MACROSCOPIC GUIDING-CENTER STABILITY THEOREM FOR NONRELATIVISTIC NONNEUTRAL ELECTRON FLOW IN A CYLINDRICAL DIODE WITH APPLIED MAGNETIC FIELD

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ABSTRACT

A sufficient condition is derived for electrostatic stability of nonrelativistic nonneutral electron flow in a cylindrical diode with applied magnetic field $B_0\hat{\mathbf{e}}_z$. The analysis is based on a cold-fluid guiding-center model that treats the electrons as a massless, strongly magnetized fluid with $\omega_{pb}^2(r)/\omega_c^2 = 4\pi n_b^0(r)mc^2/B_0^2 << 1$, and flow velocity $V_b(\mathbf{x},t) = -c\nabla\phi(x,t)\times\hat{\mathbf{e}}_z/B_0$. Making use of global conservation constraints satisfied by the fluid-Poisson equations, it is shown that $\partial n_b^0(r)/\partial r \leq 0$ over the interval $a \leq r \leq b$ is a sufficient condition for stability to small-amplitude electrostatic perturbations. Here $n_b^0(r)$ is the electron density profile, and the cathode is located at $r = a$ and the anode at $r = b$. Space-charge-limited flow with $E_r^0(r = a) = 0$ is assumed. The analysis illustrates the major generality and flexibility of using global conservation constraints to determine a sufficient condition for stability. Nowhere is it necessary to make direct use of a detailed normal-mode analysis or eigenvalue equation.

I. INTRODUCTION AND SUMMARY

In the present article, a sufficient condition is derived for electrostatic stability of nonrelativistic nonneutral electron flow[1,2] in a cylindrical diode with applied magnetic field $B_0\hat{e}_z$ (Fig. 1). The analysis is based on a cold-fluid guiding-center model (Sec. II) that treats the electrons as a massless, strongly magnetized fluid with

$$\frac{\omega_{pb}^2(r)}{\omega_c^2} = \frac{4\pi n_b^0(r)mc^2}{B_0^2} << 1 ,$$

and electron flow velocity [Eq. (2)]

$$\mathbf{V}_b(\mathbf{x},t) = -c\,\frac{\nabla\phi(\mathbf{x},t)\times\hat{\mathbf{e}}_z}{B_0} .$$

Here, $n_b^0(r)$ is the equilibrium electron density profile, and $\mathbf{E}(\mathbf{x},t) = -\nabla\phi(\mathbf{x},t)$ is the electric field. Making use of global conservation constraints satisfied by the fluid-Poisson equations (Secs. III and IV), it is shown that monotonic decreasing density profiles with [Eq. (30)]

$$\frac{\partial n_b^0(r)}{\partial r} \leq 0, \text{ for } a \leq r \leq b$$

are stable to small-amplitude electrostatic perturbations. Here, the cathode is located at r = a and the anode at r = b (Fig. 1). Space-charge-limited flow with $E_r^0(r = a) = 0$ is assumed.

The present analysis illustrates the major generality and flexibility of using global conservation constraints to determine a sufficient condition for stability. Nowhere is it necessary to make direct use of a detailed normal-mode analysis or eigenvalue equation. This work is an important generalization of the calculation of Briggs et al.[2,3] to include the presence of an internal conductor (cathode). Moreover, the stability theorem derived by Briggs et al.[2,3] was obtained directly from the electrostatic eigenvalue equation.

II. THEORETICAL MODEL AND ASSUMPTIONS

In the present analysis, we adopt a cold-fluid guiding-center model in which electron inertial effects are neglected ($m \rightarrow 0$), and the motion of a strongly magnetized electron fluid element is determined from

$$\mathbf{E}(\mathbf{x},t) + \frac{1}{c}\,\mathbf{V}_b(\mathbf{x},t) \times B_0\hat{\mathbf{e}}_z = 0. \tag{1}$$

