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ABSTRACT

The Einstein coefficient method is used to calculate the growth rate of a low gain FEL for a tenuous relativistic electron beam propagating in the combined axial and transverse helical wiggler fields $B_0\hat{e}_z - (\delta B cos k_0 z \hat{e}_x + \delta B sin k_0 z \hat{e}_y)$. The analysis assumes that the system is close to resonance between the electron cyclotron frequency in the guide field ($\omega_c = eB_0/\gamma mc$) and the wiggler frequency in the beam frame ($\omega_0 = k_0 V_b$). It is shown that the gain near resonance can be substantially enhanced relative to the gain obtained in the region far from resonance.

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I. INTRODUCTION

The growth rate for a low gain FEL has been calculated theoretically [1–5] and observed experimentally [6,7] for a tenuous electron beam far from resonance between the electron cyclotron frequency in the axial guide field ($\omega_c = eB_0/\gamma mc$) and the wiggler frequency in the beam frame ($\omega_0 = k_0 V_b$). There have also been theoretical considerations of exploiting the cyclotron resonance effect to enhance the FEL gain [8–10], although the approximations used in these studies break down close to resonance. Experiments have been performed in the Raman FEL regime near cyclotron resonance with the results that either there is no radiation emission [11–13] or the emission is enhanced, [14] indicating that conditions for FEL operation near cyclotron resonance have not been explored adequately.

In this article, we make use of the Einstein coefficient method to calculate the growth rate of a low gain FEL for a tenuous electron beam propagating in the combined axial guide and transverse helical wiggler fields

$$\begin{aligned} B^0 &= B_0 \hat{e}_z + \delta B \\ &= B_0 \hat{e}_z - \delta B(\cos k_0 z \hat{e}_x + \sin k_0 z \hat{e}_y), \end{aligned} \tag{1}$$

where B_0 and δB are assumed constant, $\lambda_0 = 2\pi/k_0$ is the wiggler wavelength, and the expression for the wiggler magnetic field is valid near the z-axis ($k_0 R_b < 1$). Near cyclotron resonance ($\omega_c \approx \omega_0$), it is found that the gain can be orders of magnitude larger than the equivalent gain obtained in the region far from resonance. It is also found that narrow band emission requires that the axial electron momentum $p_z = \gamma m v_z$ be approximately constant. To satisfy this condition <u>and</u> the condition that the system be close to cyclotron resonance ($\omega_c \approx \omega_0$) imposes the requirement that $\boldsymbol{p}_\perp \cdot \boldsymbol{\delta B} > 0$, where $\boldsymbol{p}_\perp = \gamma m(v_x \hat{e}_x + v_y \hat{e}_y)$ is the transverse momentum and $\gamma mc^2 = (m^2c^4 + c^2p^2)^{1/2}$ is the electron energy. Experimentally, the requirement $\boldsymbol{p}_\perp \cdot \boldsymbol{\delta B} > 0$ is difficult to achieve and may explain the null emission results.[11–13] Previous theoretical analyses have not considered this requirement, since in a standard FEL the electrons enter the wiggler magnetic field with negligibly small transverse momentum. Allowing finite transverse momentum has the effect of reducing the relativistic output frequency upshift. Also, since the system is near cyclotron resonance and $\boldsymbol{p}_\perp$ is finite, the cyclotron maser effect may be excited, and we obtain the condition for FEL emission to dominate over cyclotron maser emission.

