On Tearing Modes in a Resistive Medium[a]

Philip Rosenau[b]

Dept. of Nuclear Engineering and Plasma Fusion Center

Massachusetts Institute of Technology

Cambridge, Massachusetts 02139

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ABSTRACT

The theory of tearing modes in a cylindrical symmetry is reexamined. The resistive medium is embedded with the simplest Ohm's law. By considering β, the ratio of thermal to magnetic pressure, as a parameter with respect to which the sought after eigenvalues were scaled, a very simple description was obtained. Many of the classical results were obtained in a more simple way. Among the new results are: (1) the possibility of the existence of a tearing mode on a resistive time scale and which is hardly affected by drift effects, (2) for modes of an almost marginal growth rate inertia does not play an important role, slab and cylindrical modes are fundamentally different and finite-β effects play a crucial role.

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[b]On leave of absence from Dept. of Mechanical Engineering, Technion, IIT, Haifa, Israel.

I. Introduction and Statement of the Problem

Since the introduction of resistive instabilities [1] as agents of obstruction on our way toward controlled fusion, an explosive amount of work has been devoted to this subject [1-14]. (These references are only a partial sample.) The analytical work falls into two categories. The one pursues more complex geometries(e.g., Refs. 7,8); in the other, more evolved models of plasma were constructed. We shall focus our attention on tearing modes that, among the various resistive instabilities,are considered to be more dangerous for toroidal devices.

In the present work we reexamine the theory of resistive tearing modes in cylindrical symmetry. It is shown that some of the classical results may be rederived in what we believe is a simpler and more systematic way. Taking the complexity of calculations usually associated with resistive instabilities, these may be of value per se. Beyond that, the theory of tearing modes is extended and we present a complete stability analysis of modes with growth rates both on the tearing and resistive scales. Growth rates on the resistive scale behave quite differently as compared with predictions based on const-ψ approximations. In fact, for most of this paper, the relevant boundary layer equations are solved without invoking the const-ψprocedure. As a by product, this allows one to clarify the validity and limitations of that approach.

On the other side of the spectrum, we modify and extend the standard tearing theory to consider modes with an almost marginal growth rate. In this regime, on one hand, inertia is negligible; on the other, the effects due to finite-β are of paramount importance. The basic result due to Coppi *et al* [2], namely, the existence of critical $\Delta_c > 0$ at which the instability is set on, is recovered in a much simpler fashion (almost with a compass and a ruler!) It is also shown that for high-β the behavior of these modes is markedly different.

Even with some drift effects included, as done in Sec. IV, the physical model employed in admittedly a very simple one. But we believe that a coherent treatment of a simple model is of conceptual importance because the more evolved models employ as a rule a similar methodology, and measure their outcome against the simple model.

Apart from Sec. III, where an impact of finite β on almost marginal modes is considered, we shall consider a low-β regime and modes with growth rates larger than β. (A precise definition will be given shortly.) Thus, although not necessary, it is convenient, as a starting point, to take the low-β boundary layer equations in their zero-β limit, but with D_S, the average cylindrical curvature, retained. Finite-β effects will be restored in Sec. III. Following Ref. 2 or 14, the derivation of the relevant equations is summarized in Appendix A. We have in the zero-β limit

$$\psi'' = Q(\psi - x\varsigma) \tag{1}$$

$$-Q^2\varsigma'' = x\psi'' + D_s\varsigma, \quad -\infty < x < \infty \tag{2}$$

Here ψ and ς are the radial components of the magnetic field and displacement, respectively, Q is the normalized growth rate given via

$$t_H\gamma = \epsilon Q, \quad 0 < \epsilon \equiv t_H/t_D = O(\eta^{\frac{1}{3}}) \ll 1 \tag{3}$$

where t_H and t_D are the Alfven and resistive transit times, respectively (see Appendix A), and η is resistivity assumed to be scalar. x is the boundary layer variable $\epsilon x = (r - r_s)/a$. r_s is the location of the singular shear layer where $F(r_s) \equiv \boldsymbol{K} \cdot \boldsymbol{B} |_{r_s} = 0$ and a is the characteristic length of the system. Outside the layer, resistivity is neglected and to first order $\psi = F(r)\varsigma$. ς, when expanded in the vicinity of r_s, is given via

$$r\varsigma = | r - r_s |^{\nu}, \quad \nu = \frac{1}{2}[-1 \pm (1 - 4D_s)^{0.5}] \tag{4}$$

Here, as in (2)

$$D_s = -\frac{2k^2 r}{B_\theta^2 q^2}\frac{dp}{dr}|_{r=r_s}$$

q is the safety factor and it is assumed that $D_s < \frac{1}{4}$ to assure stability against localized magnetohydrodynamic oscillations.

The problem consists of solving Eqs.(1) and (2) subject to the matching conditions of the outer region as represented by Eq.(4).

In what follows, we shall not be interested in interchange modes as found for $D_s > 0$, because in the most straightforward tokamak expansion one obtains $D_s \rightarrow D_s(1 - q^2) < 0$. Therefore, our main objects are the tearing modes.

Now, rewrite Eq.(2) as

$$-Q^2\varsigma'' = Qx\psi + (D_s - x^2 Q)\varsigma \tag{5}$$

Here all terms are of equal importance if the following, optimal, scaling is adopted

$$\varsigma \simeq Q^{-\frac{1}{4}}\hat{\varsigma}, \quad \psi \simeq O(1), \quad x \simeq Q^{\frac{1}{4}}\hat{x} \tag{6a}$$

$$Q^{\frac{3}{2}} \simeq D_s \tag{6b}$$

Thus, Eq.(1) reads

$$\frac{d^2\psi}{d\hat{x}^2} = Q^{\frac{3}{2}}(\psi - \hat{x}\hat{\varsigma}) \tag{7}$$

While in Eq.(6a), we have only changed the scale, assumption (6b) relates two physically different entities and *thus cannot hold uniformly*. Though $D_s \simeq \beta \ll 1, D_s$ is a quantity given a priori as part of the information about equilibrium. With D_s stated, the spectrum $\{Q\}$ is uniquely determined. Thus, by assuming (6b), one seeks for modes in a specific range. Therefore, a resulting mode which is out of this range will not be properly calculated using this scaling. In constructing an asymptotic theory of Eqs.(1) and (2) for a given D_s, one looks for three distinctive classes of modes

I. $|D_s|/Q^{\frac{3}{2}} \ll 1$

II. $|D_s|/Q^{\frac{3}{2}} = O(1)$

III. $|D_s|/Q^{\frac{3}{2}} \gg 1$

The separation into different regimes enables one to treat analytically the simplified counterpart of equations (1) and (2). The relevant model equations for modes of class I are solved explicitly leading to a dispersion relation which extends the tearing modes into the resistive time scale. (i.e., from $\gamma \simeq \eta^{\frac{3}{5}}$ into $\gamma \simeq \eta^{\frac{1}{3}}$). To solve for modes in class II, one assumes that $|Q| \ll 1$, and expands in a small "parameter" Q. This eventually leads to the so called const-ψ approximation which should be viewed as a by-product and not, as so often stated in the literature, an a priori postulate. The resulting dispersion relation of this approximative procedure should be used only within the range of parameters that yield $Q \ll 1$. Otherwise, the predicted growth rates differ significantly from the exact ones.

Modes in different classes have the following formal overlap: growth rates in class I for small Q are the same as modes in class II expanded in large Q. (or more precisely, small $|D_S|/Q^{\frac{3}{2}}$). A similar relation holds between modes in class II and III.

We shall not consider directly modes of class II because these modes are the ones most often considered in the literature (in the $Q \ll 1$ limit). However, we call attention to Appendix D, where the modal equation of the radial displacement ς is solved using an integral representation. This will find its use in Sec. III.

II. Modes of Class I

Exploiting $|D_s| \ll Q^{\frac{3}{2}}$ we expand in small D_s:

$$\psi = \psi_0 + D_s\psi_1 + \cdots$$

$$\varsigma = \varsigma_0 + D_s\varsigma_1 + \cdots \tag{8}$$

$$Q = Q_0(1 + D_sQ_1 + \cdots)$$

to obtain

$$\psi'' - Q_0(\psi_0 - x\varsigma_0) = 0$$

$$Q_0^2\varsigma_0'' + x\psi_0'' = 0 \tag{9}$$

$$\psi_1'' - Q_0(\psi_1 - x\varsigma_1) = Q_1\psi_0''$$

$$Q_0^2\varsigma_1'' + x\psi_1'' = 2Q_1x\psi_0'' - \varsigma_0 \tag{10}$$

Our main concern will be with Eqs.(9). Dropping the zero subscript, we have to solve

$$\psi'' = Q(\psi - x\varsigma) \tag{11}$$

$$-Q^2\varsigma'' = x\psi'' \tag{12}$$

Equations (11) and (12) are equivalent to the assumption that $D_s = 0$; thus, to the leading order, the approximate boundary conditions which the solutions of (11) and (12) should blend are [compare with (4)]

$$\psi_+ \equiv \psi(x \to \infty) = A^* + B_+^*x, \quad \psi_- \equiv \psi(x \to -\infty) = A^* + B_-^*x \tag{13a}$$

$$\varsigma_+ \equiv \varsigma(x \to \infty) = B_+^* + A^*/x, \quad \varsigma_- = \varsigma(x \to -\infty) = B_-^* + A^*/x \tag{13b}$$

This is the asymptotic form of the outer solution expressed in inner variables. It is easy to see that Eqs.(11) and (12) have a solution which for large $|x|$ has the same functional behavior as boundary conditions (13). Thus, what is needed is to match the numerical coefficients.

