} \simeq k_0(\Delta v_z/c) \times (1-v_z/c)^{-2}$, where $\hat{k} = k_0(1-v_z/c)^{-1}$ and the velocity spread and momentum spread are related by $\Delta v_z \simeq \Delta p_z[1 + (e\hat{B}/mc^2 k_0)^2]/\bar{\gamma}^3 m$. Estimating $v_z \simeq v_0 = p_0/\bar{\gamma}m$, where $\bar{\gamma} = [1 + p_0^2/m^2c^2 + (e\hat{B}/mc^2 k_0)^2]^{1/2}$, we readily obtain for the characteristic width of the wave spectrum

$$\frac{\Delta\hat{k}}{\hat{k}} \simeq \left(1 + \frac{v_0}{c}\right) \frac{v_0}{c} \frac{\Delta p_z}{p_0} << 1. \tag{112}$$

Note that the spectrum is narrow by virtue of Eq. (111).

The analysis in Secs. V and VI also assumes $|\gamma_k/k\Delta v_z| \ll 1$. We estimate the maximum growth rate from Eq. (77) by evaluating $\partial G_0/\partial p_z$ at the point of maximum initial slope ($p_z = p_{z0}$) with $[\partial G_0/\partial p_z]_{p_z=p_{z0}} \approx 1/(\Delta p_z)^2$. This gives

$$\frac{\gamma_{\hat{k}}}{\hat{k}\Delta v_z} \simeq \frac{\pi}{4}\frac{\hat{\omega}_c^2\omega_p^2}{c^4k_0^4}\frac{\bar{\gamma}^2}{[1+(e\hat{B}/mc^2k_0)^2]}\left(\frac{c}{v_0}\right)^2\left(\frac{p_0}{\Delta p_z}\right)^2\left(\frac{k_0^2c^2}{\omega_{\hat{k}}\hat{k}\Delta v_z}\right)$$

where $v_0 = p_0/\bar{\gamma}m$. From $\omega_{\hat{k}} \simeq (\hat{k}-k_0)c$, $\hat{k} = k_0(1-v_z/c)^{-1}$, $\Delta v_z \simeq \Delta p_z[1 + (e\hat{B}/mc^2k_0)^2]/\bar{\gamma}^3m$, and $v_z \simeq v_0$, we obtain the characteristic value at maximum growth

$$\frac{\gamma_{\hat{k}}}{\hat{k}\Delta v_z} \simeq \frac{\pi}{4}\frac{\hat{\omega}_c^2\omega_p^2}{c^4k_0^4}\frac{1}{(1+v_0/c)^2(v_0/c)^4}\left(\frac{p_0}{\Delta p_z}\right)^3 . \tag{113}$$

Therefore, the condition $|\gamma_k/k\Delta v_z| \ll 1$ imposes the restriction that the beam density and wiggler amplitude be sufficiently small that

$$\frac{\pi}{4}\frac{\hat{\omega}_c^2\omega_p^2}{c^4k_0^4} \ll \left(1+\frac{v_0}{c}\right)^2\left(\frac{v_0}{c}\right)^4\left(\frac{\Delta p_z}{p_0}\right)^3 . \tag{114}$$

Finally, an important condition for validity of the quasilinear analysis is that the wave spectrum be sufficiently broad that the wave autocorrelation time $\tau_{ac} \simeq |\Delta(\omega_k - kv_z)|^{-1}$ be short in comparison with the time τ_{rel} for quasilinear relaxation [Eq. (45)]. Estimating $\Delta(\omega_k - kv_z) \simeq (c - v_z)\Delta\hat{k}$, and making use of Eq. (112) for $\Delta\hat{k}$ and Eq. (109) for τ_{rel}, the inequality $\tau_{ac} \ll \tau_{rel}$ can be expressed in the equivalent form

$$\frac{\omega_p^2\hat{\omega}_c^2}{c^4k_0^4} \ll \frac{24}{\pi}\left(1+\frac{v_0}{c}\right)^2\left(\frac{v_0}{c}\right)^4\left(\frac{\Delta p_z}{p_0}\right)^3 . \tag{115}$$

Apart from numerical factors, Eq. (115) is similar to the requirement in Eq. (114) that $|\gamma_k/k\Delta v_z| \ll 1$.

VII. CONCLUSIONS

In this paper, a quasilinear model was developed that describes the nonlinear evolution and stabilization of the free electron laser instability in circumstances where a broad spectrum of waves is excited. The relativistic electron beam propagates perpendicular to a helical wiggler magnetic field $\underset{\sim}{B}_0 = -\hat{B}\cos k_0 z\,\hat{\underset{\sim}{e}}_x - \hat{B}\sin k_0 z\,\hat{\underset{\sim}{e}}_y$, and the analysis is based on the Vlasov-Maxwell equations assuming $\partial/\partial x = 0 = \partial/\partial y$ and a sufficiently tenuous beam that the Compton-regime approximation is valid ($\delta\phi \simeq 0$). Coupled kinetic equations were derived that describe the evolution of the average distribution function $G_0(p_z,t)$ and spectral energy density $\mathcal{E}_k(t)$ in the amplifying electromagnetic field perturbations. Following a thorough exposition of the theoretical model and general quasilinear formalism (Secs. II - IV), we examined the stabilization process for weak resonant instability with small temporal growth rate γ_k satisfying $|\gamma_k/\omega_k| << 1$ and $|\gamma_k/k\Delta v_z| << 1$ (Secs. V and VI). Assuming that the beam electrons have small fractional momentum spread ($\Delta p_z/p_0 << 1$), we investigated the process of quasilinear stabilization by plateau formation in the resonant region of velocity space ($\omega_k - kv_z = 0$), including estimates of the saturated field energy, efficiency of radiation generation, etc. [Eqs. (101) and (103)].

