PFC/JA-84-12R

STEADY STATE SOLUTION OF THE FOKKER-PLANCK
EQUATION COMBINED WITH UNIDIRECTIONAL QUASILINEAR
DIFFUSION UNDER DETAILED BALANCE CONDITIONS*

by

Kyriakos Hizanidis

March 1985

STEADY STATE SOLUTION OF THE FOKKER-PLANCK EQUATION COMBINED WITH UNIDIRECTIONAL QUASILINEAR DIFFUSION UNDER DETAILED BALANCE CONDITIONS

Kyriakos HIZANIDIS
Plasma Fusion Center
Massachusetts Institute of Technology
Cambridge, MA 02139, U.S.A.

ABSTRACT

The collisional Fokker-Planck equation combined with an externally imposed quasilinear RF-diffusion is solved for energetic electrons under conditions of detailed balance. The detailed balance condition restricts the functional form of the quasilinear diffusion coefficient. This restriction is tightly related to the simultaneous flattening and broadening of the distribution function in the parallel and perpendicular to the magnetic field direction respectively.

Radio frequency waves (RF) are of major importance for plasmas since RF provides the means of heating toroidal plasmas and generating the current necessary for their confinement in a steady-state (as opposed to the ohmic approach) fashion. Lower hybrid waves (LH), for example, can be used for current drive since they can resonate with fast electrons moving along the magnetic field (mainly toroidal).

The Fokker-Planck (FP) equation, which normally models RF-current-drive or RF-heating, combines both collisional and quasilinear RF-diffusion of energetic (current carrying) electrons colliding with the thermal ions and electrons and interacting with the applied RF fields. Only unidirectional resonant RF-diffusion (along the toroidal magnetic field) is considered since this is the relevant dominant mechanism in the RF-current-drive problem. The normalized form of the FP equation in cartesian momentum coordinates, p_i, $i = x, y, z$, written for relativistic electrons is [1]

$$\frac{\partial f}{\partial t} + \frac{\partial}{\partial p_i}\left(F_i - \frac{\partial}{\partial p_j}D_{ij}\right)f = 0 \tag{1}$$

where the summation convention is being used. The diffusion tensor D_{ij} and the friction vector F_i are given by

$$D_{ij} = \frac{\gamma}{p^3}\left[\left(\varsigma - \frac{\gamma^2}{2p^2}\right)p^2\delta_{ij} + \left(\varsigma - \frac{3\gamma^2}{2p^2}\right)p_ip_j\right] + \delta_{iz}\delta_{jz}D_{rf} \tag{2}$$

and

$$F_i = \frac{\partial D_{ij}}{\partial p_j} - \frac{\gamma^2}{p^3}p_i \tag{3}$$

where δ_{ij} is the Kronecker's delta, $D_{rf} > 0$ is the normalized modified RF diffusion coefficient for unidirectional wave-particle interaction (z-direction), $\gamma^2 = 1 + p^2\beta_{th}^2$ with $\beta_{th} = v_{th}/c$, and $\varsigma = (Z_i + 1)/2$, Z_i being the ion charge number. The variables p_x, p_y vary in the interval 0 to $-\infty$ while p_z varies in the interval p_{min} to ∞, $(p_{min} \gg 1)$. The diffusion coefficient D_{rf} is non-zero in the interval $p_1 < p_\parallel < p_2$ $(p_1 > p_{min})$.

Appropriate boundary condition must now be provided for a quasi steady state solution of eq. (1), ($\partial f/\partial t = 0$), in the sense that this solution will be valid for time scales shorter than the time it takes the distribution of the thermal particles to be affected. It is assumed here that the distribution function matches to a Maxwellian at $p_{\parallel} = p_{min}$. This approximation is in accordance with the one used for numerical computation [2], [6], [8]. For $p_{\parallel} \to \infty$ it is assumed that the values of the distribution and all its derivations vanish. With these boundary conditions the uniqueness of a (quasi) steady-state can be assured since D_{ij} is positive definite.

The most recent attempt to anaytically solve eq. (1) for the steady state for large D_{rf}'s and $\varsigma = \gamma = 1$ is by Krapchev et al. [2]. However, the application of the boundary conditions in [2] is not clear and the method used applies restrictively only for $\varsigma = \gamma = 1$ and cannot be generalized to encompass other cases. In the regime of applicability of the theory in [2], D_{rf} is an externally imposed parameter. In this letter I follow an entirely different road posing a qualitatively different question which will eventually restrict the functional form of D_{rf} : What are the physics implications from imposing the detailed balance condition (DBC) which we know freqently lead to a closed form solution of the steady state FP equation?. Then, one may envision a variational scheme in which the DBC- solution could be used as a trial solution. First, the plausibility of the DBC assumption-approximation is investigated.