It is customary to introduce

$$\Delta = \lim_{x \to \infty} \left[\frac{\psi'(x)}{\psi - x\psi'} \right]_{-x}^{+x} \text{where} \quad [f]_\alpha^\beta = f(\beta) - f(\alpha), \tag{14a}$$

as a measure of the logarithmic jump across the layer. Let Δ' be the same quantity in terms of the normalized (with respect to a) outer variables. Then using $\partial_x = \epsilon \partial_r$ and (13a) we have

$$\epsilon \Delta' = \frac{B_+^* - B_-^*}{A^*} \tag{14b}$$

If matching is to be accomplished, they have to be equal; i.e.,

$$\epsilon \Delta' = \Delta \tag{14c}$$

Condition (14c) is also sufficient. This may be seen as follows. First, since Eqs.(11) and (12) are homogeneous, without loss of generality, we divide Eqs.(11), (12) and (13) by A^*. Then note that while Eqs. (11) and (12) are invariant under the one-parameter group of transformations

$$\Gamma_a: \quad \begin{array}{c} \psi \to \psi + ax \\ \varsigma \to \varsigma + a \end{array} \tag{15}$$

boundary conditions (13) are not. Choose $a = B_-$, then Eqs.(11) and (12) remain unchanged but the boundary conditions change to

$$\psi_- = 1, \quad \psi_+ = 1 + \epsilon \Delta' x \tag{13c}$$

$$\varsigma_- = x^{-1}, \quad \varsigma_+ = x^{-1} + \epsilon \Delta'. \tag{13d}$$

Therefore, Δ appears to be the only relevant quantity from the layer needed for the matching. For a given Δ' from the "outer world" (14c) thus becomes the matching condition.

We turn to solve Eqs.(11) and (12). Eq.(12) is integrable, yielding

$$-Q^2\varsigma' = x\psi' - \psi + A^* \tag{16}$$

The choice of constant is dictated by (13). Eq.(16) is also invariant under Γ_α. This, using the usual arguments of similarity theory, means that in terms of a new dependent variable invariant under Γ_α, the order of Eqs.(11) and (12) will be further reduced. This will result in a second order equation. Among the possible choices, note

$$\begin{aligned} &(a) \quad X \equiv x\psi' - \psi \\ &(b) \quad Y \equiv \psi - x\varsigma \\ &(c) \quad Z \equiv \varsigma' \end{aligned} \tag{17}$$

all invariant under Γ_α. In Appendix C, relations between the various variables and the resulting equations are described. Here, in order to relate to a previous work [3,4,5] we choose X in terms of which the problem to be solved is

$$Q(X'' - \frac{2}{x}X') - (Q^2 + x^2)X = x^2A^* \tag{18}$$

or, in terms of $s = x^2/Q^{\frac{1}{2}}$

$$4s\frac{d^2X}{ds^2} - 2\frac{dX}{ds} - (Q^{\frac{3}{2}} + s)X = A^*.$$

In Appendix B, a solution valid for $0 < Q \neq 1$ is constructed. It reads

$$\frac{X}{A_*} = \frac{1}{Q^{\frac{3}{2}} - 1} - \frac{Q^{\frac{3}{2}}}{Q^{\frac{3}{2}} - 1}G(s) \tag{19}$$

where

$$G(s) = \frac{1}{\sqrt{2}}\int_0^1 dy y^\omega \frac{d}{dy}\left\{(1+y)^{\frac{1}{2}} \exp\left[-s\frac{1-y}{2(1+y)}\right]\right\}, \tag{20}$$

and $4\omega = Q^{\frac{3}{2}} - 1$. If (and only if!) $Q^{\frac{3}{2}} > 1$, G may be integrated by parts, yielding

$$\frac{X}{A_*} = -1 + \frac{Q^{\frac{3}{2}}}{2^{\frac{5}{2}}} \int_0^1 dy(1+y)^{\frac{1}{2}} y^{\omega-1} \exp\left[-s\frac{1-y}{2(1+y)}\right] \tag{21}$$

One notes that properties of the solutions change dramatically according to whether $Q \lesseqgtr 1$. For $Q = 1$, which is a transitional case, Equation (18) has only a homogeneous solution,

$$X = \exp(-x^2/2) \tag{22}$$

Thus $Q = 1$ corresponds to $A_* = 0$. As one notes from (13), this, according to Newcomb [15], corresponds to a (ideally) magnetohydrodynamically marginally stable state. More about it will be said shortly.

To obtain the dispersion relation in terms of $X = x\psi' - \psi$, one uses $X' = x\psi''$ and writes (14a)

$$\Delta = \frac{2}{A_*} \int_0^\infty \frac{dX}{dx} \frac{dx}{x} \tag{23}$$

which, using (14b), (19) and (20), yields

$$\begin{aligned} \Delta' &= \frac{2Q^{\frac{5}{4}}}{\epsilon(Q^{\frac{3}{2}} - 1)} \int_0^\infty G'(s) \frac{ds}{\sqrt{s}} \\ &= \frac{2\pi Q^{\frac{5}{4}}}{\epsilon(1 - Q^3)} \frac{\Gamma[\frac{1}{4}(Q^{\frac{3}{2}} + 3)]}{\Gamma[\frac{1}{4}(Q^{\frac{3}{2}} + 1)]} \end{aligned} \tag{24}$$

While at first sight (24) indicates the possibility of two branches according to whether $Q \lesseqgtr 1$, with the exception of the $m = 1$ internal kink mode, free energy to drive instability is available only for $\Delta' > 0$.This energy in the zero-β case comes from the poloidal magnetic field out of layer [cf Ref. 16]. Thus, the upper branch must be rejected and the spectrum of possible modes is $0 < Q < 1$. This point is further illuminated if one recalls that the $m \neq 1$ kink modes are magnetohydrodynamic (MHD) stable. Therefore, for these modes the MHD marginal state wherein $A^* = 0$ is unattainable. Since for $A^* = 0$ the solution of (18) yields $Q = 1$, consequently $Q < 1$.

On the other hand, the cylindrical $m = 1$ kink mode is MHD unstable. Thus, the $Q > 1$ branch describes the resistive modifications (or rather, enhancement) of this mode. Similarly, $Q = 1$ corresponds to a resistive modification of the MHD marginally stable mode, while the $Q < 1$ branch, being MHD stable, describes a purely resistive mode. Though the instability in the upper branch may be formally described in terms of Δ', it is actually driven by finite β corrections [3,4,5,16].

The $m = 1$ mode was treated in Refs. 3,4,5 and 6. In Ref. 3, the main results were stated and their derivation was presented in Refs. 4 and 5. These derivations, however, suffer from shortcomings which preclude their use to $m \neq 1$ modes. In Ref. 4, the derivation is achieved using Laguerre polynomials and is, mathematically speaking, incorrect. When corrected the domain of validity is like that of the integral representation derived in Ref. 5, namely, it is limited to the upper $Q > 1$ branch, and, therefore, it is inapplicable to $m \neq 1$ modes which exist in the lower $Q < 1$ branch. In fact, the $Q < 1$ branch of the $m = 1$ mode was analyzed in Refs. 5 and 6 only in the $Q \ll 1$ limit using the const-ψ procedure. Thus, at least formally, previous studies of the $m = 1$ mode left an unanalyzed "window" $0 < \beta \ll Q < 1$ of possible growth rates.

The use of representation (19) derived in Appendix B, enables a uniform treatment of both $m = 1$ and $m \neq 1$ modes, with distinction between the various azimuthal numbers being needed only in the upper, $Q > 1$, branch. Further difference in the structure of the various modes is lumped into Δ', which carries all the necessary information from the "outer world" required to solve the eigenvalue problem.

For small Q ($\ll 1$), we recognize the classical tearing limit $\Delta' \simeq Q^{\frac{5}{4}}$, but whenever $Q \simeq | D_s |^{\frac{2}{3}}$,we should resort to the scaling of class II which describes appropriately this range of growth rates. Indeed, the appropriate counterpart of Eq.(24) in region II is [14]

$$\Delta' = \frac{2\pi Q^{\frac{5}{4}}}{\epsilon} \frac{\Gamma[\frac{1}{4}(3 - D_s/Q^{\frac{3}{2}})]}{\Gamma[\frac{1}{4}(1 - D_s/Q^{\frac{3}{2}})]} \tag{25}$$

and shows that for low values of Q, (24) blends into (25) expanded in high values of Q, but as Δ' is increased,Q in (24) diverges significantly from (25). From the other end, take $|D_S| \ll 1$ and expand (25) in *large parameter* $y \equiv |D_S|/Q^{\frac{3}{2}} \gg 1$ to enter the territory of marginal modes considered next.

III. Modes of Class III

Slowly growing modes are the ones most affected by finite β [2]. Therefore, in the second part of this section we incorporate the effects of finite pressure. But first we consider

A. Zero-β Modes

Exploiting $Q^{\frac{3}{2}} \ll \mid D_s \mid$ we neglect inertia in Eq.(5) to obtain

$$\varsigma = \frac{Qx}{Qx^2 - D_s}\psi \tag{26}$$

$$(Qx^2 - D_s)\psi'' + D_s Q\psi = 0 \tag{27}$$

Equation (27) is solvable explicitly in terms of the hypergeometric function. Before doing that consider first the constant-ψ approximation, that if $\mid D_s \mid \ll 1$ is certainly valid for almost marginal modes. This yields immediately

$$\Delta' = \epsilon^{-1}\pi(-D_s Q)^{\frac{1}{2}} \tag{28}$$

Thus for a given negative D_s we have $Q = -(\epsilon\Delta')^2/\pi^2 D_s$. This should be compared with modes of class II that scale like $Q \simeq \Delta'^{\frac{4}{5}}$ and thus for a given $\Delta' < 1$ predict a somehow bigger growth rate. However, this difference disappears if one formally expands (25) in the large parameter $\mid D_s \mid /Q^{\frac{3}{2}}$ (and disregards the basic assumption that for modes in class II, $\mid D_s \mid \simeq Q^{\frac{3}{2}}$).

