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A Kinetic Wave Expansion for the Plasma Fields Generated by an External Coil in a Slab Geometry

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Abstract

A solution is obtained for the plasma fields generated by an external line current in a slab geometry. The form of the solution is a summation over the various plasma waves which are defined by the hot plasma equivalent dielectric tensor. The excitation level of each plasma wave is determined by a set of boundary conditions which are derived by integrating the differential form of the equivalent dielectric tensor through the plasma-vacuum boundary layer. The boundary conditions result in a convergent plasma wave expansion.

I. Introduction

Wave propagation in a warm magnetoplasma can be described by formulating an equivalent dielectric tensor[1] which replaces the relative permitivity in Maxwell's equations. The equivalent dielectric tensor is derived by solving the linearized Vlasov equation for the plasma response due to a small electromagnetic perturbation about an equilibrium distribution function. In its usual form, a plane wave ($exp(ik_x x + ik_z z - i\omega t)$) is assumed for the EM perturbation, and the determinant of the dielectric tensor defines the dispersion relation for the various plasma modes. In many low frequency RF heating applications, the frequency (ω) and the parallel wave number (k_z) can be assumed to be fixed; and the dispersion relation can be cast into the form of a complex polynominal in k_x^2 of infinite order defining a likewise infinite set of plasma waves. For an external source such as an RF antenna, it is of interest to determine the excitation level of each of the plasma waves at the plasma-vacuum interface and hence the resultant field solution.

By introducing a sharp boundary to the plasma, the plane wave formulation of the dielectric tensor is no longer valid in the boundary layer. The inherent inhomogeneity of the plasma must be considered *a priori*. This can be accomplished by following the derivation of the equivalent dielectric tensor, however instead of assuming a plane wave perturbation for the fields, a Taylor series expansion is used [2]. The derivation results in a differential form for the equivalent dielectric tensor where $-k_x^2$ is replaced by d^2/dx^2 in which the differential operates on the product of the dielectric element and the field component. In its differential form, the equivalent dielectric tensor incorporated into Maxwell's equations can be integrated across the boundary layer to determine a set of boundary conditions for the plasma-vacuum interface. By requiring the electromagnetic fields to be finite and possess finite derivatives to all orders (i.e. the Taylor series expansion is valid in the boundary layer), an infinite set of boundary conditions can be derived. These boundary conditions uniquely determine the excitation level of each plasma wave in the equivalent dielectric formulation.

In the next section, the equivalent dielectric tensor for wave propagation in a uniform plasma is defined, and the differential form of this tensor is derived. In section 3, the differential form of the equivalent dielectric tensor is used to obtain boundary conditions at a plasma-vacuum interface. In section 4, linear wave propagation in a magnetoplasma slab is investigated for a current line source excitation. In this analysis the zero electron inertia approximation is made, and wave propagation and the excitation level for the fast wave and the infinite set of generalized Bernstein modes is examined. In the final section, a short summary and discussion of results is presented.

II. Derivation of the Differential Form of the Equivalent Dielectric Tensor

The equivalent dielectric tensor formulation can be used to describe linear plane wave propagation $exp(i(\omega t + k_x x + k_z z))$ in a homogeneous magnetoplasma [1]. Using the equivalent dielectric tensor in Maxwell's equations, the following three relations are obtained for the components of the electric field.

$$\begin{bmatrix} K_{xx} - k_z^2 & -iK_{xy} & k_x k_z(1 + K_{xz}) \\ iK_{xy} & K_{xx} + K_{yy} - k_z^2 - k_x^2 & ik_x k_z K_{yz} \\ k_x k_z(1 + K_{xz}) & -ik_x k_z K_{yz} & K_{zz} - k_x^2 \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix} = 0 \qquad (2.1)$$

In Eq. 2.1 k_x and k_z are the wave numbers, respectively, perpendicular and parallel to the confining magnetic field; and K_{ij} are the equivalent dielectric elements of the plasma which are normalized by the square of the free space wave number. The dielectric elements are defined in appendix A. We are interested in problems where ω and k_z are assumed to be fixed, and the usual procedure in solving Eq. 2.1 is to expand each dielectric element in a power series in $\lambda = k_x^2\rho_i^2/2$. The expansion is useful if the parameter λ is small, otherwise the Bessel function form of the dielectric element must be used. Appendix B contains the power series expansion of all of the dielectric elements to second order in k_x^2, and a further expansion of the K_{xx} element to fourth order in $k_x^2 = k_x^2\rho_i^2/2$ and an expression for the general term of the K_{xx} element. The notation is an extension of the cold plasma notation of Stix [1] to clearly distinguish dielectric elements in the power series expansion from the notation of the complete dielectric element.

The determinant of the equivalent dielectric tensor defines the dispersion relation for wave propagation. If we examine the cold plasma dispersion relation (only retain S, D, P in the expansion of Appendix B), we have a quadratic dispersion relation in k_x^2 defining two plasma waves. Expanding each dielectric element to first order in k_x^2 or ρ_i^2, adds one additional root to the dispersion relation resulting in a cubic in k_x^2. However, expanding each dielectric element to second order in ρ_i^2, adds three additional roots to the dispersion relation, and the process continues with three additional roots being present as the order of a dielectric element is increased by one. We note here that the boundary conditions to be derived exhibit the same structure as the expansion in k_x^2.

The equivalent dielectric tensor contained in Eq. 2.1 was derived by solving the linearized Vlasov equation for the perturbed distribution function. For an isotropic plasma, the perturbed distribution function has the form,

$$f_1(\vec{r},\vec{v},t) = \frac{-q}{m}\int_{-\infty}^{t} \vec{E}_1(r',t')\cdot\frac{\partial f_o(\vec{v}')}{\partial\vec{v}'}dt' \tag{2.2}$$

where the subscript one indicates perturbed quantities, and the integration is to be carried out along the unperturbed trajectories of the particles defined by $\vec{r}'$ and $\vec{v}'$. To illustrate the calculation of the dielectric elements, consider the K_{xx} element. This element is obtained by calculating the x-component of the plasma current density due to the x-component of the electric field. This response has the form,

$$(J_x)_x = \frac{2q^2n}{m}e^{ik_zz-i\omega t}\int_{-\infty}^{\infty} d\hat{v}\int_{o}^{\infty} dt v_x v_x' E_x(x')\frac{f_o(x',v')}{v_i^2(x')}e^{-ik_zv_zt+i\omega t} \tag{2.3}$$

where $v_i^2 = 2\kappa T(x')/m_i$, a plane wave dependence of the form $exp(ik_zz - i\omega t)$ for E_x was assumed with the x-dependence as yet unspecified, and the distribution function was assumed to be isotropic. If we assume,

$$E_x(x') = E_x^o e^{ik_xx'} \tag{2.4}$$

and that f_o/v_i^2 is independent of x, the plane wave form of the equivalent dielectric tensor is obtained. In Eq. 2.4, the prime on x indicates the unperturbed trajectory of a particle which for a uniform magnetic field has the form:

$$x' = x + \Delta, \quad \Delta = \frac{-v_x}{\omega_c}\sin\omega_ct + \frac{v_y}{\omega_c}(1-\cos\omega_ct) \tag{2.5}$$

where Δ is the projection on the x-axis of the particles gyro orbit. The origin of the Bessel function form of the plane wave dielectric elements can be readily observed by inserting

Eq. 2.5 into Eq. 2.4. and expand the exponential of a sine as an infinite power series of Bessel functions. The derivation outlined above is contained in a number of standard texts in plasma physics [1,3].

To derive a differential form for the equivalent dielectric tensor, instead of assuming the plane wave spatial dependence defined by Eq. 2.4, we expand the combination, $E_x f_o / v_i^2$, as a Taylor Series about x in the small parameter Δ [2].

$$E_x(x')\frac{f_o(x')}{v_i^2(x')} \approx E_x(x)\frac{f_o(x)}{v_i^2(x)} + \Delta\left[\frac{E_x f_o}{v_i^2}\right]'_{x'=x} + \frac{\Delta^2}{2}\left[\frac{E_x f_o}{v_i^2}\right]''_{x'=x} + ... \tag{2.6}$$

Note in Eq. 2.6, we have assumed a uniform magnetic field. Second, two additional Taylor series expansions are required with E_x replaced by E_y and E_z. Inserting Eq. 2.6 into 2.3 and performing the time and velocity space integrations, we obtain the differential form of the K_{xx} element.

$$K_{xx} = S - (S_{1x}E_x)'' + (S_{2x}E_x)^{iv} - (S_{3x}E_x)^{vi} + ... \tag{2.7}$$

The superscripts indicate differentiation with respect to x to the order indicated, and the dielectric elements are defined in appendix B. Note that odd derivatives of the Taylor series expansion are identically zero when the velocity space integration is performed. This is true for all elements of the dielectric tensor except for the xz and yz components where even derivatives are zero.

If one assumes an $exp(ik_x x)$ dependence for E_x, and that the S_{ix} elements are independent of position, the plane wave form of the equivalent dielectric tensor defined in appendix B can be obtained from Eq. 2.7. In fact, a comparison between Eq. 2.7 and Eq. B.1 suggests the differential form of the equivalent dielectric tensor can be obtained by the substitution $-\frac{d^2}{dx^2}$ for k_x^2 with the differential operating on the complete term. Using this process, the differential form of all the dielectric elements can be immediately obtained from appendix B. Appendix C contains the differential form of the dielectric tensor incorporated into Maxwell's equations (the magnetic field intensity throughout this paper is normalized by $i\omega\mu_o$). In addition that appendix contains the equivalent dielectric formulation of Maxwell's equations in the $m_e = 0$ approximation.

