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Stellarator Equilibria
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Problem of Position Control

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Abstract

A class of low- β stellarator MHD equilibria with $\beta \sim (a/R_0)^2$ is introduced and the corresponding toroidal shift is calculated. It is shown that an apparent paradox exists with regard to the problem of position control in that a vertical field is capable of shifting vacuum flux surfaces, but produces no net body force on a current free stellarator. This paradox is resolved by an analysis of the transient response of the plasma and demonstrates how the vertical field can be used as a means of position control.

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I. Introduction and Statement of the Problem

In this paper we investigate a class of toroidal magnetohydrodynamic equilibria of stellarators. We wish to obtain a better physical understanding of toroidal force balance and the corresponding toroidal shift, thus enabling us to address the problem of position control in a stellarator by means of a vertical field. However, the following two facts – (1) a vertical field shifts vacuum flux surfaces, and (2) a vertical field produces no net body force on a stellarator with zero net current – clearly indicate that in a stellarator one is confronted with the even more basic question, namely, can a vertical field be used at all for position control in a stellarator?

This somehow apparently paradoxical situation wherein the vertical field shifts the magnetic surfaces but produces no net force is manifested in the very fundamental difference in the operation of a tokamak and a stellarator. In a tokamak, the control of plasma position poses no conceptual difficulty. With the application of a vertical field a toroidal force (i.e., along the major radius) is generated due to the permanent presence of a toroidal current. Of course there is a question of a response to transients which arise during the buildup of the vertical field, but this is a minor problem, especially when the evolution of the vertical field is slow in comparison to the inertial response of the plasma. The resulting end product is a new equilibrium with shifted magnetic surfaces calculated using the well-known formula due to Shafranov.¹

In a stellarator no static toroidal force (due to the vertical field) is possible. Thus a change in the toroidal shift may only be produced by a dynamic action; in particular as a response to the buildup or decay of the vertical field. Clearly, once the transients are over, the vertical field no longer produces a body force and the plasma comes to a "new" resting position. This end state evolves adiabatically from the initial one, implying that in principle the net shift in plasma position can be determined from a static ideal theory without actually knowing or taking into consideration the dynamical mechanism which connected two static states.

Nevertheless, one has to be concerned with the dynamics in order to ensure that indeed there exists a physical process capable of transforming one static state into another. The dynamics needed to accomplish this are as follows.

(1) In a static equilibrium the vertical field as previously stated does not produce a net toroidal body force on the plasma. The outward expansion force due to the $1/R$ dependence of B_ϕ is balanced by the $\mathbf{J} \times \mathbf{B}$ force generated by the interaction of the Pfirsch-Schluter currents with the average poloidal helical magnetic field on the plasma surface, $\langle B_p \rangle \sim \hat{t}_H$ averages to zero in leading order (but is non-zero in next order, and \hat{t}_H is the helical transform $/2\pi$.) (2) When the vertical field is varied in time (as it would be for position control) the plasma, because of inertial effects, remains initially at rest. Within the context of ideal MHD a toroidal surface dipole current is induced instantaneously in order to conserve the flux in the plasma. (An additional net toroidal

surface current may or may not be induced depending on the circuits controlling the poloidal flux linked by the plasma). The induced dipole current interacts with the average poloidal field producing a body force on the plasma. The direction of this force is such that as the plasma moves, a counter surface dipole current is induced. An equilibrium is achieved when the plasma moves to a new major radius where the net surface dipole current vanishes. Consequently, the vertical field gives rise to a body force on the plasma only during transient periods when the dipole current is induced. This force persists as long as transients exist. As mentioned previously, the time scale associated with the vertical field evolution is much longer than the plasma inertial time scale, so that the plasma position essentially tracks the vertical field almost adiabatically.

Using a mechanical analogy, we may compare tokamaks and stellarators as follows. In a tokamak the vertical field acts like a (magnetic) piston pushing with a constant force against a spring with a balance achieved after the reacting spring's force counteracts the piston. In a stellarator the same piston is activated "impulsively" with a certain energy deposited but a varying force exerted by the piston. If this deposit is very short in duration as compared with the inertial response, the motion is indeed impulsively initiated. If, on the other hand, as is the case in practice, the energy deposit is slow, the spring is pushed by a series of weak impulses and the motion proceeds as a sequence of "quasi-static" steps.

Returning to our main goal, we note first that while from the physical point of view the actual calculation of plasma shift is a minor point, in practicality it constitutes the main calculative thrust of the problem and of this paper as well. Our approach which is essentially analytical, is described in some detail in the next section. For the benefit of the reader interested only in the final result, we provide the following summary.

The MHD equations are solved using a low- β (i.e., $\beta \simeq (a/R_0)^2 \equiv \epsilon^2$) variant of the Princeton stellarator expansion² where $\beta \sim (a/R_0)$ was assumed. This assumption spreads the expansion over many orders but leads to an entirely analytic theory in which simple relations are derived for the toroidal shift and the toroidal force balance as functions of the vertical field, helical sideband fields and, if included, ohmic current as well. One recalls that treating $\beta \sim (a/R_0)^2$ is equivalent to the low- β tokamak expansion and indeed, in the limit of zero helical fields our results give the Shafranov shift¹ for a tokamak. It is a useful feature of our theory that it can describe both a pure tokamak, a pure stellarator, and hybrid configurations as well.

