RADIALLY CONFINED RELATIVISTIC ELECTRON BEAM EQUILIBRIA FOR LONGITUDINAL WIGGLER FREE ELECTRON LASER APPLICATIONS

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ABSTRACT

The purpose of the present article is to establish the properties of a class of radially confined relativistic electron beam equilibria for longitudinal wiggler free electron laser applications. The theoretical model is based on the steady-state ($\partial/\partial t = 0$) Vlasov equation, assuming a thin, tenuous electron beam with $k_0^2 R_b^2 << 1$ and negligibly small equilibrium self fields. For the approximate magnetic field configuration, $B_r^0 = 0$ and $B_z^0 = B_0[1 + (\delta B/B_0)\sin k_0 z]$, the single-particle constants of the motion are: axial momentum p_z, perpendicular momentum $p_\perp = (p_r^2 + p_\theta^2)^{1/2}$, energy $\gamma mc^2 = (m^2c^4 + c^2p_z^2 + c^2p_\perp^2)^{1/2}$, and canonical angular momentum $P_\theta = r[p_\theta - (e/c)A_\theta^0(r,z)]$, where $A_\theta^0(r,z) = (r\ B_0/2)\ [1 + (\delta B/B_0)\sin k_0 z]$. Beam equilibrium properties are investigated for the class of self-consistent Vlasov equilibria $F_b^0\ (\underset{\sim}{x},\underset{\sim}{p}) = F(p_\perp^2 - 2\gamma_b m\omega_b P_\theta)G(p_z)$, where ω_b = const. is related to the mean rotation of the electron beam, and $\gamma_b mc^2$ = const. is the characteristic energy of a beam electron. Specific examples of sharp-boundary equilibria and diffuse equilibria are analyzed in detail, including the r-z modulation of the density profile $n_b^0(r,z)$ by the longitudinal wiggler field.

I. INTRODUCTION

The longitudinal wiggler configuration[1-3] has recently been proposed as an attractive magnetic field geometry for intense free electron laser (FEL) radiation generation. In the longitudinal wiggler free electron laser, an electron beam propagates along the axis of a multiple-mirror (undulator) magnetic field with axial periodicity length $\lambda_0 = 2\pi/k_0$ and axial and radial vacuum magnetic field components, $B_z^0(r,z)$ and $B_r^0(r,z)$, given by Eq. (1). For a thin pencil beam with $k_0^2R_b^2 \ll 1$, where R_b is the characteristic beam radius, the primary wiggler field is in the axial direction, and the equilibrium field components can be approximated by $B_r^0(r,z) = 0$ and [Eq. (3)]

$$B_z^0(r,z) = B_0\left[1 + \frac{\delta B}{B_0}\sin k_0z\right].$$

Here, $\delta B/B_0$ is related to the mirror ratio R by $R = (1 + \delta B/B_0)/(1 - \delta B/B_0)$.

The instability mechanism for the longitudinal wiggler free electron laser[1-3] is a hybrid of the Weibel and axial bunching mechanisms for the cyclotron maser[4-6] and standard free electron laser[7-10] instabilities. Calculations of growth rate for the longitudinal wiggler free electron laser[1-3] have been based on a very simple model in which the beam cross section is treated as infinite in extent, and the influence of finite radial geometry on stability properties is completely neglected. The purpose of the present article is to establish the properties of a class of radially confined relativistic electron beam equilibria for longitudinal wiggler free electron laser applications. The theoretical model (Sec. II) is based on the steady-state ($\partial/\partial t = 0$) Vlasov equation, assuming a thin, tenuous electron beam with $k_0^2R_b^2 \ll 1$ and negligibly small equilibrium self fields. For the approximate magnetic field configuration, $B_r^0 = 0$ and