As a final point, it should be emphasized that the analysis in Sec. VI has focussed on the (relatively fast) process of plateau formation in the resonant region of velocity space. Therefore, at least within a quasilinear model, if there is to be efficiency enhancement above the level associated with plateau formation in Eq. (103), it is necessarily associated with a long-time quasilinear degradation of the beam distribution that occurs on a time scale $t > \tau_{rel}$, which is beyond the scope of the analysis presented in Sec. VI.

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FIGURE CAPTIONS

Fig. 1. Plot of normalized growth rate γ_k/k_0c versus k/k_0 [Eq. (77)] for $e\hat{B}/mc^2k_0 = 0.718$, $\omega_p^2/c^2k_0^2 = 3.62 \times 10^{-5}$, $\bar{\gamma} = 47.1$, $p_0 = 1.286 \times 10^{-15}$ g-cm/sec, and $\Delta p_z/p_0 = 2.8 \times 10^{-3}$.

Fig. 2. Plot of normalized growth rate γ_k/k_0c versus k/k_0 [Eq. (77)] for $e\hat{B}/mc^2k_0 = 0.718$, $\omega_p^2/c^2k_0^2 = 1.28 \times 10^{-2}$, $\bar{\gamma}=47.1$, $p_0=1.286 \times 10^{-15}$ g-cm/sec, and $\Delta p_z/p_0 = 2 \times 10^{-2}$.

Fig. 3. Plots of normalized growth rate γ_k/k_0c versus k/k_0 [Eq. (77)] for $\omega_p^2/c^2k_0^2 = 8.86 \times 10^{-3}$, $\bar{\gamma} = 47.1$, $p_0 = 1.286 \times 10^{-15}$ g-cm/sec, $\Delta p_z/p_0 = 2 \times 10^{-2}$, and two values of normalized wiggler amplitude: $e\hat{B}/mc^2k_0 = 0.718$ and $e\hat{B}/mc^2k_0 = 0.823$, corresponding to $\hat{B}$=2.4 kG and $\hat{B}$=2.75 kG.

Fig. 4. Plots of normalized growth rate γ_k/k_0c versus k/k_0 [Eq. (77)] for $\omega_p^2/c^2k_0^2 = 1.60 \times 10^{-3}$, $\bar{\gamma} = 10$, $p_0 = 2.71 \times 10^{-16}$ g-cm/sec, $\Delta p_z/p_0 = 3 \times 10^{-2}$, and two values of normalized wiggler amplitude: $e\hat{B}/mc^2k_0 = 0.782$ and $e\hat{B}/mc^2k_0 = 0.879$, corresponding to $\hat{B} = 4$ kG and $\hat{B} = 4.5$ kG.

APPENDIX A

AMPLITUDE SYMMETRIES IN QUASILINEAR THEORY

The matrix dispersion equation (33), which relates the amplitudes $\delta A^{+}_{k+k_0}$ and $\delta A^{-}_{k-k_0}$ appearing in Eq. (25), is equivalent to the equations

$$D_{k+k_0}\delta A^{+}_{k+k_0} + \frac{1}{2}\frac{\hat{\omega}_c^2}{c^2k_0^2}(\alpha_3\omega_p^2+\chi_k)(\delta A^{+}_{k+k_0}+\delta A^{-}_{k-k_0}) = 0 \ , \tag{A1}$$

$$\frac{1}{2}\frac{\hat{\omega}_c^2}{c^2k_0^2}(\alpha_3\omega_p^2+\chi_k)(\delta A^{+}_{k+k_0}+\delta A^{-}_{k-k_0})+D_{k-k_0}\delta A^{-}_{k-k_0} = 0 \ . \tag{A2}$$

These equations can be rewritten in the form

$$\frac{\delta A^{+}_{k+k_0}}{\delta A^{-}_{k-k_0}} = -\frac{D_{k-k_0}}{\frac{1}{2}\frac{\hat{\omega}_c^2}{c^2k_0^2}(\alpha_3\omega_p^2+\chi_k)} - 1 \ ,$$

$$\frac{\delta A^{-}_{k-k_0}}{\delta A^{+}_{k+k_0}} = -\frac{D_{k+k_0}}{\frac{1}{2}\frac{\hat{\omega}_c^2}{c^2k_0^2}(\alpha_3\omega_p^2+\chi_k)} - 1 \ ,$$

from which it follows that

$$\delta A^{+}_{k+k_0}/\delta A^{-}_{k-k_0} = D_{k-k_0}/D_{k+k_0} \ . \tag{A3}$$

The Fourier amplitudes $\delta A^{+}_{k+k_0}$ and $\delta A^{-}_{k-k_0}$ can be expressed as

$$\delta A^{\pm}_{k\pm k_0} = \frac{e}{\bar{\gamma}mc^2}\left[\delta A_x(k\pm k_0,t)\pm i\delta A_y(k\pm k_0,t)\right]\exp\left(i\int_0^t \Omega_k(t')dt'\right) \ ,$$

and therefore obey the reality conditions

$$(\delta A^{+}_{k+k_0})^* = \delta A^{-}_{-k-k_0}, \quad (\delta A^{-}_{k-k_0})^* = \delta A^{+}_{-k+k_0} . \tag{A4}$$