III. Derivation of Boundary Conditions

In this section, boundary conditions are derived for the electromagnetic fields by integrating Maxwell's equations including the differential form of the dielectric tensor across a vanishingly thin boundary as illustrated in Fig. 1. The governing equations are given by C.1 to C.6 which determine the variation of the fields through the boundary layer ($-\delta < x < \Delta$). In the vacuum region ($x < -\delta$), all the dielectric elements in Eqs. C.4-6 vanish, except $S = P = 1$. For postions $x > \Delta$, the plasma is assumed to be uniform and the dielectric elements can be removed form the differential operator. We directly integrate Eqs. C.1 to 6 across the boundary layer as $\Delta, \delta \to 0$. Terms invloving field components or products of dielectric elements and field components vanish since the length of integration approaches zero. For the terms involving derivatives, the indefinite integral is obtained directly and evaluated at each side of the boundary layer. Designating the vacuum fields by the superscript v, the integration of Eqs. C.1 to 6 yield,

$$E_z^v = E_z \tag{3.1a}$$

$$E_y^v = E_y \tag{3.1b}$$

$$0 = S_{1x}E_x' - S_{2x}E_x''' + \ldots - i(D_1E_y' - D_2E_y''' + \ldots) + ik_z(U_1E_z - U_2E_z'' + \ldots) \tag{3.1c}$$

$$H_z - H_z^v = i(D_1E_x' - D_2E_x''' + \ldots) + S_{1y}E_y' - S_{2y}E_y''' + \ldots - k_z(V_1E_z - V_2E_z'' + \ldots) \tag{3.1d}$$

$$H_y^v - H_y = ik_z(U_1E_x - U_2E_x'' + \ldots) + k_z(V_1E_y - V_2E_y'' + \ldots) + P_1E_z' - P_2E_z''' + \ldots \tag{3.1e}$$

The set of equations defined by 3.1 relate the vacuum fields to the plasma fields: In the cold plasma limit all the subscripted dielectric elements vanish, and the well known continuity of tangential E and H boundary conditions are obtained. For a hot plasma, the tangential components of the electric field remain continuous; however, the electromagnetic perturbation induces surface currents, and the tangential magnetic intensity is no longer continuous.

To observe the form of the boundary conditions in terms of the k_x^2 expansion of the dielectric tensor, consider the expansion to first order in ρ_i^2. Retaining only the dielectric terms subscripted by one, the boundary conditions are:

$$E_z^v = E_z \tag{3.2a}$$

$$E_y^v = E_y \tag{3.2b}$$

$$0 = S_{1x}E_x' - iD_1E_y' + ik_zU_1E_z \tag{3.2c}$$

$$H_z - H_z^v = iD_1E_x' + S_{1y}E_y' - k_zV_1E_z \tag{3.2d}$$

$$H_y^v - H_y = ik_zU_1E_x + k_zV_1E_y + P_1E_z' \tag{3.2e}$$

Consider a uniform plasma half-space occupying the region $x > 0$ and plane waves propagating in the positive x direction. Assuming plane wave fields, $exp(ik_xx)$, the determinant of Eqs. C.4 to C.6 to first order in ρ_i^2 will yield three propagating plasma waves in the positive x-direction. The two incident vacuum waves will excite, at the plasma vacuum boundary, three transmitted plasma waves and two reflected vacuum waves. Exactly five boundary conditions are needed to determine the transmitted and reflected waves and these are given by Eqs. 3.2a-e.

If we continue the expansion to second order in ρ_i^2, retaining the dielectric elements in Eqs. C.4-6 subscripted by two, three new plasma waves appear, and three additional boundary conditions are needed. To obtain the boundary conditions, we set the higher order terms separately equal to zero in Eq. 3.1.

$$S_{2x}E_x''' - iD_2E_y''' + ik_zU_2E_z'' = 0 \tag{3.2f}$$

$$iD_2E_x''' + S_{2y}E_y''' - k_zV_2E_z'' = 0 \tag{3.2g}$$

$$ik_z U_2 E_x'' + k_z V_2 E_y'' + P_2 E_z''' = 0 \quad (3.2h)$$

Extending the process to order ρ_i^{2n} repeats boundary conditions Eqs. 3.2f-h with the subscript 2 being replaced by the subscripts $3 \ldots n$. For the plasma half-space problem, this process results in $3n + 2$ boundary conditions that are required for the $3n$ plasmas waves and two unknown vacuum waves.

The procedure outlined above is intuitive and was developed in the context of the k_x^2 expansion of the dielectric elements. The only constraint we have on the plasma and vacuum fields, derived directly from Maxwell's equations, is Eq. 3.1. To separately set various pairs of terms equal to zero is somewhat arbitrary. In this light, an alternative set of boundary conditions can be derived which satisfy Eq. 3.1 identically to all orders in the expansion. The first five boundary conditions are given by Eq. 3.1. The higher order boundary conditions are derived by successively subtracting the boundary conditions of given order in Eq. 3.2 from Eq. 3.1. We have,

$$j = 2, \ldots n \quad , \quad k = 2j - 1$$

$$S_{jx} E_x^k - \ldots - i(D_j E_y^k - \ldots) + ik_z(U_j E_z^{k-1} - \ldots) = 0 \quad (3.1f)$$

$$i(D_j E_x^k - \ldots) + S_{jy} E_y^k - \ldots - k_z(V_j E_z^{k-1} - \ldots) = 0 \quad (3.1g)$$

$$ik_z(U_j E_x^{k-1} - \ldots) + k_z(V_j E_y^{k-1} - \ldots) + P_j E_z^k - \ldots = 0 \quad (3.1h)$$

In the next section, we will show that the above boundary conditions (Eqs. 3.1) lead to a convergent wave expansion such that the total field solution inside the plasma can be calculated to any desired degree of accuracy. This result justifies the procedure for developing the boundary conditions, since inside the plasma the solution is exact and through the boundary layer the solution identically satisfies the only known constraint on the fields (Eqs. 3.1a-e).

It is not surprising that the boundary conditions place constraints on the derivatives of the electric field. One anticipates such conditions in order to justify the Taylor series expansion of Eq. 2.6. Starting with Maxwell's equations, we required the fields to be finite in the boundary layer and integrated Eqs. C.1-6 to obtain the boundary conditions of Eqs. 3.1a-e. To derive the next higher order boundary condition, we impose the requirement that both the fields and the first and second order derivatives of the Taylor series expansion are finite in the boundary layer. These terms (subscripted by one) have an upper-bound in the boundary layer and integrate to zero as the spatial width (δ, Δ) of the layer shrinks to zero. We obtain boundary conditions Eqs. 3.1f-h with $j = 2$. By requiring successively higher order terms in the Taylor series expansion to be finite in the boundary layer, integration of Maxwell's equations through the layer yields the remaining boundary conditions of Eqs. 3.1f-h with $j = 3, 4 \ldots$. Thus, the boundary conditions of Eq. 3.1 can be derived by requiring the existence of the Taylor series expansion that describes the plasma response. Physically, this corresponds to the requirement that the plasma currents are finite in the boundary layer.

IV. Line Current Excitation of a Plasma Slab

In this section, the plasma fields excited by an external coil are calculated for the slab geometry illustrated in Fig. 2. The slab is assumed to have a uniform density confined by a uniform magnetic field. The zero electron mass approximation is made. In this approximation, $E_z \equiv 0$ or equivalently, $P \to \infty$. Electron currents along the magnetic field are not calculable, and the H_y boundary conditions cannot be formulated. The plasma fields satisfy Faraday's law and Eqs. C.7 and 8.

Two methods of developing a perturbation expansion in the ion gyro radius are presented. The first method is a straight forward expansion of each dielectric element to a specified order, say n in the ion gyro radius. The dispersion relation is then a product of the dielectric elements yielding $2n$ polynominal in k_x^2. In the second perturbation expansion, the complete (Bessel function form) dielectric element is used, and the roots are obtained interatively. To obtain an expansion to order n, the n roots closest to the origin in the complex k_x^2 plane are retained in the calculation. We refer to this procedure as the exact dispersive expansion. In the straight forward expansion, we use the set of boundary conditions defined in Eq. 3.2, and Eq. 3.1 is used in the exact dispersive expansion. In both cases, the E_z and H_y boundary conditions are omitted, and $E_z \equiv 0$ consistent with the zero electron mass approximation.