Relation (28) very clearly displays that once the magnitudes of Q and D_s are dissociated, pressure gradients are important for the almost marginal tearing modes. In this sense, the cylindrical case is basically different from the slab model.

The negligible role of inertia for the tearing mode wherein $D_s < 0$ is of some interest as it renders the perturbed momentum equation into an equation of radial quasi-equilibrium with the flow (i.e., ς) being merely a "static" response to a time dependent diffusion of the magnetic flux. This raises the possibility that a very weak resistive instability may evolve toward a diffusive non-linear regime without intervention of inertia and thus the process described corresponds to a localized start-up of a global plasma diffusion. This should be contrasted with the idea due to Rutherford that resistive instability invokes a non-linear response that acts to cancel inertia. We claim that a different scenario of resistive evolution is possible, namely: a weakly unstable mode which does not explode may transit into a diffusion without ever being a "genuine" fluidic instability. In such a picture the role of the non-linear interaction is to slow down the growth rate to the point that no distinction can be made between the slowly evolving equilibrium and its perturbation.

Note that if $D_S > 0$, the perturbed pressure cannot counterbalance Lorentz force and there is always a sublayer in which hydrodynamic effects are essential.

Finally, let us solve (27) directly. This allows one to address the conditions in which not only $Q^{\frac{3}{2}} \ll | D_S |$, but D_S may formally be arbitrary. Although for zero-β modes this situation is somewhat academic, nevertheless, as we shall see, the results are of some interest. To this end, it is convenient to define a new variable Ω in terms of which

$$\frac{d\Omega}{dx} = \frac{\psi}{Qx^2 - D_S}, \quad \varsigma = xQ\frac{d\Omega}{dx}. \tag{29}$$

and Ω satisfies

$$[(x^2 + d)\Omega']' - dQ\Omega = 0, \quad d \equiv -D_S/Q \tag{30}$$

and its solution is expressible in terms of Legendre functions P_ν, Q_ν. The proper combination of solutions is dictated by the outer boundary conditions which when written in inner coordinate x read:

$$\psi_-(-\infty) = \epsilon^{-\nu} A_0 \mid x \mid^{-\nu} + B_- \mid x \mid^{\nu+1} \epsilon^{\nu+1} \equiv A^* \mid x \mid^{-\nu} + B^*_- \mid x \mid^{\nu+1}$$

$$\psi(+\infty) = A_0 \mid x \mid^{-\nu} + B_+ \mid x \mid^{\nu+1} \equiv A^* \mid x \mid^{-\nu} + B^*_+ \mid x \mid^{\nu+1}$$

Thus, if one represents ψ such that for large $\mid x \mid$

$$\psi = S_0 \mid x \mid^{-\nu} + (S_1 + \hat{A}_0 Sgn x) \mid x \mid^{\nu+1}, \quad \mid x \mid \to \infty \tag{31}$$

with S_0, S_1, and $\hat{A}_0$ being constants, matching is achieved, provided that

$$\epsilon^{2\nu+1}\Delta' = \Delta \tag{32}$$

where

$$\Delta' = \frac{B_- + B_+}{A_0} = \frac{B^*_- + B^*_+}{\epsilon^{2\nu+1}A^*} \tag{33a}$$

and

$$\Delta = \frac{2S_1}{S_0}. \tag{33b}$$

Exploiting the asymptotic properties of Legendre functions, P_ν, Q_ν, the symmetric part needed, [see (23)], is obtained using

$$S_\nu = P_\nu(z) - P_\nu(-z), \quad z = i\frac{x}{\sqrt{d}}$$

while for the antisymmetric part we define

$$A_\nu = R_\nu(z) + R_\nu(-z)$$

where

$$R_\nu = P_\nu - \pi^{-1}\tan(\pi\nu)Q_\nu$$

and note that $A_\nu(x \to \infty) \simeq \hat{A}_0 sgn x \mid x \mid^{\nu+1}$. In terms of S_ν and A_ν the solution is $\psi = aS_\nu + A_\nu$, with a being a constant relating S_0 to A^*. Since a appears both in S_0 and S_1 it drops out in calculation of Δ. Evaluating S_1/S_0 we obtain

$$\Delta' = \frac{(-D_S 4Q)^{\nu+\frac{1}{2}}}{\epsilon^{2\nu+1}}\left(\frac{1+2\nu}{1+\nu}\right)\left[\frac{\Gamma(\nu+\frac{1}{2})}{\sqrt{\pi}\Gamma(\nu+1)}\right]^2\left[1-\tan^2\left(\frac{\pi\nu}{2}\right)\right] \tag{34}$$

For $\nu \ll 1$ (and thus $\nu \simeq -D_S$) we recover (3.3). Now consider a given Δ' and increase $-D_S$, then the $Q(-D_S)$ curve decreases at first, *has a minimum* for some $0 < \nu < \frac{1}{2}$ and goes to infinity at $\nu = \frac{1}{2}$. Note also the change of scaling of Q versus Δ' as D_S is varied. Thus, an increase in $-D_S$ is not an indefinitely stabilizing process. At certain, impractically high, point this trend reverses and the growth rate starts to increase with $-D_S$. This indicates that the behavior of marginal high-β modes may differ considerably from their low-β counterparts.

B. Finite-β Modes

Consider now the resistive equations in the layer in which the terms due to small but finite-β are retained. The perturbed parallel component of the field $\mathcal{Y}$ now plays an active role. To proceed as in part A of this section recall that $D_S = 0(\beta)$ and assume

$$0 \leq Q \ll \beta \ll 1. \tag{35}$$

Consequently, to first order we assume $\psi'' \simeq 0$ and $\varsigma'' \simeq 0.$ in Eqs.(A14) and (A15). This yields an algebraic relation between ς and ψ ($= \psi_0 =$ const.) Using this in (A16) we have for $\mid x \mid \ll 1$ $\mathcal{Y} \simeq x$ thus $\mathcal{Y}'' \simeq 0$ for $\mid x \mid \ll 1$. Also, as $\mid x \mid \rightarrow \infty$ $\mathcal{Y}'' \rightarrow 0$. We assume therefore that $\mathcal{Y}'' \simeq 0$ everywhere, an assumption which may be inadequate for $x = O(1)$. Thus, all variables are determined algebraically, particularly

$$\varsigma \simeq \varsigma_0 = \frac{D_s + Q(Q^2 l_1 + x^2)}{Qx^2(Q^2 l_1 + x^2) - D_S Q^2 l_0} x\psi_0 \tag{36}$$

where $l_0 = l_1 - S/D_S, l_1 = 1 + 1/(\Gamma\beta)$. Note that the structure of ς changes according to whether $\beta \gtrless Q^{\frac{3}{2}}$ Note that as in the zero-β theory if $D_S > 0$, ς explodes in the layer, thus calling for restoration of inertia or viscosity. Thus, as before, $D_S < 0$ will be assumed. Correction to first order yields

$$\psi_0^{-1}\psi_1'' = \frac{-D_S(x^2 + Q^2 l_0)}{x^2(Q^2 l_1 + x^2) - D_S Q l_0} = \mathfrak{J} \tag{37}$$

leading to the usual dispersion relation

$$\epsilon\Delta' = \int_{-\infty}^{\infty} \mathfrak{J}(x)dx.$$

To calculate the integral we consult Ref. (17). Thus

$$\Delta' = \frac{-D_S\pi}{\sqrt{2}(\mathcal{L}+1)(-D_S l_0)^{\frac{3}{4}}}[Q^{\frac{5}{4}} + \sqrt{-D_S l_0}Q^{-\frac{1}{4}}] \tag{38}$$

Since $\mathcal{L} \equiv \frac{Q^{\frac{3}{2}} l_1}{2\sqrt{-D_S}} \ll 1$ it may be neglected, yielding within a factor of unity the main effect of finite-β theory [2] which is to set a minimal bound $\Delta_c > 0$ on Δ at which instability may set on. Examining $\mathfrak{J}$ one observes two parts, one of which disappears in the zero-βlimit. Indeed, fix x and let $l_0 \rightarrow \infty (l_1/l_0 = O(1))$ then

$$\mathfrak{z} \to -\frac{D_S Q}{Qx^2 - D_S}$$

which yields the zero-β dispersion (28). On the other hand, the $Q^{-\frac{1}{4}}$ branch survives any $\beta > 0$. Thus the $\Delta'(Q)|_{\beta=0} \neq \Delta'(Q)|_{\beta\to 0}$. This sensitivity of stability diagram at low-β indicates that in consideration of the almost marginal modes, not only finite compression but an *additional hydrodynamic* mechanism like viscosity should be included.