In the electrostatic approximation, $\mathbf{E} = -\nabla\phi$ and Eq. (1) gives

$$\mathbf{V}_b(x,t) = -\frac{c}{B_0}\nabla\phi(\mathbf{x},t) \times \hat{\mathbf{e}}_z \tag{2}$$

for the perpendicular motion. In cylindrical geometry, Eq. (2) reduces to

$$\begin{aligned} V_{rb}(r,\theta,t) &= -\frac{c}{B_0 r}\frac{\partial}{\partial\theta}\phi(r,\theta,t), \\ V_{\theta b}(r,\theta,t) &= \frac{c}{B_0 r}\frac{\partial}{\partial r}\phi(r,\theta,t), \end{aligned} \tag{3}$$

where $\partial/\partial z = 0$ has been assumed. The continuity equation, which relates the density $n_b(r,\theta,t)$ and flow velocity $\mathbf{V}_b(r,\theta,t)$ is given by

$$\frac{\partial}{\partial t}n_b + \frac{\partial}{\partial \mathbf{x}}\cdot(n_b\mathbf{V}_b) = 0, \tag{4}$$

$$\frac{\partial}{\partial t}n_b + \mathbf{V}_b\cdot\frac{\partial}{\partial \mathbf{x}}n_b = 0, \tag{5}$$

since $\nabla\cdot\mathbf{V}_b = 0$ for the electron flow in Eq. (2). Of course, Eqs. (2), (3) and (5) must be supplemented by Poisson's equation

$$\nabla^2\phi(r,\theta,t) = 4\pi e n_b(r,\theta,t), \tag{6}$$

which self-consistently relates the electrostatic potential $\phi(r,\theta,t)$ to the electron density $n_b(r,\theta,t)$.

Equations (2), (5) and (6) constitute a fully nonlinear description of the system evolution in the cold-fluid guiding-center approximation with $m \rightarrow 0$ and $B_0 \rightarrow \infty$. Expressing

$$n_b(r,\theta,t) = n_b^0(r) + \delta n_b(r,\theta,t), \qquad \phi(r,\theta,t) = \phi_0(r) + \delta\phi(r,\theta,t), \tag{7}$$

the boundary conditions enforced in solving Eqs. (2), (5) and (6) are

$$\phi_0(r = a) = 0 \text{ and } \phi_0(r = b) = V, \qquad \left.\frac{\partial}{\partial r}\phi_0\right|_{r = a} = 0, \tag{8}$$

$$\frac{\partial}{\partial\theta}\delta\phi = 0, \text{ at } r = a \text{ and } r = b. \tag{9}$$

The equilibrium conditions in Eq. (8) correspond to space-charge-limited flow with $E_r^0(r = a) = -\partial\phi_0/\partial r\big|_{r = a} = 0$. Moreover, Eq. (9) assures that the tangential electric field and radial flow velocity are equal to zero at the cathode and at the anode, with

$$\left.\begin{aligned} E_\theta &= -\frac{1}{r}\frac{\partial}{\partial\theta}\delta\phi = 0, \\ V_{rb} &= -\frac{c}{B_0 r}\frac{\partial}{\partial\theta}\delta\phi = 0, \end{aligned}\right\} \text{ at } r = a \text{ and } r = b\ . \tag{10}$$

Finally, because of periodicity in the θ-direction,

$$\int_0^{2\pi} d\theta \frac{\partial}{\partial\theta}\psi = 0, \tag{11}$$

where ψ represents any field of fluid variable or nonlinear combination thereof.

III. NONLINEAR GLOBAL CONSERVATION CONSTRAINTS

The macroscopic guiding center model based on Eqs. (2), (5) and (6) possesses certain global (spatially averaged) conservation constraints. Consider the quantity ΔU_G defined by

$$\Delta U_G = \int d^2x[G(n_b) - G(n_b^0)], \tag{12}$$

where $G(n_b)$ is a smooth, differentiable function, and

$$\int d^2x = \int_0^{2\pi} d\theta \int_a^b drr$$

in cylindrical geometry. From Eq. (5) and $\nabla \cdot \mathbf{V}_b = 0$, it follows that

$$\begin{aligned} \frac{\partial}{\partial t} G(n_b) &= \frac{\partial G}{\partial n_b}\frac{\partial n_b}{\partial t} = -\frac{\partial G}{\partial n_b}\mathbf{V}_b \cdot \nabla n_b \\ &= -\mathbf{V}_b \cdot \nabla G(n_b) = -\nabla \cdot [G(n_b)\mathbf{V}_b]. \end{aligned} \tag{13}$$

Therefore

$$\begin{aligned} \frac{d}{dt}\Delta U_G &= \int d^2x \frac{\partial}{\partial t} G(n_b) \\ &= -\int_0^{2\pi} d\theta \int_a^b drr \left[\frac{1}{r}\frac{\partial}{\partial r}(rV_{rb}G) + \frac{1}{r}\frac{\partial}{\partial \theta}(V_{\theta b}G)\right] = 0. \end{aligned} \tag{14}$$

