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EFFECT OF EQUILIBRIUM RADIAL ELECTRIC FIELD ON TRAPPED PARTICLE STABILITY IN TANDEM MIRRORS

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ABSTRACT

We develop the low beta tandem mirror trapped particle instability theory for arbitrary azimuthal mode number including the effects of radial equilibrium electric fields. We find a stability window exists for inward pointing electric fields even when passing electrons bounce beyond passing ions. This stability window can persist at zero radial electric field when the instability drive is sufficiently small. The theory also predicts the possibility of an additional instability when $E \times B$ rotation is present: a rotationally driven trapped particle mode.

INTRODUCTION

It has recently been suggested that trapped particle modes can be unstable with large growth rates in tandem mirrors.[1] The eigenmodes for these instabilities are flutelike throughout the tandem mirror central cell (where the bad curvature drive is located) and drop to a near zero amplitude in the minimum-B anchor cells. They are thus driven unstable by central cell trapped ions and electrons which only sample the destabilizing field line curvature of the central cell.

The stabilizing mechanisms proposed for these modes rely on the small fraction of particles that pass beyond the region where the trapped particle eigenmode rolls-off to its near zero value in the good curvature region. An important stabilizing effect results from a "charge-separation" that results because passing ions and electrons bounce at different locations and thus sample a different averaged fluctuation potential. This stabilization is analogous to the finite Larmor radius (FLR) charge separation stabilization with the important difference that it is operative for azimuthal mode number $m^o = 1$ modes whereas FLR stabilization is not. If the passing ions bounce beyond the passing electrons the charge separation bounce term will add to the FLR term and stabilization can be achieved for all mode numbers. In the opposite case these two effects subtract and instability can result at some mode number when these two charge separation terms cancel.

The former case has been used as the basis for designing a trapped particle mode stable configuration for the MFTF-B experiment.[2] The implementation of this scheme, however, requires a relatively complicated transition region design and imposes a constraint on the radial potential in this region.

A second tandem mirror arrangement that has received attention is the TARA arrangement.[3] In this device the central cell is bounded by an axisymmetric mirror "plug" termed an axicell in which all confined ions reflect. A fraction of the electrons pass through the axicell and outboard transition region, and reflect in the minimum-B anchors bounding the system.

Thus ions are confined in purely axisymmetric fields. However, in this scheme the lack of passing ions will result in trapped particle instability at some mode number.

The initial work on trapped particle instabilities utilized a high azimuthal mode number theory ($m^{o} >> 1$). In this article we extend the previous theory to permit arbitrary azimuthal mode numbers and include self-consistently the radial equilibrium electric field structure. Using this theory we propose a second scheme for stabilizing the trapped particle modes which does not require that ions bounce beyond the electrons. We will show that stabilization of trapped particle modes can result from a configuration in which the radial potential assumes a hollow profile, that is, the equilibrium radial electric field points inward. Such a scheme has been known to be stabilizing to $m^{o} > 1$ curvature driven MHD ballooning modes and rotational modes. We will show that such an arrangement can be effective for stabilizing trapped particle modes (for a sufficiently modest curvature drive). Although this stabilization becomes inoperative for $m^{o} = 1$ modes we will see that $m^{o} = 1$ trapped particle modes can be stable for a sufficient passing electron fraction.

In Section II we rederive the trapped particle mode dispersion relation for an idealized tandem mirror geometry keeping the radial gradient terms for both the perturbed and equilibrium potentials. In carrying out this task we take advantage of the flute-like nature of the trapped particle mode in the bad curvature regions and follow the method developed by Rosenbluth and Simon[(4)] in which the finite Larmor radius ordering is used to solve the Vlasov equation. In Section III we evaluate the requirements for stabilization of tandem mirrors.

SECTION II

In analyzing the stability of current tandem mirror designs we can identify four small parameters:

$$\varepsilon_i \equiv \rho_i / r_n \sim \frac{1}{10}$$

$$\varepsilon_e \equiv \rho_e/r_n \sim \frac{1}{430}$$

$$\lambda_0 \equiv \frac{\partial}{\partial z}/\frac{\partial}{\partial r} \sim \frac{1}{10}$$

$$\frac{n_p}{n_0} \sim \frac{1}{10} - \frac{1}{20}$$

where $(r_n)^{-1} = n^{-1}(dn/dr)$, n_p and n_0 are the passing and central cell density respectively and the numerical values are appropriate for TARA.(3) The perturbation expansion in λ_0 corresponds to the long-thin approximation,(5) while the expansion in ε corresponds to the finite Larmor radius ordering.

In deriving the trapped particle mode one computes the perturbed charge densities of electrons and ions driven by the perturbed E × B drift and Doppler shifted by the curvature drift frequency. If both electrons and ions bounce rapidly and turn at the same spatial point, they E × B drift equally and hence no net perturbed charge separation results to lowest order. It is the charge dependent Doppler shift induced by the particle curvature drift that leads to a higher order charge separation and hence to instability. The curvature drift frequency responsible for this change separation is of order λ_0^2 with respect to the equilibrium E × B drift frequency in the long-thin approximation. Because $\lambda_0 \sim \rho_i/r_n$ for typical mirror parameters, we must then take into account order $(\rho_i/r_n)^2$ modifications to the usual lowest order drift equation. This entails solving the ion Vlasov equation through order $(\rho_i/r_n)^4$. Note that for electrons $\rho_e/r_n \ll \lambda_0$ and hence we need only solve the electron Vlasov equation to order $(\rho_e/r_n)^2$.

In order to render the analysis tractable we replace the true TARA equilibrium by a simplified model. In the model equilibrium the correct axial variation of the magnitude of the magnetic field is retained. This variation effects the unperturbed motion of particles along field lines and in particular magnetically confines particles with appropriate pitch angle to the center cell, plug, transition or anchor. In addition the variation of B is retained in the flux tube integral $\int d\ell/B$. In the ρ/r_n expansion, however, derivatives on $\underset{\sim}{B}$ are ignored and their effect modeled by an artificial species, energy and pitch angle dependent gravity which varies along a field line like the true TARA curvature. In the real TARA geometry the center cell and plug are axisymmetric. We take the transition section and quadrupole anchor in our model to be axisymmetric as well.

In the TARA configuration a positive electrostatic potential maximum is created in the plug through the injection of neutral beams and electron-cyclotron heated electrons. This potential maximum confines center cell ions. In our model we do not treat the hot electron species but assume that the electron distribution is a single Maxwellian throughout the machine. We take the ions to be magnetically and electrostatically confined to each region. Thus the regions are only linked by transiting electrons. Further, within each region the equilibrium electrostatic potential is taken to be constant, although this constant may be different within each region. Appropriate sharp positive maxima are introduced between the regions in order to confine ions to that region. Finally, we assume that the ratio of plasma pressure to magnetic field pressure is low enough to permit us to restrict our analysis to electrostatic perturbations.

The device of ignoring derivatives on $\underset{\sim}{B}$ and the introduction of an artificial gravity simplifies the algebra and retains the important physics of the effects of curvature. The error caused by the neglect of non-axisymmetry in the transition and anchor is mitigated somewhat by the low mode amplitude which we expect in this region. More serious is the neglect of the hot species of electrons which may affect the mode significantly. Their treatment is beyond the scope of this paper.

We begin with the linearized Vlasov equation written in terms of particle velocities in the local E × B and gravity drift frame. Defining

$$\underset{\sim}{v}' \equiv \underset{\sim}{v} - \underset{\sim}{V} = v_{||}\underset{\sim}{b} + \underset{\sim}{v}'_{\perp}$$

..here

$$\underset{\sim}{V} = \underset{\sim}{V}_0 + \tilde{\underset{\sim}{V}}$$

$$\underset{\sim}{V}_0 \equiv (\frac{q}{m}\underset{\sim}{E}_0 + \underset{\sim}{g}) \times \frac{\underset{\sim}{b}}{\Omega}$$

$$\tilde{\underset{\sim}{V}} = \frac{q}{m}\frac{\tilde{\underset{\sim}{E}} \times \underset{\sim}{b}}{\Omega}$$

$$\underset{\sim}{E}_0 = -\nabla\Phi$$

$$\tilde{\underset{\sim}{E}} = -\nabla\phi$$

$$\Omega \equiv \frac{qB}{mc}$$

$$\underset{\sim}{b} \equiv \frac{\underset{\sim}{B}}{|\underset{\sim}{B}|}$$

$$\underset{\sim}{g} \equiv -(v_{||}^2\underset{\sim}{b} \cdot \nabla\underset{\sim}{b} + \frac{\mu}{m}\nabla B)$$

$$\mu \equiv m(v_{\perp}')^2/2B$$

$$\underset{\sim}{B} = \nabla\alpha \times \nabla\beta$$

$$\varepsilon' = \frac{1}{2} m(v')^2 + q\Phi$$

$$\frac{\partial}{\partial \ell} \equiv \underset{\sim}{b} \cdot \nabla \Big|_{\text{fixed } \underset{\sim}{v}'}$$

$$\frac{\partial}{\partial \ell'} \equiv \underset{\sim}{b} \cdot \nabla \Big|_{\text{fixed } \varepsilon'}$$

we write the linearized Vlasov equation for the perturbed distribution function $\tilde{f}$ as