With the more conservative taste in mind we redo the problem more carefully. Recall that the domain of interest is $0 \leq Q \ll \beta \ll 1$. This allows one to disregard the first two terms on the right–hand side of Eq.(A16). Since $Q \ll 1$ we may assume $\psi \simeq \psi_0 = \text{const}$. Define a new complex variable

$$\Omega = \mathfrak{Y} + i a_1 Q^{\frac{3}{2}} \varsigma \tag{39}$$

in terms of which the combined Eqs.(A15) and (A16) read

$$Q^2\Omega'' - Qx^2\Omega + a_2 Q^{\frac{3}{2}}\Omega = -a_3 x \psi_0 Q \tag{40}$$

where $a_1 = \sqrt{(-l_0/D_S)}, a_2 = iD_S a_1, a_3 = 1 + ia_1 Q^{\frac{3}{2}}$. The problem, mathematically speaking, is thus reduced to the one encountered for modes of class II in the zero-β theory. If, like in part A of this section, the "inertial term" $Q^2\Omega$ is neglected in (40), we recover the result of (28). Due to the frequent recurrence of Eq.(40) in the resistive theory, it is worthwhile to obtain a convenient expression for its solution. This is done in Appendix D. We have

$$\Omega = \frac{a_3}{2\sqrt{Q}} x \int_0^1 dt[t(2-t)]^{-\frac{1}{4}} \left(\frac{2-t}{t}\right)^{a_2} \exp\left[-\frac{x^2}{2\sqrt{Q}}(1-t)\right]$$

and

$$\varsigma = \frac{1}{a_1 Q^{\frac{3}{2}}} Im(\Omega), \quad \mathfrak{Y} = Re(\Omega)$$

The dispersion relation obtained as usual via $\epsilon\Delta' = Q\int_{-\infty}^{\infty}(\psi_0 - x\varsigma)dx$ reveals the presence of $Q^{-\frac{1}{4}}$ factor as predicted by (38).

IV. Some Drift Effects

The Ohm's law considered in this section is

$$\eta_{\|}J_{\|} + \eta_{\perp}J_{\perp} = E + (v_e \times B)/e + \nabla p_e/en + (0.71/e)\nabla_{\|}T_e \tag{41}$$

where e, i refers to electrons (ions), the η's are electrical resistivities and the symbols $\|$ and $\perp$ indicate the components parallel and perpendicular to the magnetic field B, respectively. In (41) we have neglected electron inertia and electron stress tensor. Also, the total pressure $p = p_e + p_i$, and $J = en(v_i - v_e)$ is the total current density.

Since we distinguish between electrons and ions, separate entropy equations are needed. We shall assume electrons and ions to behave adiabatically. For electrons, in high temperatures, adiabaticity, though consistent, is a poor assumption. But in the singular layer it is equally inadequate to assume isothermal behavior along the field lines as done in some of the previous works [9,10,11]. While away from the layer high temperature does indeed imply an isothermal behavior along the field lines, the degeneration of $B^0 \cdot \nabla$ in the layer impedes diffusion and necessitates employment of a large part of the whole electron transport equation. This in turn renders the problem analytically intractable at the present time.

We present an exact solution for modes of class I (i.e., $\beta \ll Re(Q^{\frac{3}{2}})$) which, like in the purely resistive case, may have a higher growth rate than allowed by the const-ψ theory. Our treatment admittedly is not sufficient for high temperature plasmas, but within stated limitations, it permits , using the same formalism as in Sec. II, a coherent treatment of drift effects on the tearing mode. In Refs. 5 and 6, the impact of drift on the $m = 1$ mode was considered with a similar shortcoming like in the purely resistive case. As before, our main interest is in the $m \neq 1$ modes.

With slight notational changes, we employ the equations used in Refs. 5, 6, and 9. The relevant boundary layer equations read

$$-Q(Q + iQ_*^i)\varsigma'' = x\psi'' \tag{42a}$$

$$\psi'' = (Q - iQ_*^e)(\psi - x\varsigma) \tag{42b}$$

where $Q_*^{i(e)}$ is the ion (electron) diamagnetic frequency normalized with respect to $t_{//}$ and calculations are carried in the frame of reference in which $E_r = 0$. In this frame the equilibrium ion and electron diamagnetic velocity is free from the $E \times |B^0|$ drift component and Q refers to the Doppler shifted frequency, i.e., $Q \rightarrow Q + k_z(cE_r/B_\theta)$.

Equations (42) may be solved in the same way as in Sec. II. Define $X = x\psi' - \psi$, and $\tau = x\Omega_1^{\frac{1}{2}}$. Then X satisfies

$$\frac{d^2X}{d\tau^2} - \frac{2}{\tau}\frac{dX}{d\tau} - (\tau^2 + \Omega^{\frac{3}{2}})X = A_*\tau^2 \tag{43}$$

where

$$\Omega_1^2 = (Q - iQ_*^e)/[Q(Q + iQ_*^i)] \tag{44a}$$

$$\Omega^3 = Q(Q - iQ_*^e)(Q + iQ_*^i) \tag{44b}$$

Consulting Appendix B, we write the solution of (43) as

$$\frac{X}{A_*} = -\frac{1}{1-\Omega^{\frac{3}{2}}} + \frac{2^{-\frac{1}{2}}\Omega^{\frac{3}{2}}}{1-\Omega^{\frac{3}{2}}}\int_0^1 dy y^\sigma \frac{d}{dy}\left\{(1+y)^{\frac{1}{2}}\exp\left[-\frac{\Omega_1 x^2(1-y)}{2(1+y)}\right]\right\} \tag{45}$$

and $4\sigma = \Omega^{\frac{3}{2}} - 1$. In addition, we have to require that

$$Re[\Omega_1] > 0 \tag{46}$$

to ensure that X behaves properly at infinity. Dispersion relation is obtained as in Sec. II [see Eq.(23)]. We have

$$\Delta' = \frac{2\pi\Omega^{\frac{3}{2}}\Omega_1^{\frac{1}{2}}\Gamma\left[\frac{1}{4}(\Omega^{\frac{3}{2}} + 3)\right]}{\epsilon(1-\Omega^3)\Gamma\left[\frac{1}{4}(\Omega^{\frac{3}{2}} + 1)\right]} \tag{47}$$

In our search for complex roots of (47) we must confine our interest only to those branches that satisfy $\Delta' > 0$ because irrespective of the refined physical description used in the layer, the reservoir of free energy to drive the instability remains unchanged, namely, the poloidal magnetic field. It is easy to see that unless both $|Q_*^i| \ll 1$ and $|Q_*^e| \ll 1$ the bounds of the applicability of the const-ψ procedure become more severe in the presence of drift. Indeed, in the purely resistive case, if one limits himself to $Q \ll 1$, dispersion relation (24) to first order coincides with the exact result (25) expanded in small Q. In the presence of drift, to apply const-ψ procedure, one must require that $|Q - iQ_*^e| \ll 1$. This, if $1 \ll Q_*^e$, necessitates that one limits himself to cases wherein $ImQ \simeq Q_*^e$. But even when this condition is satisfied, unless $|\Omega^3| \ll 1$ holds as well, dispersion relation (47) will differ considerably from the one obtained using const-ψ procedure. However, unless both $|Q_*^e| \ll 1$ and $Q_*^i \ll 1$, the real part of $|\Omega^3|$ may not be small. In fact, assume that $Q \simeq Q_R + iQ_*^e$ with $Q_R \ll 1$. Then $Re\Omega^3 \simeq Q_R(Q_*^i + Q_*^e)Q_*^e$, which for large drift frequencies is certainly a non-negligible quantity.

In Figs. 1 and 2 a numerical solution of (47) is displayed with $Re(Q)$ and $Im(Q)$ calculated in terms of Δ and parametric variations in Q_*^i and $\alpha \equiv Q_*^e/Q_*^i$. The parameters used are within the range of present day experiments. One observes that for a fixed α ($\alpha = 1$, in Fig. 1 and $\alpha = 2$ in Fig. 2), a jump in Q_*^i causes an almost parallel shift of the $\Delta(ReQ)$ curve, everywhere but in the vicinity of the origin. Evidently, in relative terms a given drift parameter has a larger stabilizing effect the smaller Δ is. Conversely, for a very large Δ (not displayed) the various $\Delta(ReQ)$ curves are almost indistinguishable. While our results indicate that the drift has its largest impact for the small growth rates, one cannot make a definite quantitative statement based on the presented calculations. This is so not only because of the adiabatic treatment of electrons, but firstly because the employed boundary layer Eqs.(42) correspond to modes of class I and thus are unsuitable for a proper treatment of modes of class II or III. For almost marginal modes to be treated properly, finite β effects have to be included in a similar fashion to Sec. III. This problem is in progress.

V. Summary

Using $\beta Q^{-3/2}$, where Q is the normalized growth rate with respect to the resistive time scale, as an expansion parameter, an asymptotic theory of tearing modes in a resistive medium was constructed. The classical results of previous workers as a rule correspond to modes which satisfy $\beta Q^{-3/2} = 0(1)$. In these works a low-β limit was assumed and constant-ψ procedure was employed. However, modes with growth rates which are either much smaller or much bigger than $\beta^{2/3}$ necessitate a different approach which was employed in this work. Modes with $Q \gg \beta^{2/3}$ (modes of class I) were analyzed taking the appropriate boundary layer equations in their zero-β limit. The resulting equations were completely integrated and the corresponding dispersion relation calculated. Now one may have growth rates which extend from the classical tearing time scale to the resistive time scale with a cut-off at $Q = 1$. $Q = 1$ corresponds to $\Delta' = \infty$ and thus necessitates an infinite poloidal magnetic energy to drive such a mode. Luckily, tokamak experiments indicate a much smaller Δ' than one needed to drive tearing modes with such a large growth rate. Study of drift effects on tearing modes with resistive growth rates reveals that drift has only a modestly stabilizing effect.