We outline the derivation of a straight forward perturbation expansion to first order in ρ_i in which an approximate solution to the boundary value problem illustrated in Fig. 2 is constructed. To first order in ρ_i the plasma fields satisfy,

$$(S - k_z^2)E_x - S_{1x}E_x'' - iDE_y + iD_1E_y'' = 0 \tag{4.1a}$$

$$iDE_x - iD_1E_x'' + (S - k_z^2)E_y + (1 - S_{1y})E_y'' = 0 \tag{4.1b}$$

The two second order differential equations will yield four plane wave roots, and a solution of the following form is assumed.

$$E_y(k_z) = E_3e^{ik_fx} + E_4e^{-ik_fx} + a_b(E_5e^{ik_bx} + E_6e^{-ik_bx}) \tag{4.2a}$$

$$E_x(k_z) = a_f(E_3e^{ik_fx} + E_4e^{-k_fx}) + E_5e^{ik_bx} + E_6e^{-ik_bx} \tag{4.2b}$$

The polarization factors a_b and a_f are defined by Eqs. D.1a and b of Appendix D. The wave numbers k_f and k_b refer to the smaller root (fast wave) and the larger root (ion Bernstein wave) of the quadratic dispersion relation defined by Eq. D.2.

The y-component of the vacuum electric field has the form,

$$E_y^v(x > 0) = E_2e^{-k_zx} \qquad x > 0 \tag{4.3a}$$

$$E_y^v(x < 0) = E_1e^{k_zx} \qquad x < 0 \tag{4.3b}$$

The vacuum displacement current has been neglected in Eq. 4.3. The boundary conditions at $x = \pm a$ are:

$$-S_{1x}E_x' + iD_1E_y' = 0 \tag{4.4a}$$

$$-iD_1E_x' + (1 - S_{1y})E_y' - E_y^{v\prime} = \pm i\omega\mu_o J_y \tag{4.4b}$$

$$E_y - E_y^v = \pm M_z \tag{4.4c}$$

In Eq. 4.4, the line current is repiaced by surface magnetization and electric currents at the plasma edge via the induction theorem [4]. Solving the set of Eqs. 4.1 to 4.4, the amplitudes of the fast wave modes are given by Eqs. D.3 and 4, and the amplitude of the ion Bernstein wave is given by Eqs. D.5 and 6 in terms of the fast wave amplitudes. The dispersion relation for the bounded system is given in Eq. D.7. Equations 4.2 and D.1-10 determine the field solution for a single k_z component of the Fourier spectrum of the coil. The total field solution is given by the inverse Fourier transform,

$$E_{x,y}(z) = \frac{1}{\pi} \int_0^\infty E_{x,y}(k_z) \cos(k_z z) dk_z \tag{4.5}$$

where it has been observed from Ap. D that E_x and E_y are even functions of k_z and consequently z.

It is of interest to examine the ion Bernstein wave amplitudes defined by Eqs. D.5, 6 and 10. These equations have the form,

$$E_5 \quad and \quad E_6 \sim \frac{k_f}{k_b} f(E_3, E_4) \tag{4.6}$$

The excitation level of the ion Bernstein wave is proportional to the product of a function of the fast wave amplitudes times the ratio of the wavelengths of the two modes. This result is a direct consequence of the boundary condition defined in Eq. 4.4a. The wavelength mismatch determining the excitation level of the ion Bernstein mode is suggestive of a coupling process where Eq. 4.4a determines the power transfer from the fast to the ion Bernstein mode in the boundary layer. Note near $2\omega_{ci}$, $|k_b| \approx |k_f|$, and the ion Bernstein wave strongly couples to the fast wave [5]. Also as the plasma temperature approaches zero, $k_b \sim \rho_i^{-1}$, the excitation level of the ion Bernstein wave approaches zero.

In the above, the field solution of a line current excitation of a plasma slab was outlined for a pertubation expansion to first order in the ion radius. The field solution has also been obtained to zero order in ρ_i and to second order in ρ_i. The three approximations will be denoted by $\rho_i = 0, 1, 2$ and we only briefly outline the $\rho_i = 0$ and 2 solution which closely follows the ρ_i expansion. In the ρ_i solution, only the fast wave is present with the field solution given by the first two terms in Eq. 4.2. The dispersion relation for the fast wave in this approximation is given by Eq. D.11 and only two boundary conditions are needed to obtain the field solution (Eq. 4.4b and c with $D_1 = S_{1y} = 0$). For the $\rho_i = 2$, the field solution is given by Eq. 4.2 with two additional ion Bernstein waves present. The dispersion relation in this case is a quartic in k_x^2 (Eq. D.12), and two boundary conditions are needed in addition to those in Eq. 4.4. These boundary conditions are supplied by Eqs. 3.2f and g with $E_z \equiv 0$.

Numical results of the three approximations are contrasted in Fig. 3. In that figure, the left-hand component of the electric field is plotted as a function of ion temperature for a frequency close to the ion cyclotron frequency ($\omega = 1.01\omega_{ci}$). The assumed plasma parameters and geometry are indicated in the caption, and the field is computed at a position directly under the antenna and a distance of one centimeter into the plasma. The plasma parameters approximate those of a recent ICRF heating experiment in the Phaedrus tandem mirror [6]. It can be immediately observed that the $\rho_i = 1$ and 2 theories are significant corrections to the zero gyro radius theory as the ion temperature is increased. The $\rho_i = 2$ theory appears to smooth out oscillations present in the first order theory which are due to a short wavelength propagating ion Bernstein wave. It appears that the $\rho_i = 1$ and 2 theory are in reasonably close agreement and these approximations may be close to the exact result. However, we next examine the exact form of the dispersion relation, and immediately conclude that this is not the case and the limitations of the straight forward perturbation become clear.

The dispersion relation in the $m_e = 0$ approximation has the form,

$$(K_{xx} - k_z^2)(K_{xx} + K_{yy} - k_z^2 - k_x^2) - K_{xy}^2 = 0 \tag{4.7}$$

where the Bessel function form of the dielectric elements are used as defined by Eqs. A.1-3. This dispersion relation has been solved iteratively for the first eight roots closest to the origin of the k_x^2 plane for the parameters indicated in Table 1. The roots of Eq. 4.7 are compared with the roots obtained in the $\rho_i = 0, 1, 2$ approximations where the dispersion relations are defined by Eqs. D.2, 11, and 12. It is observed that the roots of the dispersion relation are only accurate for the smallest or fast wave root. The origin of the discrepancy for the higher order roots can be seen by examining the electrostatic ion Bernstein dispersion relation which is an approximation to Eq. 4.7 [7].

$$\frac{\omega_{ci}^2}{2\omega_{pi}^2}\lambda = \frac{1}{2}[I_o(\lambda)e^{-\lambda} - 1] + \frac{\omega^2}{\omega_{ci}^2}\Sigma_{n=1}^{\infty}\frac{I_n(\lambda)e^{-\lambda}}{(\omega^2/\omega_{ci}^2 - n^2)} \tag{4.8}$$

Near $\omega = \omega_{ci}$, the $n = 1$ term is dominant in the summation, and the roots of the dispersion relation are approximated by,

$$I_1(\lambda) = 0, \qquad \lambda = \pm i j_{1,n} \tag{4.9a}$$

where $j_{1,n}$ is the n^{th} zero of the regular Bessel function of the first kind. The square of the wave number is inversely proportional to temperature.

$$k_x^2 = \pm \frac{2i}{\rho_i^2} j_{1,n} \tag{4.9b}$$

The roots of the exact dispersion relation are defined by the zeros of the modified Bessel function which is oscillatory along the imaginary axis with an asymtotic expansion which is the sum of sines and cosines. A power series expansion has a limited range in reproducing this oscillatory behavior. For example to obtain an accurate representation of K_{xx} to six significant figures for $\lambda = 8$, approximately 50 terms are needed in the power series expansion. Thus, the power series expansions are only accurate in calculating the lowest order, the fast wave root. We note that inside mode conversion zones, where the ion Bernstein root crosses the fast wave root, a power series expansion is accurate in calculating both roots [5].

To accurately represent the dispersion characteristics of the plasma modes the Bessel function form of the dispersion relation is used. A perturbation expansion to order n in the ion gyro radius squared can be constructed by retaining the smallest $2n$ roots of the dispersion relation (Eq. 4.7), and then impose the $2n + 2$ boundary conditions of Eqs. 3.1 with the inclusion of the induced surface currents from the antenna. The x and y components of the electric field have the form,

$$E_x = \Sigma_{j=1}^{n} (E_{2j-1} e^{ik_j x} + E_{2j} e^{-ik_j x}) \tag{4.10a}$$

$$E_y = \Sigma_{j=1}^{n} a_j (E_{2j-1} e^{ik_j x} + E_{2j} e^{-ik_j x}) \tag{4.10b}$$

where the wave numbers are defined by Eq. 4.7 and the polarization factor is defined in the following expression.

$$a_j = \frac{-i(K_{xx} - k_z^2)}{K_{xy}} \tag{4.10c}$$

The wave amplitudes can be determined by inserting Eq. 4.10 into the boundary conditions Eq. 3.1 with $U_i = V_i = E_z \equiv 0$ and the following set of $2n$ simultaneous equations result.

$$\Sigma_{j=1}^{n} g_{\ell j} k_j^{-1}(E_{2j-1}e^{ik_j a} - E_{2j}e^{-ik_j a}) = 0 \qquad (4.11a)$$

$$\Sigma_{j-1}^{n} g_{\ell j} k_j^{-1}(E_{2j-1}e^{-ik_j a} - E_{2j}e^{ik_j a}) = 0 \qquad (4.11b)$$

$$\Sigma_{j=1}^{n} h_{\ell j} k_j^{-1}(E_{2j-1}e^{ik_j a} - E_{2j}e^{-ik_j a}) = 0 \qquad (4.12a)$$

$$\Sigma_{j=1}^{n} h_{\ell j} k_j^{-1}(E_{2j-1}e^{-ik_j a} - E_{2j}e^{ik_j a}) = 0 \qquad (4.12b)$$

where the subscript ℓ ranges form $\ell = 1$ to $n/2$. For $\ell = 1$, the general form of Eqs. 4.11 are replaced by the following which includes the excitation coefficients of the antenna (Eq. D.8).