$B_z^0 = B_0[1 + (\delta B/B_0)\sin k_0 z]$, the single-particle constants of the motion are: axial momentum p_z, perpendicular momentum $p_\perp = (p_r^2 + p_\theta^2)^{1/2}$, energy $\gamma mc^2 = (m^2c^4 + c^2p_z^2 + c^2p_\perp^2)^{1/2}$, and canonical angular momentum $P_\theta = r[p_\theta - (e/c)A_\theta^0(r,z)]$, where $A_\theta^0(r,z) = (rB_0/2)[1 + (\delta B/B_0)\sin k_0 z]$. Beam equilibrium properties are investigated in Sec. III for the class of self-consistent Vlasov equilibria $f_b^0(\mathbf{x},\mathbf{p}) = F(p_\perp^2 - 2\gamma_b m\omega_b P_\theta)G(p_z)$, where ω_b = const. is related to the mean rotation of the electron beam, and $\gamma_b mc^2$ = const. is the characteristic energy of a beam electron. Specific examples of sharp-boundary equilibria (Sec. III.B) and diffuse equilibria (Sec. III.C) are analyzed in detail, including the r-z modulation[11] of the density profile $n_b^0(r,z)$ by the longitudinal wiggler field.

II. THEORETICAL MODEL AND ASSUMPTIONS

In the present analysis, a tenuous electron beam propagates along the axis of a multiple-mirror (undulator) magnetic field with axial periodicity length $\lambda_0 = 2\pi/k_0$ and axial and radial vacuum magnetic fields, $B_z^0(r,z)$ and $B_r^0(r,z)$, given by[12]

$$
\begin{aligned}
B_z^0(r,z) &= \left[B_0\ 1 + \frac{\delta B}{B_0} I_0(k_0 r)\sin k_0 z\right], \\
B_r^0(r,z) &= -\,\delta B\, I_1(k_0 r)\cos k_0 z,
\end{aligned}
\qquad (1)
$$

where $I_n(k_0 r)$ is the modified Bessel function of the first kind of order n, and $\delta B/B_0 < 1$ is related to the mirror ratio R by $R = (1+\delta B/B_0)/(1-\delta B/B_0)$. In circumstances where the beam radius R_b is sufficiently small with

$$k_0^2 R_b^2 \ll 1 \,, \qquad (2)$$

and the oscillatory field amplitude δB is sufficiently small with $\delta B/B_0 < 1$, then the leading-order oscillation (wiggle) in the applied field is primarily in the axial direction, with $B_r^0 = O(k_0 r)B_z^0$. Within the context of Eq. (2), the equilibrium magnetic field components can be approximated by

$$B_z^0(r,z) = B_0\left[1 + \frac{\delta B}{B_0}\sin k_0 z\right] ,$$
$$B_r^0(r,z) = 0 , \tag{3}$$

in the beam interior where $r < R_b << k_0^{-1}$.

Assuming a tenuous electron beam with negligibly small equilibrium self fields, then the electron motion in the longitudinal wiggler field specified by Eq. (3) is characterized by the four single-particle constants of the motion[1]

$$p_z ,$$
$$p_\perp^2 = p_r^2 + p_\theta^2 ,$$
$$\gamma mc^2 = \left(m^2c^4 + c^2p_\perp^2 + c^2p_z^2\right)^{1/2} , \tag{4}$$
$$P_\theta = r\left[p_\theta - \frac{e}{c}A_\theta^0(r,z)\right] ,$$

where p_z is the axial momentum, $p_\perp$ is the perpendicular momentum, γmc^2 is the electron energy, and

$$A_\theta^0(r,z) = \frac{1}{2}\, rB_0\left[1 + \frac{\delta B}{B_0}\sin k_0 z\right] \tag{5}$$

is the θ-component of vector potential consistent with Eq. (3). Here, $-e$ is the electron charge, m is the electron rest mass, and c is the speed of light in vacuo. Note that γmc^2 = const. can be constructed from the constants of the motion p_z and $p_\perp^2$, which are independently conserved.