II. ELECTRON TRAJECTORIES IN THE WIGGLER FIELD

The equation of motion for an electron moving in the combined axial guide and wiggler fields $\vec{B}^0 = B_0\hat{e}_z + \delta \vec{B}$ [Eq. (1)] is given by $d\vec{p}'/dt' = -e\vec{v}' \times \vec{B}^0(\vec{x}')/c$, where $-e$ is the electron charge, c is the speed of light in vacuo, $\vec{v}' = \vec{p}'/\gamma' m$ is the electron velocity, and m is the electron rest mass, and we assume "initial" conditions such that the particle trajectories $(\vec{x}', \vec{p}')$ pass through the phase space point $(\vec{x}, \vec{p})$ at time $t' = t$. Defining the shifted axial momentum coordinate η' by $\eta' = ck_0 p_z'/eB_0 - 1 = k_0 v_z'/\omega_c - 1$, where $\omega_c = eB_0/\gamma mc$ is the relativistic electron cyclotron frequency and $\gamma mc^2 = \gamma' mc^2 = const$ is the electron energy, it is straightforward to show that $\eta'(t')$ satisfies the nonlinear Duffing equation [15,16]

$$\frac{1}{\omega_c^2}\frac{d^2\eta'}{dt'^2} + \frac{1}{2}\eta'^3 + \left[\left(\frac{\delta B}{B_0}\right)^2 - \frac{1}{2}\left(\frac{k_0 V_z}{\omega_c} - 1\right)^2\right]\eta' + \left(\frac{\delta B}{B_0}\right)^2 = 0, \tag{2}$$

where $V_z = C_z/\gamma m$ is the exact axial invariant $\left(dV_z/dt' = 0\right)$ defined by [17]

$$\left(\frac{k_0 V_z}{\omega_c} - 1\right)^2 = \eta'^2 - \frac{2ck_0}{eB_0^2}\vec{p}_\perp' \cdot \delta\vec{B}(\vec{x}'). \tag{3}$$

For $\delta B = 0$, it follows from Eq. (3) and the definition $\eta' = k_0 v_z'/\omega_c - 1$ that $v_z' = V_z$, and the axial invariant V_z can be identified (exactly) with the axial velocity v_z'. For $\delta B \neq 0$, however, the axial invariant $V_z = const$ is generally different from the axial velocity v_z', which varies as a function of t'. We now assume that the system is very close to cyclotron resonance with

$$\frac{1}{2}\left(\frac{k_0 V_z}{\omega_c} - 1\right)^2 \ll \left(\frac{\delta B}{B_0}\right)^2 \ll 1 \tag{4}$$

in Eqs. (2) and (3). In particular, it is assumed that the beam equilibrium distribution function $f_b^0(\vec{x}, \vec{p})$ is strongly peaked about values of V_z and γ satisfying Eq. (4). Within the context of Eq. (4), the pseudopotential for the η' motion [Eq. (2)] is given by $V(\eta') = \eta'^4/8 + (\delta B/B_0)^2(\eta'^2/2 + \eta')$, which has the equilibrium point $\eta_0 \simeq -2^{1/3}(\delta B/B_0)^{2/3}$ (which solves $\partial V/\partial\eta' = 0$). Solving Eq. (2) for small-amplitude oscillations about η_0, i.e., for $|\eta' - \eta_0| \ll |\eta_0|$, we find for $v_z' = (\omega_c/k_0)(\eta' + 1)$

$$\begin{aligned} v_z' &= V_b + (v_z - V_b)\cos[\omega_\delta(t' - t)] \\ &+ \frac{\delta B}{B_0}\frac{\omega_c}{k_0}\frac{k_0 v_\perp}{\omega_\delta}\sin(k_0 z - \phi)\sin[\omega_\delta(t' - t)], \end{aligned} \tag{5}$$

where

$$\begin{aligned} \omega_\delta &= \left(\frac{3}{2}\right)^{1/2} 2^{1/3}\left(\frac{\delta B}{B_0}\right)^{2/3}\omega_c, \\ V_b &= \frac{\omega_c}{k_0}\left[1 - 2^{1/3}\left(\frac{\delta B}{B_0}\right)^{2/3}\right], \end{aligned} \tag{6}$$

and ϕ is the $t' = t$ phase of the transverse momentum $p_\perp = (p_\perp cos\phi, p_\perp sin\phi)$. The coefficient of $sin[\omega_\delta(t'-t)]$ in Eq. (5) has been determined from the boundary condition $(dv_z'/dt')_{t'=t} = -(e/\gamma mc)(v_\perp \times \delta B)_z$ at $t' = t$. Moreover, $|v_z - V_b| \ll V_b$ and $|(\delta B/B_0)(\omega_c/k_0)(k_0 v_\perp/\omega_\delta)| \ll V_b$ are assumed in Eq. (5), corresponding to small-amplitude oscilllations of v_z' about the average value V_b.