For tearing modes with almost marginal growth rates while on one hand effects of inertia are of no importance on the other hand finite-β effects together with the effects of curvature play a dominant role. Recalling that in slab geometry the parallel components of perturbations are decoupled from their perpendicular counterpart, we conclude that insofar as the almost marginal modes are concerned, tearing modes in cylindrical and slab geometries are essentially different. And this is so in spite of the fact that the boundary layer is a thin zone. The only factor that was left out and which possibly can modify our conclusion is the (classical) viscosity.

The stability diagram (Figs. 1 and 2) indicates that drift effects have their largest impact in the marginal domain. But since this diagram corresponds to modes of class I ($|Q| \gg \beta^{2/3}$) a quantitative statement is impossible. For this purpose finite-β effects have to be included. This work is in progress.

Although as a rule we have excluded the treatment of interchange (or localized) modes we mention that for these modes, inertia plays an essential role. This is exhibited in the most transparent fashion for the almost marginal modes. For such class of growth rates while inertia is negligible if one considers tearing modes, it plays an essential role for interchange modes ($D_s > 0$). In fact, neglecting of inertia will cause the eigenfunction to explode in the center of the layer [see Eqs.(26) and (29)]. In the case of interchange modes there always exists an inertial sublayer embedded in the resistive layer. Alternatively, viscosity may replace inertia as the dominant factor is the sublayer. But either viscosity or inertia play an essential role in shaping the eigenfunction.

Because the cylindrical $m = 1$ mode is unstable to (ideal) magnetohydrodynamic perturbations it is customary to separate the treatment of $m = 1$ and $m \neq 1$ modes. However, for tearing modes this distinction is artificial because as several recently done calculations reveal, toroidal correction stabilizes the ideal magnetohydrodynamic $m = 1$ kink mode. Thus, one is not interested in a resistive modification of the MHD-unstable mode, a subject addressed many times in the past, but rather in destabilization by tearing of an otherwise MHD stable equilibrium. As described in Sec. II, this aspect of the $m = 1$ mode was not sufficiently well addressed in previous works. The presented description is valid for an arbitrary azimuthal m number. The distinction between various modes is manifested, among other things, in a different location of the singular surface and different value of Δ'.

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Appendix A Equations of Motion

We summarize [2,14] the derivation of the resistive boundary layer equations. The full resistive equations are [14]

$$\partial_t \rho + \nabla \cdot \rho \boldsymbol{v} = S_M$$

$$\partial_t \rho \boldsymbol{v} + \nabla \cdot \rho \boldsymbol{v}\boldsymbol{v} + \nabla p - \boldsymbol{J} \times \boldsymbol{B} = S_p$$

$$\partial_t \boldsymbol{B} = \nabla \times (\boldsymbol{v} \times \boldsymbol{B} - \eta \boldsymbol{J}) \quad (A1)$$

$$\partial_t (p/\rho^{\Gamma-1}) + \nabla \times (\boldsymbol{v} p/\rho^{\Gamma-1}) = \frac{\Gamma - 1}{\rho^{\Gamma-1}}(\eta J^2 + S_E)$$

$$\boldsymbol{J} = \nabla \times \boldsymbol{B} \qquad \nabla \cdot \boldsymbol{B} = 0$$

These equations are derived assuming that Ohm's law is

$$\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B} = \eta J \quad (A2)$$

Here, and elsewhere in this paper, it was assumed that η is a scalar constant.

The equilibrium is assumed to be quasi-static, i.e., on the considered time scale $\boldsymbol{v} \simeq \partial_t \boldsymbol{\varsigma}$, $\quad \boldsymbol{B} = \boldsymbol{B}^\circ + \boldsymbol{b}$, where $\boldsymbol{\varsigma}$ is the perturbed displacement vector assumed in the considered cylindrical symmetry to be given by

$$\boldsymbol{\varsigma} = \boldsymbol{\varsigma}(r) \exp(\gamma t + im\theta - inkz) \quad (A3)$$

with $k = \frac{2\pi}{R} \ll 1$.

In terms of $\boldsymbol{\varsigma}$ and $\boldsymbol{b}$ the perturbed equations read

$$\rho_0 \gamma^2 \boldsymbol{\varsigma} = (\nabla \times \boldsymbol{b}) \times \boldsymbol{B}^\circ + \boldsymbol{J}^\circ \times \boldsymbol{b} + \nabla[\Gamma \rho^\circ (\nabla \cdot \boldsymbol{\varsigma}) + \boldsymbol{\varsigma} \cdot \nabla P^\circ] \quad (A4)$$

$$b = \frac{\eta}{\gamma}\nabla \times \nabla \times b + \nabla \times (\varsigma \times B^\circ) \tag{A5}$$

Everywhere but on the rational surface defined via $F(r) = 0$ where

$$F(r) \equiv \frac{B^\circ_\theta}{r}(m - nq) = 0, \quad B^\circ_\theta(r_s) \neq 0 \tag{A6}$$

and $q \equiv krB_z/B_\theta$ is the safety factor, small η and γ may be neglected. In the vicinity of the rational surface (A6) this becomes a singular perturbation manifested in a formation of a boundary layer at which both inertia and resistivity are the driving mechanisms. It is assumed that at r_s, $F'(r) \neq 0$ thus

$$B^\circ \cdot \nabla = -\frac{B^\circ_\theta}{r_s} n(r - r_s)q' + O(r - r_s)^2 \tag{A7}$$

The appropriate scaling for the variables in the layer is derived from (A4) and (A5) requiring

$$\rho_0\gamma^2 \simeq (B^\circ \cdot \nabla)(B^\circ \cdot \nabla) \simeq (F')^2(r - r_s)^2 \tag{A8}$$

and

$$\eta/\gamma \simeq \epsilon^{-2} \tag{A9}$$

where ϵ is the characteristic width of the layer. Thus

$$(r - r_s) \simeq \gamma \simeq \eta^{\frac{1}{3}} \simeq \epsilon \tag{A10}$$

Using these estimates we introduce the following dimensionless quantities: on the large domain, a as the minor radius and $t_H = \frac{a\sqrt{\rho_0}}{|B^\circ|}$ as the Alfven transit time; in the layer, the diffusive time t_D

$$t_D = \left(\frac{\rho_0 r_s^2}{\eta q'^2 B^{\circ 2}_\theta}\right)^{\frac{1}{3}} \tag{A11}$$

and the diffusive length $L_D = \sqrt{\eta t_D}$. Thus the small parameter ϵ is

$$\epsilon = \frac{L_D}{a} = \frac{t_H}{t_D} \qquad (A12))$$

and the dimensionless length x:$r - r_s = L_D x = \epsilon a x$. Finally, the normalized growth rate Q is $\gamma = Q/t_D = \epsilon Q/t_H$.

The boundary layer equations may be expressed in terms of b_r, ς_r and $\boldsymbol{b} \cdot \boldsymbol{B}^\circ$ which are normalized as follows

$$\varsigma = U_r/\gamma, \quad \psi = \frac{i r_s}{L_D q' B_\theta^\circ} b_r, \quad \Upsilon = \frac{b_\theta B_\theta^\circ + b_z B_z^\circ}{|B^\circ| P'} \tag{A13}$$

Using a well known procedure [2,14] the boundary layer equations expressed in terms of the triplet $(\varsigma, \psi, \Upsilon)$ read

$$\psi'' = Q(\psi - x\varsigma) \tag{A14}$$

$$Q^2 \varsigma'' = Q x^2 \varsigma - Q x \psi - D_S \Upsilon \tag{A15}$$

$$\Upsilon'' = Q\left(1 - \frac{2}{\Gamma\beta} + x^2/Q^2\right)\Upsilon + Q\left(\frac{s}{D_S} - 1 - \frac{2}{\Gamma\beta}\right)\varsigma - \frac{x}{Q}\psi \tag{A16}$$

where $s \equiv 4k^2/q'^2$ and $\beta \equiv \frac{2P}{(B^\circ)^2}$ is evaluated at r_s and D_s is given below equation (4).

The special zero-β limit considered in the text in obtained as $\beta \to 0$ but it is assumed that $D_s \simeq \beta' \neq 0$. In this limit from (A16)

$$\Upsilon = \varsigma$$

which used in (A15) yields Eqs.(1) and (2).