After lengthy but manageable calculations we derive a set of equations for the toroidally symmetric component of vector potential, $A_{||}(\mathbf{r}, \theta) = A(r) \cos \theta$ and the toroidal shift $\sigma(r) \equiv -\beta_1(r)/\beta'_0(r)$ where $\beta(\psi) \approx \beta_0(r) + \beta_1(r) \cos \theta$. The general form of the equations is valid for arbitrary ohmic heating profiles and finite ϵN with N the number of helical periods and $\epsilon \equiv a/R_0$ the plasma inverse aspect ratio. For the case of a current free stellarator and small ϵN these equations can be solved analytically, yielding

$$\frac{A(r)}{B_0} = r \left[b_v + \int_r^\infty \frac{dx}{x^3} \int_0^x \frac{y^2 \beta'}{\hat{l}_{IJ}} dy \right] \quad (1)$$

$$\frac{\sigma(a)}{a} = -\frac{1}{\hat{l}_{IJ}(a)} \left\{ \frac{b_v}{\epsilon} + \frac{1}{2\hat{l}_{IJ}(a)} \int_0^1 z^{6-2\ell} \left(\frac{\beta}{\epsilon} \right)' dz - \frac{b_\ell}{N\epsilon^2} \left[b_{\ell+1} + b_{\ell-1} + \frac{\epsilon}{2} b_\ell \right] \right\}, \quad (2)$$

where B_0 is the toroidal field, \hat{l}_{IJ} , as before, is the helical transform $/2\pi$, N is the number of helical periods, $z = r/a$ is the normalized radius, and $b_v = B_r^{(V)}(a)/B_0$, $b_\ell = B_r^{(\ell)}(a)/B_0$, and $b_{\ell\pm 1} = B_r^{(\ell\pm 1)}(a)/B_0$ are the normalized amplitudes of the applied vertical field, main helical field and helical sideband fields respectively.

In passing we note that while Eq. (2) for the shift is model dependent, the concepts leading to it are not. This is to imply that while in practical situations one may have to calculate the actual shift numerically (as is usually the case with Grad-Shafranov equation)^{3,4} or by using a different perturbational scheme, the arguments presented assure the possibility of a transit from one stable stage to another.

II. Stellarator Equilibria and the Corresponding Toroidal Shift

We consider the equilibrium of a diffuse toroidal stellarator as described by the ideal magnetohydrodynamic model

$$\mathbf{J} \times \mathbf{B} = \nabla p$$

$$\mathbf{J} = \nabla \times \mathbf{B} \tag{3}$$

$$\nabla \cdot \mathbf{B} = 0$$

A. Assumptions

As mentioned in Sec. I, the calculation is carried out using a low β modification of the well-known stellarator expansion introduced by Greene and Johnson.² In the present variant, β is assumed one order smaller in inverse aspect ratio, $\epsilon \equiv a/R_0$, than in the original expansion; that is, $\beta \simeq \epsilon^2$ rather than $\beta \simeq \epsilon$. This has several important consequences:

1. The physics of the modified expansion is such that a stellarator is treated comparably to a conventional tokamak, $\beta \simeq \epsilon^2$ (as opposed to a high β tokamak, $\beta \simeq \epsilon$, in the original expansion).
2. In the modified expansion the flux surfaces are nearly circular with small helical and toroidal shifts. (The toroidal shift is of order unity in the original expansion).
3. Because of the small shifts it is possible to analytically calculate the equilibrium fields and the corresponding "Shafranov" shift.¹ (In the original expansion one must solve a two dimensional "Grad Shafranov" partial differential equation to obtain the equilibrium fields).
4. Because of the small shifts, the expansion, expressed in terms of the helical field amplitude, $\delta_\ell \simeq \epsilon^2$, is spread out over many orders. In fact, parts of the calculation must be carried out to sixth order to satisfy the parallel current constraint⁵ associated with guaranteeing $\nabla \cdot \mathbf{J} = 0$.
5. The calculation is carried out for arbitrary helical multipolarity, ℓ , but difficulties arise near the magnetic axis for $\ell \geq 3$ where the shifts can be finite. (No such difficulty occurs in the original expansion).

6. A comparison of the results of the modified expansion with the low β limit of the original expansion indicates that many terms are common, but that each expansion also contains effects not included in the other.

The analysis is carried out in toroidal coordinates (r, θ, z) which are related to the usual cylindrical coordinates (R, ϕ, Z) as follows

$$R = R_0 + r \cos \theta$$

$$Z = r \sin \theta \quad (4)$$

$$\phi = -z/R_0,$$

where R_0 is the major radius to the magnetic axis of the vacuum helical magnetic field. The geometry is illustrated in Fig. 1.

The appropriate expansion for the field variables is given in terms of the ordering parameter δ_ℓ , characterizing the amplitude of helical fields.

$$B = B_0 e_z + B_1(r, \alpha) + B_2(r, \theta) + [\tilde{B}_3(r, \alpha \pm \theta) + \tilde{B}_3(r, \alpha)] \dots$$

$$J = J_2(r) + J_3(r, \alpha) + [\tilde{J}_4(r, \theta) + \tilde{J}_4(r, 2\alpha)] \dots \quad (5)$$

$$p = p_4(r) + p_5(r, \alpha) + [\tilde{p}_6(r, \theta) + \tilde{p}_6(r, 2\alpha)] \dots$$

where $\alpha = \ell\theta + hz$, ℓ is the helical multipolarity of the stellarator field, and $2\pi/h$ is the length of a single helical period. The subscripts on each expansion quantity represent the corresponding ordering in δ_ℓ : $B_n \simeq \delta_\ell^n$.

The motivation for choosing the functional dependencies of B , J , and p given by Eq.(5) will become apparent shortly. At this point it is sufficient to note that Eq.(5) is consistent with the basic physics requirements of the modified stellarator expansion summarized as follows.

- (a) helical period, $2\pi/h$, comparable to the plasma radius, a

$$ha \simeq 1$$

- (b) total helical transform, $\iota_{II}/2\pi \equiv \tilde{\iota}_{II}$ of order unity

$$\hat{v}_H \simeq 1$$

$$\epsilon \equiv a/R_0 \simeq \delta_\ell^2$$

(c) total ohmic transform, $\iota_J/2\pi \equiv \hat{\iota}_J$ of order unity

$$\hat{\iota}_J \simeq 1$$

$$aJ_{\parallel}/B_0 \simeq \delta_\ell^2$$

(d) small distortions of the flux surfaces, $r = r_0 + \sigma_H(r_0, \alpha) + \sigma_T(r_0, \theta) \dots$

$$\text{helical shift: } \sigma_H/a \simeq \delta_\ell$$

$$\text{toroidal shift: } \sigma_T/a \simeq \delta_\ell^2$$

(e) small effects distorting the flux surfaces

$$\text{pressure, } p: \beta \equiv 2p_A/B_0^2 \simeq \delta_\ell^4$$

$$\text{vertical field, } B_v: B_v/B_0 \simeq \delta_\ell^4$$

$$\text{helical sideband fields, } \delta_{\ell \pm 1}: \delta_{\ell \pm 1} \simeq \delta_\ell^3 \dots$$

B. Calculation of Perpendicular Pressure Balance

In what follows we describe order by order the relevant expansion describing perpendicular pressure balance and the corresponding solution. A critical feature of the solution is the introduction of appropriate homogeneous solutions at certain orders of the expansion which are required to satisfy periodicity constraints (i.e., the parallel current constraint) appearing in higher order.