With regard to the perpendicular motion, we note from Eq. (3) that $p_\perp' \cdot \delta B(x')/B_0$ is required to satisfy the approximate condition $p_\perp' \cdot \delta B(x')/B_0 = (eB_0/2ck_0)\eta'^2 = (1/2)\gamma m(\omega_c/k_0)\eta'^2 > 0$ close to resonance [Eq. (4)], which places a strong constraint on the perpendicular electron motion in the wiggler field. If we estimate the characteristic size of η' by its average value $\eta_0 = -2^{1/3}(\delta B/B_0)^{2/3}$, then the condition $p_\perp' \cdot \delta B(x')/B_0 \approx (1/2)\gamma m(\omega_c/k_0)\eta_0^2$ gives $p_\perp' > (1/2)\gamma m(\omega_c/k_0)2^{2/3}(\delta B/B_0)^{1/3} \simeq \gamma m V_b(\delta B/2B_0)^{1/3}$ for the characteristic size of $p_\perp'$. This is a relatively large value of transverse momentum compared with that assumed in a standard FEL far from resonance.

Eq. (5) can be integrated to determine $z'(t')$ with boundary condition $z'(t' = t) = z$, and the result substituted into the perpendicular equations of motion for $v_x'(t')$ and $v_y'(t')$. Defining, $w' = v_x' + iv_y'$, the perpendicular motion satisfies $dw'/dt' = i\omega_c w' + i\omega_c(\delta B/B_0)v_z' exp(ik_0 z') = i\omega_c w' + (\omega_c/k_0)(\delta B/B_0)(d/dt')exp(ik_0 z')$. It is convenient to rewrite Eq. (5) in the form $v_z' = V_b + \epsilon V_b cos(\omega_\delta \tau + a)$ so that z' can be expressed as

$$z' = z + V_b\tau + \epsilon\frac{V_b}{\omega_\delta}[sin(\omega_\delta\tau + a) - sin a], \tag{7}$$

where $\tau = t' - t$, and ϵ and a are defined by

$$\begin{aligned}
\epsilon &= \left[\left(\frac{v_z - V_b}{V_b}\right)^2 + \left(\frac{\delta B}{B_0}\frac{\omega_c}{k_0 V_b}\frac{k_0 v_\perp}{\omega_\delta}\right)^2 sin^2(k_0 z - \phi)\right]^{1/2}, \\
\epsilon sin a &= -\frac{\delta B}{B_0}\frac{\omega_c}{k_0 V_b}\frac{k_0 v_\perp}{\omega_\delta} sin(k_0 z - \phi), \\
\epsilon cos a &= \frac{v_z - V_b}{V_b}.
\end{aligned} \tag{8}$$

The orbit factor $exp(ik_0 z')$ occurring in the equation for dw'/dt' can then be expressed as

$$\begin{aligned}
exp(ik_0 z') &= exp i[k_0 z - \epsilon(k_0 V_b/\omega_\delta) sin a] \\
&\times \sum_{m=-\infty}^{\infty} J_m\left(\epsilon\frac{k_0 V_b}{\omega_\delta}\right) exp i[k_0 V_b + m\omega_\delta]\tau \quad exp(ima),
\end{aligned} \tag{9}$$

where use has been made of $exp(ibsina) = \sum_{m=-\infty}^{\infty} J_m(b)exp(ima)$, and $J_m(b)$ is the Bessel function of the first kind of order m.