As for R_2, taking into consideration that σ may be negative $(-1 < \sigma)$ we multiply (B11) by $y^{\sigma-1}$ and integrate between $(\epsilon, 1)$, $\quad 0 < \epsilon \ll 1$;

$$\sum_{n=0}^{\infty} \frac{(-1)^n L_n^{\alpha}(s) y^{n+\sigma}}{n+\sigma} \Big|_{\epsilon}^{1} = \int_{\epsilon}^{1} y^{\sigma-1}(1+y)^{-\delta} \exp\left(\frac{ys}{1+y}\right) dy$$

$$= \frac{y^{\sigma}}{\sigma}(1+y)^{-\delta} \exp\left(\frac{ys}{1+y}\right) \Big|_{\epsilon}^{1} - \int_{\epsilon}^{1} dy \frac{y^{\sigma}}{\sigma} \frac{d}{dy}\left[(1+y)^{-\delta} \exp\left(\frac{ys}{1+y}\right)\right] \quad (B13)$$

The two diverging terms in ϵ in (B13) cancel each other, indeed, fix any s, then

$$\frac{\epsilon^{\sigma}}{\sigma} L_0^{\alpha} - \frac{\epsilon^{\sigma}}{\sigma}(1+\epsilon)^{-\delta} \exp\left(\frac{\epsilon s}{1+\epsilon}\right) = O(\epsilon^{1+\sigma}) \quad (B14)$$

All other ϵ-dependent terms vanish with ϵ. Thus, taking $\epsilon \to 0$ we obtain

$$\sum_{n=0}^{\infty} \frac{(-1)^n L_n^{\alpha}(s)}{n+\sigma} = \frac{2^{-\delta}}{\sigma} \exp\left(\frac{s}{2}\right) - \frac{1}{\sigma} \int_0^1 dy y^{\sigma} \frac{d}{dy}\left[(1+y)^{-\delta} \exp\left(\frac{ys}{1+y}\right)\right] \quad (B15)$$

Thus

$$R_2 = \frac{-\omega}{\omega-1} + \frac{2^{\delta}\omega}{\omega-1} \int_0^1 dy y^{\sigma} \frac{d}{dy}\left[(1+y)^{-\delta} \exp\left(-s\frac{1-y}{2(1+y)}\right)\right]$$

and finally,

$$\hat{Z} = \frac{-1}{\omega-1} + \frac{2^{\delta}\omega}{\omega-1} \int_0^1 dy y^{\sigma} \frac{d}{dy}\left[(1+y)^{-\delta} \exp\left(-x^2\frac{1-y}{2(1+y)}\right)\right]. \quad (B16)$$

$\hat{Z}$ is an integral representation of the solution to (B2′), which is a regular function of δ. Thus, we may choose $\delta \equiv a+1 = -0.5$ to obtain the needed representation of (B2):

$$Z = \frac{1}{1-\omega}\left(1 - 2^{\frac{1}{2}}\omega \int_0^1 dy y^{\sigma} \frac{d}{dy}\left\{(1+y)^{0.5} \exp\left[-s\frac{1-y}{2(1+y)}\right]\right\}\right). \quad (B17)$$

If $\omega > 1$, (B17) may be integrated by parts to yield

$$Z = 1 - 2^{-\frac{5}{2}}\omega \int_0^1 dy y^{\sigma-1}(1+y)^{\frac{1}{2}} \exp\left[-x^2\frac{1-y}{2(1+y)}\right]. \quad (B18)$$

As a closing remark, we note that in Refs. 4 and 5, representation (B18), which is valid only for $\omega > 1$, was derived. In Ref. 4, this was done using directly an integral representation while in reference 5, generalized Laguerre polynomials $L_n^{-\frac{3}{2}}(s)$ were employed. The derivation in the latter case, even though limited to $\omega > 1$, is, strictly mathematically speaking, incorrect because $\left\{L_n^{-\frac{3}{2}}\right\}$ cannot form a complete orthogonal set on $(0, \infty)$: the needed weight function $x^{-\frac{3}{2}}e^{-x}$ has a non-integrable singularity at the origin.

Appendix C Some Properties of Alternative State Variables

In the bulk of the paper we have employed $\underline{X} \equiv x\psi' - \psi$ as the dependent variable in terms of which modes of class I were expressed. As mentioned there, $\underline{X}$ is not the only choice and moreover, not necessarily the most convenient one when high temperature effects or more generalized Ohm's law are considered. Here we shall review some of the other candidates.

First note that in addition to X, Y, Z, defined in (17), the invariance property (15) renders second order differential equation for ψ. This is so because if (ψ_0, ς_0) is a solution pair, so is $(\psi_1, \varsigma_1) = (\psi_0 + ax, \varsigma_0 + a)$. Take $\psi_0 = \varsigma_0 = o$. Thus $(\psi_1, \varsigma_1) = (ax, a)$ is a particular integral of equations of motion. Indeed, eliminating ς from (11), (12) and (16) we find that ψ satisfies

$$Qx\psi''' - Q\psi'' - x(Q^2 + x^2)\psi' + (Q^2 + x^2)\psi = x^2 A^* \qquad (C1)$$

and has x as a particular solution of the homogeneous part. Define

$$\phi = \psi/x \quad \text{and} \quad V = \frac{d\phi}{dx} \qquad (C2)$$

to find that

$$Qx^2 V'' + 2xQV' - [2Q + x^2(Q^2 + x^2)]V = x^2 A^* \qquad (C3)$$

is the needed second order differential equation. Solution of the homogeneous is expressible in terms of the confluent hypergeometric functions. This may be used to construct the Green function in terms of which the solution of the inhomogeneous equation is readily stated.

Now consider the choice of $Y \equiv \psi - x\varsigma$ as a state variable. Y is related to the parallel component of the electric field. Using (11) and (12) Y is found to satisfy

$$QY''' = 4xY + (Q^2 + x^2)Y' \qquad (C4)$$

Dispersion relation (14) expressed in terms of Y reads

$$\epsilon\Delta' = 2Q.\int_0^\infty Y dx/[Y(0) - \int_0^\infty xY dx] \qquad (C5)$$

Note that Y is subject to the natural boundary conditions

$$Y(0) = 1, \quad Y'(0) = 0 \tag{C6}$$

in addition to

$$Y(\infty) = 0 \tag{C7}$$

This completes statement of the problem in terms of Y.

Further simplification arises either using (16) or noting that (C4) has $(x^2 + Q^2)$ as an integrating factor. This allows one to integrate once Y to obtain

$$Q(Q^2 + x^2)\left[\frac{y'}{(x^2 + Q^2)}\right]' + 2QY - (Q^2 + x^2)Y = C_0 \tag{C8}$$

where C_0 is a constant. Alternatively, introduce

$$y = \int_{-\infty}^{\infty} Y(x)e^{ix\mu}d\mu$$

to find

$$\frac{d^2y}{dS^2} - \frac{2}{S}\frac{dy}{dS} - (S^2 + Q^{\frac{3}{2}}) = \frac{\delta(S)}{Q^{\frac{1}{4}}} \tag{C9}$$

where $S \equiv \mu Q^{\frac{1}{4}}$.Thus, the homogeneous part of (C9) is exactly the one satisfied by $\underline{X}$ in the configuration space [see Eq. (18)]. Solution of (C9) is facilitated by expanding it in Laguerre polynomials. Of course, since $\underline{X}' = QxY$ and $\underline{X}$ has been found explicitly, Y may be considered known. But our aim here is to show the viability of alternative approaches, like, say, the one that uses Y as a prime variable. In fact, in Appendix D of their paper [1], Furth *et al.*, recognizing the limitation of the const-ψ approximation, solve *numerically* the problem of a symmetric sheet-pinch. For that purpose, they employ a new dependent variable which essentially is equal to Y and which satisfies an equation analogous to (C4) with dispersion relation (C5) and boundary conditions (C6) and (C7).

Irrespective of whether equation (C4) with (C6) and (C7) is addressed analytically or numerically, it should have a proper solution for each value of Q. Before presenting their numerical solution, Furth *et al.* conjecture this to be the case, which is plausible because Y has two well behaved solutions at infinity. Since we have solved for Y we know that this conjecture is correct. Alternatively, this conjecture may be demonstrated directly as follows:

Consider the space of thrice smoothly differentiable functions over $\Re^1_+$, which in addition satisfy (C6) and (C7). Let $(Y_1, Y_2) = \int_o^\infty Y_1 Y_2 dx$ define the inner product. Then, integrating by part we show that $L \equiv -Q\partial_x^3 + Qx + (Q^2 + x^2)\partial_x$ satisfies $(Y_1, LY_2) = (LY_1, Y_2)$ and thus this is a self-adjoint operator. Next, we claim that if to $Q = Q_0$ corresponds a solution function Y_0 then in its ϵ vicinity, $0 < \epsilon \ll 1$, to every $Q_0 + \epsilon$ corresponds a solution function Y_ϵ. This property is shown by perturbation; let $Q = Q_0 + \epsilon Q_1$ and $Y = Y_0 + \epsilon Y_1$. Then to first order

$$L(x, Q_0)Y_1 = Q_1[2Y_0' - Y_0'''] \qquad (C10)$$

and therefore,

$$(Y_0, L(x; Q_0)Y_1) = Q_1(Y_0, 2Y_0' - Y_0'''). \qquad (C11)$$

Integrating by parts, we have $(Y_0, 2Y_0' - Y_0''') = 0$. Since $LY_0 = 0$,(C11) is identically satisfied and thus the perturbed Q_1 may be arbitrarily chosen. To each choice of Q_1, (C10) yields appropriate Y_1. Since Q_0 is arbitrarily chosen, using ϵ-patching, we may cover the whole Q domain provided that at least one pair $\{Q_0, Y_0\}$ exists which satisfies (C4), (C6), and (C7). This is afforded by $Q_0 = 1$ and $Y_0 = \exp(-x^2/2)$ and we are done.

Appendix D Solution of Equation for Radial Displacement

In dealing with modes of class II, one has to solve an equation for the perturbed radial displacement ς, which is of the form

$$\frac{d^2Z}{d\mathring{x}^2}+(b_1-b_0\mathring{x}^2)Z=b_2\mathring{x} \tag{D1}$$

where a_i are constants (possibly complex).

