NONLINEAR COUPLING OF LOWER HYBRID WAVES

AT THE EDGE OF TOKAMAK PLASMAS*

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Abstract

We solve the steady state coupling problem of lower hybrid waves excited by a waveguide array. The theory takes into account the pondermotive density modulation to all orders in the electric field amplitude but assumes that the nonlinear effects are important only along the magnetic field lines. It is shown that the important new feature is the appearance of a resonant term in the transverse refractive index, which is due to the finite size of the excitation structure. A calculation for a two waveguide array, linear density profile and constant temperature is presented, and we make a comparison with the experimental results.

The success of heating and steady state current drive by lower hybrid (LH) waves depends critically on the coupling at the plasma edge. This determines the reflection coefficient and the power spectrum of the excited wave. The waveguide array was recognized as the most efficient source, and the first experiments were in good agreement with the linear coupling theory [1,2]. Since the available area for the RF ports is small, significant tokamak heating can be achieved only when the power density is large ($> 1\ kW/cm$). At these power levels significant deviations from linear coupling have been observed. The reflection coefficient depends on the power density and becomes less sensitive to the phasing of adjacent waveguides [3,4]. In other experiments one has observed a change in the power spectrum, which is shifted to larger values of the longitudinal refractive index (n_z), [5]. The purpose of this paper is to present a theory which will explain, at least qualitatively, these results. The problem is treated analytically, and the method may serve as a guide toward an accurate numerical scheme.

High frequency, spatially modulated waves can produce a ponderomotive effect, which is usually measured by the density depression. This occurs when $V_t\tau/L >> 1$, where V_t is the electron thermal velocity, $\tau(L)$ are the time (space) scales of modulation of the electric field amplitude [6,7]. The pondermotive force acts on the electrons and the charge separation leads to an ambipolar potential. Its time variation is determined by the ion-acoustic frequency: $\omega_a \approx V_t/L\sqrt{m/M}$, where M is the ion mass. The balance of the pondermotive and ambipolar potential gives for the time rate of change of the electric field amplitude: $\tau \approx \omega_a^{-1}$. The spatial scale L at the edge of the plasma is of the order of the waveguide width. It is clear that the requirement for a ponderomotive effect is satisfied.

Since the coupling region is localized, the problem can be treated in a two dimensional geometry. The waveguide excitation produces at low densities, electric field components $E_z, E_x >> E_y$. To evaluate the non-linear interaction we define the amplitudes of the oscillating velocities: $V_{0z} = e|E_z|/m\omega$, $V_{0y} = e|E_x|/m\Omega_e$ where ω is the frequency of the source and Ω_e is the electron cyclotron frequency. For LH waves $\omega << \Omega_e$ and at low density $V_{0z} << V_{0y}$. Therefore the wave-particle interaction can be taken as linear in the direction perpendicular to B_T. For realistic values of $|E_z|$ in high power experiments at the plasma edge $V_{0z}/V_t \approx 1$. We can apply the one dimensional theory of the Vlasov equation to find the distribution function to all orders in the field amplitude [7]. Typically at the edge $\omega/k_z \simeq 20V_t$ and $V_{TR} \equiv 2\sqrt{\omega V_{0z}/k_z} \simeq 10V_t$, where k_z is a wavevector along B_T and V_{TR} is the half width of the island of trapped particles. It is clear that the interaction involves only the particles from the bulk. In that case the exponential profile modification is accurate [6]. We assume quasinentiality and the average density can be written as:

$$n = n_0 \exp\left[-\frac{1}{4m(T_e+T_i)}\left(\frac{e|E_z|}{\omega}\right)^2\right] \tag{1}$$

where $T_e(T_i)$ are the electron (ion) temperature. Recently there have been measurements of the density depression at high powers [8] and the exponential profile modification was confirmed. Particle simulations for a lower hybrid wave [9] also show the important role of pondermotive effects in the coupling.