Furthermore, from Eqs. (24), (34), and (35), the dielectric functions D_{k+k_0} and D_{k-k_0} satisfy the conjugate symmetries (since $\omega_{-k} = -\omega_k$ and $\gamma_{-k} = \gamma_k$)

$$D^*_{k+k_0} = D_{-k-k_0} , \quad D^*_{k-k_0} = D_{-k+k_0} . \tag{A5}$$

Making use of Eqs. (A3) and (A4), the left-hand side of Eq. (B4) can be expressed as

$$|\delta A^{+}_{k+k_0} + \delta A^{-}_{k-k_0}|^2 \frac{D_{k+k_0} D_{k-k_0}}{(D_{k+k_0} + D_{k-k_0})}$$

$$= \left(|\delta A^{+}_{k+k_0}|^2 + |\delta A^{-}_{k-k_0}|^2 + \delta A^{+}_{k+k_0} \delta A^{+}_{-k+k_0} + \delta A^{-}_{k-k_0} \delta A^{-}_{-k-k_0}\right) \frac{D_{k+k_0} D_{k-k_0}}{(D_{k+k_0} + D_{k-k_0})}$$

$$= \left(|\delta A^{+}_{k+k_0}|^2 + |\delta A^{-}_{k-k_0}|^2 + |\delta A^{+}_{k+k_0}|^2 \frac{D_{-k-k_0}}{D_{-k+k_0}} + |\delta A^{-}_{k-k_0}|^2 \frac{D_{-k+k_0}}{D_{-k-k_0}}\right)$$

$$\times \frac{D_{k+k_0} D_{k-k_0}}{(D_{k+k_0} + D_{k-k_0})} .$$

Substituting Eqs. (A3) and (A5), the right-hand side of the above equation becomes

$$\left(|\delta A^{+}_{k+k_0}|^2 + |\delta A^{-}_{k-k_0}|^2\right) \frac{D_{k+k_0} D_{k-k_0}}{(D_{k+k_0} + D_{k-k_0})}$$

$$+ |\delta A^{-}_{k-k_0}|^2 \frac{D^2_{k-k_0}}{(D_{k+k_0} + D_{k-k_0})} + |\delta A^{+}_{k+k_0}|^2 \frac{D^2_{k+k_0}}{(D_{k+k_0} + D_{k-k_0})} .$$

Hence we obtain

$$|\delta A^{+}_{k+k_0} + \delta A^{-}_{k-k_0}|^2 \frac{D_{k+k_0} D_{k-k_0}}{(D_{k+k_0} + D_{k-k_0})} = D_{k+k_0} |\delta A^{+}_{k+k_0}|^2 + D_{k-k_0} |\delta A^{-}_{k-k_0}|^2 . \tag{A6}$$

It can also be shown, by a similar derivation, that

$$(\delta A^{+}_{k+k_0}+\delta A^{-}_{k-k_0})(\delta A^{+}_{k+k_0}-\delta A^{-}_{k-k_0})^{*}\frac{D_{k+k_0}D_{k-k_0}}{(D_{k+k_0}+D_{k-k_0})}$$
$$= D_{k+k_0}|\delta A^{+}_{k+k_0}|^{2}-D_{k-k_0}|\delta A^{-}_{k-k_0}|^{2} . \qquad \text{(A7)}$$

Equations (A6) and (A7) are used in Sec. IV.B and Appendix B. As can be seen from Eq. (40), employing the dispersion relation to eliminate $\chi_k(\Omega_k)$ leads to the appearance of the combination of dielectric functions which occurs on the left-hand side of Eqs. (A6) and (A7). We then make use of Eqs. (A1) - (A5) in order to reduce the expressions in Eqs. (B3) and (B8) to the forms given in Eqs. (B5) and (B11).

APPENDIX B

PROOF OF ENERGY AND MOMENTUM CONSERVATION RELATIONS FROM QUASILINEAR THEORY

Here, we make use of the quasilinear kinetic equations (29) and (44) and the dispersion relation (40) to verify energy conservation [Eqs. (61) and (62)] and momentum conservation [Eqs. (65) and (66)]. All terms are expressed in terms of sums over k involving the dimensionless amplitudes $\delta A^{\pm}_{k\pm k_0}$, the wavenumbers $k\pm k_0$, and the complex oscillation frequency $\Omega_k(t)$ that solves the dispersion relation (40) adiabatically in time.

Energy Conservation: To verify Eq. (61), we proceed by taking the derivative of Eq. (62) with respect to time, and substituting Eq. (29) for $\partial G_0/\partial t$ into the first term on the right-hand side of Eq. (62). The perturbed distribution function δG appearing in the second term is obtained from Eqs. (21) and (28). In the three last terms on the right-hand side of Eq. (62), δA_x and δA_y are expanded according to Eq. (25), and the spatial average is taken, making use of $\int_{-L}^{L} \exp(ikz)\exp(ik'z)dz = 2L\delta_{k,-k'}$, where $\delta_{k,-k'}$ is the Kronecker delta. Moreover, we employ the symmetries $\omega_{-k} = -\omega_k$, $\gamma_{-k} = \gamma_k$ [Eq. (24)] and $(\delta A^{+}_{k+k_0})^* = \delta A^{-}_{-k-k_0}$, $(\delta A^{-}_{k-k_0})^* = \delta A^{+}_{-k+k_0}$ [Eq. (A4)]. Further, in the last two terms, use is made of the definitions in Eq. (37). After simplifying, this gives

$$\frac{\partial}{\partial t}\langle KED\rangle = -n_0 i \left(\frac{e\hat{B}}{2k_0}\right)^2 \sum_k |\delta A^{+}_{k+k_0} + \delta A^{-}_{k-k_0}|^2 \exp\left(2\int_0^t \gamma_k(t')dt'\right)$$

$$\times\ \Omega_k \bar{\gamma}^2 \int_{-\infty}^{\infty} \frac{dp_z}{\gamma^2} \frac{k\partial G_0/\partial p_z}{\Omega_k - kv_z} - n_0 \left(\frac{e\hat{B}}{2k_0}\right)^2 \frac{\partial}{\partial t} \sum_k |\delta A^{+}_{k+k_0} + \delta A^{-}_{k-k_0}|^2$$