The $\partial/\partial\theta$ contribution in Eq. (14) integrates to zero by virtue of periodicity in the θ-direction [Eq. (11)]. The $\partial/\partial r$ contribution in Eq. (14) integrates to zero because the radial flow velocity is equal to zero at the cathode $(r = a)$ and at the anode $(r = b)$ [Eq. (7)]. From Eq. (14), we conclude that

$$\Delta U_G = \int d^2x[G(n_b) - G(n_b^0)] = \text{const.} \tag{15}$$

A special case of Eq. (15) is the conservation of total charge

$$\Delta U_q = -e\int d^2x(n_b - n_b^0) = \text{const.} = 0, \tag{16}$$

where the constant in Eq. (16) has been taken equal to zero. This is consistent for all time t provided zero net charge is introduced into the system by the initial density perturbation, i.e., provided $\int d^2x\delta n_b(r,\theta, t = 0) = 0$.

A further global conservation constraint is related to the density-weighted average radial location of guiding centers. Defining

$$\Delta U_r = \int d^2x r^2 (n_b - n_b^0), \tag{17}$$

and making use of $\partial n_b/\partial t = -\nabla \cdot (n_b \mathbf{V}_b)$ gives

$$\begin{aligned}\frac{d}{dt}\Delta U_r &= -\int_0^{2\pi} d\theta \int_a^b drr \cdot r^2 \left[\frac{1}{r}\frac{\partial}{\partial r}(rn_b V_{rb}) + \frac{1}{r}\frac{\partial}{\partial \theta}(n_b V_{\theta b})\right] \\ &= -\int_0^{2\pi} d\theta \int_a^b dr \left[\frac{\partial}{\partial r}(r^3 n_b V_{rb}) - 2r^2 n_b V_{rb}\right],\end{aligned} \tag{18}$$

where the $\partial/\partial\theta$ contribution in Eq. (18) vanishes by virtue of periodicity in the θ-direction [Eq. (11)]. Moreover, making use of $V_{rb} = 0$ at $r = a$ and $r = b$ [Eq. (7)], the $(\partial/\partial r)(r^3 n_b V_{rb})$ term in Eq. (18) integrates to zero, which gives

$$\frac{d}{dt}\Delta U_r = -\frac{2c}{B_0}\int_0^{2\pi} d\theta \int_a^b drrn_b \frac{\partial}{\partial \theta}\phi . \tag{19}$$

In Eq. (19), use has been made of $V_{rb} = -(c/rB_0)\partial\phi/\partial\theta$ [Eq. (3)] to eliminate the radial flow velocity V_{rb} from the final term in Eq. (18). From $\nabla^2\phi = 4\pi e n_b$, Eq. (19) can be expressed as

$$\begin{aligned}\frac{d}{dt}\Delta U_r &= -\frac{c}{2\pi e B_0}\int_0^{2\pi} d\theta \int_a^b drr \left[\frac{1}{r}\frac{\partial}{\partial r} r\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}\right]\frac{\partial\phi}{\partial\theta} \\ &= -\frac{c}{2\pi e B_0}\int_0^{2\pi} d\theta \int_a^b dr \left[\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right)\right]\left[\frac{\partial\phi}{\partial\theta}\right],\end{aligned} \tag{20}$$

where the $\partial^2/\partial\theta^2$ contribution in Eq. (20) integrates to zero because of periodicity in the θ-direction. Equation (20) can also be expressed (exactly) as

$$\frac{d}{dt}\Delta U_r = -\frac{c}{2\pi e B_0}\int_0^{2\pi} d\theta \int_a^b dr \left[\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\frac{\partial\phi}{\partial\theta}\right) - r\frac{\partial\phi}{\partial r}\frac{\partial}{\partial\theta}\frac{\partial\phi}{\partial r}\right] . \tag{21}$$