$$\Sigma_{j=1}^{n}\left\{[(-k_z + ik_j)a_j - ig_{\ell j}k_j^{-1}]E_{2j-1}e^{-ik_j a} + [(-k_z - ik_j)a_j + ig_{\ell j}k_j^{-1}]E_{2j}e^{ik_j a}\right\} = 0 \qquad (4.11c)$$

$$\Sigma_{j=1}^{n}\left\{[(-k_z - ik_j)a_j + ig_{\ell j}k_j^{-1}]E_{2j-1}e^{ik_j a} + [(-k_z + ik_j)a_j - ig_{\ell j}k_j^{-1}]E_{2j}e^{-ik_j a}\right\} = E^{+} \qquad (4.11d)$$

The functions $g_{\ell j}$ and $h_{\ell j}$ are defined by,

$$g_{\ell j} = \Sigma_{m=\ell}^{\infty} k_j^{2m}(iD_m + S_{my}a_j) \qquad (4.12a)$$

$$h_{\ell j} = \Sigma_{m=\ell}^{\infty} k_j^{2m}(S_{mx} - iD_m a_j) \qquad (4.12b)$$

or evaluated in terms of the complete dielectric element we have,

$$g_{\ell j} = i(K_{xy} - D) + (K_{xx} + K_{yy} - S)a_j - \Sigma_{m=1}^{\ell-1} k_j^{2m}(iD_x + S_{my}a_j) \qquad (4.13c)$$

$$h_{\ell j} = K_{xx} - S - i(K_{xy} - D)a_j - \Sigma_{m=1}^{\ell-1} k_j^{2m}(S_{mx} - iD_m a_j) \qquad (4.13d)$$

For accurate numerical evaluation of these coefficients, Eqs. 4.13a and are used for $\lambda < .5$, and the latter expressions for larger values of λ. The set of $2n$ simultaneous equations defined by Eqs. 4.11 to 4.13 can be solved numerically using standard matrix inversion techniques.

As n becomes large, we expect convergence of the summation of the $2n$ plasma waves. The convergence can be investigated by examining the excitation level of the last two waves retained in the summation. The excitation level of the $2n$ and $2n-1$ waves can be expressed in terms of the lower order modes.

$$E_{2n-1} \sim \Sigma_{m=1}^{2n-2}\left(\frac{k_m}{k_{2n-1}}\right)^{2n-1} e_m(E_m) \qquad (4.14a)$$

$$E_{2n} \sim \Sigma_{m=1}^{2n-2}\left(\frac{k_m}{k_{2n}}\right)^{2n-1} f_m(E_m) \qquad (4.14b)$$

where the functions e_m and f_m are well behaved functions of the dielectric elements (defined by Eqs. 4.11 and 12), and the polarization of the modes. The excitation level of the next two higher order waves in terms of the lower order modes is dependent upon the ratio of the wave numbers to the $2n-1$ power. Since the magnitude of the wave numbers montotonically increase away from the origin of the k_x^2 plane, the power series expansion in k_x^2 converges by the ratio test [8]. Thus, the boundary conditions developed in the previous section yield a convergent power series expansion in $k_x^2\rho_i^2$. Numerically, this is demonstrated in Table 2 which displays 4-place numerical values of the field solution for an increasing number of plasma waves retained in the summation. The fields are computed for the position and plasma parameters indicated in the table. As the number of plasma waves increases from 2 to 30, accuracy to 4 significant digits is obtained for E_x, E_y, and E_+, with slower convergence being observed for the B_z magnetic field component.

Figure 4 displays a comparison results of the straight forward $\rho_i = 2$ expansion to the exact dispersion expansion retaining 2, 6 and 14 plasma waves. As the temperature increases from 30 to 200eV, the results of the $\rho_i = 2$ expansion diverges from the exact dispersive theories. In the exact dispersive theories, as the number of waves increases from 2 to 6, there is a significant decrease in the slope of $|E_+|$ as a function of temperature. Retaining 14 plasma waves alters the $|E_+|$ value slighting, suggesting 6 plasma waves would provide sufficient numerical accuracy for this particular example. Figure 5 displays a transverse profile of $|E_+|$ across the plasma slab comparing $\rho_i = 0$ theory to the exact dispersive theory retaining 2 to

10 plasma waves. Inclusion of the ion Bernstein waves in exact dispersive theory significantly alters the $|E_+|$ fast wave polarization characteristics predicted by $\rho_i = 0$ theory. The $|E_+|$ field is enhanced on the antenna side of the plasma over a distance of approximately a gyro radius. The penetration of the ion Berstein waves into the plasma the order of a gyro radius is quantitatively predicted by the approximate dispersion relation Eq. 4.9b. As observed in the figure, there is a large variation $|E_+|$ radial structure in going from 2 to 6 plasma waves, while retaining 10 plasma waves is a small correction to the six plasma wave result.

For the particular example chosen, $\omega = 1.01\omega_{ci}$ the wave numbers of the ion Bernstein waves are closely defined by the zeros of the complex I_n Bessel function. The location of these zeros are on the imaginary k_x^2 axis suggesting a ρ_i^{-1} attenuation length for all of these waves. As such, these modes can be referred to as surface ion waves or surface ion Bernstein waves. These modes are convective in the sense that a perturbation of the ion motion at the surface of the plasma carries the disturbance one gyro radius into the interior of the plasma. The surface ion Bernstein modes are clearly different than the usual ion Bernstein waves which defines a single propagating mode between the harmonics of the ion cyclotron frequency [7]. The propagating Bernstein modes are included in the theory that has been developed, however, these waves have negligible effect in fundamental heating, since $k_b \to \infty$ as $\omega \to \omega_c$, and the excition level of the propagating ion Bernstein wave approaches zero.

If we relax the zero electron mass approximation, electron motion along the magnetic field is significant, and there is an appearance of a new class of waves referred to as ordinary modes [2]. In the $k_z \to 0$ limit, the dispersion relation for the ordinary waves decouples from Eq. 4.7, and has the form

$$K_{zz} - k_x^2 = 0 \tag{4.15}$$

These waves have an electric polarization vector along the magnetic field. In the antenna calculation, as the k_z integration proceeds (Eq. 4.5), the ordinary waves are coupled to the extraordinary waves defined by Eq. 4.7 through the complete dispersion relation defined by the determinant of matrix defined in Eq. 2.1.

A numerical solution including the ordinary modes has been developed. Construction of the solution follows the development outlined above in Eqs. 4.7-14 with the inclusion of the E_z component of the electric field. The E_z and H_z boundary conditions of Eqs. 3.1

a, e, and h provide the additional constraints necessary to determine the excitation level of the additional waves. Figure 6 displays some numercal results comparing the finite electron mass theory to the zero electron mass theory. A cross-sectional variation of $|E_+|$ and $|E_z|$ is shown for an axial positon 5 cm away from the antenna and for the same parameters as Fig. 5. Six (nine) plasma waves were included in the zero (finite) electron mass calculation. It is observed that the inclusion of finite electron mass provides a small correction to the profile of $|E_+|$ predicted by the $m_e = 0$ theory. The new physics in the $m_e \neq 0$ theory is the penetration of the E_z field which can lead to electron heating via Landau damping. The $|E_z|$ field is magnified by a factor of 2 in Fig. 6, and is typically a factor of 10 below the value of $|E_+|$ and peaks near the surface of the plasma.

V. Summary and Discussion

A wave perturbation expansion has been developed that is a solution to the boundary value problem of a line current exciting a magnetoplasma slab. The waves in the perturbation expansion are defined by the roots of the complete hot plasma dispersion relation derived from kinetic theory. Boundary conditions for these modes are derived by integrating Maxwell's equations including the differential form of the dielectric tensor across the abrupt plasma-vacuum boundary. The resulting boundary conditions placed requirements on the derivatives of the plasma electric field. This insures the existance of the Taylor series expansion of the electric field in the boundary layer which was the basis for the differential form of the equivalent dielectric tensor. The wave perturbation expansion results in a convergent power series of a character similar to a Fourier series. For increasing wave number, there is a decrease in the excitation level of the wave components.

The numerical example presented modelled ICRF heating at the fundamental frequency in the central cell of a tandem mirror. The results showed significant finite gyro radius corrections to the $|E_+|$ field near the surface of the plasma over a skin depth of approximately one gyro radius. As such, the theory developed is particularly important for fundamental ICRF heating of mirror machines where the ion gyro radius is comparable to the size of the plasma column. Second, the theory self-consistantly treats ICRF second and higher harmonic heating where a correct treatment of finite gyro radius effects is essential.

One obvious deficiency of the modelling is the abrupt plasma-vacuum boundary. A stratification of the plasma density and temperature profile would clearly be an improvement. For this case, one could directly solve Maxwell's equations including the differential form of the dielectric tensor for inhomogeneous profiles using numerical methods such as invariant inbedding or finite elements. At the surface of the plasma where the density drops to zero, the boundary conditions of Sec. 3 could be used to initiate the numerical solution for the plasma fields.