The DBC are associated with the existence of a stationary state and lead to a particular class of steady state solutions of eq.(1). Such states are local equilibrium states and are likely to be found on the flux surfaces which are away from the external source of energy, such as wave guides. Accelerating forces such as toroidal electric fields as well as radiative processes are excluded otherwise the DBC will not be valid [3]. There are two main points on the basis of which one can justify the DBC assumption at the microscopic level as a valid approximation. First, the energetic electrons, which

eq.(1) deals with, are circulating particles (as opposed to trapped particles). These electrons interact periodically with the several spatially localized RF-field "islands", which they encounter during their circulation, in an incoherent fashion. These "islands" are the multiple crossings of the flux surfaces of the toroidally confined plasma by the resonance cones. The electrons resonate as long as they are in an "island". When circulating electrons reenter a different "island" they do not carry any memory of their previous resonance since they suffer randomizing collisional interactions between resonances. The second point is the fact that eq. (1) has only a flux surface average meaning; therefore this loss of memory is actually spread over the entire flux surface the equation is written for. One can consider this flux suface averaged randomization as a stationary process which obeys detailed balance in momentum space. Implication of DBC is the following relation

$$f_2(\vec{p}, \vec{p}_o, \tau; \vec{B}) = f_2(-\vec{p}, -\vec{p}_o, \tau; -\vec{B}) \,, \tag{4}$$

where $f_2(\vec{p}, \vec{p}_o, \tau; \vec{B})$ is the flux surface averaged stationary joint distribution for the points $\vec{p}$ and $\vec{p}_o$ in momentum space at times t and t_o $(\tau = |t - t_o|)$ respectively.

Marginal particles with momenta very close to p_1 and p_2 are excluded from this analysis since, being able to jump from the state of being resonant with the waves to the state of not being resonant (during their crossing of a resonance cone), they constitute a two state system. The joint distribution function, $f_2(\vec{p}, \vec{p}_o, \tau; \vec{B})$, of eq. (4) is not well defined (or, at least conventionally defined) if, for example, $\vec{p}_o$ and $\vec{p}$ fall inside and outside the resonance region respectively. According to numerical evidence [6], the boundary layers the marginal particles constitute are very thin. In the present work the effect of these layers has been ignored on the basis of the smallness of their thickness.

One can now introduce the vectors R_i and I_i [4] such that

$$F_i = R_i + I_i \tag{5}$$

with

$$R_i(\vec{p}) = [F_i(\vec{p}; \vec{B}) + F_i(-\vec{p}; -\vec{B})]/2, \quad I_i(\vec{p}) = [F_i(\vec{p}; \vec{B}) - F_i(-\vec{p}; -\vec{B})]/2 \tag{6}$$

where $\vec{B}$ is the magnetic field. The vectors R_i and I_i are called "reversible" and "irreversible" drift, respectively. These vectors have the following property

$$\mathcal{T} \cdot \left(\frac{\partial}{\partial p_i} R_i \right) = -\frac{\partial}{\partial p_i} R_i \, , \quad \mathcal{T} \cdot \left(\frac{\partial}{\partial p_i} I_i \right) = \frac{\partial}{\partial p_i} I_i \tag{7}$$

where $\mathcal{T}$ is the time reversal operator. From eqs. (2-3) and using eq. (6) one simply has,

$$R_i = 0 \tag{8}$$

since D_{rf} does not change sign under time reversal. The necessary and sufficient conditions for detailed balance, namely the conditions equivalent to eq. (4), are

$$D_{ij}(\vec{p}; \vec{B}) = D_{ij}(-\vec{p}; -\vec{B}) \tag{9}$$

$$\frac{\partial}{\partial p_i}(f_o R_i) = 0 \tag{10}$$

$$\frac{\partial \phi}{\partial p_i} = (D^{-1})_{ik}\left(\frac{\partial D_{kl}}{\partial p_l} - I_k \right) \tag{11}$$

where $f_o = \exp(-\phi)$ is the steady state solution of eq. (1). Equations (9-10) are automatically satisfied by virtue of eq. (8) and since D_{rf} does not change sign under time reversal. Equation (11), on the other hand, will provide the steady state solution of the FP equation and is equivalent, in our case, to the DBC, eq. (4). Solving eq. (11) requires that the integrability conditions

$$\frac{\partial}{\partial p_j}\left[(D_{ik})^{-1}\left(\frac{\partial D_{kl}}{\partial p_l} - I_k \right) \right] = \frac{\partial}{\partial p_i}\left[(D_{jk})^{-1}\left(\frac{\partial D_{kl}}{\partial p_l} - I_k \right) \right] \tag{12}$$

be satisfied. As it is readily clear only those D_{rf}'s which satisfy eq. (12) are associated with the existence of a stationary steady state satisfying the DBC. The interpretation of these restrictive integrability conditions is that the normalized RF diffusion coefficient, which is related to the imposed RF spectrum, evolves in time and becomes eventually a functional of the stationary distribution function.