$$\left\{ \frac{\partial}{\partial t} + (\underset{\sim}{v}' + \underset{\sim}{V}_0) \cdot \nabla + \left[\underset{\sim}{v}' \times \underset{\sim}{b}\Omega + \frac{q}{m} E^0_{||}\underset{\sim}{b} - (\underset{\sim}{v}' + \underset{\sim}{V}_0) \cdot (\nabla \underset{\sim}{V}_0) \right] \cdot \frac{\partial}{\partial \underset{\sim}{v}'} \right\} \tilde{f}$$

$$= - \left\{ \tilde{\underset{\sim}{V}} \cdot \nabla + \left[\frac{q}{m} \tilde{E}_{||}\underset{\sim}{b} - \frac{\partial \tilde{\underset{\sim}{V}}}{\partial t} - (\underset{\sim}{v}' + \underset{\sim}{V}_0) \cdot (\nabla \tilde{\underset{\sim}{V}}) - \tilde{\underset{\sim}{V}} \cdot \nabla \underset{\sim}{V}_0 \right] \cdot \frac{\partial}{\partial \underset{\sim}{v}'} \right\} F , \quad (1)$$

where the spatial derivatives are taken at constant $\underset{\sim}{v}'$. The equilibrium distribution function F satisfies the equation

$$\left\{ (\underset{\sim}{v}' + \underset{\sim}{V}_0) \cdot \nabla + \left[\underset{\sim}{v}' \times \underset{\sim}{b}\Omega + \frac{q}{m} E^0_{||}\underset{\sim}{b} - (\underset{\sim}{v}' + \underset{\sim}{V}_0) \cdot (\nabla \underset{\sim}{V}_0) \right] \cdot \frac{\partial}{\partial \underset{\sim}{v}'} \right\} F = 0$$

We define $f = \tilde{f} + F$. Because of axisymmetry we may choose F to be a function only of the single flux function α and velocity $\underset{\sim}{v}'$.

Let us examine the size of the terms in these equations. The lowest order term is $\underset{\sim}{v}' \times \underset{\sim}{b}\Omega \cdot (\partial/\partial \underset{\sim}{v}')f$ which we denote as $\varepsilon^0 \Omega$. The perpendicular spatial derivatives are of order ε^1

$$\underset{\sim}{v}_\perp \cdot \nabla \sim \frac{v_t}{r_n} \sim \varepsilon\Omega$$

The parallel spatial derivative is also of order ε, but because the equilibrium varies more slowly axially than radially we expect this term to be smaller. We will address this problem as we proceed.

The equilibrium E × B drift frequency is of order $\varepsilon^2\Omega$

$$\underset{\sim}{V}_\theta^E \cdot \nabla \equiv \frac{q}{m}\frac{\underset{\sim}{E}\times\underset{\sim}{b}}{\Omega} \cdot \nabla \sim \frac{qE_\theta}{m\Omega r} \sim \frac{q\Delta\Phi}{m\Omega r^2} \sim \frac{T}{m\Omega r^2} \sim \varepsilon^2\Omega$$

where $\Delta\Phi$, the equilibrium potential difference between the axis and the plasma edge, is typically of order $5(T_e/e)$. Notice that since $\varepsilon^2\Omega$ is independent of mass, the electron and ion equations are of equal magnitude at this order. The gravity drift frequency, however, is of order $\varepsilon^2\lambda_0^2\,\Omega$ and is thus of higher order. The equilibrium electrostatic potential, Φ, is a function of α in each region. The mode eigenfrequency, ω, we take to be of the order of the equilibrium E × B drift frequency.

We now expand the equilibrium and perturbed distribution functions in each region, n, in a power series in ε and a Fourier series in the gyro-angle ξ,

$$f^{(n)} = \sum_{r=0}^{\infty} \sum_{m=-r}^{r} \varepsilon^r f_{rm}^{(n)} e^{im\xi}. \tag{2}$$

With this choice the lowest order equation in ε is satisfied identically

$$\Omega\underset{\sim}{v}' \times \underset{\sim}{b} \cdot \frac{\partial}{\partial \underset{\sim}{v}'} f = 0 \tag{3}$$

For electrons only the m = 0 component of the ε_e^1 and ε_e^2 equations are needed. The ε_e^1 equation is

$$v_{||} \frac{\partial}{\partial \ell'} \tilde{f}_{0,0} = q v_{||} \frac{\partial}{\partial \ell'} \phi \frac{\partial F_0}{\partial \varepsilon'} \quad , \tag{4}$$

where the prime on ℓ signifies that ε' and μ are to be held fixed in the spatial derivative. The lowest order equilibrium distribution function F_0 is a Maxwellian, $F_0 = \hat{n}(\alpha)\exp(-\varepsilon'/T_e(\alpha))/(2\pi T_e(\alpha)/m)^{3/2}$. Because the axial dependence of $\tilde{f}_{0,0}$ differs between trapped and passing species we consider each separately.

We expect the perturbed potential to make an order unity change between regions. Within each region, however, the perturbed potential will vary slowly along a field line on the scale length of the equilibrium. We thus expand the perturbed potential in each region in a power series in a small parameter η which we will determine self-consistently at a later stage in the calculation,

$$\phi^{(n)}(\alpha,\beta,\ell) = \phi_0^{(n)}(\alpha,\beta) + \eta\phi_1^{(n)}(\alpha,\beta,\ell) + \eta^2\phi_2^{(n)}(\alpha,\beta,\ell) + \ldots \tag{5}$$

where $\phi_0^{(n)} = \phi_0^{(n)}(\alpha,\beta)$ is a constant along a field line within each region and (n) labels the region (central cell, axi-cell, anchor, etc.).

We first consider trapped particles. Substituting this expansion for ϕ in Eq. (4) gives for trapped particles

$$v_{||} \frac{\partial}{\partial \ell'} \tilde{f}_{0,0}^{(n)} = q v_{||} \frac{\partial}{\partial \ell'} (\eta\phi_1^{(n)} + \eta^2\phi_2^{(n)}) \frac{\partial F_0}{\partial \varepsilon'} \tag{6}$$

We now expand $\tilde{f}_{0,0}$ in a subsidiary power series in η

$$\tilde{f}^{(n)}_{0,0}(\alpha,\beta,\ell') = h^{(n)}_0(\alpha,\beta,\ell') + \eta h^{(n)}_1(\alpha,\beta,\ell') + \eta^2 h^{(n)}_2(\alpha,\beta,\ell') \qquad (7)$$

The lowest order equation is

$$v_{||} \frac{\partial}{\partial \ell'} h^{(n)}_0 = 0 \qquad (8)$$

with solution $h^{(n)}_0 = h^{(n)}_{0,0}(\alpha,\beta)$, a constant we determine at a higher order. The next order in η gives

$$v_{||} \frac{\partial}{\partial \ell'} \eta h^{(n)}_1 = q v_{||} \frac{\partial}{\partial \ell'} \eta \phi^{(n)}_1 \frac{\partial F_0}{\partial \varepsilon'} \qquad (9)$$

with solution

$$h^{(n)}_1(\alpha,\beta,\ell') = q\phi^{(n)}_1(\alpha,\beta,\ell') \frac{\partial F_0}{\partial \varepsilon'} + h^{(n)}_{1,0}(\alpha,\beta) \qquad (10)$$

where $h^{(n)}_{1,0}$ is a constant along a field line.

The next order equation in η duplicates Eq. (9) and yields no new information. We will determine the appropriate size of η by requiring these terms to appear in the ε^2 equation in a self-consistent manner. We thus turn to the ε^2 equation and again consider only electrons confined to the center cell.

The lowest order equation is

$$\left(\frac{\partial}{\partial t} + \underset{\sim}{V}^E_0 \cdot \nabla\right) h^{(n)}_0 + v_{||} \frac{\partial}{\partial \ell'} \eta^2 h^{(n)}_2$$

$$= -\frac{q}{m}\frac{\tilde{b}}{\Omega} \times \nabla\phi_0^{(n)} \cdot \nabla F_0 + qv_{||}\frac{\partial}{\partial \ell'}(\eta^2\phi_2^{(n)})\frac{\partial F_0}{\partial \varepsilon'} . \qquad (11)$$