Integrating $dw'/dt' = i\omega_c w' + (\omega_c/k_0)(\delta B/B_0)(d/dt')exp(ik_0 z')$ with respect to t', and imposing $v'_x(t'=t) = v_x = v_\perp cos\phi$ and $v'_y(t'=t) = v_y = v_\perp sin\phi$, we find

$$
\begin{aligned}
v'_x(t') + iv'_y(t') &= v_\perp cos(\omega_c\tau + \phi) + iv_\perp sin(\omega_c\tau + \phi) \\
&+ \frac{\omega_c}{k_0}\frac{\delta B}{B_0} expi[k_0 z - \epsilon(k_0 V_b/\omega_\delta)sina] \\
&\times \sum_{m=-\infty}^{\infty} J_m\left(\epsilon\frac{k_0 V_b}{\omega_\delta}\right)\frac{(k_0 V_b + m\omega_\delta)}{(k_0 V_b - \omega_c + m\omega_\delta)} exp(ima) \\
&\times [expi(k_0 V_b + m\omega_\delta)\tau - exp(i\omega_c\tau)]. \qquad (10)
\end{aligned}
$$

II. SPONTANEOUS EMISSION COEFFICIENT

The spontaneous emission coefficient $\eta_\omega(x,p)$ is the energy radiated by an electron per unit frequency interval per unit solid angle divided by the time $T = L/V_b$ that the electron is being accelerated in the wiggler field. (Here L is the length of the interaction region.) We assume a circularly polarized radiation field $\delta E = \delta E(\hat{e}_x \pm i\hat{e}_y)exp(ikz - i\omega t)$ propagating in the z-direction, where ω and k are related by $\omega = kc$ in the tenuous beam limit. For observation along the z-axis, the spontaneous emission coefficient $\eta_\omega(x,p)$ is given by [18]

$$
\eta_\omega = \frac{1}{T}\frac{d^2 I}{d\omega d\Omega} = \frac{e^2\omega^2}{4\pi^2 c^3 T}\left|\int_0^T d\tau \hat{e}_z \times (\hat{e}_z \times v'_\perp)expi(kz' - \omega\tau)\right|^2. \qquad (11)
$$

In Eq. (11), the transverse orbit $v'_\perp$ is given by Eq. (10), and the factor $expi(kz' - \omega\tau)$can be expressed as

$$
\begin{aligned}
expi(kz' - \omega\tau) &= expi[kz - \epsilon(k/k_0)(k_0 V_b/\omega_\delta)sina] \\
&\times \sum_{\ell=-\infty}^{\infty} J_\ell\left(\epsilon\frac{k}{k_0}\frac{k_0 V_b}{\omega_\delta}\right) expi(kV_b - \omega + \ell\omega_\delta)\tau \quad exp(i\ell a). \qquad (12)
\end{aligned}
$$

Making use of Eqs. (10) and (12), the τ-integration in Eq. (11) can be carried out in a straightforward manner. This leads to two types of resonant factors contributing to η_ω:

$$
(\omega - kV_b \mp \omega_c - \ell\omega_\delta)^{-1} \qquad (Cyclotron Resonance),
$$

and

$$
[(\omega - (k \pm k_0)V_b - (\ell + m)\omega_\delta)]^{-1} \qquad (Beam Resonance).
$$

For present purposes, we examine the spontaneous emission coefficient η_ω for frequency ω near the fundamental beam resonance ($\ell + m = 0$ and $\omega - kV_b - k_0V_b \approx 0$) and the fundamental cyclotron resonance corresponding to right-hand circular polarization ($\ell = 0$ and $\omega - kV_b - \omega_c \approx 0$), and neglect those terms in η_ω corresponding to sideband emission at harmonics of ω_δ. Substituting Eqs. (10) and (12) into Eq. (11) then gives