The Maxwell equations for the steady state problem are:

$$\overline{\nabla} \times (\overline{\nabla} \times \overline{E}) = \frac{\omega^2}{c^2}\overline{\overline{K}} \cdot \overline{E}. \tag{2}$$

One can assume that at the edge $E_y \sim 0$ and $\overline{\overline{K}}$ is the dielectric tensor: $K_\perp = 1$, $K_x = 0$ and $K_\parallel = 1 - \tilde{\omega}_{pe}^2/\omega^2$. $\tilde{\omega}_{pe}$ is the nonlinear frequency: $\tilde{\omega}_{pe}^2 = 4\pi e^2 n/m$ and n is given by (1). From (2) one can write an equation for E_z, where we have introduced dimensionless space variables by normalizing to c/ω.

$$\frac{\partial^2 E_z}{\partial x^2} + \left(\frac{\partial^2}{\partial z^2} + 1\right)\left(1 - \frac{\tilde{\omega}_{pe}^2}{\omega^2}\right)E_z = 0. \tag{3}$$

This nonlinear Klein-Gordon equation was derived for the first time in Ref. 10 and has been the subject of intensive analytical and numerical work [11–14]. The work in [11–13] is essentially on a one-dimensional problem. These authors do not take into account the modulation along B_T, and their analysis is valid for a very long excitation structure (narrow spectrum in k_z). Besides the fact that this assumption is far from the experimental situation, it misses completely the fundamental point about the pondermotive effect. It is the modulation of the wave along B_T which leads to the density (1). A travelling plane wave in the z direction gives no pondermotive effect. Another point of a more technical nature is that one should not expand the density (1) in terms of V_{0z}/V_T. As we mentioned earlier, this ratio can be larger than 1. In Ref. 14 the problem was treated numerically, but no results on the reflection coefficient and power spectrum were presented.

We shall reduce Eq. (3) to a linear form by using the global properties of the electric field amplitude $|E_z|$. Note that the coupling is localized just in front of the waveguides. The modulational length along B_T is then simply the waveguide width (b). The linear theory [2] shows that the form of $|E_z|$ is simple, except for narrow spikes originating at the wall of adjacent waveguides. The coupling problem is not very sensitive to the density gradient in the x direction (the reflection is a cubic root of the scale length). Therefore, any x dependence in $|E_z|$ which modifies the scale length is not of great importance. It is reasonable to assume that the nonlinear density change is x independent.

Let us calculate the case of a two waveguide excitation. The scheme can be extended to many waveguides, with the obvious technical complications. The nonlinear density can be written as:

$$\frac{n}{n_0} = \exp\left[-\frac{1}{4m(T_e + T_i)}\left(\frac{eE_0}{m}\cos\frac{\pi z}{2b}\right)^2\right], \qquad |z| \le b$$
$$\frac{n}{n_0} = 1, \qquad |z| > b. \tag{4}$$

For E_0 one can use the value of the waveguide field amplitude in the fundamental mode. The Fourier transform of (3) in z is:

$$\frac{d^2\tilde{E}_z(x,n_z)}{dx^2} - \left(n_z^2 - 1\right)\left[1 - \frac{\omega_{pe}^2}{\omega^2}\int_{-\infty}^{\infty}\tilde{E}_z(x, n_z - n)\varphi(n)\,dn\right] = 0. \tag{5}$$

ω_{pe} is the linear plasma frequency. $\varphi(n)$ is the Fourier transform of the r.h.s. of (4):

$$\varphi(n) = \frac{1}{2\pi}\left(\int_{-\infty}^{-b} + \int_{b}^{\infty}\right)\exp(-inz)\,dz + \frac{1}{2\pi}\int_{-b}^{b}\exp(-\beta - inz - \beta\cos\frac{\pi z}{b})\,dz \tag{6}$$

where $\beta = \frac{1}{8m(T_e+T_i)}(eE_0/\omega)^2$. The integrals in (6) can easily be calculated and the result is:

$$\varphi(n) = \delta(n) - \frac{1}{\pi}\frac{\sin(nb)}{n} + \frac{1}{\pi}e^{-\beta}\sum_{p=-\infty}^{\infty} I_p(\beta)\frac{\sin(nb)}{n - \frac{p\pi}{b}} \tag{7}$$