1. Zero Order (δ^0)

Zeroth order describes a vacuum toroidal field

$$B = B_0 e_z, \quad J_0 = p_0 = 0. \quad (6)$$

2. First Order (δ^1)

First order describes the main vacuum helical field. Here $\nabla \times B_1 = 0$ and $\nabla \cdot B_1 = 0$ yields

$$B_1 = B_0 \nabla [\phi_\ell \sin \alpha], \quad J_1 = p_1 = 0, \quad (7)$$

with $\phi_\ell(r)$ satisfying

$$\frac{1}{r} \frac{d}{dr} r \frac{d\phi_\ell}{dr} - \left(\frac{\ell^2}{r^2} + h^2 \right) \phi_\ell = 0.$$

3. Second Order (δ^2)

The second order fields describe the basic $1/R$ correction to the toroidal field and allow a small unidirectional ohmic heating current as would flow in a tokamak. After a straightforward calculation we find

$$B_2 = -\frac{B_0}{R_0} r \cos \theta e_z + B_{\theta 2}(r) e_\theta$$

$$J_2 = J_{\parallel}(r) e_z \quad (8)$$

$$p_2 = 0,$$

where $J_{\parallel} = (rB_{\theta 2})'/r$ can be considered to be a free function.

4. Third Order (δ^3)

Third order describes a number of effects: the coupling of the ohmic current with the helical field; the application of external helical sideband fields; and the $1/R$ toroidal correction to the main helical field. Straightforward analysis yields

$$B_3 = B_0 \left[\nabla \phi_s - \frac{B_{z1}}{B_0} \frac{r \cos \theta}{R_0} e_z + r J_{\parallel} e_{\parallel} \times \nabla (\phi_\ell \cos \alpha) + e_{\perp} \times \nabla (\psi_\ell \cos \alpha) \right]$$

$$J_3 = J_{\parallel} \nabla (\phi_\ell \sin \alpha) + \frac{J'_{\parallel} \phi'_\ell}{h} \cos \alpha e_z \quad (9)$$

$$p_3 = 0,$$

where $e_{\perp} = (\ell e_z - h r e_{\theta})/\Delta$, $e_{\parallel} = (\ell e_{\theta} + h r e_z)/\Delta$, $\Delta = \ell^2 + h^2 r^2$. The first two terms in B_3 represent the coupling of toroidal geometry with the main helical field. The potential, ϕ_s , satisfies

$$\nabla^2 \phi_s = \frac{1}{B_0 R_0} \left[\frac{\partial}{\partial z} (B_{z1} r \cos \theta) - \nabla_{\perp} \cdot (B_{\perp 1} r \cos \theta) \right]. \quad (10)$$

These terms give rise to helical sideband harmonics $(\ell \pm 1)\theta + hz$. External sideband fields are allowed by applying source boundary conditions on ϕ_s as $r \rightarrow \infty$.

The last two terms in B_3 describe the coupling of the main helical field with the ohmic current and generate modifications to the basic helical harmonic, $\ell\theta + hz$. The quantity ψ_{ℓ} is a helical flux function satisfying

$$\left(\frac{r\psi'_{\ell}}{\Delta} \right)' - \frac{\psi_{\ell}}{r} = \frac{\ell}{h} (r\phi'_{\ell}) \left(\frac{J_{\parallel}}{\Delta B_0} \right)'. \quad (11)$$

5. Fourth Order (δ^4)

The fourth order fields contain the following contributions: the $1/R$ toroidal corrections to the second order fields; second harmonic fields caused by the non-linear interaction of the main helical field, B_1 ; pure transverse dipole fields to satisfy higher order periodicity requirements, and the first appearance of the plasma pressure, $p(r)$. The second harmonic fields [i.e., those proportional to $2(\ell\theta + hz)$] are not explicitly required in the higher order constraint equations, and hence, for simplicity, are not included here.

The fourth order fields can be written as

$$\begin{aligned} B_4 &= e_z \times \nabla A_{\parallel} - \left(B_{\theta 2} e_{\theta} + B_0 \frac{r \cos \theta}{R_0} e_z \right) \frac{r \cos \theta}{R_0} + B_{z4}(r) e_z + \tilde{B}_4(r, 2\alpha) \\ J_4 &= \left[\nabla_{\perp}^2 A_{\parallel} - \frac{(r^2 B_{\theta 2})'}{r R_0} \cos \theta \right] e_z + \frac{1}{B_0} \left[p' + J_{\parallel} B_{\theta 2} + \frac{\ell B_0}{2hr} J'_{\parallel} \phi_{\ell} \phi'_{\ell} \right] e_{\theta} + \tilde{J}_4(\ell, 2\alpha) \end{aligned} \quad (12)$$

$$p_4 = p(r).$$

Here, $A_{\parallel}(r, \theta)$ is a vector potential, satisfying the homogeneous equations, which gives rise to purely transverse fields and which is required to satisfy the parallel current constraint. The quantity $B_{z4}(r)$ represents the diamagnetic response to the plasma pressure $p(r)$. These quantities are related by the one dimensional radial pressure balance equation

$$(p + B_0 B_{z4})' + \frac{B_{\theta 2}}{r} (r B_{\theta 2})' + \frac{\ell R_0}{2hr} J_{\parallel}' \phi_e \phi_e' = 0. \quad (13)$$

At this point in the calculation, if one specifies the free functions $J_{\parallel}(r)$ and $p(r)$, and selects the external amplitudes of the main helical fields and its sidebands, the solutions are in principle known to fourth order with the exception of the as yet undetermined vector potential $A_{\parallel}(r, \theta)$.