In TARA the electrons confined to the central cell see an axi-symmetric equilibrium and an electrostatic equilibrium potential Φ which is constant except for a sharply localized confining minimum in the thermal barrier mode. Under these circumstances the terms $((\partial/\partial t) + \tilde{V}_0^E \cdot \nabla)h_0^{(n)}$ and $-(q/m\Omega)\tilde{b} \times \nabla\phi_0^{(n)} \cdot \nabla F_0$ are constants along a magnetic field line within each region. We write

$$\left(\frac{\partial}{\partial t} + \tilde{V}_0^E \cdot \nabla\right) h_0^{(n)} = \left(-i\omega + \frac{d\Phi^{(n)}}{d\alpha}\, im^0 c\right)h_0^{(n)} \equiv -i\,(\omega - \omega_E^{(n)})h_0^{(n)} \qquad (12)$$

where $\omega_E^{(n)} \equiv cm^0(d\Phi^{(n)}/d\alpha)$ and

$$-\frac{q}{m}\frac{\tilde{b}}{\Omega} \times \nabla\phi_0^{(n)} \cdot \nabla F_0 = im^0\phi_0^{(n)} c\,\frac{\partial F_0}{\partial \alpha} \qquad (13)$$

where the perturbed quantities vary like $(-i\omega t + im^0\beta)$. A self consistent solution to Eq. (11) is therefore

$$h_0^{(n)} = \frac{-\,m^0 c\,\frac{\partial F_0}{\partial \alpha}}{\omega - \omega_E^{(n)}}\;\phi_0^{(n)} \qquad (14)$$

with η chosen to be small enough to force the parallel derivatives to appear in the next order equation. This equation is

$$\left(\omega - \omega_E^{(n)}\right)\left(\eta q\phi_1^{(n)}\frac{\partial F_0}{\partial \varepsilon} + \eta h_{1,0}^{(n)}\right) + iv_{||}\frac{\partial}{\partial \ell}\,\eta^2 h_2^{(n)}$$

$$+ \omega_g h_{0,0}^{(n)} = - m^0 \eta\phi_1^{(n)} c \frac{\partial F_0}{\partial \alpha} + iqv_{||} \frac{\partial}{\partial \ell'} \eta^2\phi_2^{(n)} \frac{\partial F_0}{\partial \varepsilon'} \qquad (15)$$

where

$$\omega_g \equiv - m^0 \nabla\beta \cdot \underset{\sim}{V}_g = - \nabla\alpha \cdot (mv_{||}^2 \underset{\sim}{b} \cdot \nabla \underset{\sim}{b} + \mu\nabla B) \frac{m^0 |\nabla\beta|^2}{m\Omega B} .$$

The terms proportional to the parallel spatial derivative enter into this order in a natural way, but we must ask whether the ordering of η is self-consistent. The first term on the left hand side of Eq. (15) is of order $\varepsilon^2\eta$, the second is of order $\varepsilon\eta^2(r/\tilde{L}_t)$ where $\tilde{L}_t$ is the scale length of the perturbed distribution function for trapped electrons, and the gravity drift term is of order $\varepsilon^2\lambda_0^2$. Thus $\eta \sim \lambda_0^2$ in order to balance the first term with the gravity drift term. In other words, the axial variation of the eigenfunction within each region is caused by the axial variation of the gravity drift which is of order λ_0^2. Thus the parallel derivative is of order $\varepsilon\lambda_0^4(r/\tilde{L}_t)$. For this term to enter into this equation and not in the next higher order equation implies that $\tilde{L}_t$ be such that

$$\varepsilon^2\lambda_0^2 \gtrsim \varepsilon\lambda_0^4(r/\tilde{L}_t) >> \varepsilon^2\eta^2 \sim \varepsilon^2\lambda_0^4 \qquad (16)$$

which implies in turn that

$$(\varepsilon/\lambda_0^2) \gtrsim (r/\tilde{L}_t) >> \varepsilon.$$

Since $\varepsilon \sim 1/430$ and $\varepsilon/\lambda_0^2 \sim 1/4$ the inequality becomes

$$1/4 \gtrsim (r/\tilde{L}_t) \gg 1/430. \qquad (17)$$

We expect $\tilde{L}_t$ to be larger than the transition length $L_t \backsim 100$ cm and smaller than the central cell length $L_{cc} \backsim 10^3$ cm. The hot plasma radius is 10-15 cm and thus in TARA the quantity $r/\tilde{L}_t$ lies in the range: $1/10 > (r/\tilde{L}_t) > 1/100$. The inequality (17) is therefore satisfied and the ordering is consistent.

Returning to Eq. (15) for the perturbed distribution function for particles trapped in region (n), we bounce average the equation by applying the operator

$$\sum_{sgn(v_{||})} \int_{-\ell_b}^{+\ell_b} \frac{d\ell}{|v_{||}|}$$

where the sum is over the sign of $v_{||}$ and ε' and μ are to be held constant during the field line integration. The bounce point ℓ_b is found from $v_{||}(\varepsilon', \mu, \pm \ell_b) = 0$. Using the boundary condition that

$$\tilde{f}^{+}(\varepsilon', \mu, \ell_b(\varepsilon', \mu)) = \tilde{f}^{-}(\varepsilon', \mu, \ell_b(\varepsilon', \mu))$$

where +(-) refer to positive (negative) electron velocities, this operator annihilates the parallel spatial derivative on h_2. The term containing the parallel spatial derivative on ϕ_2

$$iq\, v_{||} \frac{\partial}{\partial \ell'} \eta^2 \phi_2^{(n)} \frac{\partial F_0}{\partial \varepsilon'}$$

vanishes in the sum over the sign of $v_{||}$ since it is odd in the sign of $v_{||}$. We thus obtain the following constraint for particles trapped in region (n):

$$\sum_{sgn(v_{||})} \int_{-\ell_b}^{+\ell_b} \frac{d\ell}{|v_{||}|} \left[(\omega - \omega_E^{(n)}) \left(\eta q \phi_1^{(n)} \frac{\partial F_0}{\partial \varepsilon'} + \eta h_{1,0}^{(n)} \right) + \omega_g h_{0,0}^{(n)} \right.$$

$$\left. + m^0 \ \eta \phi_1^{(n)} \ c \frac{\partial F_0}{\partial \alpha} \right] = 0 \tag{18}$$

where $\omega_E^{(n)} \equiv cm^0 (d\Phi^{(n)}/d\alpha)$ which equals $-m^0 cE_0^{(n)}/(rB)$ in a cylindrical geometry. Note that the spatial integration extends to the particle bounce point.

We turn now to the passing electrons. The order ε equation is

$$v_{||} \frac{\partial}{\partial \ell'} \tilde{f}^p_{0,0} = qv_{||} \frac{\partial}{\partial \ell'} (\phi_0^{(n)} + \eta \phi_1^{(n)}) \frac{\partial F_0}{\partial \varepsilon'} \quad . \tag{19}$$

The lowest order equation in η however differs from the corresponding equation for trapped particles because the passing particles transverse different regions and hence see an axially varying perturbed potential to zero order in η. For trapped particles the correction term to the lowest order equation reflected the axial variation of the perturbed potential within a region which in turn reflected the axial variation of the gravity drift and was thus of order λ_0^2. For passing particles, however, the correction term reflects the zero order axial variation of the perturbed potential from region to region, and thus we must allow for this term to be larger than λ_0^2 compared to the zero order term. Thus we expand $\tilde{f}^p_{0,0}$ in a new parameter η_p,

$$\tilde{f}^p_{0,0} = h_0^p + \eta_p h_1^p$$

with $\eta_p > \eta$.

The lowest order equation is

$$v_{||} \frac{\partial}{\partial \ell'} h_0^p = qv_{||} \frac{\partial}{\partial \ell'} \phi_0 \frac{\partial F_0}{\partial \varepsilon'} \quad , \tag{20}$$

with solution

$$h_0^p = q\phi_0 \frac{\partial F_0}{\partial \varepsilon'} + h^p_{0,0} \tag{21}$$

where we recall that ϕ_0 is a constant along a field line in each region but differs from region to region (Eq. (5)).

The next order term in η_p, $v_{||}(\partial/\partial \ell')\eta_p h_1$, however, cannot be balanced by any term in the ε equation, we thus turn to the order ε^2 equation

$$(\omega - \omega_E^{(n)}) \left(q\phi_0 \frac{\partial F_0}{\partial \varepsilon'} + h^p_{0,0}\right) + iv_{||} \frac{\partial}{\partial \ell'} \eta_p h_1 = - m^0 \phi_0 c \frac{\partial F_0}{\partial \alpha} \; . \tag{22}$$

We note that $F_0 = F_0(\alpha, \varepsilon')$ where $\varepsilon' = 1/2\, m(\underset{\sim}{v}')^2 + q\Phi(\alpha)^{(n)}$ and therefore that the derivative $\partial F_0/\partial \alpha$ at constant $\underset{\sim}{v}'$ will vary from region to region as Φ varies. We therefore write $(\partial F_0/\partial \alpha) = (\partial F_0/\partial \alpha') + (\partial F_0/\partial \varepsilon')(dq\Phi/d\alpha)$ where the prime on α signifies holding ε' fixed. The quantity $(\partial F_0/\partial \alpha')$ is constant from region to region because of the assumption that all species of electrons have Maxwellian distributions characterized by the same temperature.