It is important to keep in mind that the validity of the single-particle constants of the motion in Eq. (4) assumes that $k_0^2r^2 << 1$, $\delta B/B_0 < 1$

and that the oscillatory radial magnetic field $B_r^0 \simeq -(\delta B/2)k_0 r\cos k_0 z$ can be approximated by $B_r^0 = 0$ [Eq. (3)]. To determine the range of validity of this approximation, we have also calculated[1] (in an iterative sense) the leading-order corrections to the longitudinal and transverse orbits, treating the magnetic force $(-e/c)\mathbf{v}\times B_r^0\hat{\mathbf{e}}_r$ as a small correction. It is found that the quantities p_z and $p_\perp^2$ remain good single-particle constants of the motion provided the inequalities

$$\frac{1}{k_0^2 v_z^2}, \frac{1}{k_0^2 v_\perp^2} \gg \frac{1}{2} \left| \frac{\omega_c}{k_0 v_z}\frac{\delta B}{B_0} \right. \times \sum_{m=-\infty}^{\infty}\sum_{q=-\infty}^{\infty} J_m\left(\frac{\omega_c}{k_0 v_z}\frac{\delta B}{B_0}\right) \times J_q\left(\frac{\omega_c}{k_0 v_z}\frac{\delta B}{B_0}\right)(mk_0 v_z + \omega_c)^{-1} \times \left. [k_0 v_z + (m+q)k_0 v_z + \omega_c]^{-1} \right|$$

are satisfied. Of course, this requires that $\delta B/B_0 < 1$, $k_0^2 v_z^2/\omega_c^2 < 1$, and that the axial motion be removed from cyclotron resonance ($\omega_c + Nk_0 v_z \neq 0$).

III. RADIALLY CONFINED BEAM EQUILIBRIA

A. General Considerations

Any distribution function $f_b^0(\mathbf{x},\mathbf{p})$ that is a function only of the single-particle constants of the motion in Eq. (4) is a solution to the steady-state ($\partial/\partial t = 0$) Vlasov equation. Previous analyses[1-3] of the longitudinal wiggler free electron laser instability have considered perturbations about the class of uniform beam equilibria $f_b^0(p_\perp^2, p_z)$. In order to construct radially confined self-consistent beam equilibria with

$n_b^0(r \to \infty) = 0$, which is the subject matter of this article, it is necessary that $f_b^0(\underset{\sim}{x},\underset{\sim}{p})$ depend explicitly on the canonical angular momentum P_θ.

For present purposes, we consider the class of radially confined beam equilibria of the general form

$$f_b^0(\underset{\sim}{x},\underset{\sim}{p}) = F(p_\perp^2 - 2\gamma_b m\omega_b P_\theta)G(p_z) \ , \tag{6}$$

where $\int_{-\infty}^{\infty} dp_z\, G(p_z) = 1$, ω_b is a constant, and $\gamma_b mc^2 = \text{const.}$ is the characteristic energy of a beam electron. Making use of Eq. (4), it is readily shown that the argument of the function F in Eq. (6) can be expressed as

$$p_\perp^2 - 2\gamma_b m\omega_b P_\theta = p_r^2 + (p_\theta - \gamma_b m\omega_b r)^2 + \psi(r,z) \tag{7}$$

where the <u>envelope function</u> $\psi(r,z)$ is defined by

$$\psi(r,z) = \gamma_b^2 m^2 r^2\ \omega_b \left\{ \omega_{cb}\left[1 + \frac{\delta B}{B_0}\sin k_0 z\right] - \omega_b \right\} . \tag{8}$$

Here, $\omega_{cb} = eB_0/\gamma_b mc$ is the relativistic cyclotron frequency. Therefore, from Eqs. (6) - (8), the electron beam density profile $n_b^0(r,z) = 2\pi \int_0^\infty dp_\perp p_\perp \int_{-\infty}^{\infty} dp_z\ f_b^0(p_\perp^2, P_\theta, p_z)$ can be expressed as

$$n_b^0(r,z) = 2\pi \int_0^\infty dp_\perp' p_\perp'\ F[p_\perp'^2 + \psi(r,z)], \tag{9}$$

where $p_\perp'^2 = p_r^2 + (p_\theta - \gamma_b m\omega_b r)^2$, and use has been made of $\int_{-\infty}^{\infty} dp_z G(p_z) = 1$. It is clear from Eq. (9) that $\psi(r,z) = \text{const.}$ contours correspond to constant-density contours.