6. Fifth Order (δ^5)

The fifth order equations are quite complicated. Our task is greatly simplified by noting that we need only J_5 and not B_5 or p_5 for the parallel current constraint. Furthermore, only those terms in J_5 corresponding to helical sidebands are explicitly required. These terms can be easily found from the fifth order pressure balance relation

$$B_0 J_5 \times e_z + J_4 \times B_1 + J_3 \times B_2 + J_2 \times B_3 - \nabla p_5 = 0. \quad (14)$$

Straightforward calculation gives the perpendicular sideband component of J_5 , denoted by $\hat{J}_{\perp 5}$ as

$$\hat{J}_{\perp 5} = J_{\parallel} \nabla_{\perp} \phi_s + \frac{B_{\perp 1}}{B_0} \left[\nabla_{\perp}^2 A - \frac{B_{\theta 2}}{R_0} \cos \theta \right]. \quad (15)$$

The parallel component of \hat{J}_5 follows from $\nabla \cdot \hat{J} = 0$

$$\frac{\partial J_{\parallel 5}}{\partial z} = -\nabla_{\perp} \cdot \hat{J}_{\perp 5}. \quad (16)$$

7. Sixth Order (δ^6)

It is in sixth order that the parallel current constraint first becomes non-trivial. This constraint arises when trying to solve for the sixth order fields in the pressure balance relation

$$B_0 J_6 \times e_z + J_5 \times B_1 + J_4 \times B_2 + J_3 \times B_3 + J_2 \times B_4 - \nabla p_6 = 0. \quad (17)$$

If we perform the operation $e_z \cdot \nabla \times \langle \quad \rangle$ on this equation, where $\langle \quad \rangle$ denotes average over one helical period, then all the explicit sixth order quantities vanish, yielding the constraint

$$e_z \cdot \nabla \times \langle J_5 \times B_1 + J_4 \times B_2 + J_3 \times B_3 + J_2 \times B_4 \rangle = 0. \quad (18)$$

Similar relations, which appear in lower orders, are trivially satisfied by symmetry. The constraint equation can be satisfied only by properly choosing the as yet unspecified homogeneous solution, $A_{\parallel}(r, \theta)$. After a lengthy calculation we obtain

$$\nabla_{\perp}^2 \left(\frac{A_{\parallel}}{B_0} \right) - \left(\frac{R_0 J'_{\parallel}}{r B_0 \hat{t}} \right) \left(\frac{A_{\parallel}}{B_0} \right) = - \left[\frac{\beta'}{\hat{t}} - \frac{r}{R_0^2} \hat{t}_I \right] \cos \theta - \left(\frac{R_0 J'_{\parallel}}{r B_0 \hat{t}} \right) R_H$$

$$R_H = \frac{1}{h} \langle \mathbf{e}_z \cdot \nabla_{\perp} (\phi_{\ell} \cos \alpha) \times \nabla_{\perp} \left(\phi_s + \frac{3\ell \phi_{\ell} r}{R_0} \sin \theta \cos \alpha \right) \rangle, \quad (19)$$

where $\beta(r) \equiv 2p(r)/B_0^2$, $\hat{t}_I(r) = \iota_I/2\pi$ is the ohmic transform, $\hat{t}_H(r) = \iota_H/2\pi$ is the helical transform and $\hat{t} = \hat{t}_I + \hat{t}_H$. If an external vertical field is desired, it can be applied by means of the boundary condition on $A_{\parallel}(r, \theta)$ as $r \rightarrow \infty$.

This equation represents the basic force balance relation for the general stellarator/tokamak hybrid. The terms driving A_{\parallel} can be described as follows: the β' term represents the outward toroidal force caused by the $1/R$ dependence of the toroidal field; the \hat{t}_I term represents the outward hoop force associated with the ohmic current; the $J'_{\parallel} R_H$ term represents the force associated with the interaction of the helical field with the ohmic current; the $J'_{\parallel} A_{\parallel}$ term gives rise to a net body force if the externally applied vertical field is non-zero; and the $\nabla_{\perp}^2 A_{\parallel}$ term represents the net dipole current induced in balancing the forces. Note that the "helical sideband force", the dominant restoring force in a high β stellarator,^{6,7,8} does not appear in Eq. (19). The reason is that this force is of order $\beta \delta_{\ell} \delta_{\ell \pm 1} \sim \delta_{\ell}^8$ and thus would first appear two orders further in our expansion.

C. Calculation of Parallel Pressure Balance

In the previous section we demonstrated the need for a vector potential, $A_{\parallel}(r, \theta)$, in order to solve the sixth order perpendicular pressure balance equations. In this section we solve the parallel pressure balance relation, $\mathbf{B} \cdot \nabla p = 0$. The solution to $\mathbf{B} \cdot \nabla p = 0$ gives the equation for the pressure contours (i.e., the flux surfaces). Under the assumptions of our expansion, the flux surfaces are circles with small helical and toroidal shifts. It is of particular interest to calculate the toroidal shift, σ , as a function of the applied vertical and helical fields and the free functions $p(r)$, $J_{\parallel}(r)$. We now describe the order by order expansion of the flux surfaces, starting with fourth order, the first non-trivial contribution.