$$\eta_\omega(x,p) = \frac{e^2\omega^2}{8\pi^2c^3T}\left|\left\{\sum_{m=-\infty}^{\infty} J_m\left(\epsilon\frac{k_0V_b}{\omega_\delta}\right)exp(ima)\left[\left(v_\perp expi(\phi - k_0z)\right.\right.\right.\right.$$
$$\left. - \frac{\omega_c}{k_0}\frac{\delta B}{B_0}\frac{k_0V_b + m\omega_\delta}{k_0V_b - \omega_c + m\omega_\delta}\right) J_0\left(\frac{k}{k_0}\epsilon\frac{k_0V_b}{\omega_\delta}\right)\frac{[expi(\omega_c + kV_b - \omega)T - 1]}{i(\omega_c + kV_b - \omega)}$$
$$+\left(\frac{\omega_c}{k_0}\frac{\delta B}{B_0}(-1)^m exp(-ima)J_m\left(\frac{k}{k_0}\epsilon\frac{k_0V_b}{\omega_\delta}\right)\frac{k_0V_b + m\omega_\delta}{k_0V_b - \omega_c + m\omega_\delta}\right)$$
$$\left.\left.\left.\times\frac{[expi(k_0V_b + kV_b - \omega)T - 1]}{i(k_0V_b + kV_b - \omega)}\right]\right\}\right|^2, \tag{13}$$

where $T = L/V_b$ is the length of time the beam spends in the interaction region, and use has been made of the identity $exp[i\epsilon(k_0V_b/\omega_\delta)sina] = \sum_{m=-\infty}^{\infty} J_m(\epsilon k_0V_b/\omega_\delta)exp(ima)$. In obtaining Eq. (13), we have retained only the $\ell = 0$ contribution in the cyclotron resonance term and the $\ell + m = 0$ contribution in the beam resonance term. We now consider Eq. (13) for $\epsilon k_0V_b/\omega_\delta < 1$ and retain only the lowest-order $m = 0$ contribution. (Typically, if we estimate $v_z \approx V_b$ and $v_\perp \approx V_b(\delta B/2B_0)^{1/3}$, then $\epsilon k_0V_b/\omega_\delta < 1/3$.) This gives the simplified expression for η_ω

$$\eta_\omega(x,p) = \frac{e^2\omega^2}{8\pi^2c^3T}J_0^2\left(\epsilon\frac{k_0V_b}{\omega_\delta}\right)J_0^2\left(\frac{k}{k_0}\epsilon\frac{k_0V_b}{\omega_\delta}\right)$$
$$\times\left|\left[v_\perp expi(\phi - k_0z) + \left(\frac{\delta B}{2B_0}\right)^{1/3}V_b\right]\frac{[expi(\omega_c + kV_b - \omega)T - 1]}{i(\omega_c + kV_b - \omega)}\right.$$
$$\left. - \left[\left(\frac{\delta B}{2B_0}\right)^{1/3}V_b\right]\frac{[expi(k_0V_b + kV_b - \omega)T - 1]}{i(k_0V_b + kV_b - \omega)}\right|^2 \tag{14}$$

where use has been made of $(\delta B/B_0)(\omega_c/k_0)k_0V_b/(k_0V_b - \omega_c) = -(\delta B/2B_0)^{1/3}V_b$ [Eq. (6)].

It is important to note that the wiggler field contributes a relatively large transverse velocity component $(\delta B/2B_0)^{1/3}V_b$ to the cyclotron maser term in Eq. (14). Indeed, it is evident from Eq. (14) that the cyclotron maser and FEL beam resonance contributions to the spontaneous emission η_ω are generally comparable in size and represent competing processes when the system is close to beam-cyclotron resonance ($k_0V_b \approx \omega_c$). For simplicity, we now consider Eq. (14) in circumstances where the system is either close to cyclotron resonance ($\omega - kV_b - \omega_c \simeq 0$) or close to beam resonance ($\omega - kV_b - k_0V_b \simeq 0$) and neglect the cross terms in Eq. (14). This gives