B. <u>Sharp-Boundary Beam Equilibrium</u>

As a specific example that is analytically tractable, we consider the case where

$$F = \frac{\hat{n}_b}{(2\pi\gamma_b m\hat{V}_{\perp b})}\ \delta(p_\perp^2 - 2\gamma_b m\omega_b r P_\theta - 2\gamma_b m\hat{T}_{\perp b}), \tag{10}$$

where $\hat{T}_{\perp b}$ = const. is related to the transverse temperature of the beam electrons, and $\hat{V}_{\perp b}^2 \equiv 2\hat{T}_{\perp b}/\gamma_b m$. Substituting Eq. (10) into Eq. (9) readily gives

$$n_b^0(r,z) = \begin{cases} \hat{n}_b = \text{const.} \ , & \psi(r,z) \le (\gamma_b m \hat{V}_{\perp b})^2 , \\ \\ 0 \ , & \psi(r,z) > (\gamma_b m \hat{V}_{\perp b})^2 . \end{cases} \tag{11}$$

From Eq. (11), the electron density is constant (equal to $\hat{n}_b$) in the beam interior, and the outer envelope of the electron beam is determined from

$$\psi(r,z) = (\gamma_b m \hat{V}_{\perp b})^2 \ , \tag{12}$$

where $\hat{V}_{\perp b}^2 = 2\hat{T}_{\perp b}/\gamma_b m$. Substituting Eq. (8) into Eq. (12), and solving for the radius $r = R_b(z)$ of the electron beam gives

$$R_b^2(z) = \frac{\hat{V}_{\perp b}^2}{\omega_b\{\omega_{cb}[1 + (\delta B/B_0)\sin k_0 z] - \omega_b\}} \tag{13}$$

where $n_b^0(r,z) = \hat{n}_b$ = const. for $0 \le r \le R_b(z)$, and $n_b^0(r,z) = 0$ for $r > R_b(z)$.

From Eq. (13), it is clear that radially confined equilibria exist only for angular rotation velocity ω_b in the range

$$0 < \omega_b < \omega_{cb}[1 - \delta B/B_0] \ , \tag{14}$$

where $0 < \delta B/B_0 < 1$ has been assumed. In this regard, it is important to keep in mind that $\langle p_\theta \rangle = (\int d^3p\, p_\theta f_b^0)/(\int d^3p\, f_b^0) = \gamma_b m \omega_b r$ for the entire class of self-consistent Vlasov equilibria in Eq. (6). That is, ω_b is directly related to the mean angular rotation of the electron beam. It is also clear from Eq. (13) that the outer envelope of the constant-density electron beam undulates axially between a maximum outer radius $[R_b]_{MAX}$,

$$[R_b^2]_{MAX} = \frac{\hat{V}_{\perp b}^2}{\omega_b\{\omega_{cb}[1 - \delta B/B_0] - \omega_b\}} \quad , \tag{15}$$

where the axial magnetic field is weakest [at $k_0 z = (2n + 1)\pi/2$, $n = \pm 1, \pm 3, \pm 5, \ldots$], and a minimum outer radius $[R_b]_{MIN}$,

$$[R_b^2]_{MIN} = \frac{\hat{V}_{\perp b}^2}{\omega_b\{\omega_{cb}[1 + \delta B/B_0] - \omega_b\}} \quad , \tag{16}$$

where the axial magnetic field is strongest [at $k_0 z = (2n + 1)\pi/2$, $n = 0, \pm 2, \pm 4, \ldots$].

Equation (2) is an important condition for validity of the present equilibrium theory. From Eq. (15), it is readily shown that $k_0^2 [R_b^2]_{MAX} << 1$ can be expressed in the equivalent form

$$k_0^2 r_{Lb}^2 << \frac{\omega_b}{\omega_{cb}} \left[1 - \frac{\delta B}{B_0} - \frac{\omega_b}{\omega_{cb}}\right] \tag{17}$$

where $0 < \omega_b < \omega_{cb}(1 - \delta B/B_0)$ is assumed, and $r_{Lb} \equiv (\hat{V}_{\perp b}/\omega_{cb})$ is the effective thermal Larmor radius of a beam electron. Equations (2) and (17) of course restrict the validity of the present analysis to a thin pencil beam propagating down the axis of the multiple-mirror system.