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Appendix A

The equivalent dielectric tensor elements of a hot Maxwellian plasma emmersed in a uniform magnetic field have the following form for an assumed $exp(ik_x x + ik_z z - i\omega t)$ plane wave perturbation.

$$K_{xx} = k_o^2 + \Sigma_\alpha \epsilon \lambda^{-1} \quad \Sigma_{n=1}^{\infty} n^2 M_n \mu_n \qquad A.1$$

$$K_{xy} = \Sigma_\alpha \epsilon \quad \Sigma_{n=1}^{\infty} n M'_n \nu_n \qquad A.2$$

$$K_{yy} = -2\Sigma_\alpha \epsilon \lambda \quad \{M'_o Z_o + \Sigma_{n=1}^{\infty} M'_n \mu_n\} \qquad A.3$$

$$K_{zz} = k_o^2 - \Sigma_\alpha \epsilon \quad \{M_o \varsigma_o Z'_o + \Sigma_{n=1}^{\infty} M_n \eta_n\} \qquad A.4$$

$$K_{xz} = \frac{1}{2} \Sigma_\alpha \epsilon \delta \lambda^{-1} \quad \Sigma_{n=1}^{\infty} n M_n \nu'_n \qquad A.5$$

$$K_{yz} = \frac{1}{2} \Sigma_\alpha \epsilon_\alpha \delta \quad \{M'_o Z'_o + \Sigma_{n=1}^{\infty} M'_n \mu'_n\} \qquad A.6$$

In the above equations, $M_n = I_n e^{-\lambda}$ where I_n is the modified Bessel function of the first kind of argument λ, Z is the plasma dispersion function of argument ς, the prime designates derivative of these functions with respect to their argument, and Σ_α designates a summation over the various plasma constitutents (electrons, ions, etc.). The remaining notation is defined below:

$$\epsilon = \frac{\omega_{p\alpha}^2}{\omega k_z v_\alpha} k_o^2 \qquad A.7$$

$$\delta = \frac{v_\alpha}{k_z \omega_{c\alpha}} \qquad A.8$$

$$\lambda = \frac{v_\alpha^2 k_x^2}{2\omega_{c\alpha}^2} = \frac{\rho_\alpha^2 k_x^2}{2} \qquad A.9$$

$$\zeta_n = \frac{\omega + n\omega_{ca}}{k_z v_a} \quad \text{A.10}$$

$$M_n = I_n(\lambda)e^{-\lambda} \quad \text{A.11}$$

$$\mu_n = Z_n(\zeta_n) + Z_n(\zeta_{-n}) \quad \text{A.12}$$

$$\nu_n = Z_n(\zeta_n) - Z_{-n}(\zeta_{-n}) \quad \text{A.13}$$

$$\eta_n = \zeta_n Z'_n(\zeta_n) - \zeta_{-n} Z'_{-n}(\zeta_{-n}) \quad \text{A.14}$$

In the above, ω_{pa} is the plasma frequency, ω_{ca} is the cyclotron frequency, v_a is the thermal velocity $= (2kT_a/m_a)^{\frac{1}{2}}$, and $k_o = \omega/c$ is the free space wave number.

Appendix B

The power series expanison in k_x^2 of the dielectric elements of appendix A is contained in this appendix. Much of the notation used below is defined in appendix A.

$$K_{xx} = S + S_{1x}k_x^2 + S_{2x}k_x^4 + \ldots \tag{B.1}$$

$$S = k_\circ^2 + \frac{1}{2}\Sigma_\alpha \epsilon \mu_1 \tag{B.2a}$$

$$S_{1x} = -\frac{1}{4}\Sigma_\alpha \epsilon \rho_\alpha^2(\mu_1 - \mu_2) \tag{B.2b}$$

$$S_{2x} = \frac{1}{64}\Sigma_\alpha \epsilon \rho_\alpha^4(5\mu_1 - 8\mu_2 + 3\mu_3) \tag{B.2c}$$

$$K_{xy} = D + D_1 k_x^2 + D_2 k_x^4 + \ldots \tag{B.3}$$

$$D = \frac{1}{2}\Sigma_\alpha \epsilon \mu_1 \tag{B.4a}$$

$$D_1 = -\frac{1}{4}\Sigma_\alpha \epsilon \rho_\alpha^2(2\nu_1 - \nu_2) \tag{B.4b}$$

$$D_2 = \frac{1}{64}\Sigma_\alpha \epsilon \rho_\alpha^4(15\nu_1 - 12\nu_2 + 3\nu_3) \tag{B.4c}$$

$$K_{xx} + K_{yy} = S + S_{1y}k_x^2 + S_{2y}k_x^4 + \ldots \tag{B.5}$$

$$S_{1y} = -\frac{1}{4}\epsilon \rho_\alpha^2(-2\mu_\circ + 3\mu_1 - \mu_2) \tag{B.5a}$$

$$S_{2y} = \frac{1}{64}\Sigma_\alpha \epsilon \rho_\alpha^4(-24\mu_\circ + 37\mu_1 - 16\mu_2 + 3\mu_3) \tag{B.5b}$$

$$K_{zz} = P + P_1 k_x^2 + P_2 k_x^4 ... \tag{B.6}$$

$$P = k_o^2 - \frac{1}{2}\Sigma_\alpha \epsilon \eta_o \tag{B.6a}$$

$$P_1 = -\frac{1}{4}\Sigma_\alpha \epsilon \rho_\alpha^2(-\eta_o + \eta_1) \tag{B.6b}$$

$$P_2 = \frac{1}{64}\Sigma_\alpha \epsilon \rho_\alpha^4(-6\eta_o + 8\eta_1 + 2\eta_2) \tag{B.6c}$$

$$K_{xz} = U_1 + U_2 k_x^2 + U_3 k_x^4 + ... \tag{B.7}$$

$$U_1 = \frac{1}{4}\Sigma_\alpha \epsilon \delta \nu'_1 \tag{B.7a}$$

$$U_2 = -\frac{1}{16}\Sigma_\alpha \epsilon \delta \rho_\alpha^2(2\nu'_1 - \nu'_2) \tag{B.7b}$$

$$U_3 = \frac{1}{128}\Sigma_\alpha \epsilon \delta \rho_\alpha^4(5\nu'_1 - 4\nu'_2 + \nu'_3) \tag{B.7c}$$

$$K_{yz} = V_1 + V_2 k_x^2 + V_2 k_x^4 + ... \tag{B.8}$$

$$V_1 = \frac{1}{4}\Sigma_\alpha \epsilon \delta(-\mu'_o + u'_1) \tag{B.8a}$$

$$V_2 = -\frac{1}{16}\Sigma_\alpha \epsilon \delta \rho_\alpha^2(-3\mu'_o + 4\mu'_1 - \mu'_2) \tag{B.8b}$$

$$V_3 = \frac{1}{128}\Sigma_\alpha \epsilon \delta \rho_\alpha^4(-10\mu'_o + 15\mu'_1 - 6\mu'_2 + \mu'_3) \tag{B.8c}$$

The general term in the expansion of one of the dielectric elements is obtained from the power series representation of M_n.

$$M_n = e^{-\lambda} I_n(\lambda) = \Sigma_{i=n}^{\infty} a_n^i \lambda^i \tag{B.9a}$$

$$a_n^i = \Sigma_{\ell,m} \frac{(-1)^\ell}{\ell!} \frac{(\frac{1}{2})^{2m+n}}{m!(m+n)!} \tag{B.9b}$$

where the sum is over all $\ell, m > 0$ such that $i = n + 2m + \ell$. The K_{xx} element takes the form,

$$K_{xx} = k_\circ^2 + \Sigma_\alpha \epsilon \Sigma_{n=1}^{\infty} \Sigma_{i=n}^{\infty} a_n^i \lambda^{i-1} n^2 \mu_n \tag{B.10a}$$

or the next two higher terms in the expansion are,

$$S_{3x} = \frac{1}{384} \Sigma_\alpha \epsilon \rho_\alpha^6 (-7\mu_1 + 14\mu_2 - 9\mu_3 + 2\mu_4) \tag{B.10b}$$

$$S_{4x} = \frac{1}{12288} \Sigma_\alpha \epsilon \rho_\alpha^8 (42\mu_1 - 96\mu_2 + 81\mu_3 - 32\mu_4 + 5\mu_5) \tag{B.10c}$$