where I_p is the Bessel function of imaginary argument. For a certain class of functions $\tilde{E}_z$, one can replace in the integral of Eq. (5) the resonant functions of the form $\frac{1}{\pi}\frac{\sin(bx)}{x}$ with the Dirac δ-function. This does not mean that we have gone to the limit $b \to \infty$, when the replacement is correct for any function $\tilde{E}_z$. If the electric field near the cutoff $x \sim x_c$ is of the form: $\tilde{E}_z \approx \sum_m [A_m \sin(a_m n_z) + B_m \cos(b_m n_z)]$ and all $a_m, b_m < b$, then one can show that this procedure is valid. The reader may verify the following identity for $a < b$:

$$\int_{-\infty}^{\infty} dn \sin[a(n_z - n)]\delta(n - \frac{p\pi}{b}) = \frac{(-1)^p}{\pi}\int_{-\infty}^{\infty} dn \sin[a(n_z - n)]\frac{\sin(nb)}{n - \frac{p\pi}{b}}. \tag{8}$$

The same identity holds when one replaces $\sin a(n_z - n)$ by $\cos a(n_z - n)$. It was checked by a numerical integration that the relationship (8) is reasonably well satisfied when one uses instead of $\sin a(n_z - n)$ the linear $\tilde{E}_z$ field, excited by a two waveguide array [2]. On the basis of these considerations, one can approximate Eq. (5) by substituting a new function $F(n)$ for $\varphi(n)$,

$$F(n) = e^{-\beta}\sum_{p=-\infty}^{\infty}(-1)^p I_p(\beta)\delta\left(n - \frac{p\pi}{b}\right). \tag{9}$$

Now (5) becomes a nonlocal second order differential equation:

$$\frac{d^2\tilde{E}_z(x,n_z)}{dx^2} - ({n_z}^2 - 1)\left[\tilde{E}_z(x,n_z) - \frac{\omega_{pe}^2}{\omega^2}e^{-\beta}\sum_{p=-\infty}^{\infty}(-1)^p I_p(\beta)\tilde{E}_z(x, n_z + \frac{p\pi}{b})\right] = 0. \qquad (10)$$

The important role of the pondermotive effect is that it couples different harmonics of the wave spectrum and is similar to a parametric process. Due to the modulation of the wave by the size of the waveguides (b), the nonlinear coupling is an interaction between a wave and its sidebands. If the width of the spectrum $\Delta n < 2\pi/b$, one may take into account only the nearest sideband. Thus the infinite sum in (10) will be reduced to three terms. In such a case we can find a local differential equation which corresponds to (10). Let us define a function $n_x(x, n_z)$:

$$\frac{d^2\tilde{E}_z(x,n_z)}{dx^2} = -{n_x}^2(x,n_z)\tilde{E}_z(x,n_z). \qquad (11)$$

For WKB soltions n_x has the meaning of a refractive index. However, our method is valid in general and no slow x dependence is required. With (11) substituted in (10) we can write three homogeneous equations for $\tilde{E}_z(x, n_z + p\pi/b)$, $p = 0, \pm 1$. (Note that higher harmonics $\tilde{E}(n, n_z \pm 2\pi/b)$ are neglected.) The system of equations will have a nonvanishing solution only when $n_x(x, n_z)$ satisfies a certain equation. It is important to stress that n_x is a function and the resulting equation is a functional equation. This procedure is similar to a diagonalization of a 3×3 matrix. The corresponding determinant is zero when:

$$n_x^2(x,n_z) = (n_z^2 - 1)\left\{A + B^2 \sum_{p=\pm 1} \frac{(n_z + \frac{p\pi}{b})^2 - 1}{n_z^2(x, n_z + \frac{p\pi}{b}) - A\left[(n_z + \frac{p\pi}{b})^2 - 1\right]}\right\} \qquad (12)$$

$$A = \frac{\omega_{pe}^2}{\omega^2}e^{-\beta}I_0(\beta) - 1, \qquad B = \frac{\omega_{pe}^2}{\omega^2}e^{-\beta}I_1(\beta).$$

The solution of (12) is:

$${n_x}^2(x,n_z) = ({n_z}^2 - 1)\left[A + \sqrt{2}B\left|\tan\left(\frac{n_z b}{2}\right)\right|\right]. \qquad (13)$$