The θ-derivative term in Eq. (21) vanishes because of periodicity in the θ-direction [Eq. (11)]. The term $(\partial/\partial r)[r(\partial\phi/\partial r)(\partial\phi/\partial\theta)]$ in Eq. (21) integrates to zero by virtue of $E_\theta = -(1/r)(\partial\phi/\partial\theta) = 0$ at the cathode ($r = a$) and at the anode ($r = b$) [Eq. (10)]. This gives

$$\frac{d}{dt}\Delta U_r = 0 , \tag{22}$$

or equivalently,

$$\Delta U_r = \int d^2x r^2(n_b - n_b^0) = \text{const.} \tag{23}$$

IV. SUFFICIENT CONDITION FOR STABILITY

A sufficient condition for stability follows directly from Eqs. (15) and (23). Defining an effective free energy function ΔF by

$$\Delta F = \Delta U_r + \Delta U_G , \tag{24}$$

it follows that

$$\Delta F = \int d^2x[r^2(n_b - n_b^0) + G(n_b) - G(n_b^0)] = \text{const.} \tag{25}$$

is an exact (nonlinear) global constraint. Expressing $\delta n_b = n_b - n_b^0$, and Taylor expanding

$$G(n_b) = G(n_b^0) + G'(n_b^0)(\delta n_b) + \frac{1}{2}G''(n_b^0)(\delta n_b)^2 + \dots , \tag{26}$$

it follows that Eq. (25) can be expressed as

$$\Delta F = \int d^2x \left\{ [r^2 + G'(n_b^0)](\delta n_b) + \frac{1}{2} G''(n_b^0)(\delta n_b)^2 \right\} = \text{const.}, \tag{27}$$

correct to quadratic order in the density perturbation $\delta n_b = n_b(r,\theta,t) - n_b^0(r)$. The function $G(n_b^0)$ has been arbitrary up to this point. We now choose $G(n_b^0)$ to satisfy

$$G'(n_b^0) = -r^2 \quad , \tag{28}$$

so that $G''(n_b^0) = -(\partial n_b^0/\partial r)^{-1}$. Equation (27) then becomes

$$\Delta F = \frac{1}{2} \int d^2x \frac{1}{[-\partial n_b^0/\partial r^2]} (\delta n_b)^2 = \text{const.} \tag{29}$$

It follows trivially from Eq. (29) that for monotonic decreasing density profiles with

$$\frac{1}{r} \frac{\partial n_b^0}{\partial r} \leq 0, \text{ for } a \leq r \leq b, \tag{30}$$

the density perturbation $\delta n_b(r,\theta,t)$ cannot grow without bound, and the system is linearly stable. That is to say, Eq. (30) is a sufficient condition for stability in the context of the cold-fluid guiding-center model based on Eqs. (2), (5) and (6).

We therefore conclude that a necessary condition for instability is that $\partial n_b^0/\partial r$ change sign on the interval $a \leq r \leq b$, or equivalently that

$$\frac{\partial}{\partial r} \frac{1}{r} \frac{\partial}{\partial r} [r^2 \omega_E(r)]$$

change sign on the interval $a \leq r \leq b$. Here, $\omega_E(r) = -cE_r^0(r)/B_0 r$ is the equilibrium angular velocity of a fluid element, and $n_b^0(r)$ is related to $E_r^0(r)$ by $(1/r)(\partial/\partial r)(rE_r^0) = -4\pi e n_b^0$. When $\partial n_b^0/\partial r$ changes sign in the interval $a \leq r \leq b$, the corresponding shear in angular velocity can provide the free energy to drive the diocotron instability.[1,2]

V. CONCLUSIONS

The present analysis illustrates the major generality and flexibility of using global conservation constraints to determine a sufficient condition for stability. Nowhere was it necessary to make direct use of a detailed normal-mode analysis or eigenvalue equation.

The sufficient condition for stability developed here can be extended to the case of relativistic nonneutral electron flow in a planar diode with equilibrium flow velocity $V_y^0(x) = -cE_x^0(x)/B_z^0(x)$. Within the context of a relativistic guiding-center model that treats the electrons as a cold massless fluid ($m \to 0$), it is found[4] that the sufficient condition for the electron flow to be stable to small-amplitude extraordinary-mode perturbations is given by

$$\frac{\partial}{\partial x}\left[\frac{n_b^0(x)}{\gamma_b^0(x)}\right] \leq 0 \ , \quad 0 \leq x \leq d,$$

where $\gamma_b^0(x) = [1 - E_x^{0\,2}(x)/B_z^{0\,2}(x)]^{-1/2}$. Here, the cathode is located at $x = 0$ and the anode at $x = d$.

ACKNOWLEDGMENTS

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3. cf. pp 67-78 of Ref. 1.