1. Fourth Order

The fourth order equation $B_0 \cdot \nabla p_4 = B_0 \partial p / \partial z = 0$ is automatically satisfied if

$$p_4 = p(r). \quad (20)$$

2. Fifth Order

The fifth order contribution describes the helical shift associated with the vacuum fields. A simple calculation yields

$$p_5 = \frac{p'\phi'_\ell}{h} \cos \alpha. \quad (21)$$

3. Sixth Order

The sixth order contribution to parallel pressure balance has second harmonic helical terms [i.e., $\tilde{p}_6(r, 2\alpha)$] plus the toroidal shift term $\bar{p}_6(r, \theta)$ which appears as a homogeneous solution. As before, the homogeneous term $\bar{p}_6(r, \theta)$ is not determined until higher order and the second harmonic contribution is not explicitly required. Thus, in sixth order we have

$$p_6 = \bar{p}_6(r, \theta) + \tilde{p}_6(r, 2\alpha). \quad (22)$$

4. Seventh Order

The seventh order pressure is quite complicated. However, in order to calculate $\bar{p}_6(r, \theta)$, only the sideband contribution to p_7 , denoted by \hat{p}_7 , is required. A simple calculation yields

$$\frac{\partial \hat{p}_7}{\partial z} = -p' \frac{\partial \phi_6}{\partial r} + 2 \frac{\partial p_5}{\partial z} \frac{r \cos \theta}{R} - \frac{1}{B_0} B_{\perp 1} \cdot \nabla \bar{p}_6. \quad (23)$$

5. Eighth Order

The eighth order parallel pressure balance relation must satisfy a periodicity constraint which determines \bar{p}_6 . The basic equation for p_8 is given by

$$B_0 \frac{\partial p_8}{\partial z} + B_1 \cdot \nabla p_7 + B_2 \cdot \nabla p_6 + B_3 \cdot \nabla p_5 + B_4 \cdot \nabla p = 0. \quad (24)$$

If we now compute $\langle \quad \rangle$ of this equation, the quantity p_8 no longer explicitly appears and we are left with the constraint

$$\langle B_1 \cdot \nabla \hat{p}_7 + B_2 \cdot \nabla \bar{p}_6 + B_3 \cdot \nabla p_5 + B_4 \cdot \nabla p \rangle = 0. \quad (25)$$

Substituting into Eq. (25) leads to the functional dependence $\bar{p}_6(r, \theta) = \bar{p}_6(r) \cos \theta$. Defining the helically averaged pressure contours as $\bar{p}(r, \theta) \equiv \langle p \rangle$, we find $\bar{p}(r, \theta) = p(r) + \bar{p}_6(r) \cos \theta \dots$ corresponding to toroidally shifted circular flux surfaces.

If the equation of a flux surface is written as $r = r_0 + \sigma(r_0) \cos \theta \dots$, then the shift $\sigma(r_0) = -\bar{p}_6(r_0)/p'(r_0)$, is found to be

$$\alpha(r) \cos \theta = -\frac{R_0}{ri(r)} \left[\frac{A_{||}(r, \theta)}{B_0} - R_{H}(r, \theta) \right]. \quad (26)$$

[Straightforward calculation shows that both $A_{||}(r, \theta)$ and $R_H(r, \theta)$ are proportional to $\cos \theta$.]

The first term in Eq. (26) represents the shift due to the vertical field, the hoop force and the toroidal $1/R$ effects, while the second term represents the shift due to helical sideband fields.

D. Summary of Equations

For convenience we summarize the basic equations describing arbitrary hybrid stellarator/tokamak equilibria. In these equations we assume that $p(r)$ and $J_{||}(r)$ are two specified free functions. Similarly we assume the vacuum vertical field, B_v , and externally applied helical fields $B_\ell, B_{\ell \pm 1}$ are also specified. Here $B_\ell \equiv B_{r1}(a, \alpha = \pi/2)$ is the amplitude of the radial component of helical magnetic field on the zeroth order plasma surface $r = a$. Analogous definitions apply for $B_{\ell \pm 1}$.

To find an equilibrium first solve

$$\frac{1}{r} \frac{d}{dr} r \frac{d\phi_\ell}{dr} - \left(\frac{\ell^2}{r^2} + h^2 \right) \phi_\ell = 0$$

$$\phi_\ell(0) = 0; \quad \phi'_\ell(a) = B_\ell/B_0 \quad (27)$$

for $\phi_\ell(r)$ in terms of modified Bessel functions $I_\ell(hr)$.

Second, calculate the sideband potential $\phi_s(r, \alpha \pm \theta)$ by solving

$$\nabla^2 \phi_s = \frac{1}{B_0 R_0} \left[\frac{\partial}{\partial z} (B_{z1} r \cos \theta) - \nabla_\perp \cdot (B_{\perp 1} r \cos \theta) \right]$$

$$\phi_s(0, \theta, z) = 0; \quad \frac{\partial \phi_s}{\partial r}(a, \theta, z) = \frac{B_{\ell+1}}{B_0} \sin(\alpha + \theta) + \frac{B_{\ell-1}}{B_0} \sin(\alpha - \theta) \quad (28)$$

where $B_1 = B_0 \nabla \phi_\ell \sin \alpha$.

Knowing $\phi_\ell(r)$ and $\phi_s(r, \theta, z)$, solve for the vector potential, $A_{||}(r, \theta)$, from

$$\nabla_\perp^2 \hat{A}_{||} - \frac{(r^3 \ell_J')}{r^3 \ell} \hat{A}_{||} = - \left[\frac{\beta'}{\ell} - \frac{r}{R_0^2 \ell} \right] \cos \theta - \frac{(r^3 \ell_J')}{r^3 \ell} R_{H}$$

$$R_{H} = \frac{1}{h} \langle e_z \cdot \nabla_\perp (\phi_\ell \cos \alpha) \times \nabla_\perp \left(\phi_s + \frac{3\ell \phi_\ell r}{R_0} \sin \theta \cos \alpha \right) \rangle \quad (29)$$

$$\hat{A}_{\parallel}(0, \theta) = 0; \quad \hat{A}_{\parallel}(r = b, \theta) = b \frac{B_v}{B_0} \cos \theta.$$

Here, $\hat{A}_{\parallel} \equiv A_{\parallel}/B_0$ and $\hat{i}_I(r)$ and $\hat{i}_H(r)$ are related to J_{\parallel} and ϕ_e as follows

$$\frac{(r^2 \hat{i}_I)'}{r} = \frac{R_0 J_{\parallel}}{B_0} \quad (30)$$

$$\hat{i}_H = -\frac{\ell R_0}{2hr} \left(\frac{\phi_e \phi_e'}{r} \right)' \quad (32)$$

Finally, from \hat{A}_{\parallel} the shift $\sigma(r)$ is calculated as follows

$$\sigma(r) \cos \theta = -\frac{R_0}{r \hat{i}} [\hat{A}_{\parallel} - R_H]. \quad (32)$$

This completes the formulation of the equilibrium problem.