Utilizing eqs. (5), (8) and (11) one readily obtains

$$\frac{\partial \phi}{\partial p_i} = (D^{-1})_{ik}\left(\frac{\partial D_{kl}}{\partial p_l} - F_k\right) \tag{13}$$

This equation implies that in our particular case the flux, S_{oi}, associated with the steady state distribution function f_o :

$$S_{oi} = -f_o F_i + \frac{\partial}{\partial p_k} D_{ik} f_o \tag{14}$$

is zero. Therefore the stationary steady state solution of the FP equation as in eq. (1) is associated with zero flux.

Equation (11) is now replaced by

$$\mathcal{D}\frac{\partial \phi}{\partial p_\alpha} = \frac{\gamma^3}{p^3}\left(\varsigma - \frac{\gamma^2}{2p^2}\right)\left[D_{rf} + \frac{\gamma}{p}\left(\varsigma - \frac{\gamma^2}{2p^2}\right)\right]\frac{p_\alpha}{p}, \quad \mathcal{D}\frac{\partial \phi}{\partial p_z} = \frac{\gamma^4}{p^4}\left(\varsigma - \frac{\gamma^2}{2p^2}\right)\frac{p_z}{p} \tag{15}$$

where $\alpha = x, y$ and $\mathcal{D} = \det D_{ij}$ is given by

$$\mathcal{D} = \frac{\gamma^2}{p^2}\left(\varsigma - \frac{\gamma^2}{2p^2}\right)\left\{D_{rf}\left[\left(\varsigma - \frac{3\gamma^2}{2p^2}\right)\frac{p_z^2}{p^2} + \frac{\gamma^2}{p^2}\right] + \frac{\gamma^3}{p^3}\left(\varsigma - \frac{\gamma^2}{2p^2}\right)\right\}. \tag{16}$$

For the case of $D_{rf} = 0$ one can recover the relativistic Maxwellian distribution since then eq. (15) becomes,

$$\frac{\partial \phi}{\partial p_i} = \frac{p_i}{\gamma} \; , \; i = x, y, z \tag{17}$$

which implies that

$$\phi = \text{constant} + \frac{p^2}{\gamma+1} = \text{constant} + \frac{mc^2(\gamma-1)}{T_B} \tag{18}$$

where the bulk temperature is: $T_B = mv_{th}^2$.

In the large p limit, that is, when the driving RF spectrum is located away from the thermal bulk one can drop the $\gamma^2/p^2 = 1/p^2 + \beta_{th}^2$ terms in eq. (14). Then, in cylindrical coordinates, $p_\parallel = p_z$, $p_\perp = \sqrt{p_x^2 + p_y^2}$, one has

$$\frac{\partial \phi}{\partial p_\perp} = \frac{p_\perp}{\gamma} \frac{pD_{rf}/\varsigma\gamma + 1}{pp_\parallel^2 D_{rf}/\gamma^3 + 1}, \quad \frac{\partial \phi}{\partial p_\parallel} = \frac{p_\parallel}{\gamma} \frac{1}{pp_\parallel^2 D_{rf}/\gamma^3 + 1} \tag{19}$$

along with the associated integrability conditions which are meaningfull only in the region $p_1 < p_\parallel < p_2$. Equation (19) can be used to compute expectation values of various physical quantities (like momentum, energy and current) by an alternative to the standard Monte Carlo technique [5].