$$\begin{aligned}\eta_\omega(x,p) &= \eta_\omega^{CM}(x,p) + \eta_\omega^{FEL}(x,p)\\ &= \frac{e^2\omega^2 T}{8\pi^2 c^3} J_0^2\left(\epsilon\frac{k_0V_b}{\omega_\delta}\right) J_0^2\left(\frac{k}{k_0}\epsilon\frac{k_0V_b}{\omega_\delta}\right)\end{aligned}$$
$$\times\left\{\left[v_\perp^2 + \left(\frac{\delta B}{2B_0}\right)^{2/3} V_b^2 + 2v_\perp V_b\left(\frac{\delta B}{2B_0}\right)^{1/3} cos(\phi - k_0 z)\right]\frac{sin^2[(\omega - kV_b - \omega_c)T/2]}{[(\omega - kV_b - \omega_c)T/2]^2}\right.$$
$$\left. + \left(\frac{\delta B}{2B_0}\right)^{2/3} V_b^2 \frac{sin^2[(\omega - kV_b - k_0V_b)T/2]}{[(\omega - kV_b - k_0V_b)T/2]^2}\right\}. \tag{15}$$

We reiterate that the cyclotron maser and FEL contributions in Eq. (15) are generally competing processes. It is important to note, however, that one or other of the emission processes can be made to dominate by judicious choice of the length $L = V_bT$ of the interaction region. For example, for the FEL contribution to dominate with $\omega - kV_b - k_0V_b \simeq 0$, the cyclotron maser contribution in Eq. (15) is negligibly small whenever $(\omega - kV_b - \omega_c)(L/2V_b) \simeq n\pi$ where n is an integer and the first sin^2 factor in Eq. (15) is approximately zero. For $\omega - kV_b - k_0V_b \simeq 0$, this condition becomes $(k_0V_b - \omega_c)(L/2V_b) = -(\delta B/2B_0)^{2/3}(L\omega_c/V_b) \simeq n\pi$, which determines the critical length for suppression of the cyclotron maser effect. Here, use has been made of Eq. (6) to evaluate $k_0V_b - \omega_c$.

IV. AMPLITUDE GAIN IN TENUOUS BEAM LIMIT

With the expression for the spontaneous emission coefficient η_ω given in Eq. (15) the amplitude gain Γ can be determined from the classical limit of the Einstein coefficient method.[18] The amplitude gain per unit length is given by ($\Gamma > 0$ for amplification)

$$\Gamma = \frac{4\pi^3 c}{\omega^2} F \int_0^{2\pi} d\phi \int_{-\infty}^{\infty} dp_z \int_0^{\infty} dp_\perp p_\perp \eta_\omega$$
$$\times \frac{\gamma m}{p_\perp}\left[\left(\frac{\omega}{k} - v_z\right)\frac{\partial f_b^0}{\partial p_\perp} + v_\perp \frac{\partial f_b^0}{\partial p_z}\right], \tag{16}$$

where $f_b^0(p_\perp^2, p_z)$ is the equilibrium distribution function for the beam electrons, $\omega \simeq kc$ is assumed in the tenuous beam limit, $v_z = p_z/\gamma m$ and $v_\perp = p_\perp/\gamma m$ are the axial and transverse velocities, and $\gamma mc^2 = (m^2c^4 + c^2p_z^2 + c^2p_\perp^2)^{1/2}$ is the electron energy. In Eq. (16), we have introduced a phenomenological geometric filling factor F which is related to the ratio of the electron beam cross sectional area to the area of the emitted radiation.[3]

For present purposes, we consider an electron beam that is cold in the axial direction and has constant transverse momentum, i.e.,

$$f_b^0 = \frac{n_b}{2\pi p_\perp}\delta(p_\perp - \gamma_b m V_\perp^0)\delta(p_z - \gamma_b m V_b), \tag{17}$$

where $n_b = 2\pi \int_{-\infty}^{\infty} dp_z \int_0^\infty dp_\perp p_\perp f_b^0 = const$ is the beam density, and $\gamma_b = (1 - V_b^2/c^2 - {V_\perp^0}^2/c^2)^{-1/2}$. Substituting Eqs. (15) and (17) into Eq. (16) and integrating by parts with respect to p_z and $p_\perp$, we obtain for the gain Γ