C. Diffuse Beam Equilibrium

Depending on the choice of F, the class of beam equilibria in Eq. (6) also allows the possibility of a diffuse (bell-shaped) density profile. As a second example, consider the case where F is specified by

$$F = \frac{\hat{n}_b}{(2\pi\gamma_b m\hat{T}_{\perp b})} \exp\left\{-\frac{p_\perp^2 - 2\gamma_b m\omega_b r P_\theta}{2\gamma_b m\hat{T}_{\perp b}}\right\}, \tag{18}$$

where $\hat{n}_b$ and $\hat{T}_{\perp b}$ are positive constants. Substituting Eq. (18) into Eq. (9) gives directly

$$n_b^0(r,z) = \hat{n}_b \exp\left\{-\frac{\psi(r,z)}{2\gamma_b m\hat{T}_{\perp b}}\right\}, \tag{19}$$

which can be expressed in the equivalent form

$$n_b^0(r,z) = \hat{n}_b \exp\left\{-\frac{r^2}{R_b^2(z)}\right\}, \tag{20}$$

where $R_b^2(z)$ is defined in Eq. (13) with $\hat{V}_{\perp b}^2 = 2\hat{T}_{\perp b}/\gamma_b m$. As for the case of the rectangular density profile in Eq. (11), the necessary condition for a radially confined equilibrium in Eq. (20) is that ω_b be in the interval $0 < \omega_b < \omega_{cb}(1-\delta B/B_0)$. Note from Eq. (20) that the choice of distribution function in Eq. (18) gives a Gaussian density profile with peak density ($\hat{n}_b$) at $r = 0$ and characteristic radial width of the density profile equal to $R_b(z)$. Moreover, $r/R_b(z)$ = const. contours correspond to constant-density contours with $n_b^0(r,z)$ = const. Finally, the condition $k_0^2R_b^2(z) << 1$ is required, where $R_b^2(z)$ is defined in Eq. (13).

IV. CONCLUSIONS

Calculations of growth rate for the longitudinal wiggler free electron laser[1-3] have heretofore been based on a very simple model in which the beam cross section is treated as infinite in extent, and the influence of finite radial geometry on stability properties is completely neglected. The purpose of the present article was to establish the properties of a class of radially confined relativistic electron beam equilibria for longitudinal wiggler free electron laser applications. The theoretical model (Sec. II) was based on the steady-state ($\partial/\partial t = 0$) Vlasov equation, assuming a thin, tenuous electron beam with $k_0^2R_b^2 << 1$ and negligibly small equilibrium self fields. For the approximate magnetic field configuration, $B_r^0 = 0$ and $B_z^0 = B_0[1 + (\delta B/B_0)\sin k_0 z]$, the single-particle constants of the motion are: axial momentum p_z, perpendicular momentum $p_\perp = (p_r^2 + p_\theta^2)^{1/2}$, energy $\gamma mc^2 = (m^2c^4 + c^2p_z^2 + c^2p_\perp^2)^{1/2}$, and canonical angular momentum $P_\theta = r[p_\theta - (e/c)A_\theta^0(r,z)]$, where $A_\theta^0(r,z) = (rB_0/2)[1 + (\delta B/B_0)\sin k_0 z]$. Beam equilibrium properties were investigated in Sec. III for the class of self-consistent Vlasov equilibria $f_b^0(\mathbf{x},\mathbf{p}) = F(p_\perp^2 - 2\gamma_b m\omega_b P_\theta)G(p_z)$, where ω_b = const. is related to the mean rotation of the electron beam, and $\gamma_b mc^2$ = const. is the characteristic energy of a beam electron. Specific examples of sharp-boundary equilibria (Sec. III.B) and diffuse equilibria (Sec. III.C) were analyzed in detail, including the r - z modulation of the density profile $n_b^0(r,z)$ by the longitudinal wiggler field.

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