$$\times\ \exp\left(2\int_0^t \gamma_k(t')dt'\right)\bar{\gamma}^2 \int_{-\infty}^{\infty} \frac{dp_z}{\gamma^2} \frac{k\partial G_0/\partial p_z}{\Omega_k - kv_z}$$

$$+ n_0\alpha_1 \frac{\bar{\gamma}mc^2}{4} \frac{\partial}{\partial t} \sum_k \left[|\delta A^+_{k+k_0}|^2 + |\delta A^-_{k-k_0}|^2 \right] \exp\left(2\int_0^t \gamma_k(t')dt' \right)$$

$$- \frac{1}{2} n_0 \left(\frac{e\hat{B}}{2k_0}\right)^2 \frac{\alpha_3}{\bar{\gamma}mc^2} \frac{\partial}{\partial t} \sum_k |\delta A^+_{k+k_0} + \delta A^-_{k-k_0}|^2 \exp\left(2\int_0^t \gamma_k(t')dt' \right) . \tag{B1}$$

In evaluating the right-hand side of Eq. (62) and obtaining Eq. (B1), it has not been assumed that[33]

$$<[\delta A_x(z,t)]^2> = <[\delta A_y(z,t)]^2> , \quad <\delta A_x(z,t)\delta A_y(z,t)> = 0 . \tag{B2}$$

The conditions in Eq. (B2) correspond to the requirement that the field fluctuations are excited in an axisymmetric manner. However, it can be shown from Eq. (25) that this assumption would not be justified in the present analysis. The dispersion relation [Eq. (40)] is substituted into Eq. (B1) in order to eliminate the integrals over momentum in favor of the dielectric functions D_{k+k_0} and D_{k-k_0}. Further, combining all terms involving α_3, we find that these terms make zero net contribution to the sum over k. Collecting terms then gives

$$\frac{\partial}{\partial t} \langle KED \rangle = \left(\frac{\bar{\gamma}mc^2}{2e}\right)^2 \frac{1}{2\pi c^2} \sum_k (i\omega_k + \gamma_k) |\delta A^+_{k+k_0} + \delta A^-_{k-k_0}|^2$$

$$\times \exp\left(2\int_0^t \gamma_k(t')dt' \right) \frac{D_{k+k_0} D_{k-k_0}}{(D_{k+k_0} + D_{k-k_0})} \tag{B3}$$

$$+ n_0\alpha_1 \frac{\bar{\gamma}mc^2}{4} \sum_k 2\gamma_k \left(|\delta A^+_{k+k_0}|^2 + |\delta A^-_{k-k_0}|^2 \right) \exp\left(2\int_0^t \gamma_k(t')dt' \right) .$$

We now make use of the relation

$$|\delta A^{+}_{k+k_0} + \delta A^{-}_{k-k_0}|^2 \frac{D_{k+k_0} D_{k-k_0}}{(D_{k+k_0} + D_{k-k_0})} = D_{k+k_0} |\delta A^{+}_{k+k_0}|^2 + D_{k-k_0} |\delta A^{-}_{k-k_0}|^2, \tag{B4}$$

which is proved in Appendix A [Eq. A6)]. Substituting Eqs. (B4), (34), (35) and (24) into Eq (B3), and eliminating terms in the k-summations which are odd functions of k, yields

$$\frac{\partial}{\partial t} \langle KED \rangle = - \left(\frac{\bar{\gamma} mc^2}{2e}\right)^2 \frac{1}{4\pi c^2} \frac{\partial}{\partial t} \sum_k \exp\left(2\int_0^t \gamma_k(t')dt'\right)$$

$$\times \left\{ |\delta A^{+}_{k+k_0}|^2 \left[|\Omega_k|^2 + c^2 (k+k_0)^2 \right] + |\delta A^{-}_{k-k_0}|^2 \right. \tag{B5}$$

$$\left. \times \left[|\Omega_k|^2 + c^2 (k-k_0)^2 \right] \right\} .$$

The right-hand side of Eq. (61) can be evaluated by making use of Eqs. (3), (21) and (25). It is straightforward to show that

$$- \frac{\partial}{\partial t} < \frac{1}{8\pi} \left[(\delta \underset{\sim}{E}_T)^2 + (\underset{\sim}{B}_0 + \delta \underset{\sim}{B}_T)^2 \right] >$$

$$= - \frac{1}{8\pi} \frac{\partial}{\partial t} \sum_k \left(\frac{1}{c^2} \frac{\partial}{\partial t} \delta A_x(k,t) \frac{\partial}{\partial t} \delta A^{*}_x(k,t) + \frac{1}{c^2} \frac{\partial}{\partial t} \delta A_y(k,t) \frac{\partial}{\partial t} \delta A^{*}_y(k,t) \right.$$

$$\left. + k^2 |\delta A_x(k,t)|^2 + k^2 |\delta A_y(k,t)|^2 \right) \tag{B6}$$

$$= - \left(\frac{\bar{\gamma} mc^2}{2e}\right)^2 \frac{1}{4\pi c^2} \frac{\partial}{\partial t} \sum_k \exp\left(2\int_0^t \gamma_k(t')dt'\right)$$

$$\times \left\{ |\delta A^{+}_{k+k_0}|^2 \left[|\Omega_k|^2 + c^2 (k+k_0)^2 \right] + |\delta A^{-}_{k-k_0}|^2 \left[|\Omega_k|^2 + c^2 (k-k_0)^2 \right] \right\} .$$

Comparing Eqs. (B5) and (B6) completes the proof of Eq. (61). Note that the applied field $\underset{\sim}{B}_0$ does not occur explicitly on the right-hand side of Eq. (B6). This is because of the identify $(8\pi)^{-1}\int_{-L}^{L} dz(\underset{\sim}{B}_0 + \delta\underset{\sim}{B}_T)^2 = (8\pi)^{-1}\int_{-L}^{L} dz(\delta\underset{\sim}{B}_T)^2 + (8\pi)^{-1}\,\hat{B}^2(2L)$, which follows from the form of the wiggler field in Eq. (2), for $k \neq \pm k_0$.