III. Applications

The calculation of the shift in actual cases has to be carried out numerically. However, there are two special cases of interest which may be calculated explicitly: (1) the pure tokamak corresponding to zero applied helical fields (i.e., $B_\ell, B_{\ell \pm 1} = 0$) and (2) the pure stellarator corresponding to zero ohmic heating current (i.e., $J_{\parallel} = 0$).

A. Pure Tokamak

Consider first the case of the pure tokamak. Setting the helical fields to zero in Eq. (29) and letting $\hat{A}_{\parallel}(r, \theta) = A(r) \cos \theta$ leads to

$$\left[r^3 \hat{\ell}_I^2 \left(\frac{A}{r \hat{\ell}_I} \right)' \right]' = -r^2 \beta' + \frac{r^3}{R_0^2} \hat{\ell}_I^2 \quad (33)$$

Upon integrating we find

$$A(r) = r \hat{\ell}_I(r) \int_r^b \frac{dx}{x^2 \hat{\ell}_I(x)} \int_0^x \left[y^2 \beta'(y) - \frac{y^3}{R_0^2} \hat{\ell}_I^2(y) \right] dy + \frac{r \hat{\ell}_I(r) B_v}{\hat{\ell}_I(b) B_0} \quad (34)$$

where for simplicity we have assumed a conducting shell located at $r = b$. The shift, σ , is related to A via the relation $\sigma(r) = -R_0 A(r) / [r \hat{\ell}_I(r)]$. If we evaluate the shift of plasma surface, $\sigma(a)$ we arrive at the well known formula first derived by Shafranov³

$$\frac{\sigma(a)}{b} = \frac{b}{2R_0} \left[\left(\beta_p + \frac{\ell_i - 1}{2} \right) \left(1 - \frac{a^2}{b^2} \right) + \ln \frac{b}{a} \right] - \frac{B_v}{B_{\theta 2}(b)} \quad (35)$$

Here, ℓ_i is the internal inductance per unit length and β_p is the volume averaged poloidal β . Note that the vertical field produces a uniform shift of the flux surfaces. Also, if we integrate the inward restoring force due to the vacuum vertical field, $\mathbf{J} \times \mathbf{B}_v \cdot \mathbf{e}_R$ over the plasma volume, we obtain the familiar result

$$F_v \equiv \int \mathbf{J} \times \mathbf{B}_v \cdot \mathbf{e}_R dr = 2\pi R_0 B_v I \quad (36)$$

This leads to the intuitive picture of position control in a tokamak, in which the flux surfaces and the plasma shift simultaneously and adiabatically as the externally applied vertical field is slowly varied. For any given applied vertical field there is a unique equilibrium position depending upon b/R_0 , β_p , ℓ_i and b/a . In particular, for a fixed plasma current there is a unique vertical field proportional to $1/R_0$ which keeps the plasma centered [i.e., $\sigma(a) = 0$].

While this picture is correct in an "adiabatic" sense, it does not provide an accurate description of the actual dynamics. The crucial point is that the flux surfaces in the vacuum region can change instantaneously (in the low frequency Maxwell equation model) whereas the plasma itself can only respond on its inertial time scale. Since the inertial time scale is very short (on the order of microseconds), in most cases of interest it is sufficient to neglect these transients and to assume that adiabatic evolution is valid. We shall see that in the case of a pure stellarator, the dynamics play a far more critical role.

B. Pure Stellarator

Consider now the case of a pure stellarator. Setting $J_{\parallel} = 0$ leads to the following set of equations for $A_{\parallel}(r, \theta) = A(r) \cos \theta$ and $\sigma(r)$.

$$\left[r^3 \left(\frac{A}{r} \right)' \right]' = - \frac{r^2 \beta'}{i_H} \quad (37)$$

$$\frac{\sigma}{R_0} = - \frac{1}{ri_H} [A - \hat{R}_h] \quad (38)$$

where $R_H(r, \theta) = \hat{R}_H(r) \cos \theta$ is given by Eq. (29).

The equation for A can easily be integrated, yielding

$$A(r) = r \left[\frac{B_v}{B_0} + \int_r^{\infty} \frac{dx}{x^3} \int_0^x \frac{y^2 \beta'}{i_H} dy \right] \quad (39)$$

An analytic expression for the shift is obtained by assuming the helical wavelength is long compared to the plasma radius: $ha \ll 1$. In this case

$$\begin{aligned} \phi_{\ell} &\approx \frac{a B_{\ell}}{\ell B_0} \left(\frac{r}{a} \right)^{\ell} \\ i_H &\approx -i_{\ell} \left(\frac{r}{a} \right)^{2\ell-4} \\ i_{\ell} &\approx \frac{(\ell-1)N}{h^2 a^2} \left(\frac{B_{\ell}}{B_0} \right)^2 \\ \phi_{\ell \pm 1} &\approx \frac{a}{\ell \pm 1} \frac{B_{\ell \pm 1}}{B_0} \left(\frac{r}{a} \right)^{\ell \pm 1} \end{aligned} \quad (40)$$

where, as before, B_{ℓ} , $B_{\ell \pm 1}$ are the amplitudes of the radial helical magnetic field evaluated on the plasma surface, $r = a$.