Appendix C

Maxwell's equations incorporating the differential form of the equivalent dielectric tensor have the following form (the magnetic field has been normalized by multiplicative constant $i\omega\mu_o$),

$$-ik_z E_y = H_x \tag{C.1}$$

$$ik_z E_x - E_z' = H_y \tag{C.2}$$

$$E_y' = H_z \tag{C.3}$$

$$-ik_z H_y = SE_x - (S_{1x}E_x)'' + (S_{2x}E_x)^{iv} + \dots$$

$$-iDE_y + i(D_1E_y)'' - i(D_2E_y)^{iv} + \dots$$

$$-ik_z(U_1E_z)' + ik_z(U_2E_z)''' - ik_z(U_3E_z)^{v} + \dots \tag{C.4}$$

$$ik_z H_x - H_z' = iDE_x - i(D_1E_x)'' + i(D_2E_x)^{iv} + \dots$$

$$+SE_y - (S_{1y}E_y)'' + (S_{2y}E_y)^{iv} + \dots$$

$$+k_z(V_1E_z)' - k_z(V_2E_z)''' + k_z(V_3E_z)^{v} + \dots \tag{C.5}$$

$$H_y' = -ik_z(U_1E_x)' + ik_z(U_2E_x)''' - ik_z(U_3E_x)^{v} + \dots$$

$$-k_z(V_1E_y)' + k_z(V_2E_y)''' - k_z(V_3E_y)^{v} + \dots$$

$$+PE_z - (P_1E_z)'' + (P_2E_z)^{iv} + \dots \tag{C.6}$$

To second order in ρ_i^2, the plasma fields satisfy Eqs. C.1-3 along with the following two equations in the zero electron mass approximation ($m_e \rightarrow 0, P \rightarrow \infty$).

$$(S - k_z^2)E_x - (S_{1x}E_x)'' + (S_{2x}E_x)^{iv} - \dots$$

$$-iDE_y + i(D_1E_y)'' - i(D_2E_y)^{iv} + \dots = 0 \tag{C.7}$$

$$iDE_x - i(D_1E_x)'' + i(D_2E_x)^{iv} - \dots$$

$$+(S - k_z^2)E_y + E_y'' - (S_{1y}E_y)'' + (S_{2y}E_y)^{iv} - \dots = 0 \tag{C.8}$$

Appendix D

A single Fourier component of the field solution to first order in $\hat{\rho}_i$ for the boundary value problem illustrated in Fig. 2 is given by Eqs. 4.2a and b. The constants defined by that solution are contained in Eqs. D.1 to D.10.

$$a_f = \frac{iD + iD_1 k_f^2}{S - k_z^2 + S_{1x} k_f^2} \tag{D.1a}$$

$$a_b = \frac{-iD - iD_1 k_b^2}{S - k_z^2 - k_b^2 + S_{1y} k_b^2} \tag{D.1b}$$

The unbounded dispersion relation for the fast wave and the first ion Bernstein wave is,

$$a_2 k_x^4 + a_1 k_x^2 + a_o = 0 \tag{D.2}$$

$$a_o = (S - k_z^2)^2 - D^2 \tag{D.2a}$$

$$a_1 = (S_{1x} + S_{1y} - 1)(S - k_z^2) - 2DD_1 \tag{D.2b}$$

$$a_2 = S_{1x}(S_{1y} - 1) - D_1^2 \tag{D.2c}$$

The fast wave amplitudes are,

$$E_3 = b_1 E^+ / D(\omega, k_z) \tag{D.3}$$

$$E_4 = -b_2 E^+ / D(\omega, k_z) \tag{D.4}$$

The ion Bernstein wave amplitudes are,

$$E_5 = (c_1 E_3 + c_2 E_4) \tag{D.5}$$

$$E_6 = (c_2 E_3 + c_1 E_4) \tag{D.6}$$

the bounded dispersion relation is,

$$D(\omega, k_z) = b_1^2 - b_2^2 \tag{D.7}$$

In the above equations we have,

$$E^+ = i\omega\mu_o I[e^{-k_z(x_1-a)} - e^{-k_z(2x_2+x_1-a)}] \tag{D.8}$$

$$b_1 = -(a_2 + c_3)e^{ik_f a} - (c_4 + a_4)c_1 e^{ik_b a} + (c_4 - a_3)c_2 e^{-ik_b a} \tag{D.9a}$$

$$b_2 = (-a_1 + c_3)e^{-ik_f a} - (c_4 + a_4)c_2 e^{ik_b a} + (c_4 - a_3)c_1 e^{-ik_b a} \tag{D.9b}$$

$$c_3 = ik_f(S_{1y} + ia_f D_1)$$

$$c_4 = ik_b(S_{1y}a_b + iD_1)$$

$$a_1 = -k_z + ik_f \qquad a_2 = -k_z - ik_f$$

$$a_3 = (-k_z + ik_b)a_b \qquad a_4 = (-k_z - ik_b)a_b$$

$$c_1 = \delta\frac{\sin[(k_f + k_b)a]}{\sin(2k_b a)} \tag{D.10a}$$

$$c_2 = \delta\frac{\sin[(k_f - k_b)a]}{\sin(2k_b a)} \tag{D.10b}$$

$$\delta = \frac{-(S_{1x}a_f - iD_1)}{(S_{1x} - iD_1 a_b)}\frac{k_f}{k_b} \tag{D.10c}$$

The unbounded dispersion relation for the fast wave alone is

$$k_f^2 = \frac{(S-k_z^2)^2 - D^2}{S-k_z^2} \tag{D.11}$$

The unbounded disperison relation for the fast wave and the first three ion Bernstein waves is,

$$0 = a_\circ + a_1 k_x^2 + a_2 k_x^4 + a_3 k_x^6 + a_4 k_x^8 \tag{D.12}$$

where $a_\circ$ and a_1 are defined by $D.2a$ and b, and the remaining coefficients are defined below.

$$a_2 = (S-k_z^2)(S_{2x}+S_{2y}) + S_{1x}(S_{1y}-1) - 2D_2D - D_1^2 \tag{D.12a}$$

$$a_3 = (S_1S_{2y} + S_{2x}(S_{1y}-1) - 2D_1D_2 \tag{D.12b}$$

$$a_4 = S_{2x}S_{2y} - D_2^2 \tag{D.12c}$$

Roots of the Dispersion Relations

$k_z = 1m^{-1}, \omega/\omega_{ci} = 1.05, T_i = 30eV$, and the other plasma parameters the same as Fig. 4

k_x^2 (m^{-1}) (Real, Imaginary)

$\rho_i = 0$	$\rho_i = 1$	$\rho_i = 2$	Eq. 4.7
105., .003	99.5, 1.	99.2,1.0	99.0, 1.0
	558., -5.6	393., 689.	-1. $\times 10^4$, 31.
		394., -691.	920., $\pm 2.5 \times 10^4$
		4.4×10^4, -340.	$-1.2 \times 10^4, \pm 2.4 \times 10^4$
			920., $\pm 4.6 \times 10^4$

Table 1

Convergence of the Fields

$T_i = 30$eV and the other parameters are the same as Fig. 3.

# Rts	B_z(g)	E_x (v/cm)	E_y (v/cm)	E_+ (v/cm)
2	27.72, -4.132	-2.316, 11.75	-11.94, -.8617	.7330
6	30.08, -3.251	-2.820, 11.76	-11.93, -1.061	.8831
10	30.87, -3.149	-2.900, 11.77	-11.95, -1.089	.9096
18	31.30, -2.999	-2.930, 11.77	-11.97, -1.104	.9182
26	31.43, -2.889	-2.933, 11.77	-11.98, -1.105	.9196
30	31.46, -2.848	-2.933, 11.78	-11.98, -1.105	.9198

Table 2

Figure Captions

1. The boundary layer. A sharp boundary is obtained in the limiting process of letting $\Delta, \delta \rightarrow 0$.

2. Geometry of the line current excitation of a plasma slab. The plasma is unbounded and uniform in the y and z directions.

3. Results of the straight-forward theory, $|E_+|$ at $r = 6cm, z = 0$; $a = 7cm, x_o = 22cm, x_1 = 48cm, I = 2000A, B = 450G, n = 5 \times 10^{12}cm^{-3}, T_e = 30eV$, hydrogen plasma, $\omega/\omega_{ci} = 1.01$.

4. Results of the exact dispersive theory, $|E_+|$ at $r = 6cm, z = 0$, plasma parameters same as Fig. 3.

5. Results of the exact dispersive theory, $|E_+|$ radial scan, plasma parameters same as Fig. 3 except, $n = 10^{13}cm^{-3}$ and $T_i = 150eV$.