Momentum Conservation: Rearranging the terms in Eq. (66), employing Eqs. (21), (25), (28), and (37), and taking the spatial average yields

$$\frac{\partial}{\partial t}\langle PMD\rangle = -\,n_0 i\left(\frac{e\hat{B}}{2k_0}\right)^2 \sum_k \exp\left(2\int_0^t \gamma_k(t')dt'\right)$$
$$\times\left[k(\delta A^+_{k+k_0} + \delta A^-_{k-k_0})(\delta A^-_{-k-k_0} + \delta A^+_{-k+k_0}) + k_0(\delta A^+_{k+k_0} + \delta A^-_{k-k_0})\right. \tag{B7}$$
$$\times\left.(\delta A^-_{-k-k_0} - \delta A^+_{-k+k_0})\right]\bar{\gamma}^2\int_{-\infty}^{\infty}\frac{dp_z}{\gamma^2}\,\frac{k\partial G_0/\partial p_z}{\Omega_k - kv_z}$$
$$-n_0 i\left(\frac{e\hat{B}}{2k_0}\right)^2 \frac{\alpha_3}{\bar{\gamma}mc^2}\sum_k k_0(\delta A^+_{k+k_0} + \delta A^-_{k-k_0})(\delta A^-_{-k-k_0} - \delta A^+_{-k+k_0})\exp\left(2\int_0^t \gamma_k(t')dt'\right).$$

With the use of the dispersion relation [Eq. (40], and combining and simplifying terms, Eq. (B7) can be rewritten as

$$\frac{\partial}{\partial t}\langle PMD\rangle = \left(\frac{\bar{\gamma}mc^2}{2e}\right)^2 \frac{i}{2\pi c^2}\sum_k \exp\left(2\int_0^t \gamma_k(t')dt'\right)\frac{D_{k+k_0}D_{k-k_0}}{(D_{k+k_0} + D_{k-k_0})}$$
$$\times\left[k(\delta A^+_{k+k_0} + \delta A^-_{k-k_0})(\delta A^-_{-k-k_0} + \delta A^+_{-k+k_0}) + k_0(\delta A^+_{k+k_0} + \delta A^-_{k-k_0})(\delta A^-_{-k-k_0} - \delta A^+_{-k+k_0})\right]. \tag{B8}$$

To proceed further, we substitute into Eq. (B8) the relations [Eqs. (A6) and (A7) in Appendix A]

$$(\delta A^{+}_{k+k_0} + \delta A^{-}_{k-k_0})(\delta A^{-}_{-k-k_0} + \delta A^{+}_{-k+k_0}) \frac{D_{k+k_0} D_{k-k_0}}{(D_{k+k_0} + D_{k-k_0})}$$

$$= D_{k+k_0} |\delta A^{+}_{k+k_0}|^2 + D_{k-k_0} |\delta A^{-}_{k-k_0}|^2 , \tag{B9}$$

$$(\delta A^{+}_{k+k_0} + \delta A^{-}_{k-k_0})(\delta A^{-}_{-k-k_0} - \delta A^{+}_{-k+k_0}) \frac{D_{k+k_0} D_{k-k_0}}{(D_{k+k_0} + D_{k-k_0})}$$

$$= D_{k+k_0} |\delta A^{+}_{k+k_0}|^2 - D_{k-k_0} |\delta A^{-}_{k-k_0}|^2 . \tag{B10}$$

Finally, Eq. (66) reduces to

$$\frac{\partial}{\partial t} \langle PMD \rangle = - \left(\frac{\bar{\gamma} mc^2}{2e}\right)^2 \frac{1}{2\pi c^2} \frac{\partial}{\partial t} \sum_k \exp\left(2\int_0^t \gamma_k(t')dt'\right)\omega_k$$

$$\times \left\{ (k+k_0) |\delta A^{+}_{k+k_0}|^2 + (k-k_0) |\delta A^{-}_{k-k_0}|^2 \right\} , \tag{B11}$$

where ω_k is the real part of the oscillation frequency $\Omega_k (=\omega_k + i\gamma_k)$. We now evaluate the right-hand side of Eq. (65). After some straightforward algebraic manipulation, which makes use of Eqs. (3), (21), and (25), we find