We bounce average Eq. (22) over the trajectory of a passing particle (see Eq. (18)) and obtain

$$h^p_{0,0} = -\left(\omega q \frac{\partial F_0}{\partial \varepsilon'} + m^0 c \frac{\partial F_0}{\partial \alpha'}\right) \frac{\bar{\phi}_0}{\omega - \bar{\omega}_E} \qquad (23)$$

where

$$\bar{\phi}_0 = \frac{\sum_n \phi_0^{(n)} \tau^{(n)}}{\sum_n \tau^{(n)}}$$

where $\tau^{(n)}(\varepsilon', \mu)$ is the transit time for each region,

$$\tau^{(n)} = \int_{(n)} \frac{d\ell}{|v_{||}|} \quad .$$

Substituting this expression for $h^p_{0,0}$ into Eq. (22) gives

$$\left(\omega q \frac{\partial F_0}{\partial \varepsilon'} + m^0 c \frac{\partial F_0}{\partial \alpha'}\right) \left(\phi_0 - \bar{\phi}_0 \left(\frac{\omega - \omega_E}{\omega - \bar{\omega}_E}\right)\right) + i v_{||} \frac{\partial}{\partial \ell'} \eta_p h^p_1 = 0 \qquad (24)$$

We can integrate this to obtain $\eta_p h^p_1$

$$\eta_p h^p_1 = A_0 + i \int_0^z \frac{d\ell}{v_{||}} \left(\omega q \frac{\partial F_0}{\partial \varepsilon'} + m^0 \; c \frac{\partial F_0}{\partial \alpha'}\right) \left[\phi_0 - \bar{\phi}_0 \left(\frac{\omega - \omega_E}{\omega - \bar{\omega}_E}\right)\right] \qquad (25)$$

where A_0 is a constant of integration. In order to estimate the size of η_p we take $z = L_c$, $\omega_E = \bar{\omega}_E$, and estimate $\phi_0 - \bar{\phi}_0 \sim L_a/(L_a + L_c)\phi_0$ to obtain

$$\eta_p h^p_1 \sim \frac{\omega}{v_{||}} L_a h^p_0 \qquad (26)$$

if $A_0 = 0$. This implies $\eta_p \sim \omega L_a/v_{||} \sim \epsilon(L_a/r) \sim (1/20)$. This is in fact larger than $\eta \sim \lambda_0^2 \sim 1/100$ and thus our ordering is self consistent.

We show that the constant A_0 is zero by evaluating the next order equation in η_p which is

$$(\omega - \omega_E)\eta_p h_1^p + iv_{||}\frac{\partial}{\partial \ell'}\eta_p^2 h_2^p = 0 \tag{27}$$

Note that since $\eta_p > \eta$ neither the term $\omega_g h_0^p$ nor the term proportional to $\eta\phi_1^{(n)}$ appear. Applying the bounce average operator gives

$$\sum_{\mathrm{sgn}\,(v_{||})} \int_{-\ell_b}^{+\ell_b} \frac{d\ell}{|v_{||}|} (\omega - \omega_E)\eta_p h_1^p = 0 \tag{28}$$

and thus $A_0 = 0$ and this term will not contribute to our dispersion relation. Note that Eq. (28) is valid even if ω_E varies from region to region.

We now consider how these results for the perturbed electron distribution functions will enter into the quasi-neutrality condition. Since the expressions for the low order contributions to the perturbed distribution function are constants in each region and since the higher order corrections to the trapped electron distribution function involve spatial integrals we consider the set of coupled equations derived from the quasi-neutrality condition,

$$\sum_q q \int_{(n)} \frac{d\ell}{B} (\omega - \omega_E^{(n)}) d^3v' \tilde{f} = 0 \tag{29}$$

where n denotes the region over which the line integral is taken, central cell, axicell or anchor. We divide the integral into the trapped and passing species and interchange orders of integration

$$0 = \sum_q \frac{q}{m^2} \int_T d\xi d\varepsilon d\mu \sum_{sgn(v_{||})} \int_{-\ell_b}^{+\ell_b} \frac{d\ell}{|v_{||}|} (\omega - \omega_E^{(n)}) \tilde{f}^{(n)}$$

$$+ \frac{q_e}{m_e^2} \int_p d\xi d\varepsilon d\mu \sum_{sgn(v_{||})} \int_{(n)} \frac{d\ell}{|v_{||}|} (\omega - \omega_E^{(n)}) \tilde{f}^{(p)} . \qquad (30)$$

Substituting the electron perturbed distribution function from Eqs. (7, 10, 21 and 28) we obtain,

$$0 = \frac{2\pi q_e}{m_e^2} \int_T d\varepsilon d\mu \sum_{sgn(v_{||})} \int_{-\ell_b}^{+\ell_b} \frac{d\ell}{|v_{||}|} (\omega - \omega_E^{(n)}) \left(h_{0,0}^{(n)} + \eta q_e \phi_1^{(n)} \frac{\partial F_0}{\partial \varepsilon'} + h_{1,0}^{(n)} \right)$$

$$+ \frac{2\pi q_i}{m_i^2} \int_T d\varepsilon d\mu \sum_{sgn(v_{||})} \int_{-\ell_b}^{+\ell_b} \frac{d\ell}{|v_{||}|} (\omega - \omega_E^{(n)}) \left(\tilde{f}_{0,0_i} + \tilde{f}_{2,0_i} \right)$$

$$+ \frac{2\pi q_e}{m_e^2} \int_p d\varepsilon d\mu \sum_{sgn(v_{||})} \int_{(n)} \frac{d\ell}{|v_{||}|} (\omega - \omega_E^{(n)}) \left(q_e \phi_0^{(n)} \frac{\partial F_0}{\partial \varepsilon'} + h_{0,0}^{(p)} \right) \qquad (31)$$

Substituting for $h_{0,0}^{(n)}$, $h_{1,0}^{(n)}$ and $h_{0,0}^{(p)}$ (Eqs. 14, 18 and 23) yields

$$0 = \frac{2\pi q_e}{m^2} \int_T d\varepsilon d\mu \sum_{sgn(v_{||})} \int_{-\ell_b}^{+\ell_b} \frac{d\ell}{|v_{||}|} \Bigg[- m^0 c \frac{\partial F_0}{\partial \alpha} (\phi_0^{(n)} + \eta \phi_1^{(n)})$$

$$+ \frac{\omega_g^{(\ell)}}{\omega - \omega_E^{(n)}} m^0 c \frac{\partial F_0}{\partial \alpha} \phi_0^{(n)} \Bigg]$$

$$+ \frac{2\pi q_e}{m_e^2} \int_p d\epsilon d\mu \sum_{sgn(v_{||})} \int_{(n)} \frac{d\ell}{|v_{||}|} \Bigg\{ (\omega - \omega_E^{(n)}) \; (q_e \phi_0^{(n)} - q_e \frac{\omega}{\omega - \bar{\omega}_E} \bar{\phi}_0) \; \frac{\partial F_0}{\partial \epsilon'}$$

$$- \left(\frac{\omega - \omega_E}{\omega - \bar{\omega}_E} \right) m^0 \bar{\phi}_0 \, c \frac{\partial F_0}{\partial \alpha'} \Bigg\}$$

$$+ \frac{2\pi q_i}{m_i^2} \int_T d\epsilon d\mu \sum_{sgn(v_{||})} \int_{(n)} \frac{d\ell}{|v_{||}|} (\omega - \omega_E^{(n)}) \; (\tilde{f}^{\,i}_{0,0} + \tilde{f}^{\,i}_{2,0}) \qquad (32)$$

Ions

As with electrons we expand the equilibrium and perturbed ion distribution functions in a power series in ϵ and a Fourier series in ξ, the gyro-angle,

$$f_i^{(n)} = \sum_{r=0}^{\infty} \sum_{m=-r}^{r} \epsilon^r f_{r,m}^{(n)} e^{im\xi} \qquad (33)$$

and write $v_x' = v_\perp \cos\xi$ and $v_y' = v_\perp \sin\xi$.

In our model equilibrium, the ions are confined to each region. They therefore see no axial variation of the equilibrium electrostatic potential except for sharp maxima at the boundaries of each region. Since we have assumed a single Maxwellian for all electrons, the particle density within each region must be a constant. Therefore unless the magnetic field is a constant within a region, the equilibrium ion distribution, F, must also be Maxwellian, although each region may be characterized by a different temperature and density.

The axial variation of the perturbed ion distribution function $\tilde{f}$ results from the axial variation of the curvature drift and of the perturbed potential $\phi = \phi_0^{(n)} + \eta\phi_1^{(n)}$. The ions, however, are confined to each region and therefore the axial variation of the perturbed potential seen by the ions is only due to the term $\eta\phi_1^{(n)}$. Since the curvature drift is of order λ_0^2 which we showed was comparable to η, and since in TARA $\lambda_0 \sim \epsilon_i$, the axial variation of $\tilde{f}$ is of order ϵ_i^2. Thus $\tilde{f}_{0,0}$ and $\tilde{f}_{1,\pm 1}$ are constants along a field line and the axial variation only enters in the second order contribution to the perturbed distribution function. This implies that we can follow the Simon and Rosenbluth[4] analysis directly except for the addition of parallel derivatives in the ϵ^4 equation. Specifically the ϵ^2 equation becomes

$$(\omega - \omega_E)\, \tilde{f}_{0,0} = - m^0 c\, \phi_0^{(n)} \frac{\partial F_{0,0}}{\partial \alpha} \tag{34}$$

and the ϵ^4 equation becomes

$$-i\left(\omega - \omega_E + v_{||}\frac{\partial}{\partial \ell'}\right)\tilde{f}_{2,0} + \underset{\sim}{V}_g \cdot \nabla \tilde{f}_{0,0} - i m^0 c \eta \phi_1^{(n)} \frac{\partial F_{0,0}}{\partial \alpha}$$

$$- i m^0 c \phi_0^{(n)} \epsilon^2 \frac{\partial F_{2,0}}{\partial \alpha} + \frac{\epsilon^3 v_\perp}{2}\left[\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)\tilde{f}_{3,-1} + \left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)\tilde{f}_{3,1}\right]$$

$$- \frac{\partial}{\partial \ell}(q\tilde{\phi}_1 \eta)\, v_{||}\frac{\partial}{\partial \epsilon'} F_{0,0} + \frac{1}{v_\perp}\frac{\partial}{\partial v_\perp}[\tilde{A}] = 0 \tag{35}$$

where $\tilde{A}$ is given in Eq. (2.18) of Rosenbluth and Simon. We note that the terms $\underset{\sim}{V}_E \cdot \nabla$ and $v_{||}(\partial/\partial\ell')$ are comparable in TARA since