Eq. (10) for the nearest sidebands can be written in the local form:

$$\frac{d^2 E_z}{dx^2} + ({n_z}^2 - 1)\left[A + \sqrt{2}B\left|\tan\left(\frac{n_z b}{2}\right)\right|\right]E_z = 0. \qquad (14)$$

The nonlinear process brings a resonant term in the refractive index at $n_z \approx \pi/b$. This will result in a change of the wave spectrum, which may lead to fillamentation: the original wave can be split into wavepackets with central values of n_z where the spectrum peaks. Furthermore, the power spectrum will be enhanced at those values of n_z. This is consistent with the experimental observations on Alcator-A (5). One may proceed as in linear theory [2] to determine the physically relevant quantities: R, the reflection coefficient and $S_x(n_z)$, the power spectrum. We solve the case of two waveguides and assume linear density gradient and homogenous temperature at the plasma edge.

R is defined as the reflection coefficient of the electric field. By following Ref. 2 we can write the following equation for R in the case of two waveguides with a phase difference of π.

$$\frac{1-R}{1+R} = \frac{a_2}{a_1 b}\left(\alpha e^{-\beta} I_0(\beta)\right)^{1/3}\left\{ i\int_0^1 dn \frac{K(n)}{(1-n^2)^{2/3}} + e^{-i\frac{\pi}{6}} \int_1^\infty dn\, \frac{K(n)}{(n^2-1)^{2/3}} \right\} \tag{15}$$

where

$$K(n) = \frac{(n - \frac{\pi}{2b}\sin nb)(\cos nb - 1)}{n\left[\left(\frac{\pi}{2b}\right)^2 - n^2\right]}\left[1 + \sqrt{2}\frac{I_1(\beta)}{I_0(\beta)}\left|\tan\left(\frac{nb}{2}\right)\right|\right]^{1/3},$$

$a_1 = .355$, $a_2 = .259$, and α is the density scale length at the cutoff, $\alpha = 1/\omega^2 \cdot d\omega_{pe}^2(x)/dx\Big|_{x_c}$.

Up to a trivial normalization factor, the power spectrum can be written as:

$$S_x(n_z) \simeq \left\{ \frac{n_z - \frac{\pi}{2b}\sin(n_z b)}{\left[n_z{}^2 - \left(\frac{\pi}{2b}\right)^2\right](n_z{}^2 - 1)^{1/3}} \right\}^2 \left[1 + \sqrt{2}\frac{I_1(\beta)}{I_0(\beta)}\left|\tan\frac{n_z b}{2}\right|\right]^{1/3}. \tag{16}$$

Note the peak which is introduced by the resonant term at $n_z = \pi/b$. Similar formulas can be written for any phase difference by simply following the linear theory.

On Fig. 1 we compare the results from the Petula experiment with two different limiters and the theory outlined above. The agreement is quite good for $\Delta\varphi = \pi$ up to very high power levels. For $\Delta\varphi = 0$ only a qualitative agreement can be claimed in the fact that $|R|^2$ decreases with increasing power.

On Fig. 2 we have plotted the power spectrum, and a peak at $n_z = \pi/b$ appears. Experimentally it was observed on the Alcator-A. We find the ponderomotive effects as the most natural explanation of this experimental result.

The present theory does not solve the basic equation (3). The lack of self-consistency is compensated by a relatively simple scheme, which emphasizes the relevant physics, explains qualitatively the observations, and can be generalized to the case of a more complicated array of waveguides. It is hoped that the many efforts to solve Eq. (3) numerically will cast more light on the limits of validity of various approximations. At the present time one can verify the assumptions *a posteriori.* The electric field which we obtain from (14) has a form consistent with the assumption for the amplitude modulation and the reduction of the integral Eq. (5) to the local form (14). We believe that the analytical theory, just described, may serve as a guide toward the fuller understanding of the coupling process.

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Figure Captions

Fig. 1. Reflection coefficient for PETULA parameters and comparison with some experimental data. We have $f = 1.25\ GHz$, $b = 1.8\ cm$ (= 0.47 in normalized units). Two kinds of limiters (Alumina and Tungten) were used in the experiments.