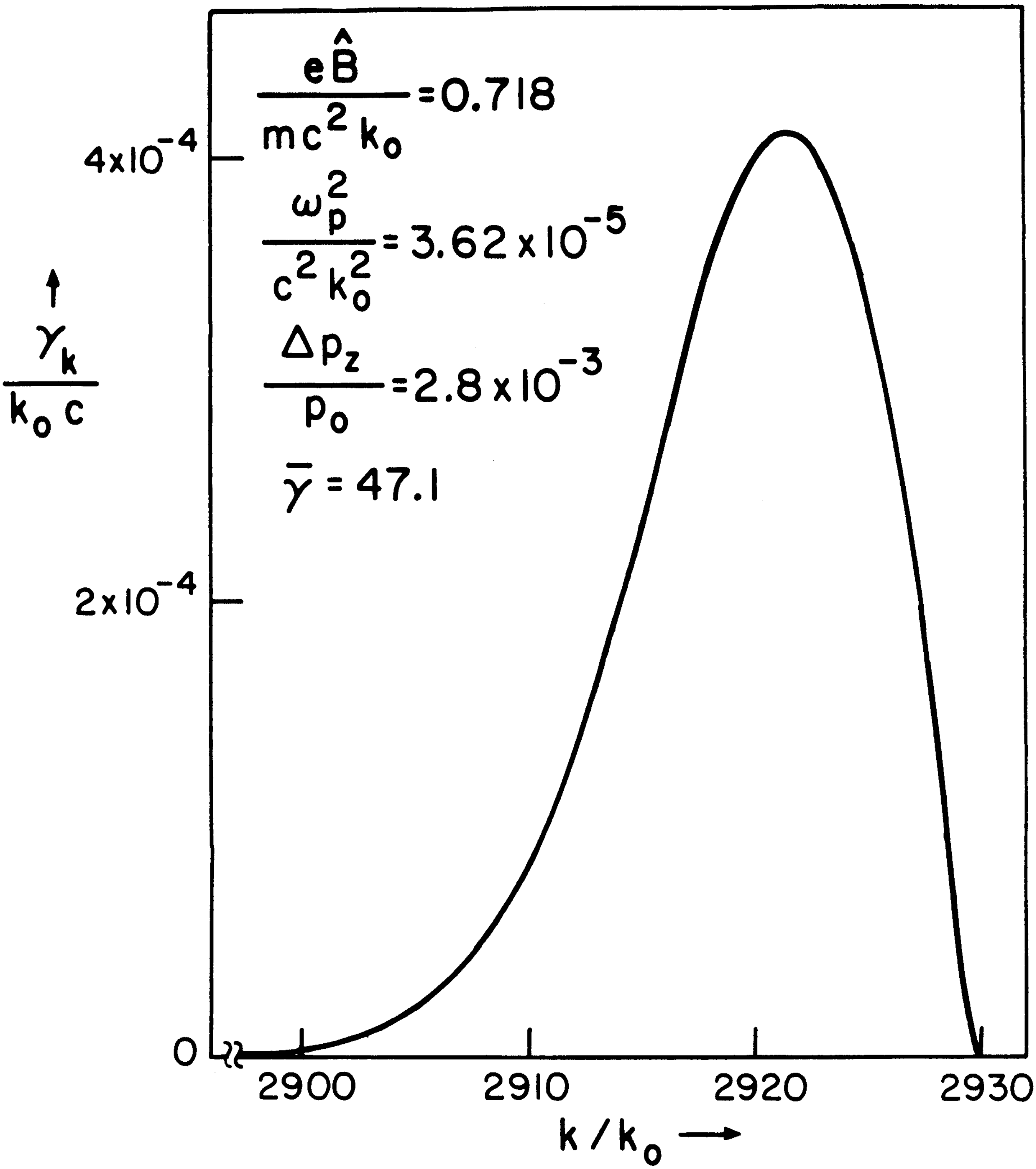
$$- \frac{\partial}{\partial t} \left\langle \frac{1}{4\pi c} (\delta \underset{\sim}{E}_T \times \delta \underset{\sim}{B}_T)_z \right\rangle$$

$$= - \frac{\partial}{\partial t} \frac{1}{4\pi c} \sum_k \frac{ik}{c} \left[\delta A^{*}_{x}(k,t) \frac{\partial}{\partial t} \delta A_x(k,t) + \delta A^{*}_{y}(k,t) \frac{\partial}{\partial t} \delta A_y(k,t)\right] \tag{B12}$$

$$= -\left(\frac{\bar{\gamma} mc^2}{2e}\right)^2 \frac{1}{2\pi c^2} \frac{\partial}{\partial t} \sum_k \exp\left(2\int_0^t \gamma_k(t')dt'\right) \omega_k$$

$$\times \left\{ (k+k_0) |\delta A^{+}_{k+k_0}|^2 + (k-k_0) |\delta A^{-}_{k-k_0}|^2 \right\} .$$

Equation (65) follows directly upon comparing Eqs. (B11) and (B12).



$\frac{e\hat{B}}{mc^2 k_o} = 0.718$
$\frac{\omega_p^2}{c^2 k_o^2} = 3.62 \times 10^{-5}$
$\frac{\Delta p_z}{p_o} = 2.8 \times 10^{-3}$
$\bar{\gamma} = 47.1$
$\frac{\gamma_k}{k_o c}$
4×10^{-4}
2×10^{-4}
0
2900
2910
2920
2930
k/k_o