$v_{||}\,(\partial/\partial\ell') \sim \epsilon\lambda_0\Omega$ and $\underset{\sim}{V}_E \cdot \nabla \sim \epsilon^2\Omega$ and $\lambda_0 \sim \epsilon$. We therefore cannot neglect one with respect to the other. We can, however, obtain a constraint by applying the operator $\int d\ell/B \int d^3v'$ where the spatial integration extends over a single region (n). This gives

$$0 = \int_{(n)} \frac{d\ell}{B}\int d^3v' \left\{\left(\frac{\partial}{\partial t} + \underset{\sim}{V}_E \cdot \nabla\right)\epsilon^2\tilde{f}_{2,0} + \underset{\sim}{V}_g \cdot \nabla\tilde{f}_{0,0} - im^0c\left(\eta\phi_1^{(n)}\frac{\partial F_{0,0}}{\partial\alpha} + \phi_0^{(n)}\epsilon^2\frac{\partial F_{2,0}}{\partial\alpha}\right) + \frac{\epsilon^3 v_\perp}{2}\left[\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)\tilde{f}_{3,-1} + \left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)\tilde{f}_{3,1}\right]\right\}. \quad (36)$$

Substituting for $\tilde{f}_{3,1}$ and $\tilde{f}_{3,-1}$ from Rosenbluth and Simon,[(4)] Eq. (2.17), defining n, $P_\perp$ and $P_{||}$

$$n \equiv \int f d^3v' = n_0 + \epsilon^2 n_2 = \int f_{0,0} d^3v' + \int \epsilon^2 f_{2,0} d^3v'$$

$$P_\perp \equiv \frac{m}{2}\int v_\perp^2 f d^3v' = \pi m\left[\int v_\perp^2 f_{0,0} v_\perp dv_\perp dv_{||} + \epsilon^2 \int v_\perp^2 f_{2,0} v_\perp dv_\perp dv_{||}\right]$$

$$P_{||} \equiv m\int v_{||}^2 f d^3v' = 2\pi m\left[\int v_{||}^2 f_{0,0} dv_{||} v_\perp dv_\perp + \epsilon^2 \int v_{||}^2 f_{2,0} dv_{||} v_\perp dv_\perp\right]$$

combining the ϵ^2 and ϵ^4 equations and substituting for $\tilde{f}_{0,0}$ from the ϵ^2 equation in the gravity drift term gives

$$0 = \int_{(n)} \frac{d\ell}{B}\left\{-i(\omega - \omega_E^{(n)})(\tilde{n}_0^{(n)} + \epsilon^2\tilde{n}_2^{(n)}) + i\frac{m^{0^2}c}{m_i rR_c\hat{\omega}\Omega}\frac{\partial(P_{||}^{(n)} + P^{(n)})}{\partial\alpha} - im^0c\left[\phi_0^{(n)}\frac{\partial}{\partial\alpha}(n_0^{(n)} + \epsilon^2 n_2^{(n)}) + \eta\phi_1\frac{\partial n_0^{(n)}}{\partial\alpha}\right]\right.$$

$$+ \frac{1}{\Omega^2} \nabla \cdot \left[\rho \frac{D}{Dt} \frac{q}{m} \underset{\sim}{E} - \frac{D}{Dt} \nabla P_\perp \right]^{(n)}_{\text{perturbed}} \Bigg\} \tag{37}$$

where
$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \underset{\sim}{V}_E \cdot \nabla$$

and

$$R_c^{-1} \equiv \frac{-\nabla\alpha \cdot (\underset{\sim}{b} \cdot \nabla \underset{\sim}{b})}{|\nabla\alpha|} .$$

The term proportional to Ω^{-2} is

$$\frac{1}{\Omega^2} \nabla \cdot \left[\rho \frac{D}{Dt} \frac{q_i}{m_i} \underset{\sim}{E} - \frac{D}{Dt} \nabla P_\perp \right]^{(n)}_{\text{perturbed}} = \frac{-im^\circ}{\Omega} (\Phi' c)^2 \tilde{n}_\circ^{(n)}$$

$$+ i \frac{\hat{\omega}^{(n)}}{\Omega} c \left\{ \frac{2}{\alpha^{1/2}} \frac{1}{\hat{\omega}^{(n)}} \frac{\partial}{\partial\alpha} \left[S^{(n)} \hat{\omega}^{(n)\,2} n_\circ^{(n)} \alpha^2 \frac{\partial}{\partial\alpha} \left(\frac{\phi_\circ^{(n)}}{\alpha^{1/2} \hat{\omega}^{(n)}} \right) \right] \right.$$

$$\left. - S^{(n)} \frac{(m^{\circ 2} - 1)}{2\alpha} n_\circ^{(n)} \phi_\circ^{(n)} + \frac{\omega + \omega_E^{(n)}}{\hat{\omega}^{(n)}} \frac{\partial n_\circ^{(n)}}{\partial\alpha} \phi_\circ^{(n)} \right\} \tag{38}$$

where

$$\hat{\omega}^{(n)} \equiv \omega - \omega_E^{(n)}$$

$$S^{(n)} \equiv 1 - \frac{m^\circ B}{m_i n_\circ^{(n)} \Omega_i \hat{\omega}^{(n)}} \frac{\partial P_i^{(n)}}{\partial\alpha} = 1 - \frac{m^\circ c}{n_\circ^{(n)} e \hat{\omega}} \frac{\partial P_i^{(n)}}{\partial\alpha}$$

and where $P_{i_\perp}^{(n)}$ is the perpendicular equilibrium ion pressure in region (n).

Substituting for $\tilde{n}_0^{(n)}$ from the ϵ^2 equation, $\tilde{n}_0^{(n)} = - m^0 c\phi_0^{(n)}(\partial n_0^{(n)}/\partial\alpha)/\hat{\omega}^{(n)}$ in Eq. (38), and combining Eq. (38) and Eq. (37) with the quasi-neutrality relation, Eq. (32) gives

$$0 = - e \int_n \frac{d\ell}{B} \Bigg\{ - m^0 c \frac{\partial n_T^e}{\partial\alpha} (\phi_0^{(n)} + \eta\phi_1^{(n)}) - \frac{2m^{0^2} c}{\hat{\Omega}_i \omega m_i r R_c} \frac{\partial P_e}{\partial\alpha} \phi_0^{(n)}$$

$$+ \int_p d^3v' \left[\hat{\omega}(\phi_0^{(n)} - \frac{\omega}{\hat{\omega}} \bar{\phi}_0) e \frac{F_0^e}{T_e} - \frac{\omega}{\hat{\omega}} m^0 c\bar{\phi}_0 \frac{\partial F_0^e}{\partial\alpha'} \right] \Bigg\}$$

$$+ e \int_{(n)} \frac{d\ell}{B} \Bigg\{ - m^0 c \left[\phi_0^{(n)} \frac{\partial}{\partial\alpha} (n_0^i + \epsilon^2 n_2^i) + \eta\phi_1 \frac{\partial n_0^i}{\partial\alpha} \right]$$

$$+ \frac{m^0 c\phi_0^{(n)}}{\hat{\omega}\Omega_i} \left[\frac{m^0}{m_i r R_c} \frac{\partial(P_i + P_{i||})}{\partial\alpha} + m^0 (\phi' c)^2 \frac{\partial n_0^i}{\partial\alpha} \right]$$

$$+ \frac{\hat{\omega}}{\Omega_i} c \left[\frac{2}{\alpha^{1/2}\hat{\omega}} \frac{\partial}{\partial\alpha} \left(S\hat{\omega}^2 n_0^i \alpha^2 \frac{\partial}{\partial\alpha} \left(\frac{\phi_0^{(n)}}{\alpha^{1/2}\hat{\omega}} \right) \right) \right.$$

$$\left. - S \frac{(m^{0^2} - 1)}{2\alpha} n_0^i \phi_0^{(n)} + \frac{\omega + \omega_E}{\hat{\omega}} \frac{\partial n_0^i}{\partial\alpha} \phi_0^{(n)} \right] \Bigg\} \qquad (39)$$

where in the ion terms we have suppressed the superscript (n) on all equilibrium terms for notational simplicity. Using the equilibrium relation $n_T^e + n_p^e = n_0^i + \epsilon^2 n_2^i$, dropping the terms $\eta\phi_1(\partial n_p^e/\partial\alpha)$ and $\eta\phi_1(\partial\epsilon^2 n_2^i/\partial\alpha)$ compared to $(\phi_0^{(n)} - \bar{\phi}_0)(\partial n_p^e/\partial\alpha)$, and defining the variable $\theta^{(n)}(\alpha) = \phi_0^{(n)}/\hat{\omega}^{(n)}$ yields the following expression:

$$
\begin{aligned}
0 = \int_{(n)} \frac{d\ell}{B} \Bigg\{ & \int_p d^3v' \hat{\omega} \left(e\hat{\omega} \frac{\partial F_0^e}{\partial \varepsilon'} - m^* c \frac{\partial F_0^e}{\partial \alpha} \right) \left(\theta^{(n)} - \frac{\overline{\theta \hat{\omega}}}{\hat{\omega}} \right) \\
& + \frac{m^{*2} c}{\Omega_i} \left[\frac{2}{m_i r R_c} \frac{\partial P}{\partial \alpha} + \left(\frac{\omega_E}{m^*} \right)^2 \frac{\partial n_0}{\partial \alpha} \right] \theta^{(n)} \\
& + \frac{c}{\Omega_i} \left[\frac{2}{\alpha^{1/2}} \frac{\partial}{\partial \alpha} \left(S\hat{\omega}^2 n_0 \alpha^2 \frac{\partial}{\partial \alpha} \left(\frac{\theta^{(n)}}{\alpha^{1/2}} \right) \right) \right. \\
& \left. - S\hat{\omega}^2 \frac{n_0 (m^{*2} - 1)}{2\alpha} \theta^{(n)} + \hat{\omega}^2 \frac{\partial n_0}{\partial \alpha} \theta^{(n)} \right] \\
& + \frac{2c}{\Omega_i} \hat{\omega} \omega_E \frac{\partial n_0}{\partial \alpha} \theta^{(n)} \Bigg\}
\end{aligned} \tag{40}
$$

where $P \equiv P_e + P_i \equiv P_e + (P_{i\perp} + P_{i||})/2$

and $S\hat{\omega}^2 = \hat{\omega}^2 - \hat{\omega} \frac{m^* c}{e n_0} \frac{\partial P_{i\perp}}{\partial \alpha}$

contains both a term quadratic and a term linear in $\hat{\omega}^{(n)}$, the mode frequency in the frame rotating with the local E × B drift velocity. In all equilibrium quantities we have again suppressed the superscript (n).

The first term in this expression comes from passing electrons and is non-zero for a mode with differing mode amplitudes in each region. The second term contains the curvature drive and centrifugal rotation drive due to the radial electric field. The third term contains the charge perturbations due to the ion polarization drift and the incomplete cancellation of electron and ion E × B drifts due to the ion finite Larmor radius. The polarization drift multiplies $\hat{\omega}^2$ and thus enters as an inertial term; the finite Larmor radius term is linear in $\hat{\omega}$ and accounts for the usual FLR stabilization of modes which are unstable in the MHD zero-Larmor radius limit. The last term is due to the charge perturbation caused by the Coriolis drift. In the rotating frame the Coriolis force is in the azimuthal direction and thus produces an ion drift in the radial direction. Since this term is linear in $\hat{\omega}$ it will either subtract or add to the usual linear FLR term depending on the sign of ω_E. For positive ω_E, which implies a radially inward pointing electric field the term is stabilizing.

In the limit of large m' the infinite Larmor radius terms due to the incomplete cancellation of ion and electron drifts dominate all other terms. Since these are linear in $\hat{\omega}$, a quadratic form derived from Eq. (40) predicts stability. This is the well known finite Larmor radius stabilization effect. As we examine progressively smaller azimuthal mode numbers, the linear finite Larmor radius term can be cancelled by the electron passing particle term for a mode localized to the central cell. For TARA parameters this occurs at $m' \sim 2$. For a configuration in which ions bounce beyond electrons, the two terms add and the device can be made stable for all m'. This is the physical basis for the stabilization scheme incorporated in the MFTF-B design. For a TARA configuration, we find that the additional stabilization due to the Coriolis drift resulting from an inward pointing equilibrium electric field can stabilize the $m' = 2$ mode. In the following section we examine this scheme quantitatively.

We note that Eq. (40) is in fact a set of N coupled differential equations in the radial variable α, where N is the number of regions. The coupling is through the passing density. We now construct a set of coupled quadratic forms by multiplying each equation by $(\theta^{(n)})^*$ and integrating over α.

In order to simplify the analysis we further restrict our equilibrium model by requiring that $\Phi(\alpha)$ be the same in all regions. Combined with the previous assumption of Maxwellian electrons this implies that the density is equal in all regions. The ion equilibrium distribution function is assumed to be Maxwellian with appropriate temperatures for each region. Since ω_E is in general a function of α we write a variational form for each region n as a quadratic in ω

$$A\omega^2 + B\omega + C = 0 \tag{41}$$

where

$$A = \int_{(n)} \frac{d\ell}{B^2}\left\{\int d\alpha \frac{dn}{d\alpha} \left|\theta^{(n)}\right|^2 - m_i \frac{\Omega_i^2}{B} \int d\alpha \int_p d^3v' \frac{F_0^e}{T_e} (\theta^{(n)} - \bar{\theta})(\theta^{(n)})^* \right.$$

$$\left. - \int d\alpha \left[2\alpha^2 \left|\frac{d(\theta\alpha^{-1/2})}{d\alpha}\right|^2 + \frac{(m^{0^2} - 1)\left|\theta^{(n)}\right|^2}{2\alpha}\right] n_0 \right\} \tag{42}$$

$$B = \int_{(n)} \frac{d\ell}{B^2}\left\{\int d\alpha \int_p d^3v' \left(2\omega_E m_i \frac{\Omega_i^2}{B} \frac{F_0^e}{T_e} - m^0 \Omega_i \frac{\partial F_0}{\partial \alpha}\right) (\theta^{(n)} - \bar{\theta})(\theta^{(n)})^* \right.$$

$$+ \int d\alpha n_0 \left[2\alpha^2 \left|\frac{d(\theta^{(n)}\alpha^{-1/2})}{d\alpha}\right|^2 + \frac{(m^{0^2} - 1)\left|\theta^{(n)}\right|^2}{2\alpha}\right]$$

$$\left. \times \left(2\omega_E + \frac{m^0 c}{e n_0} \frac{dP_i}{d\alpha}\right)\right\} \tag{43}$$

$$C = \int_n \frac{d\ell}{B^2} \left\{\frac{2m^{0^2}}{m_i r R_c} \int d\alpha \frac{dP}{d\alpha} \left|\theta^{(n)}\right|^2 \right.$$

$$- \int d\alpha \int_p d^3v' \left(m_i \frac{\Omega_i^2}{B} \omega_E^2 \frac{F_0^e}{T_e} - m^0 \Omega_i \omega_E \frac{\partial F_0^e}{\partial \alpha}\right) (\theta^{(n)} - \bar{\theta})(\theta^{(n)})^*$$

$$- \int d\alpha n_0 \left[2\alpha^2 \left|\frac{d(\theta\alpha^{-1/2})}{d\alpha}\right|^2 + \frac{(m^{0^2} - 1)}{2\alpha} \left|\theta^{(n)}\right|^2\right]$$

$$\times \left[\omega_E^2 + \omega_E \frac{m^\circ c}{en_0} \frac{dP_i}{d\alpha}\right]\Bigg\} \tag{44}$$

Note that transforming the quadratic form into the laboratory frame in order to permit further analysis has scrambled the terms of Eq. (40) making the physical origin of the various terms less transparent.

Stability Criteria

a. Inward Pointing Equilibrium Electric Field

We consider first modes with $m^\circ > 1$. We propose a stabilization mechanism for these modes based on reversing the radial equilibrium electric field. Examining the drive term, C, we note that all the terms are negative (destabilizing) for the center cell except for the term proportional to

$$- \omega_E m^\circ \frac{dP_i}{d\alpha} = - m^{\circ 2} c \frac{d\Phi}{d\alpha} \frac{dP_i}{d\alpha}$$

If E_0 points inward ($E_0 = - \nabla\Phi < 0$) this term is stabilizing. For $m^\circ > 1$ we need to compare the term to the centrifugal drive term proportional to $- \omega_E^2$ and the gravity drive term proportional to $m^{\circ 2}(1/rR_c)(dP/d\alpha)$.

For comparison with the centrifugal term we will assume parabolic ion density, temperature and potential profiles (linear in flux, α).; e.g. $n_0 = n_{0,0} g(\alpha)$, $T = T_{0,i}^{(n)} g(\alpha)$ with $g = 1 - \alpha/\alpha_0$ and $\Phi = \Phi_0 + (\alpha/\alpha_0)\delta\Phi$, with α_0 the value of the flux function at the plasma edge. We note that $\theta^{(n)}$ is independent of position along the field line within each region. Within this model we then obtain a criteria for rotational stability of the form

$$e\delta\Phi < \frac{\sum\limits_n T_{0,i}^{(n)} \int\limits_{(n)} \frac{d\ell}{B^2}}{\sum\limits_n \int\limits_{(n)} \frac{d\ell}{B^2}} \qquad (45)$$

As an example we consider the TARA tandem mirror. Noting that the numerator is dominated by the axicell (where neutral beams maintain an energetic component), we can estimate

$$\frac{e\delta\Phi}{T_{0,i}^{c}} \lesssim 2\left[\frac{L_{ax}}{L_{cc}}\left(\frac{B_c}{B_a}\right)^2 \frac{T_{0,i}^{(ax)}}{T_{0,i}^{(c)}} + 1\right] \qquad (46)$$

with L_{ax}/L_{cc} the axicell to central cell length ratio (= 0.4), B_c/B_a the central cell to axicell field ratio (0.4) and $T_{0,i}^{(ax)}/T_{0,i}^{(c)}$ the axicell to central cell temperature ratio ($\sim$ 12). Thus we get $(e\delta\Phi/T_{0,i}^{(c)}) < 4$.