From these relations a straightforward calculation leads to the following expression for $\sigma(r)$

$$\frac{\sigma(r)}{R_0} = -\frac{1}{i_H} \left[b_v + \int_r^\infty \frac{dx}{x^3} \int_0^x \frac{y^2 \beta'}{i_H} dy \right] - \frac{b_\ell}{h a i_\ell} \left[b_{\ell+1} \left(\frac{r}{a} \right)^2 + b_{\ell-1} + \frac{a}{2R_0} b_\ell \left(\frac{r}{a} \right)^2 \right] \quad (41)$$

where $b_v = B_v/B_0$, $b_\ell = B_\ell/B_0$ and $b_{\ell\pm 1} = B_{\ell\pm 1}/B_0$. Two useful limits of this expression correspond to the surface shift, $\sigma(a)$, given by

$$\frac{\sigma(a)}{R_0} = \frac{1}{i_\ell} \left[b_v - \frac{1}{2i_\ell} \int_0^1 u^{8-2\ell} \frac{d\beta}{du} du - \frac{b_\ell}{h a} \left(b_{\ell+1} + b_{\ell-1} + \frac{a}{2R_0} b_\ell \right) \right] \quad (42)$$

and the axis shift, $\sigma(0)$, which for $\ell = 2$ is given by

$$\frac{\sigma(0)}{R_0} = \frac{1}{i_\ell} \left[b_v + \frac{1}{2} \frac{\beta(0)}{i_\ell} - \frac{b_\ell b_{\ell-1}}{h a} \right] \quad (43)$$

Note that the axis shift, $\sigma(0)$, becomes infinite for $\ell \geq 3$ if the vertical field is non zero. This is a consequence of the "strong" vanishing of helical field on axis for $\ell \geq 3$ leading to a finite shift even though B_v is treated as a small quantity.

There are a number of conclusions to be drawn from these results.

1. The terms appearing in Eq. (41) can be interpreted as follows: The first term represents the flux surface shift due to the vertical field. The second term represents the outward shift due to the $1/R$ dependence of B_ϕ in toroidal geometry. The next two terms represent shifts due to the externally applied helical sideband fields. The last term represents a small toroidal correction to the cylindrical helical field (i.e., an a/R_0 correction to the modified Bessel function solutions).
2. The problem of position control is quite different in a stellarator than in a tokamak. In the latter case the vertical field is adjusted to center the last flux surface within the vacuum chamber (or limiter). An incorrect value of vertical field causes the plasma to drift into the wall. In contrast, the outer surfaces of a stellarator are essentially held fixed by the external helical coils. These coils are carefully designed so that the last surface is approximately centered in the vacuum chamber. The problem of position control in a stellarator thus corresponds to that of moving the center of the plasma, $\sigma(0)$, keeping the last surface $\sigma(a)$ fixed. Equations (42) and (43) indicate that for $\ell = 2$ a combination of an upper helical sideband and either a vertical field or lower sideband can accomplish this task, although admittedly in an actual experiment this may be difficult to implement. For $\ell \geq 3$ a vertical field is quite effective for position control since the axis shift is much larger than the surface shift.
3. Unlike a tokamak, a centered stellarator, $\sigma(a) = 0$, does not lead to a unique relationship between the vertical field and the major radius. The reason is associated with the fact that the externally applied helical

fields define a unique center to the magnetic geometry, even in the absence of plasma currents. The vertical field produces shifts with respect to this center. In a tokamak there is no "magnetic" center without plasma currents, and the plasma comes to rest at that major radius at which the outward expansion forces balance the inward B_v force. In a stellarator the plasma automatically induces the appropriate toroidal dipole currents (i.e., Pfirsch-Schluter currents) which produce an inward force to counteract the $1/R$ outward expansion force at whatever major radius the helical center and hence the plasma happen to be located. This inward force represents the interaction of the Pfirsch-Schluter currents with the "average" helical field on the plasma surface. (Note that to leading order the average helical field is zero, but to next order there is a non zero contribution proportional to \hat{l}_H .)

4. In equilibrium the vertical field produces no net body force on the plasma, although, paradoxically, it contributes to the toroidal shift. The vanishing of F_v follows directly from the integration of the body force over the plasma

$$F_v = \int \mathbf{J} \times \mathbf{B}_v \cdot \mathbf{e}_R dr = 0 \quad (44)$$

5. The apparent paradox is resolved as follows. The vertical field produces a force on the plasma only during transient periods when $B_v(t) \neq 0$. In fact, it is these transients which are similar to but (incorrectly) ignored in a tokamak that are critical in the stellarator. To see this assume a stellarator plasma is initially at rest in an equilibrium position. The vertical field is then suddenly increased from B_v to $B_v + \Delta B_v$. The magnetic field changes instantaneously (in the low frequency Maxwell equation model). The plasma, however, cannot respond faster than its inertial time scale. Since the plasma is a perfect conductor, the change in B_v can only be accommodated by the induction of a skin current on the plasma surface. This surface current has a toroidal dipole component (and may or may not have a net toroidal component depending upon the external circuitry and the net vertical flux linking the torus. For simplicity we assume the circuits are programmed so that the net toroidal current is instantaneously zero, leaving only the toroidal dipole current.) It is the interaction of this surface dipole current with the average helical field on the plasma surface that produces a net body force. The body force exists only during transients and is in a direction to produce a plasma shift which reduces the surface current.

A simple heuristic model demonstrating this point can be formulated for the more realistic case where the vertical field changes slowly with respect to the plasma inertial scale. In this case as $B_v(t)$ varies, the plasma shift $\sigma(a, t)$ varies and a surface current is induced whose value can be calculated as follows.