6. Results of the exact disispersive theory $m_e = 0$ and $m_e \neq 0$, plasma parameters same as Fig. 5.

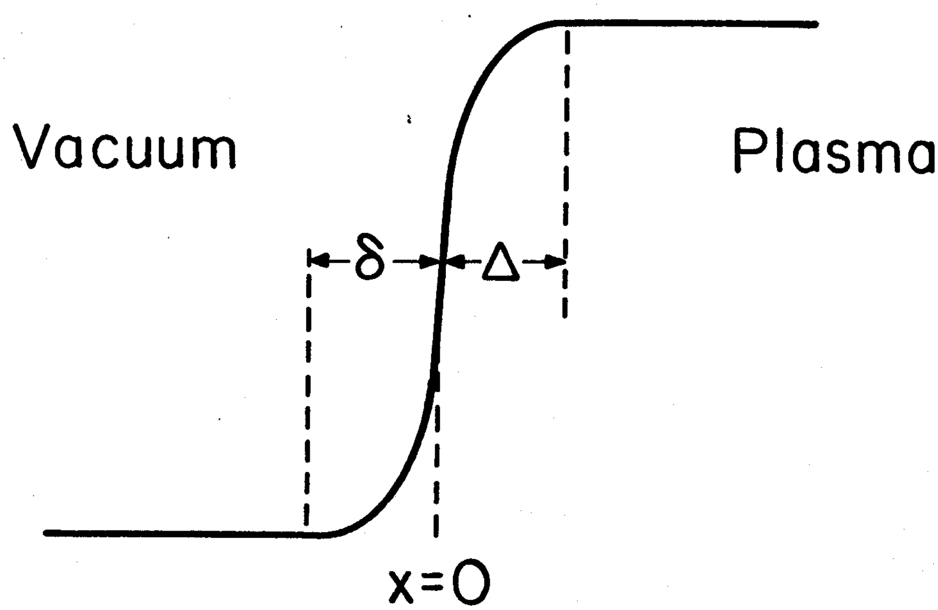

Fig. 1

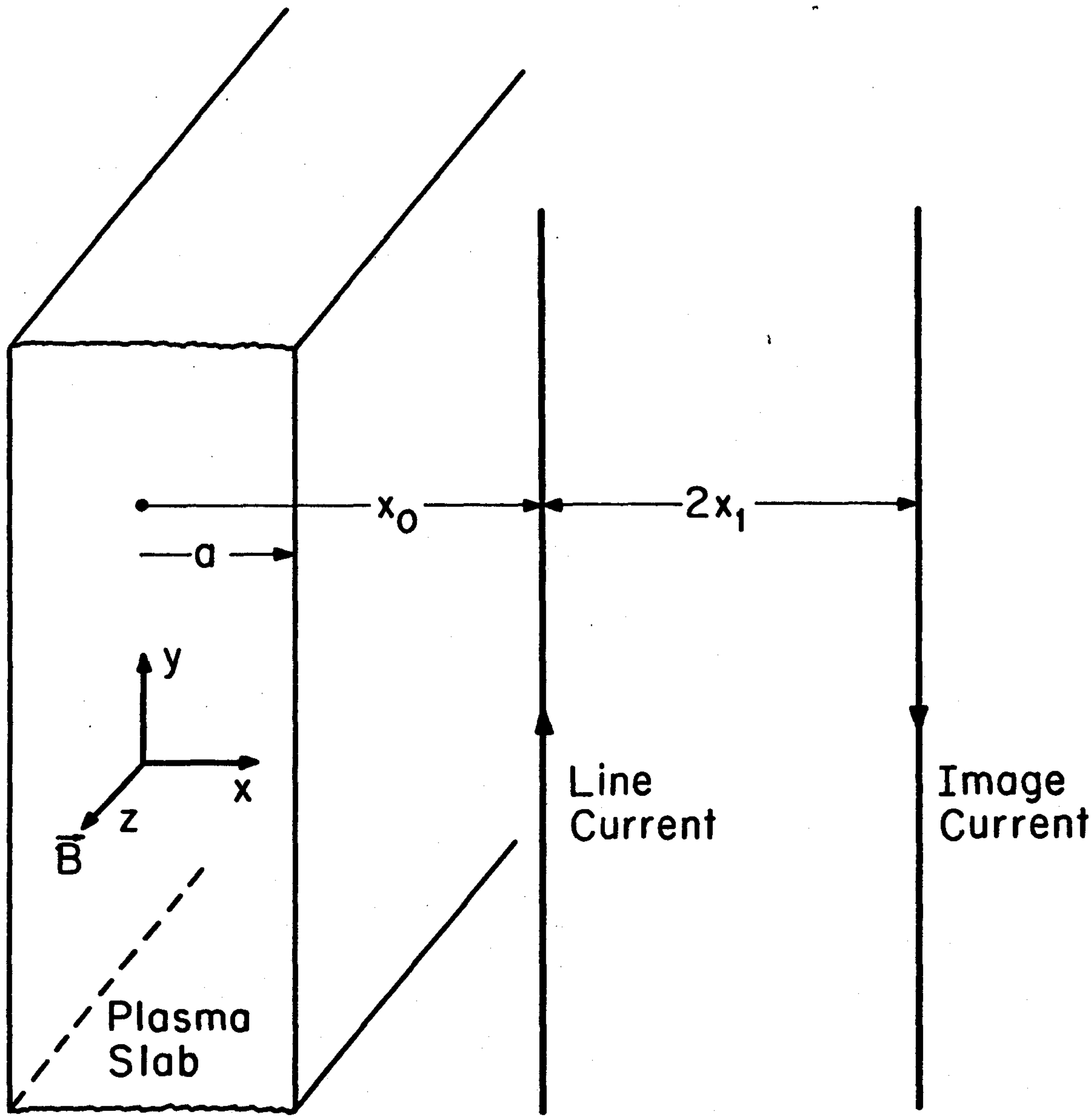