The requirement that the stabilizing term be greater than the curvature drive requires that

$$\int\frac{d\ell}{B^2}\frac{1}{rR_c}\int d\alpha\,\theta^2\,\frac{\partial P}{\partial\alpha} - \frac{(m_0^2 - 1)}{4}\;\frac{B}{\Omega_i}\frac{\omega_E}{m^0}\int\frac{d\ell}{B^2}\int\frac{d\alpha}{\alpha}\,\theta^2\,\frac{\partial P_i}{\partial\alpha} > 0 \qquad (47)$$

The stabilizing term proportional to ω_E is diminished if the α in the denominator is replaced by the flux at the plasma boundary, α_0. This yields a sufficient condition for stability. Furthermore, if we set $P(\alpha,\ell) = P_0^{(n)}g^2(\alpha)$ we obtain the inequality

$$\sum_n P_0^{(n)}\int\limits_{(n)}\frac{d\ell}{B^2}\,1/(rR_c) < \frac{(m^{0^2} - 1)}{4}\,\frac{\omega_E}{m^0\alpha_0}\,\frac{B}{\Omega}\,\sum_n P_0^{(n)}\int\limits_{(n)}\frac{d\ell}{B^2} \qquad (48)$$

which may be cast in the form

$$\frac{e\delta\Phi}{T_i} > \frac{1}{(m^{\theta 2} - 1)} \frac{r_c^4}{\rho_{ic}^2} \frac{\sum_n P_0^{(n)} \int_{(n)} \frac{d\ell}{B^2} 1/(rR_c)}{\sum_n P_0^{(n)} \int_{(n)} \frac{d\ell}{B^2}} \quad . \tag{49}$$

For TARA the integral in the numerator is dominated by the axicell ion pressure and the denominator by the central cell. Thus in the numerator we integrate only over the axicell. The integral $\int_{(a)} d\ell/(B^2 rR_c)$ in the axicell is found numerically to be 2.8×10^{-4} cm^{-1}-kG^{-2} and taking $2P_{0,i}^{(ax)}/P_{0,i}^{(c)} \approx 24$, $\int d\ell/B^2 \sim 125$ cm-kG^{-2} (which comes predominately from the central cell), $r_c = 14$ cm, $\rho_{ic} \equiv (T_{0,i}^{(c)}/m_i)^{1/2}/\Omega_i = 1.4$ cm and noting that the most unstable mode is $m^\theta = 2$ we obtain $e\delta\Phi/T_{0,i}^{(c)} > .05$.

Thus there exists the possibility of a stability window for modes with $m^\theta > 1$. For TARA parameters the requirement on the potential rise from axis to plasma edge is $.05 < e\delta\Phi/T_{0,i}^{(c)} < 4$.

Assuming that the electric field lies within this window, we next turn to the stability of the $m^\theta = 1$ mode. An examination of the "C" term, Eq. (44), shows that the stabilizing term is multiplied by $(d(\theta\alpha^{-1/2})/d\alpha)^2$. Thus a mode for which $\theta = \alpha^{1/2}\Psi$ with Ψ a constant independent of α and ℓ is not stabilized by this mechanism. This is the so-called rigid mode. The theory thus predicts that one cannot stabilize the $m^\theta = 1$ interchange mode in a simple mirror by radial potential control. In a tandem mirror configuration however, we may stabilize the $m^\theta = 1$ mode by the charge separation that results from passing density.

We now assume that axially the eigenfunction is zero in the good curvature anchor and flute-like elsewhere so that $\bar{\theta} \sim \theta_c\tau_c/\tau$ with τ_c and τ half the center cell and half the total bounce time respectively. With this assumption we examine the stability condition: $B^2 > 4AC$. To evaluate flux integrals we again assume parabolic density and temperature profiles and we evaluate n_{pass} assuming only magnetic trapping (no thermal barrier) using the large mirror ratio approximation $n_{pass} = n_o B(\ell)/2B_{max}$. After some algebra these approximations lead to the result

$$1/B_{max} > \frac{16}{3} B_c r_c^2 \left(\frac{\tau}{\tau_a}\right) \frac{1}{L_c N_c T_e} \sum_{(n)} P_0^{(n)} \int_{(n)} \frac{d\ell}{B^2} \frac{1}{rR_c} (1 + 1/A_p)$$

$$+ 16 \frac{\tau}{\tau_a L_c} \frac{e\delta\Phi}{T_i^{(c)}} \left(\frac{\rho_{ic}}{r_c}\right)^2 B_c \sum_n \int_{(n)} \frac{d\ell}{B^2} \quad 1 + \frac{e\delta\Phi}{T_e} \tag{50}$$

with A_p defined in Eq. (52). The first term on the right-hand side of Eq. (50) represents the curvature drive. The second term contains the Coriolis and centrifugal forces due to plasma rotation. Since $A_p \propto 1/B_{max}$ this inequality can be cast into the form

$$A_p > \Gamma + 2\delta + ((\Gamma + 2\delta)^2 + 2\Gamma))^{1/2}. \tag{51}$$

The quantity δ is a measure of radial potential given by

$$\delta \equiv \frac{e\delta\Phi}{T_e} \left(1 + \frac{e\delta\Phi}{T_e}\right)$$

and Γ a measure of curvature drive is given by

$$\Gamma = \frac{1}{3} \frac{r_c^4}{\rho_{ic}^2} \frac{T_{ic}}{n_c T_e^2} \frac{\int d\ell 2P/rR_c B^2}{\int d\ell/B^2} \equiv \frac{1}{2}\left(\frac{r_c}{\rho_i}\right)^4 \left(\frac{T_i}{T_e}\right)^2 \left(\frac{\gamma_{MHD}}{\Omega_i}\right)^2$$

where γ_{MHD} is the MHD growth rate.

A_P is given by[1]

$$A_P = \frac{1}{4}\frac{T_i^{(c)}}{T_e}\left(\frac{r_c}{\rho_i}\right)^2 \frac{\tau_a L_c}{\tau}\frac{1}{B_c B_{max}}\left(\int_c d\ell/B^2\right)^{-1} \tag{52}$$

$$\approx \frac{1}{4}\frac{T_i^{(c)}}{T_e}\left(\frac{r_c}{\rho_i}\right)^2 \frac{L_a}{(L_a + L_c)}\quad\frac{B_c}{B_{max}}$$

A_P relates the bounce averaged passing density charge separation term to the FLR stabilization term. Typically $A_P \gtrsim 1$. Ratios of bounce times can be replaced by geometric ratios ($\tau/\tau_a \approx L/L_a$)(This is a good approximation since passing particles have small pitch angles and thus to lowest order the bounce times are unaffected by cell-to-cell magnetic field variation). Eq. (51) indicates $m^\circ = 1$ instability can result from both curvature ($\Gamma > 0$) and rotation ($\delta > 0$). Thus even with neutral central cell curvature a tandem mirror can be subject to trapped-particle instability. This instability is a rotationally driven trapped particle mode, not discussed previously in the literature.

b. <u>Zero Radial Equilibrium Field</u>

When the radial electric field is zero the rotational terms discussed above will not enter the stability criterion. We now show that for a small enough drive the $m^\circ = 1$ mode can be stabilized by the passing density charge separation term as discussed above and the $m^\circ > 1$ modes can be stabilized by finite Larmor radius effects. This can provide stability even when electrons bounce beyond ions.

To investigate this stability window we will set $\omega_E = 0$ and assume rigid eigenmodes for m° values of 1 and 2. Although the assumption of a rigid eigenmode is strictly valid only for the $m^\circ = 1$, we consider a rigid trial function as an approximation to the $m^\theta > 1$ eigenmode. This gives a stability criteria

$$\left(A_P \mp (m^{\theta^2} - 1)\right)^2 > \Gamma\left(1 + m^{\theta^2} + 2A_P\right) \qquad (53)$$

with A_P defined by Eq. (52).

The minus sign in the left hand side is appropriate when electrons bounce beyond ions and vice versa. In the situation that ions bounce beyond electrons we assume that the ion transit time is short compared to the mode frequency. The $m^\theta = 1$ mode stability condition now becomes

$$A_P > \Gamma + (\Gamma(\Gamma + 2))^{1/2} \qquad (54)$$

(which also follows from Eq. (51). For $m^\theta = 2$ the requirement that the FLR stabilization term dominates the passing density charge separation term as well as the curvature drive requires

$$A_P < \Gamma + 3 - (\Gamma(11 + \Gamma))^{1/2}. \qquad (55)$$

Comparing Eqs. (54) and (55) one finds that a stability window exists for $\Gamma < 0.37$. This criteria can also be put into the form

$$(\gamma_{MHD}/\Omega_i) < 0.86\ (T_i/T_e)^{1/2}(\rho_i/r_c)^2. \qquad (56)$$

For TARA we estimate $\Gamma = 0.23$ which yields a stability window for $0.9 < A_P < 1.7$ since the estimated value of A_P is $A_P = 0.9$, TARA can operate at the edge of the stable regime for $\omega_E = 0$.

When ions bounce beyond electrons we obtain the old result[1] that when the $m^\theta = 1$ mode is stable, higher m^θ modes are also stable.

c. Presence of a Thermal Barrier

When a thermal barrier is present within the TARA axicell (See dashed curve in Fig. 1) the equilibrium potential will have a local depression. Thus, in addition to the throttle coil which only permits the passage of electrons with small pitch angles, the passing density will be further diminished by the reflection of low energy electrons. This would appear to destabilize trapped particle modes.