To find the contribution due to the change in B_v , assume the plasma remains initially at rest. Since the

plasma is a perfect conductor, there can be no change in the flux in the plasma. Furthermore, since $\mathbf{n} \cdot \mathbf{B} = 0$ on the plasma surface the perturbed vector potentials in the vacuum and plasma have the form

$$\begin{aligned}\delta\hat{A} &= \delta B_v(r - a^2/r) \cos\theta & (\text{vacuum}) \\ \delta A &= 0 & (\text{plasma})\end{aligned}\quad (45)$$

where $\delta B_v(t) = B_v(t) - B_v(0)$. This gives rise to a surface current

$$K_v(t) = 2\delta B_v \cos\theta \mathbf{e}_z \quad (46)$$

As the plasma moves in response to the force generated by $K_v(t)$, the plasma shift changes by an amount $\delta\sigma(t) = \sigma(a, t) - \sigma(a, 0)$ which in turn changes the value of the surface current. This change can be calculated by noting that the poloidal flux, averaged over one helical period, is given by

$$\frac{\langle\psi_p\rangle}{2\pi R_0} = \frac{1}{2}\sigma_h(a)B_{h\theta}(a) \quad (47)$$

If the plasma now undergoes a virtual shift $a \rightarrow a + \delta\sigma \cos\theta$, the average poloidal flux changes by an amount

$$\frac{\langle\delta\psi_p\rangle}{2\pi R_0} = \frac{1}{2}[\sigma_h(a)B_{h\theta}(a)]'\delta\sigma \cos\theta \quad (48)$$

Since the plasma surface must remain a flux surface during the displacement, a surface current must be induced to cancel this flux. The perturbed vector potentials giving rise to the shift induced surface current are given by

$$\begin{aligned}\delta\hat{A} &= c\frac{a^2}{r} \cos\theta & (\text{vacuum}) \\ \delta A &= cr \cos\theta & (\text{plasma})\end{aligned}\quad (49)$$

The constant c is determined by setting $\delta\hat{A}(a, \theta) = \delta A(a, \theta) = -\langle\delta\psi_p\rangle/2\pi R_0$, so that the net average dipole flux cancels. This leads to a surface current given by

$$K_\sigma(t) = -2B_0\hat{e}_t\left(\frac{\delta\sigma}{R_0}\right)\cos\theta\mathbf{e}_z \quad (50)$$

where we have used the relation $(\sigma_h B_\theta) = -2aB_0\hat{e}_t/R_0$.

The two surface current contributions combine and produce a net body force on the plasma which can be written as

$$F_R \approx -a \int (K_v + K_o) B_{\theta h} \cos \theta d\theta dz \quad (51)$$

where $B_{\theta h}$ is to be evaluated on the plasma surface $r \approx a + \sigma_h \cos \alpha$. Upon integrating we find

$$\frac{F_R}{2\pi R_0} = -2\pi a \langle (K_v + K_o) \cos \theta \rangle_{\theta} \langle B_{\theta h} \rangle_z \quad (52)$$

Here, the subscripts on the $\langle \rangle$ symbols denote the variable over which averaging is carried out. Note that to leading order $\langle B_{\theta h} \rangle_z \approx 0$. To first order, however, there is a non zero contribution. A straightforward calculation shows that

$$\langle B_{\theta h} \rangle_z = \frac{B'_{\theta h} \sigma_h}{2} \approx -\frac{B_0 a i_{\ell}}{2R_0} \quad (53)$$

Combining terms, we obtain

$$\frac{F_R}{2\pi R_0} = -\frac{\pi a^2 B_0^2 i_{\ell}}{R_0} \left(\frac{\delta \sigma}{R_0} - \frac{\delta B_v}{i_{\ell} B_0} \right) \quad (54)$$

Note that the system is in equilibrium, $F_R = 0$, when $\delta \sigma / R_0 = \delta B_v / i_{\ell} B_0$. Furthermore, the shift that is induced is identical to that predicted by the general shift relation defining the flux contours, Eq. (42).

The dynamical equations are obtained by equating F_R to the inertial force $M \delta \ddot{\sigma}$ with $M = 2\pi^2 a^2 R_0 \langle \rho \rangle$, yielding

$$\begin{aligned} \left(\frac{\delta \ddot{\sigma}}{R_0} \right) + \omega_a^2 \left(\frac{\delta \sigma}{R_0} \right) &= \frac{\omega_a^2}{i_{\ell}} \frac{\delta B_v}{B_0} \\ \omega_a^2 &= v_a^2 i_{\ell}^2 / R_0^2 = B_0^2 i_{\ell}^2 / \langle \rho \rangle R_0^2 \end{aligned} \quad (55)$$

To see how the plasma motion evolves as the vertical field changes, assume the plasma is initially at rest in equilibrium: $\delta \sigma(0) = \delta \dot{\sigma}(0) = 0$. At $t = 0$, a slowly changing external vertical field is applied of the form

$$\delta B_v(t) = \Delta B_v (1 - e^{-t/\tau_v}) \quad (56)$$

The motion of the plasma is easily calculated and is given by

$$\frac{\delta \sigma}{R_0} = \frac{\Delta B_v}{i_{\ell} B_0} \frac{1}{1 + \alpha^2} [1 + \alpha^2 - \alpha^2 e^{-t/\tau_v} - \cos \omega_a t - \alpha \sin \omega_a t] \quad (57)$$

where $\alpha \equiv \omega_a \tau_v$.

$\delta\sigma(t)$ is illustrated in Fig. 2 for $\alpha \gg 1$, the typical experimental situation. Note that $\delta\sigma(t)$ essentially tracks the applied vertical field $\delta B_v(t)$ with small superimposed oscillations of relative amplitude $\sim 1/\alpha$. A small dissipation would cause these oscillations to damp. Once the vertical field reaches its final value, the net force on the plasma, F_R , vanishes. The vertical field has produced a force only during the transient period, causing the plasma to shift to its new equilibrium position, consistent with the flux surface shift relation Eq. (42). Once in this position the vertical field no longer produces a net body force on the plasma.

IV. Conclusions

By solving the MHD equations using a low β expansion we have obtained physical insight into the questions of force balance and position control in tokamaks and stellarators. Specifically, we have resolved the apparent paradox regarding the use of vertical fields for position control in current free stellarators. The paradox arose because the vertical field could shift the plasma flux surfaces and yet produce no net body force on the plasma when in equilibrium. The resolution shows that the vertical field does indeed produce a body force on the plasma, but only during transient periods when $\dot{B}_v(t) \neq 0$. This force vanishes when the plasma comes to its new equilibrium position, given by the vertical field shift of the flux surfaces.

References

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Figure Captions

Fig. 1 Toroidal Geometry

Fig. 2 Time Evolution of $\delta B_v(t)$ and $\delta\sigma(t)$.