Fig. 2. Linear and nonlinear power spectra for ALCATOR-A parameters. Units are arbitrary, the spectra are normalized to be the same area. We have $b = 0.66$ and for the nonlinear case $V_{0z}^2/2V_T^2 = 10$.

(A.1.A)

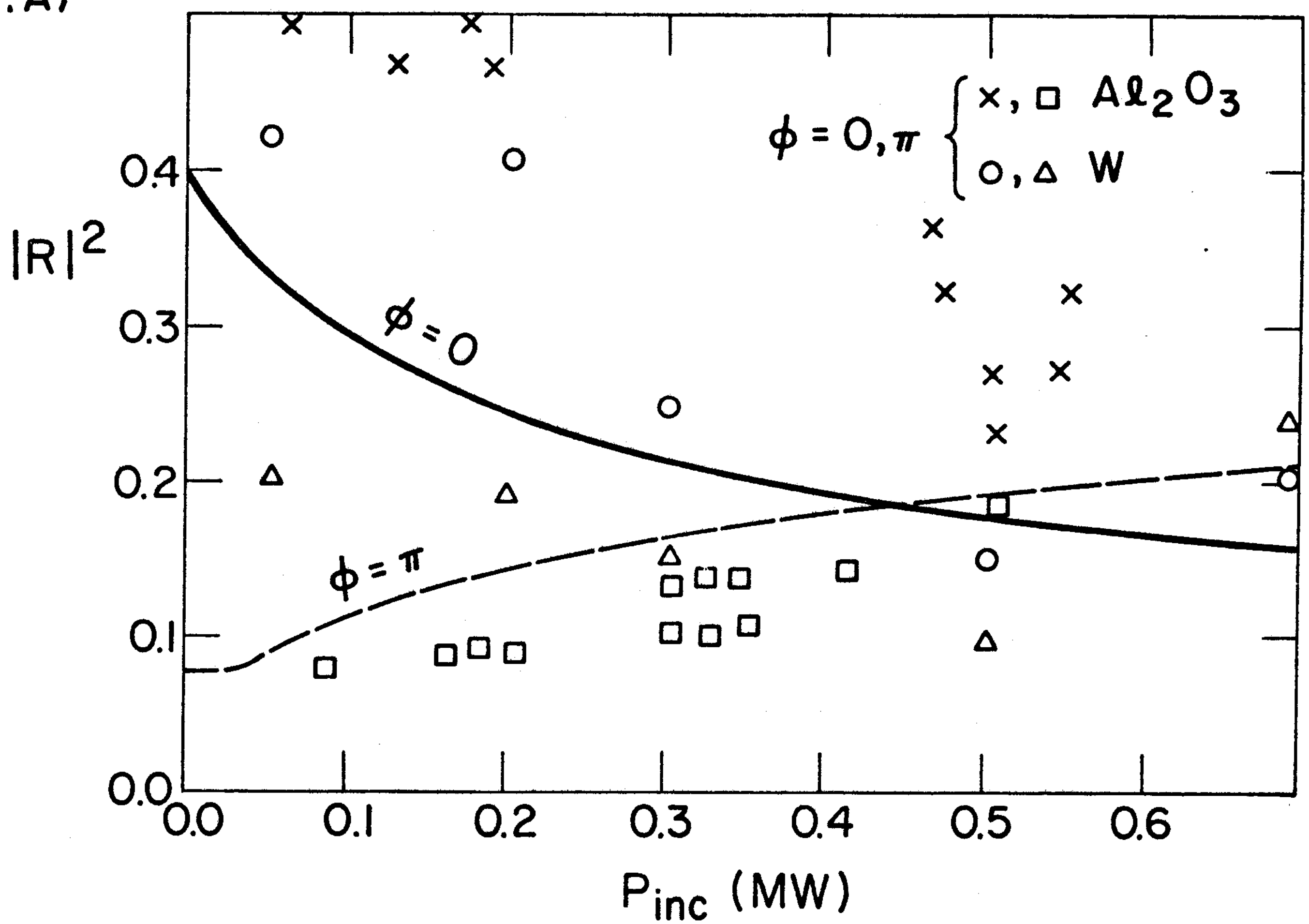

Figure 1

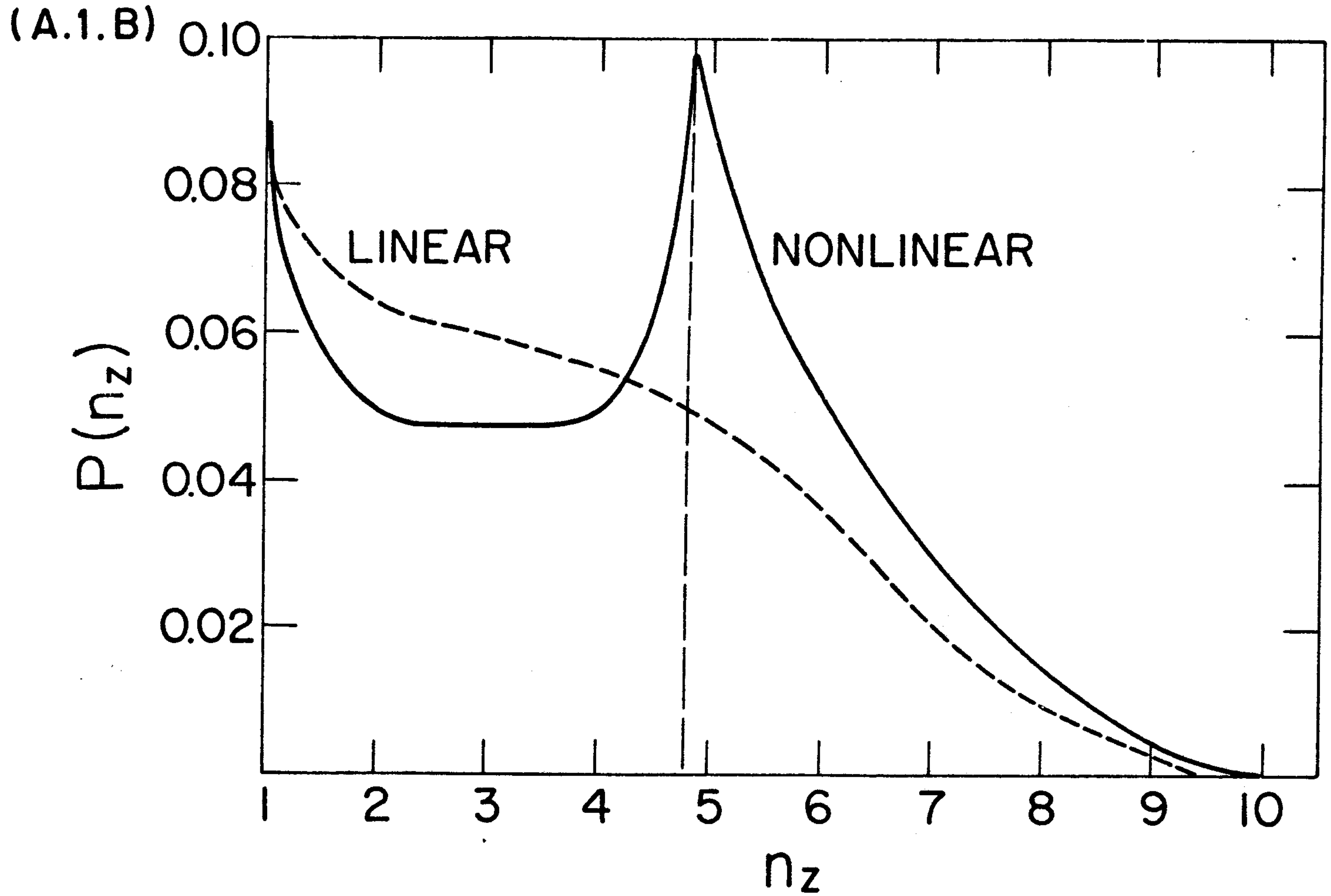

Figure 2