Define

$$C_0\equiv b_1b_0^{-\frac{1}{2}}$$

$$C_1\equiv b_2b_0^{-\frac{3}{4}} \tag{D2}$$

$$x\equiv b_0^{\frac{1}{4}}\mathring{x}$$

To obtain the standard form

$$\frac{d^2Z}{dx^2}+(C_0-x^2)Z=C_1x \tag{D3}$$

for both sides of (D3) to have even symmetry, use $y=z/x$. Further define

$$\begin{aligned}\mu&=x^2/2\\ M(\mu)&=\tfrac{Z}{x}\exp\mu\end{aligned} \tag{D4}$$

in terms of which

$$2\mu\frac{d^2M}{d\mu^2}+(3-4\mu)M'+(C_0-3)M=C_1e^{\mu}$$

The solution of M is obtained using the integral representation

$$M(\mu)=\int_{\Omega}N(t)\exp(\mu t)dt \tag{D5}$$

where $N(t)$ and the contour Ω are determined by the auxiliary conditions

$$(2t^2 - 4t)N\exp(\mu t)|_\Omega = C_1 \exp\mu \tag{D6}$$

and

$$\frac{\partial}{\partial t}(2t^2 - 4t)N = (C_0 - 3 + 3t)N. \tag{D7}$$

From (D7)

$$N(t) = N_0(2-t)^{\frac{C_0-1}{4}} t^{-\frac{(1+C_0)}{4}}, \quad N_0 = \text{const} \tag{D8}$$

Using (D8) we find that (D6) can be satisfied if: (1) the contour Ω is chosen to lie on the real axis between $t = 0$ and $t = 1$, and (2)

$$Re[C_0] < 3. \tag{D9}$$

Thus

$$M = -\frac{C_1}{2}\int_0^1 dt[(2-t)^{\frac{C_0-1}{4}} t^{-(\frac{1+C_0}{4})}]e^{\mu t}$$

and, with $\tau = 1 - t$

$$Z = -\frac{C_1}{2}x\int_0^1 dt(1-t^2)^{-\frac{1}{4}}(\frac{1+t}{1-t})^{\frac{C_0}{4}}\exp(-\frac{x^2}{2}t) \tag{D10}$$

The simplicity of representation (D10) of the solution should be compared with its representation by Hermite polynomials used in earlier works.

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Figure Captions

Fig.1 Normalized growth rate Q as a function of Δ. In Fig. (1a) the real part is displayed while in (1b) the imaginary part is given. Four values of Q_*^i, the normalized ions drift, are considered: $Q_*^i = 0.00, 0.04, 0.10,$ and 0.20. Electrons drift parameter $a = 1$. ($a \equiv Q_*^e/Q_*^i$). Note that an increase in Q_*^i has a stabilizing effect which, in relative terms, is most pronounced for a weakly unstable tearing mode.

Fig.2 Same as in Fig. (1) but here $a = 2$. Note that increase in a has a stabilizing effect; it causes, for a given Δ, a decrease in ReQ but it also increases the oscillatory part of the mode.

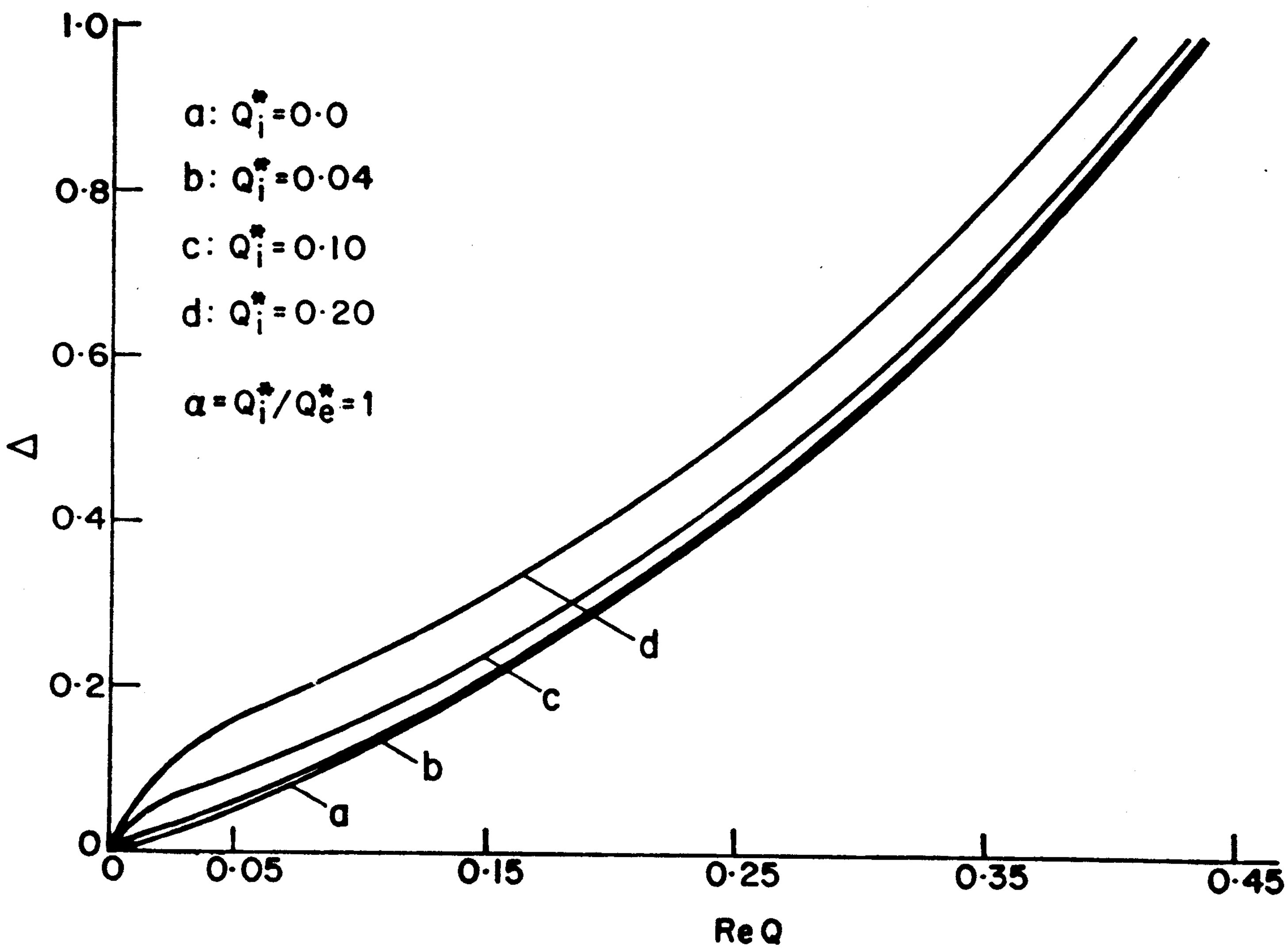

1·0
0·8
0·6
0·4
0·2
0
Δ
a: $Q_i^* = 0·0$
b: $Q_i^* = 0·04$
c: $Q_i^* = 0·10$
d: $Q_i^* = 0·20$
$\alpha = Q_i^*/Q_e^* = 1$
a
b
c
d
0
0·05
0·15
0·25
0·35
0·45
Re Q

Fig. 1a.