Fig. 1

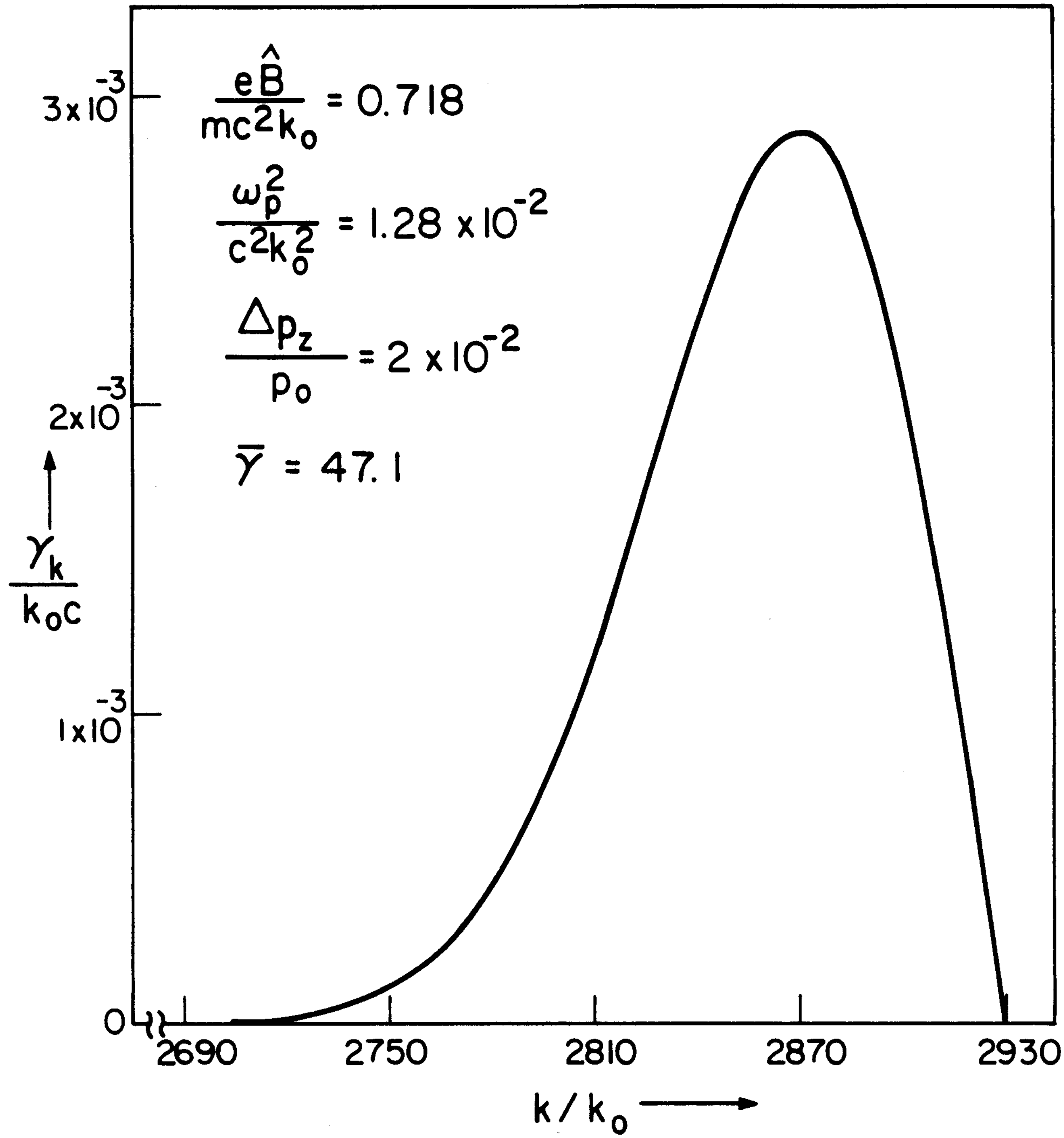
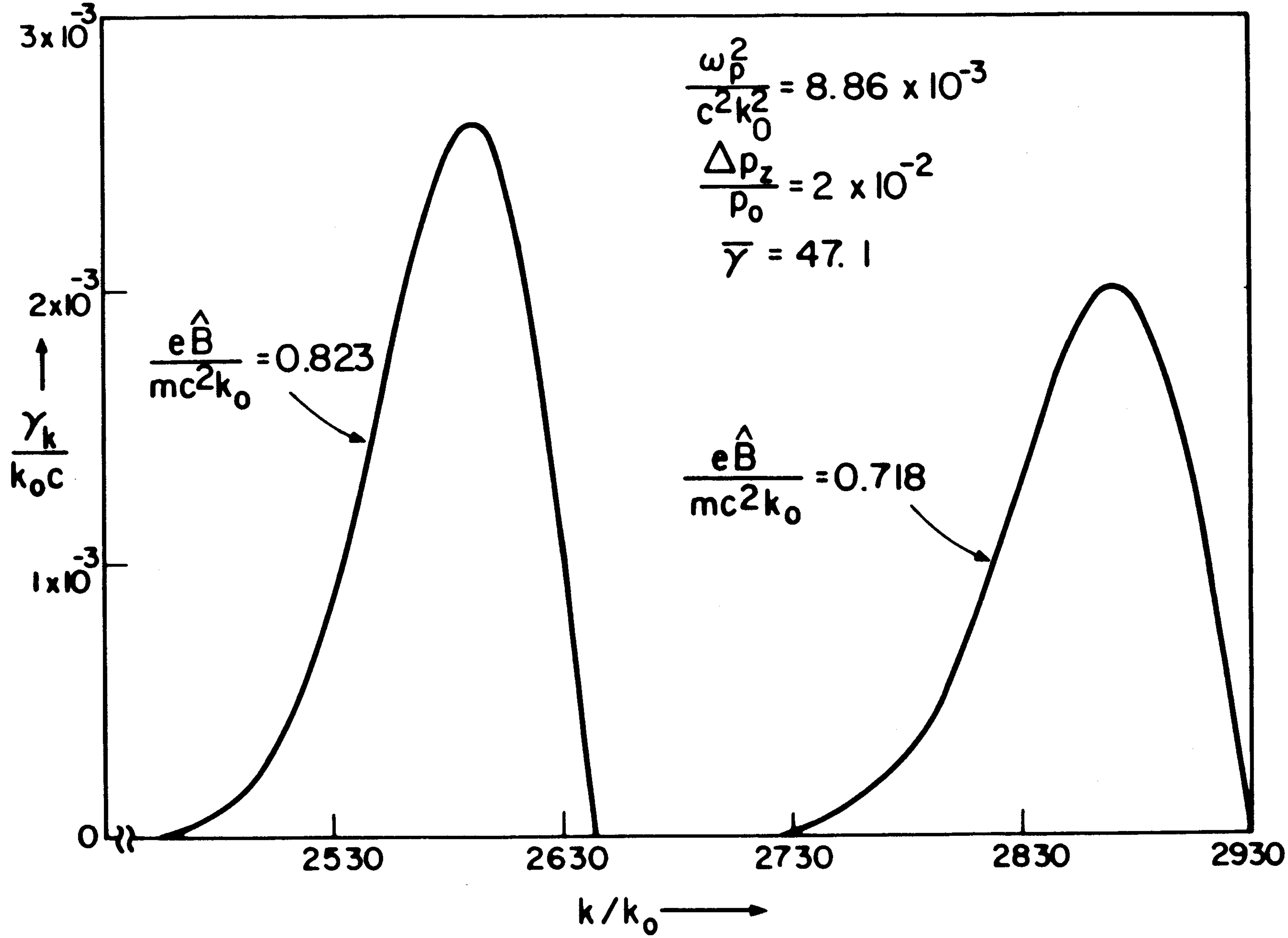
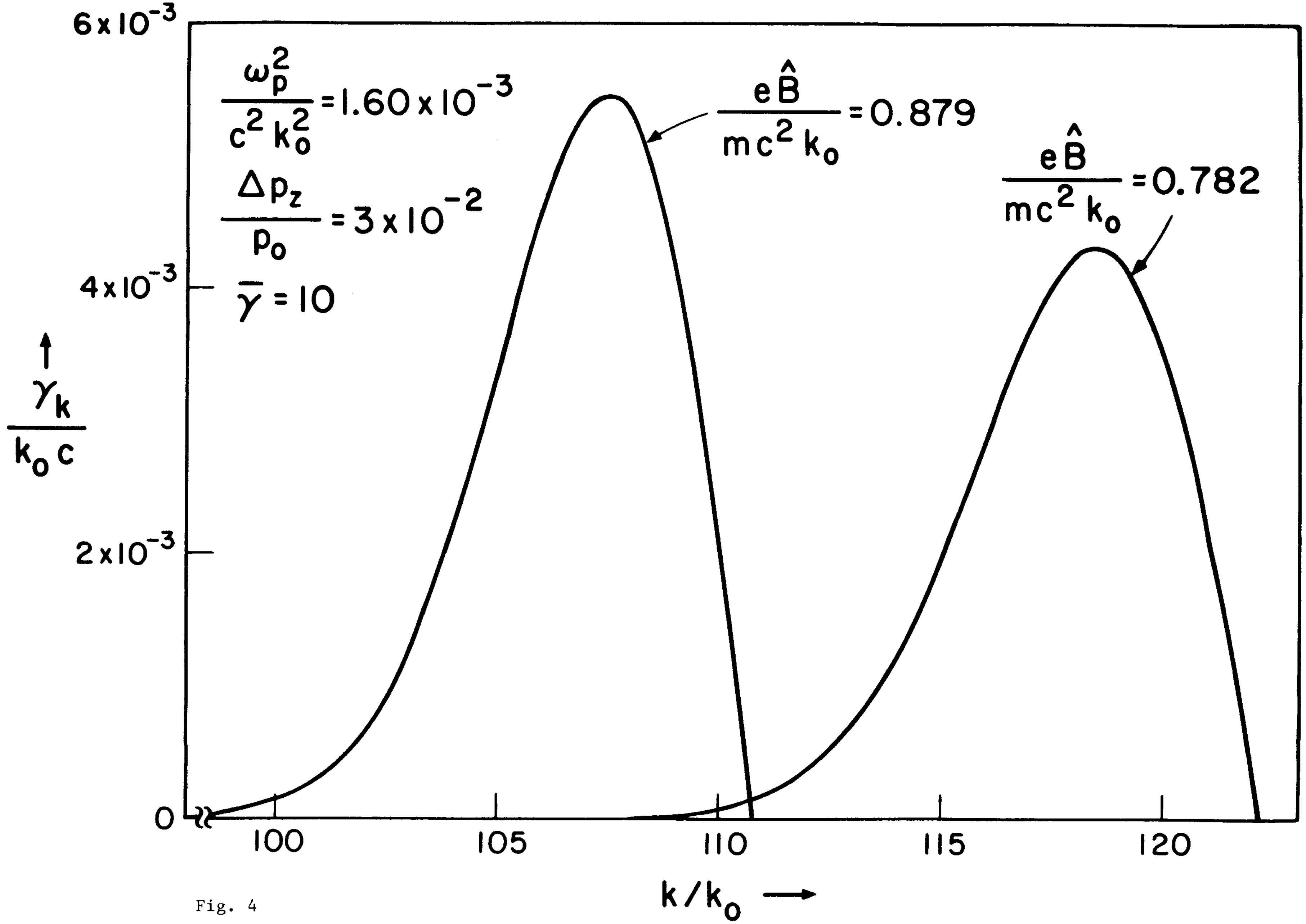