An analytic solution to eq. (19) can be found for the nonrelativistic case. For $\gamma = 1$ eq. (19) along with the integrability condition, eq. (12), becomes

$$\frac{\partial \phi}{\partial p_\perp^2} = \frac{1}{2T} \quad \frac{\partial \phi}{\partial p_\parallel^2} = \frac{1}{2T} \frac{\varsigma p_\parallel^2 - T}{\varsigma p_\parallel^2 - 1} \tag{20}$$

where T is a function of the general form

$$T(p_\parallel, p_\perp) = T(p_\parallel^2 + p_\perp^2 + 1/\varsigma \ln(\varsigma p_\parallel^2 - 1)), \qquad p_\parallel \epsilon(p_1, p_2) \tag{21}$$

arising from the integrability conditions for D_{rf} which now becomes

$$D_{rf}(p_\parallel, p_\perp) = \frac{\varsigma}{p} \frac{T-1}{\varsigma p_\parallel^2 - T}. \tag{22}$$

The positivity requirement for D_{rf} implies that

$$1 < T < \varsigma p_\parallel^2. \tag{23}$$

The interpretation of the function $T(p_\parallel, p_\perp)$ is the following: in the absence of the RF drive, $T = 1$ and thus $T(p_\parallel, p_\perp)$ plays the role of the normalized temperature of the distribution of the energetic electrons which, in this case, becomes part of the bulk distribution with a temperature equal to the bulk temperature. When the RF drive is present there is evidence from the experimental data [6] as well as from numerical simulations [7] that the distribution of the energetic electrons is much "hotter" in the perpendicular direction than it is for the bulk electrons, that is, $\frac{\partial \phi}{\partial p_\perp^2} \ll \frac{1}{2}$. It is also very well known [8] that the distribution of energetic electron flattens in the parallel direction. From eq. (20) it is clear that for $T(p_\parallel, p_\perp) \gg 1$ one has a simultaneous broadening and a flattening of the distribution function in the perpendicular and parallel direction respectively.

In actuality only the RF spectrum of the wave guide is quite accurately known. The spectrum inside the plasma, and therefore D_{rf}, can be calculated by solving the entire self-consistent problem in toroidal geometry, a quite arduous task. There are simpler methods [9], of course, but they are based on many, not yet well-founded, simplifications. Assuming that D_{rf} is known on a given flux surface inside the plasma, then, if DBC have been achieved, the function T will satisfy eq.(22) and (23). The simplest possible model for the function T for $p_1 < p_\parallel < p_2$, supported also by numerical evidence [6], is

$$T(p_\parallel, p_\perp) = \text{constant} = T \gg 1 \tag{24}$$

where T satisfies eq. (23). The solution for $f(p_\parallel, p_\perp)$ based on eq. (24) can easily be obtained and is given by,

$$f(p_\parallel, p_\perp) = \exp\left(-\frac{p_\parallel^2 + p_\perp^2}{2T}\right)(\varsigma p_\parallel^2 - 1)^{\frac{T-1}{2T\varsigma}}. \tag{25}$$

8

The average perpendicular energy $T_\perp = \int_0^\infty dp_\perp\, p_\perp^3 f / 2 \int_0^\infty dp_\perp\, p_\perp f$ coincides with T in eq. (24). Matching to a Maxwellian distribution at p_1 and/or p_2, leads to discontinuity of the solution, eq. (25), and a slight underestimation in the number of particles, resonant or not, involved in the actual distribution of the energetic electrons. One has to solve the Fokker Planck equation, eq. (1), in the left boundary layer in order continuously match the solution, eq. (25), to a Maxwellian distribution function at $p_\parallel = p_{min}$. The numerical results, [6], suggests also that the distribution goes to a Maxwellian function at a point $p_\parallel = p_1'$, $p_{min} < p_1' < p_1$ (actually p_1' is very close to p_1) Finally, eq.(22) suggests that the diffusion coefficien, D_{rf}, has its highly localized maximum value in the vicinity of $p_\parallel = p_1$. This feature is in accordance with the one used in the numerical solution for entirely different reasons (Bonoli-Englade's ray tracing results for Alcator C [9]).

One can calculate the current density, J, associated with the resonance region, as well as the power density dissipated, P_d, required to raise that current in the regime $\varsigma p_\parallel^2 \gg 1$. These quantities are of major importance in the RF current-drive theory. The current density J which is defined by $2\pi \int_{p_1}^{p_2} dp_\parallel \int_0^\infty dp_\perp\, p_\perp p_\parallel f$ is given by

$$J = 2\pi T^2 (2T\varsigma)^{\frac{T-1}{2T\varsigma}} \left\{ \left(\frac{p_\parallel^2}{2T} \right)^{\frac{T-1}{2T\varsigma}} \exp\left(-\frac{p_\parallel^2}{2T} \right) - \frac{T-1}{2T\varsigma} \Gamma\left(\frac{T-1}{2T\varsigma}, \frac{p_\parallel^2}{2T} \right) \right\}_2^1 \qquad (26)$$