$$\Gamma = -\frac{\omega\omega_{pb}^2 F L^2}{16\gamma_b c^3} G_0(b_\perp) \left\{\left(\frac{\delta B}{2B_0}\right)^{2/3}\left(\frac{\partial}{\partial\theta}\frac{sin^2\theta}{\theta^2}\right) + \left[\frac{{V_\perp^0}^2}{V_b^2} + \left(\frac{\delta B}{2B_0}\right)^{2/3}\right]\left(\frac{\partial}{\partial\psi}\frac{sin^2\psi}{\psi^2}\right)\right\} - \frac{\omega\omega_{pb}^2 F L^2}{16\gamma_b c^3}\left[a_1\left(\frac{sin^2\theta}{\theta^2}\right) + a_2\left(\frac{sin^2\psi}{\psi^2}\right)\right], \tag{18}$$

where $\omega_{pb}^2 = 4\pi n_b e^2/m$, and

$$\theta = (\omega - kV_b - k_0 V_b)T/2,$$

$$\psi = (\omega - kV_b - \omega_{cb})T/2,$$

$$G_0(b_\perp) = J_0^4(b_\perp) + 2\sum_{n=1}^{\infty} J_n^4(b_\perp), \tag{19}$$

with $T = L/V_b$, $\omega_{cb} = eB_0/\gamma_b mc$ and $b_\perp = (k/2k_0)(\omega_c/\omega_\delta)(\delta B/B_0)(k_0 V_\perp^0/\omega_\delta)$. In obtaining Eq. (18), we have approximated $J_0^2(\epsilon k_0 V_b/\omega_\delta) \simeq 1$ in Eq. (15) since $\epsilon k_0 V_b/\omega_\delta < 1$. On the other hand, the factor $J_0^2(\epsilon k V_b/\omega_\delta)$ has been retained since $(k/k_0)\epsilon(k_0 V_b/\omega_\delta)$ is typically of order unity because of the upshift in k/k_0. The factors a_1 and a_2 occurring in Eq. (18) are defined by

$$a_1 = \frac{2}{3}\left(1 - \frac{V_b}{c}\right)\frac{c}{\omega_{cb} L}\frac{V_b}{V_\perp^0}\left(\frac{\delta B}{2B_0}\right)^{1/3} G_1(b_\perp),$$

$$a_2 = 4\left(1 - \frac{V_b}{c}\right)\frac{c}{V_b}\frac{c}{\omega L}G_0(b_\perp) + \left(1 - \frac{V_b}{c}\right)\frac{2V_b}{3(\delta B/2B_0)^{1/3} V_\perp^0}\frac{c}{\omega_{cb} L}\left[\frac{{V_\perp^0}^2}{V_b^2} + \left(\frac{\delta B}{2B_0}\right)^{2/3}\right] G_1(b_\perp), \tag{20}$$

where $G_1(b_\perp) = \int_0^{2\pi} da/2\pi \sin a \quad J_0(2b_\perp sina)J_1(2b_\perp sina)$.

Equation (18) is valid for low gain ($\Gamma L < 1$) and $c/\omega L \ll 1$. A careful examination of Eqs. (18) - (21) shows that the terms proportional to a_1 and a_2 can be neglected for typical parameters of experimental interest, provided $G_0(b_\perp)$ is not too small ($b_\perp < 1.5$, for example). In this case, Eq. (18) can be approximated by

$$\Gamma = -\frac{\omega\omega_{pb}^2 FL^2}{16\gamma_b c^3}G_0(b_\perp)\left\{\left(\frac{\delta B}{2B_0}\right)^{2/3}\left(\frac{\partial}{\partial\theta}\frac{sin^2\theta}{\theta^2}\right)+\left[\frac{{V_\perp^0}^2}{V_b^2}+\left(\frac{\delta B}{2B_0}\right)^{2/3}\right]\left(\frac{\partial}{\partial\psi}\frac{sin^2\psi}{\psi^2}\right)\right\} \tag{21}$$