$\frac{e\hat{B}}{mc^2k_o} = 0.718$
$\frac{\omega_p^2}{c^2k_o^2} = 1.28 \times 10^{-2}$
$\frac{\Delta p_z}{p_o} = 2 \times 10^{-2}$
$\bar{\gamma} = 47.1$
3×10^{-3}
2×10^{-3}
1×10^{-3}
0
$\frac{\gamma_k}{k_oc}$
2690
2750
2810
2870
2930
k/k_o

Fig. 2



3×10^{-3}
2×10^{-3}
1×10^{-3}
0
$\frac{\gamma_k}{k_o c}$
$\frac{\omega_p^2}{c^2k_0^2} = 8.86\times10^{-3}$
$\frac{\Delta p_z}{p_o} = 2\times10^{-2}$
$\bar{\gamma} = 47.1$
$\frac{e\hat{B}}{mc^2k_o} = 0.823$
$\frac{e\hat{B}}{mc^2k_o} = 0.718$
2530
2630
2730
2830
2930
k/k_o

Fig. 3



6 x 10^-3
4 x 10^-3
2 x 10^-3
0
γ_k / k_o c
ω_p^2 / c^2 k_o^2 = 1.60 x 10^-3
Δp_z / p_o = 3 x 10^-2
γ̄ = 10
eB̂ / mc^2 k_o = 0.879
eB̂ / mc^2 k_o = 0.782
100
105
110
115
120
k / k_o

Fig. 4