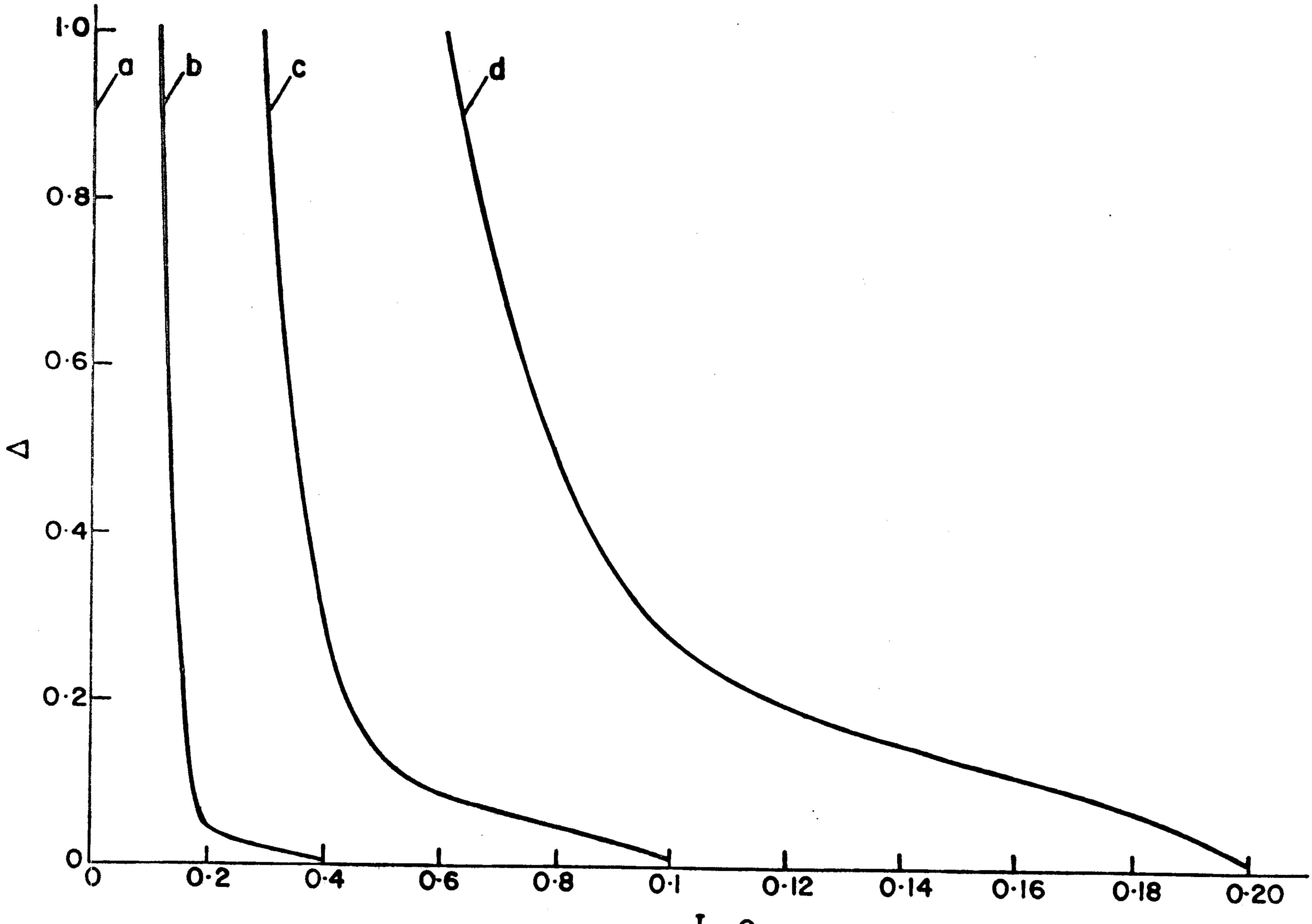
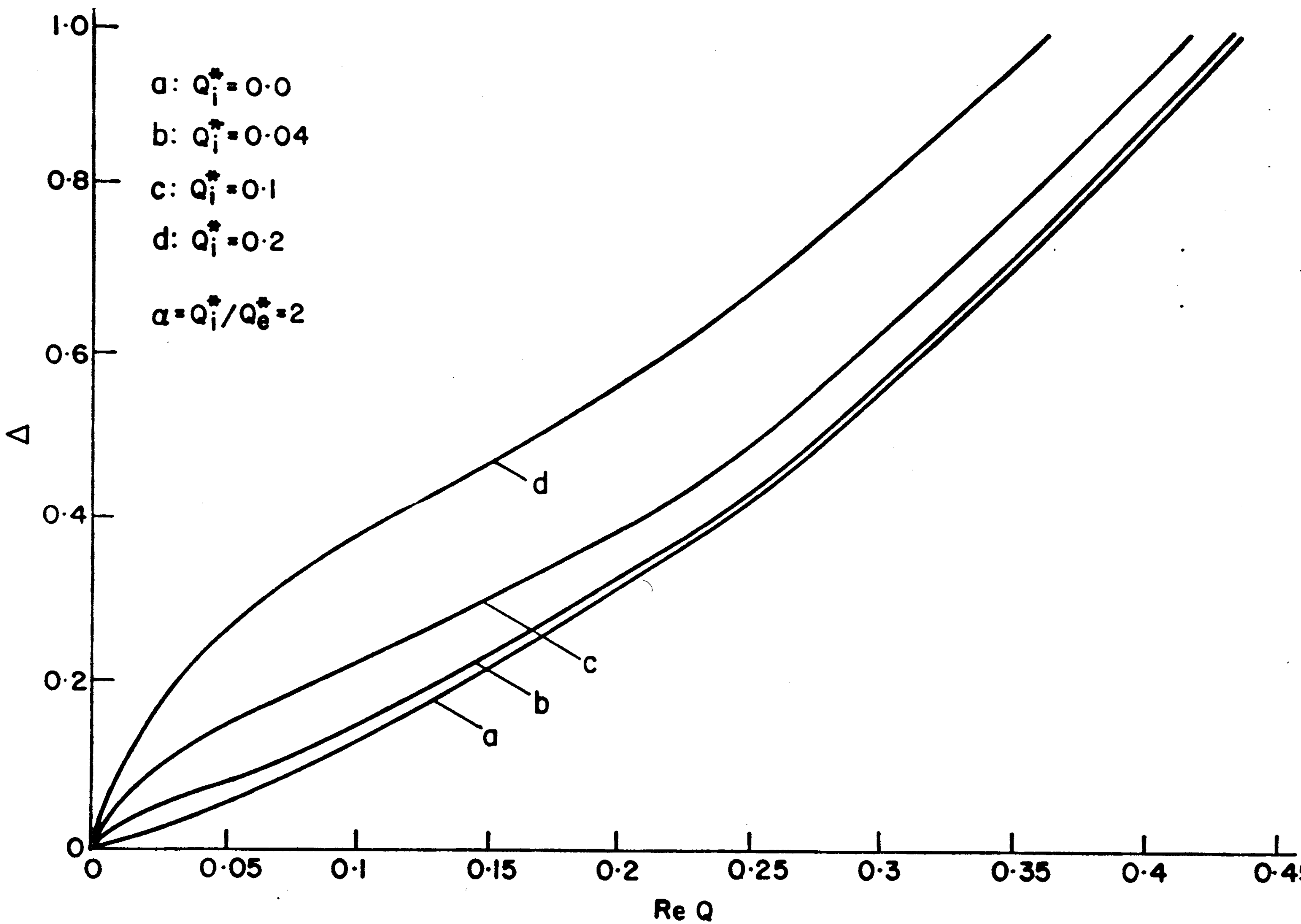


Fig. 1b.



a: $Q_i^* = 0{\cdot}0$
b: $Q_i^* = 0{\cdot}04$
c: $Q_i^* = 0{\cdot}1$
d: $Q_i^* = 0{\cdot}2$
$\alpha = Q_i^*/Q_e^* = 2$
Δ
1·0
0·8
0·6
0·4
0·2
0
0·05
0·1
0·15
0·2
0·25
0·3
0·35
0·4
0·45
Re Q
a
b
c
d

Fig. 2a.

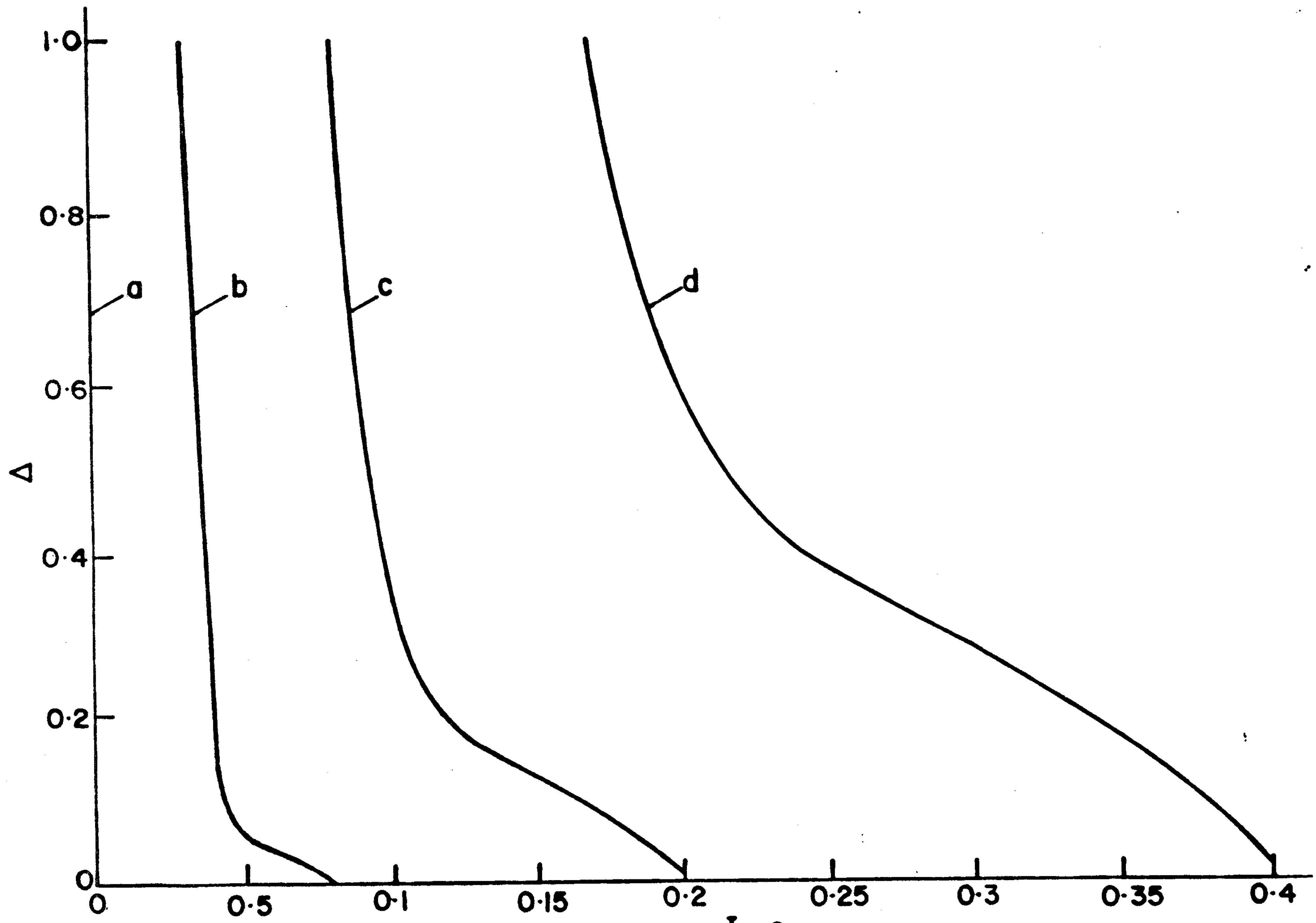


Fig. 2b.