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FIGURE CAPTIONS

Fig. 1: Cylindrical diode configuration with cathode at $r = a$ and anode at $r = b$, and applied axial magnetic field $B_0(\mathbf{x}) = B_0\hat{\mathbf{e}}_z$. Equilibrium electron flow is in the θ-direction.

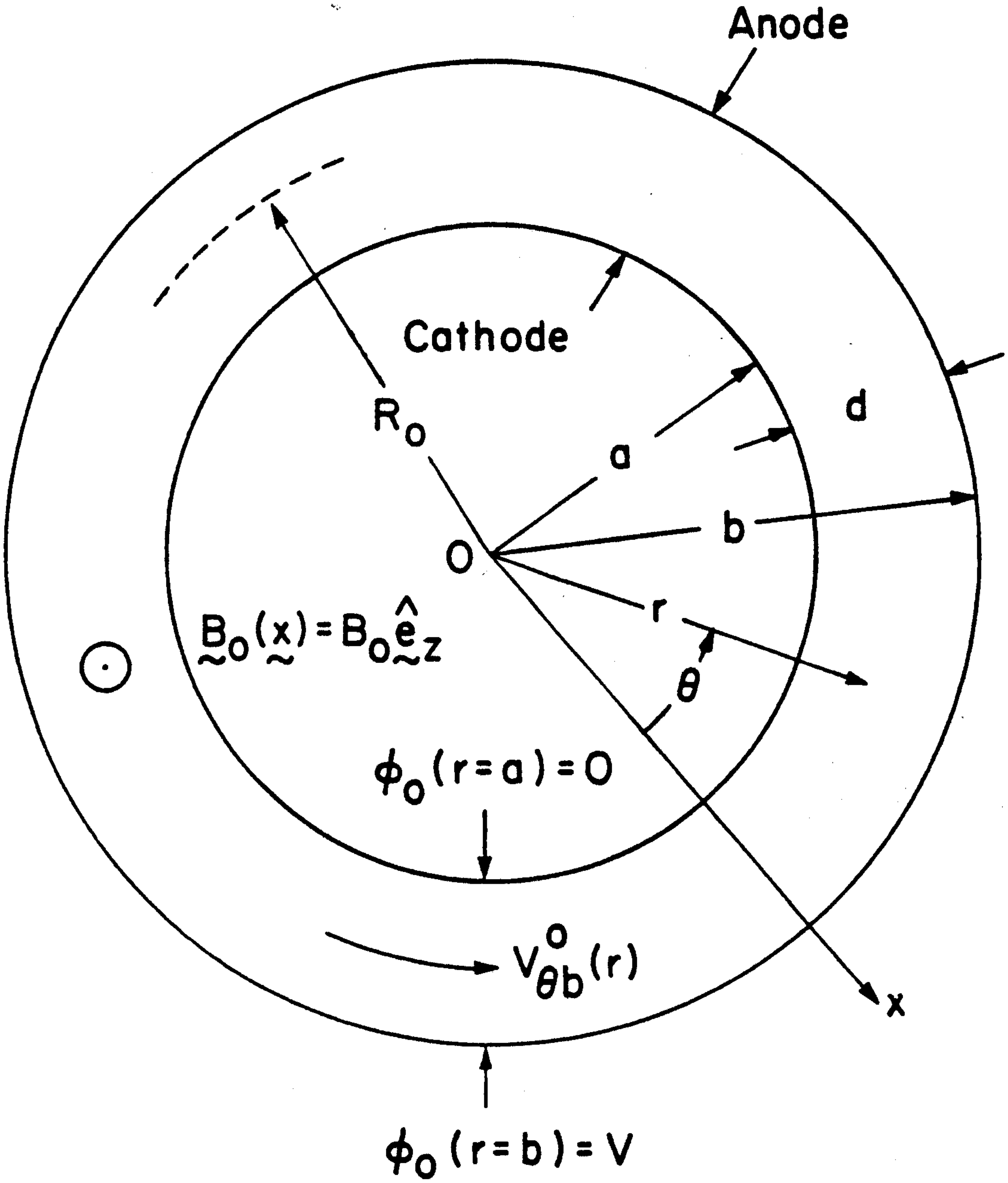

Figure 1: Cylindrical Diode Configuration with cathode at $r = a$ and anode at $r = b$, and applied axial magnetic field $\underset{\sim}{B}_0(\underset{\sim}{x}) = B_0\hat{\underset{\sim}{e}}_z$. Equilibrium electron flow is in the θ-direction.