Fig. 2

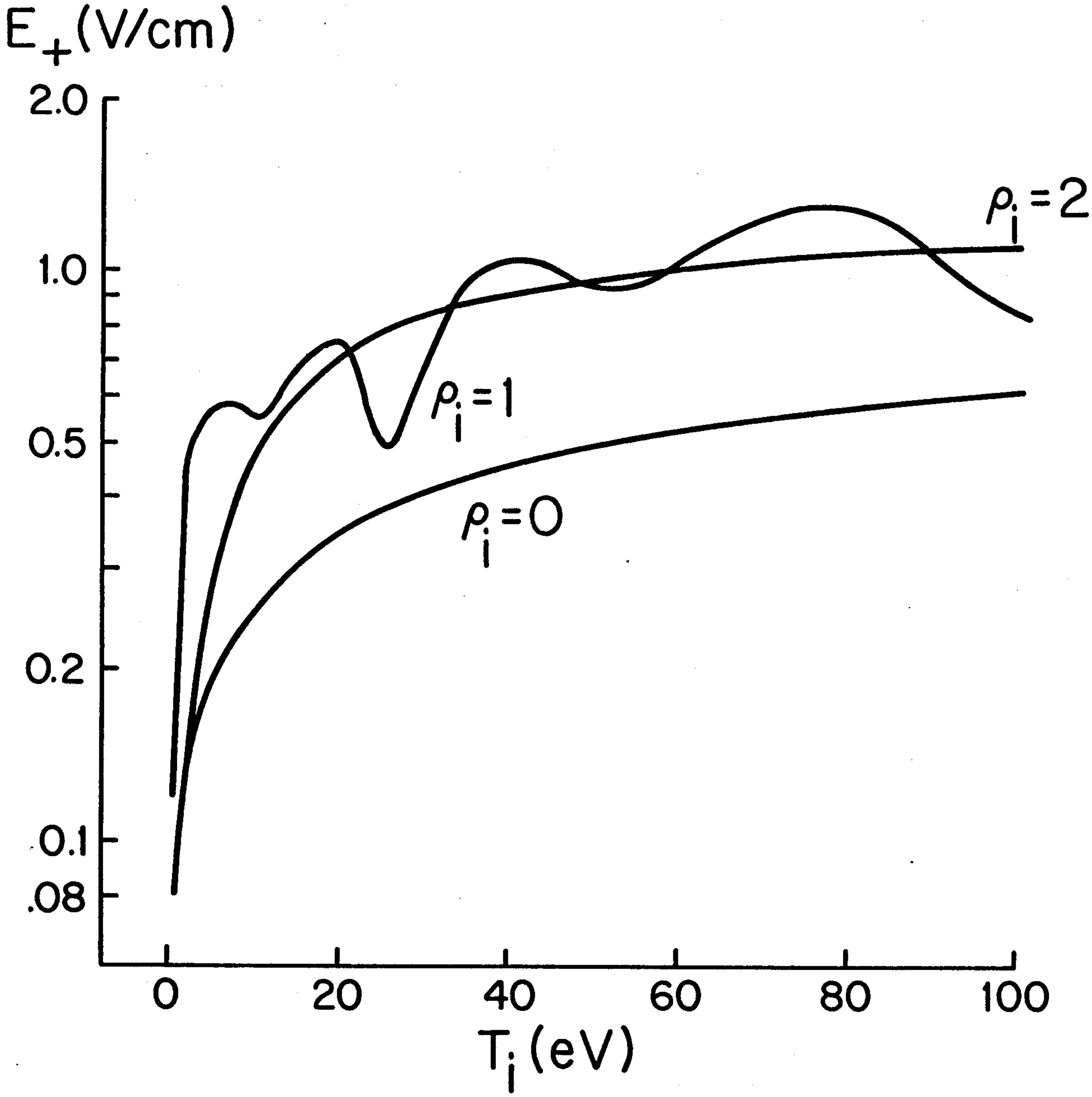

Fig. 3

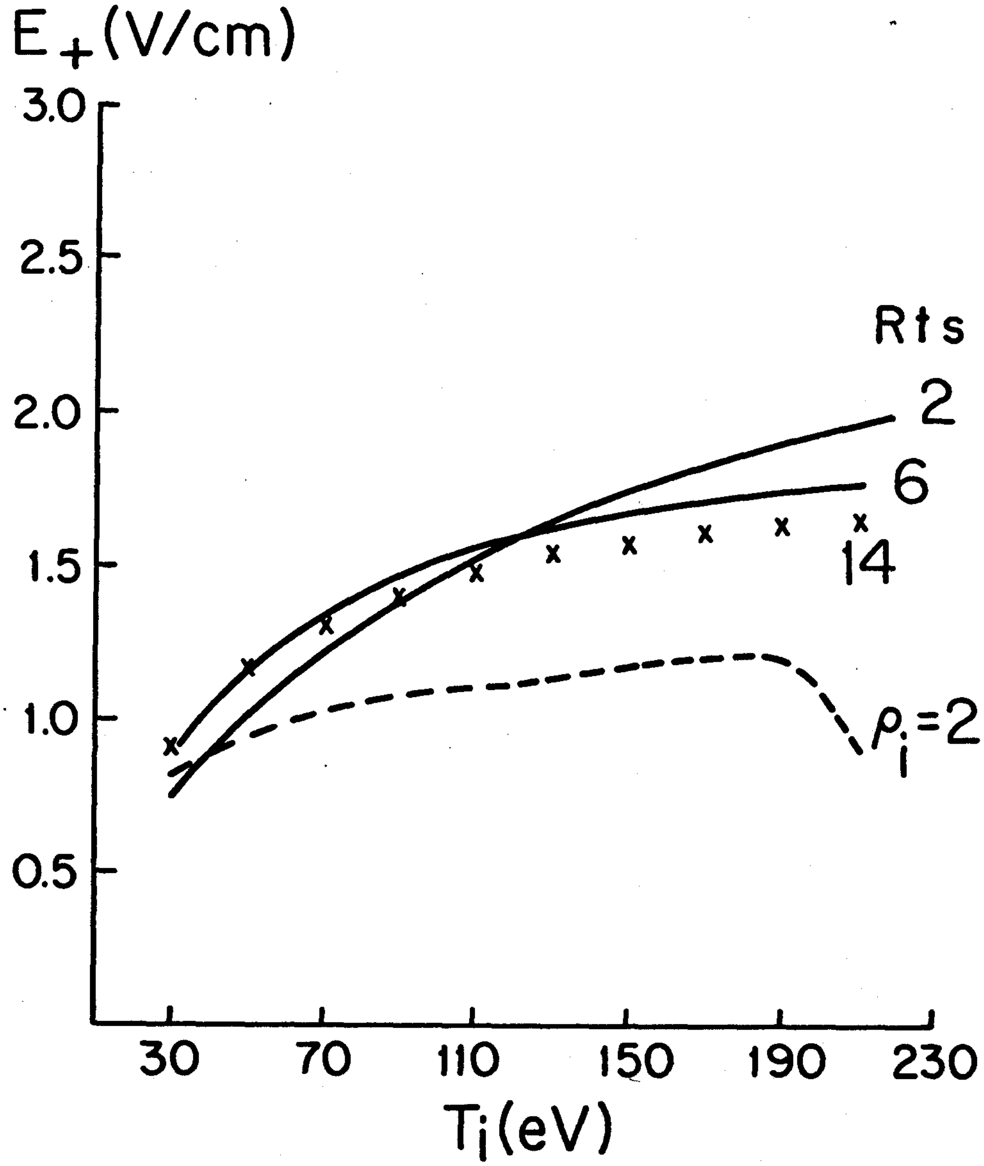

Fig. 4

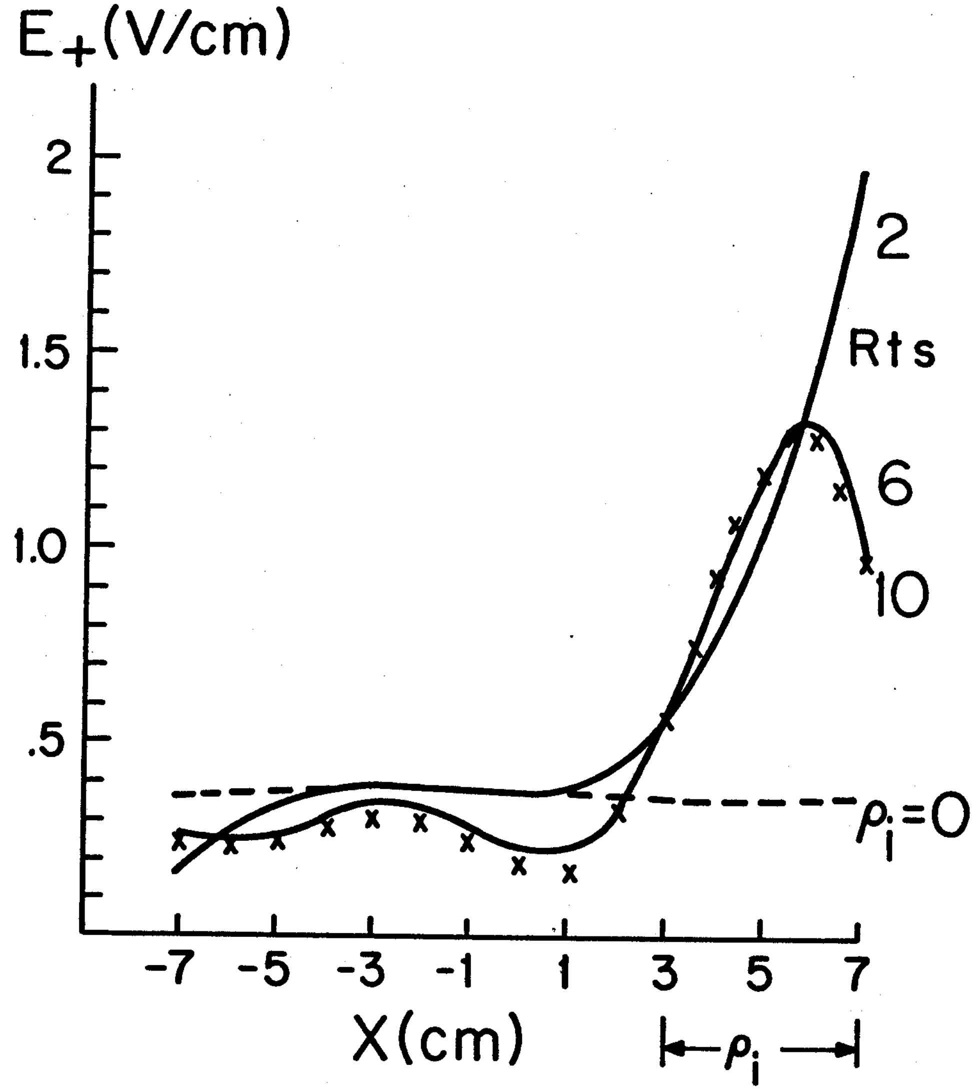

Fig. 5

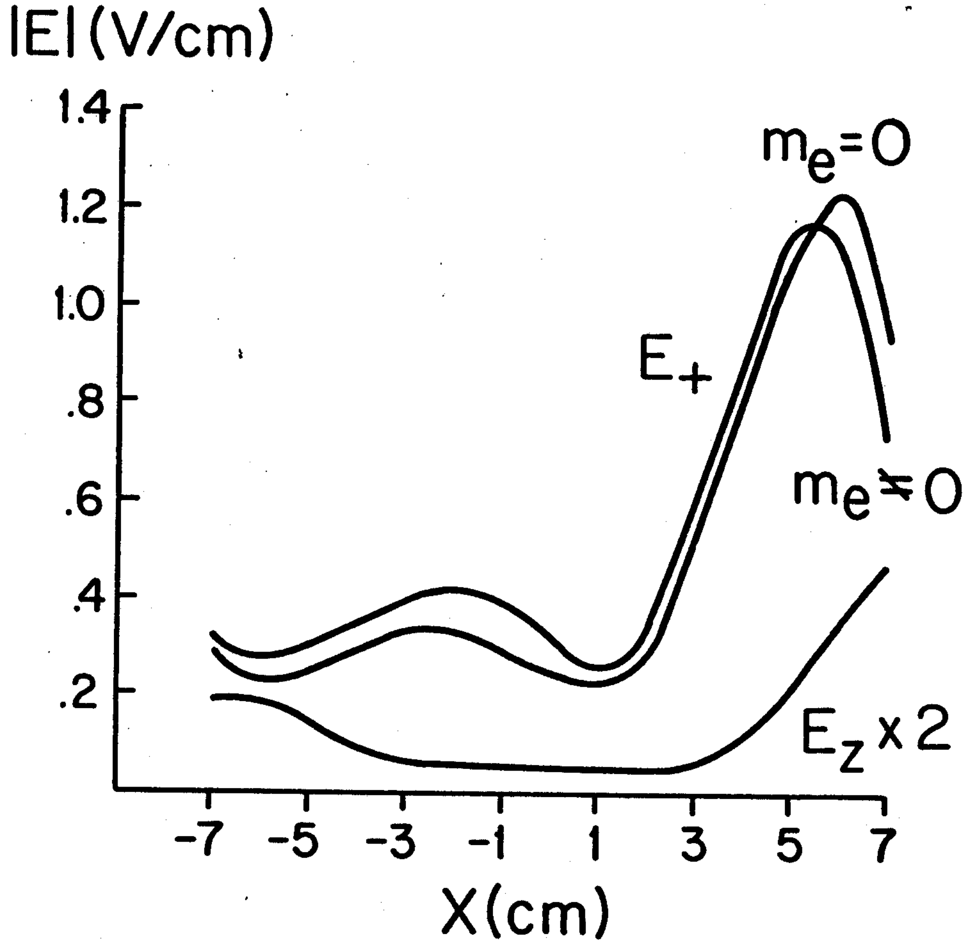

Fig. 6