R.H. Cohen has however pointed out that an additional stabilizing effect enters.(6) In the presence of a potential barrier it can be shown that a temperature gradient will result in an enhancement of the transition region passing density gradient. This can be seen as follows:

The term that gives rise to stabilization of the $m^{\theta} = 1$ modes derives from the first B term in Eq. 43, which may be written as

$$B_{pass} = \frac{m^{\theta}\Omega_i}{B} \int_{(n)} \frac{d\ell}{B} \int d\alpha \; (\bar{\theta}-\theta^{(n)})\theta^{(n)} \int_p d^3v' \frac{\partial F_o}{\partial \alpha} \tag{57}$$

with

$$\int_P d^3v' \, \frac{\partial F_o}{\partial \alpha} = \frac{4\pi}{m^2} \; B \int_{-e\Phi_b}^{\infty} d\varepsilon \int_o^{(\varepsilon + e\Phi_b)/B_{max}} d\mu$$

$$\times \left(\frac{1}{\left(\frac{2}{m}\right)^{1/2} (\varepsilon - \mu B + e\Phi_c)^{1/2}} \quad \frac{\partial F_o}{\partial \alpha} \right) \tag{58}$$

where Φ_b is the potential at the barrier, Φ_c is the potential in the center cell, and B_{max} is the maximum value of the magnetic field. Placing the thermal barrier at the field maximum does not correspond to a realistic configuration but has been chosen to be consistent with our overall model configuration and to simplify the integrals.

The electron distribution function, F_o, is a Maxwellian $F_o = n_o(\alpha)$ $(2\pi T(\alpha)/m_e)^{-3/2} \exp(-(\varepsilon + e\Phi_c)/T(\alpha))$. Noting that the derivative with respect to α is taken at constant $\underset{\sim}{v}'$ we write

$$\frac{\partial F_o}{\partial \alpha} = \left[\frac{\partial n_o}{\partial \alpha} + \frac{n_o}{T} \frac{\partial T}{\partial \alpha} \left(\frac{\varepsilon + e\Phi_c}{T} - \frac{3}{2}\right)\right] \frac{e^{-(\varepsilon + e\Phi_c)/T}}{\left(\frac{2\pi T}{m}\right)^{3/2}}. \tag{59}$$

setting $\theta = \theta_c \tau_a/\tau$ with $\tau = \tau_a + \tau_c$ and defining $\chi = (e\Phi_c - e\Phi_b)/T$ and $\xi = \partial \ln T/\partial \ln n_o$, we obtain for a rigid $m^\theta = 1$ mode

$$B_{pass} = \frac{\tau_a}{\tau} \frac{\Omega_i}{B} \psi^2 \int \frac{d\ell}{B} \int d\alpha\, \alpha \frac{\partial n_{pass}}{\partial \alpha} g(\chi,\xi) \tag{60}$$

with $\psi = \theta\sqrt{\alpha}$, $n_{pass} = n_o B_c/(2 B_{max})$ and

$$g(\chi,\xi,R) = 2R \left[\operatorname{erfc}(\sqrt{\chi}) - \sqrt{\frac{R-1}{R}} \exp(-\chi/(R-1)) \operatorname{erfc}\left(\sqrt{\frac{R\chi}{R-1}}\right)\left(1 - \frac{\chi\xi}{R-1}\right)\right] \tag{61}$$

with erfc the complimentary error function defined by

$$\operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty dy e^{-y^2}.$$

In the limit of large mirror ratio, R, this becomes

$$g(\chi,\xi) = \left(1 + 2(\xi-1)\ \chi\right) \operatorname{erfc}\ (\chi^{1/2}) + \frac{2}{\sqrt{\pi}}\ \chi^{1/2}\ e^{-\chi} \qquad (62)$$

For $\xi = 0$, χ corresponds to a flat temperature profile and g decreases with increased thermal barrier height, reflecting the fact that less electrons pass over the barrier. For a temperature profile that is more peaked than the density profile, that is $\xi > 1$, g (χ,ξ) can increase with increasing barrier height.

Figure 2 shows g as a function of ξ for varying χ values. If for an arbitrary density profile $n(\alpha)$ the temperature is assumed to have the form $T = T_0(n(\alpha)/n(\alpha=0))^p$ with p a measure of the relative peaking of temperature then ξ becomes independent of the flux coordinate, α (i.e. $\xi \equiv \partial(\ell n\ T)\ /\ \partial(\ell n\ n) = p$). If we assume χ is also independent of α then g becomes a multiplicative factor in Eq. 58. For g = 1 no degradation in stabilization will accompany the thermal barrier formation. From Fig. 2 we see for example for a thermal barrier characterized by $\phi_b = 1.5T_e$ a peaking factor of p = 2.3 is required not to degrade trapped particle mode stability.

Raising the value of p should be possible since gas puffing has been observed to flatten radial density profiles and steepen temperature. In addition, the presence of turbulence would tend to flatten central cell density profiles.

Thus it appears that given sufficient control of radial equilibrium potentional, temperature and density profiles, stability of the $m^\theta = 1$ trapped particle mode can accrue for a modest inside thermal barrier design.

CONCLUSION

We have extended the theory of trapped particle instability in tandem mirrors to arbitrary azimuthal mode numbers and have included the effect of ExB rotation. This has been done using a finite Larmor radius ordering of the Vlasov equation.

We find that, when the radial equilibrium electric field is small and points inward a stabilizing Coriolis term appears that can add to the FLR term and stabilize low m^θ ($m^\theta > 1$) modes. At high m^θ FLR stabilization is dominant. For the $m^\theta = 1$ rigid mode the FLR terms do not appear and the stability requirement does not depend on which species bounces farther out. The stability criteria can then be put in a form that imposes an upper limit on the peak "throttle" coil field that bounds the central cell.

When the rotational frequency is zero we show that a window of stability can exist when electrons bounce beyond ions for sufficiently small curvature drive. This window results from the $m^\theta = 1$ mode being barely stabilized by the passing species and the $m^\theta \geq 2$ modes being FLR stabilized.

This theory is seen to uncover a new instability, a rotationally driven trapped particle mode. Existence of this instability does not require bad curvature. Stability against this mode requires small radial potential gradients ($e\delta\Phi \ll T_e$). This instability is the trapped particle analogue of the MHD rotationally driven instabilities discussed in the literature.(7)

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$\chi = e\phi_b / T_e$

$\xi = \delta \ln T / \delta \ln n$

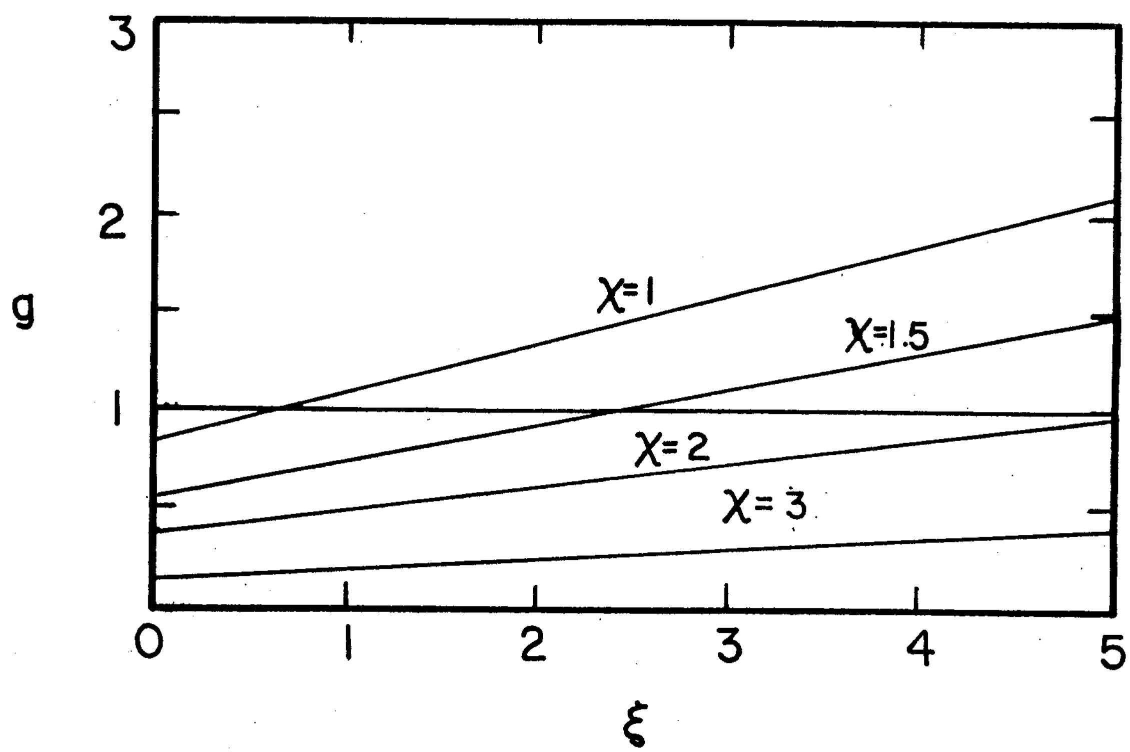

FIG. 2