where Γ is the incomplete gamma function and $\{h(p_\parallel)\}_2^1 = h(p_1) - h(p_2)$. The power density, P_d, which is defined by, $2\pi \int_{p_1}^{p_2} dp_\parallel \int_0^\infty dp_\perp\, p_2 \frac{p_\parallel^2 + p_\perp^2}{2} \frac{\partial}{\partial p_\parallel}\left(D_{rf} \frac{\partial f}{\partial p_\parallel} \right)$ is given by

$$P_d = -\frac{\pi(T-1)(2T\varsigma)^{\frac{T-1}{2T\varsigma}+1}}{T(\varsigma+1)-1}\left\{\left(\frac{p_\parallel^2}{2T}\right)^{\frac{T-1}{2T\varsigma}}e^{-\frac{p_\parallel^2}{2T}}\right.$$

$$-\frac{T-1}{2T\varsigma}\Gamma\left(\frac{T-1}{2T\varsigma},\frac{p_\parallel^2}{2T}\right)+\left(\frac{p_\parallel^2}{2T}\right)^{\frac{T-1}{2T\varsigma}}\left(\frac{T(\varsigma+1)-1}{2T\varsigma}e^{-\frac{p_\parallel^2}{2T}}\right.\tag{27}$$

$$\left.\left.-\sqrt{\pi}\left(-\frac{p_\parallel^2}{2T}\right)^{\frac{1}{2}}\frac{2\varsigma p_\parallel^2-T(\varsigma+1)+1}{2\varsigma p_\parallel^2}\left[1-\varphi\left(\frac{p_\parallel}{\sqrt{2T}}\right)\right]\right)\right\}_2^1$$

where φ is the error function.

An important quantity, which determines the efficiency of the current drive is the figure of merit, $|J/P_d|$. From eqs. (26) and (27) one has

$$\left|\frac{J}{P_d}\right|=T\left(\frac{1}{\varsigma}+\frac{T}{T-1}\right)\frac{1}{1+\dfrac{\left\{\left(\frac{p_\parallel^2}{2T}\right)^{\frac{T-1}{2T\varsigma}}\left(\frac{T(\varsigma+1)-1}{2T\varsigma}e^{-\frac{p_\parallel^2}{2T}}-\sqrt{\pi}\left(-\frac{p_\parallel^2}{2T}\right)^{\frac{1}{2}}\frac{2\varsigma p_\parallel^2-T(\varsigma+1)+1}{2\varsigma p_\parallel^2}\left[1-\varphi\left(\frac{p_\parallel}{\sqrt{2T}}\right)\right]\right)\right\}_2^1}{\left\{\left(\frac{p_\parallel^2}{2T}\right)^{\frac{T-1}{2T\varsigma}}e^{-\frac{p_\parallel^2}{2T}}-\frac{T-1}{2T\varsigma}\Gamma\left(\frac{T-1}{2T\varsigma},\frac{p_\parallel^2}{2T}\right)\right\}_2^1}}$$

$$\tag{28}$$

This equation exhibits the leading order dependencies of the efficiency on ς and T. Since marginal particles have been excluded from this analysis, eqs. (26), (27) and (28) are approximately valid. The approximation is based on the smallness of the thickness of the boundary layers. Solution of the Fokker Planck equation in the left boundary may provide the value of the constant T.

The method of solution investigated in this paper may serve as a trial solution in a variational scheme in which the DBC are only approximatelly satisfied. In such a case the fluxes S_{oi} will be finite but small. This scheme may provide an alternative method, more appropriate for arbitrary D_{rf}, to those existing in the literature, for example, [2] and [10] for strong and weak RF diffusion respectively.

References

[1] K. Hizanidis and A. Bers, Phys. of Fluids **27**, 2669(1984).

[2] V. S. Krapchev, D. W. Hewett and A. Bers, Phys. of Fluids **28**, 522(1985).

[3] K. Molvig and K. Hizanidis, Phys. Fluids **27**, 2847(1984).

[4] H. Risken, Z. Physik **251**, 231 (1972).

[5] G. Parisi, Nucl. Phys. **B180**, 378 (1981).

[6] K. Hizanidis, D. W. Hewett and A. Bers, Proc. Fourth Varenna/Grenoble International Symposium on Heating in Toroidal Plasmas, Rome, Italy, March 1984.

[7] S. Von Goreler et al., Bull. Amer. Phys. Soc. **27**, 1069 (1982).

[8] C. F. F. Karney and N. J. Fisch, Phys. Fluids **22**, 1817 (1979).

[9] P. C. Englade, P. T. Bonoli and M. Porkolab, Proceedings of the IAEA Technical Committee Meeting, Culham Laboratory Report CLM-CD, p. 178, April 1983.

[10] T. M. Antonsen and K. R. Chu, Phys. Fluids **25**, 1295 (1982).