The first term in Eq. (21) corresponds to the FEL (beam resonance) contribution calculated near cyclotron resonance. The second term in Eq. (21) corresponds to the cyclotron maser contribution, including the important influence of the wiggler field. As indicated earlier, both effects are generally competing, although radiation generation by one process can be made to dominate over the other by judicious choice of the length L of the interaction region. For example, if the FEL contribution in Eq. (21) dominates, then the maximum gain occurs for $(\partial^2/\partial\theta^2)(sin^2\theta/\theta^2) = 0$, which gives $\theta \simeq 1.3$, The corresponding value of $(\partial/\partial\theta)(sin^2\theta/\theta^2)$ is -0.54, and the maximum FEL gain from Eq. (21) is

$$\Gamma_{MAX} = 0.54\frac{\omega\omega_{pb}^2 FL^2}{16\gamma_b c^3}G_0(b_\perp)\left(\frac{\delta B}{2B_0}\right)^{2/3}. \tag{22}$$

It is important to note that the FEL gain far from cyclotron resonance ($\omega - kV_b \simeq k_0V_b \gg \omega_c$) can be calculated for a tenuous beam in the low gain regime using entirely similar techniques [1—5]. The basic modification of the present analysis is to replace Eq. (7) by $z' = z+v_z\tau$, Eq. (10) by $v_x'+iv_y' = v_\perp expi(\omega_c\tau+\theta)+\omega_c(\delta B/B_0)(k_0v_z-\omega_c)^{-1}exp(ik_0z)\times [exp(ik_0v_z\tau) - exp(i\omega_c\tau)]$, and to assume that the beam has small transverse momentum with ${V_\perp^0}^2 \ll V_b^2$ in Eq. (17). The resulting value for the maximum FEL gain Γ_0 far from cyclotron resonance ($\omega - kV_b \simeq k_0V_b \gg \omega_c$) is given by [1—5]

$$\Gamma_0 = 0.54\frac{\omega_{pb}^2\omega FL^2}{16\gamma_b^3c^3}\left(\frac{\omega_{cb}}{ck_0}\right)^2\left(\frac{\delta B}{B_0}\right)^2. \tag{23}$$

Comparing Eqs. (22) and (23), we see that the FEL gain close to cyclotron resonance is enhanced relative to its value far from resonance by the factor

$$\frac{\Gamma_{MAX}}{\Gamma_0} = \frac{1}{4}\left(\frac{2B_0}{\delta B}\right)^{4/3}\gamma_b^2\left(\frac{ck_0}{\omega_{cb}}\right)^2 G_0(b_\perp). \tag{24}$$

For $G_0(b_\perp)$ of order unity, the gain enhancement in Eq. (24) can be several orders of magnitude, depending on the values of $\delta B/B_0, \gamma_b$, etc. The frequency upshift close to cyclotron resonance will be somewhat lower, however, because of the sizeable transverse energy. Solving $\omega - kV_b \simeq k_0V_b$ and $\omega \simeq kc$ gives

$$\omega = \frac{k_0V_b}{(1-V_b/c)} = \frac{(1+V_b/c)\gamma_b^2k_0V_b}{1+\gamma_b^2{V_\perp^0}^2/c^2} \qquad (25)$$

where $\gamma_b = \left(1 - V_b^2/c^2 - {V_\perp^0}^2/c^2\right)^{-1/2}$. If we estimate $V_\perp^0 \approx (\delta B/2B_0)^{1/3}V_b$ close to cyclotron resonance, then the ${V_\perp^0}^2$ reduction factor in Eq. (25) is negligibly small provided $(\delta B/2B_0)^{2/3}(V_b/c)^2 < 1/\gamma_b^2$.

Finally, we point out that the expression for the growth rate derived here is valid only when the small longitudinal and transverse spreads in electron momenta Δp_z and $\Delta p_\perp$, respectively, satisfy the inequalities $c/L\omega \gg (V_b/c)^2\Delta p_z/\gamma_b m V_b$ and $c/L\omega \gg (V_\perp^0/c)^2\Delta p_\perp/\gamma_b m V_\perp^0$.

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