THEORY OF MODE-CONVERSION IN

WEAKLY INHOMOGENEOUS PLASMA*

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Theory of Mode-Conversion in Weakly Inhomogeneous Plasma

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Abstract

The theory of pair-wise coupled modes excited at a real frequency ω in a plasma which is weakly inhomogenous in one spatial dimension x, is developed on the basis of local Vlasov dispersion relations $D(k,z) = 0$, which define a many valued mapping of the real axis $x = \mathrm{Re}\, z$ onto the complex plane of the wavenumber k. Mode coupling is, by definition, the analytical continuation of a branch of the mapping, and only occurs at the branch points. This requires that the z-plane be cut along contours C_b given by $D(k_c, C_b) = 0$, $\frac{\partial D(k_c,z)}{\partial k} = 0$, where z traces a contour passing through the branch points. The coupled modes can be analyzed by expanding the dispersion relation to second order in k around the saddle points and along the lines $k_c(x)$, yielding a system of embedded dispersion relations corresponding to second-order differential equations possessing turning points at the appropriate branch points of the dispersion relation.

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I. Introduction

During the past few years the interest in heating tokamak plasmas to thermonuclear temperatures has accelerated the development and implementation of radio-frequency supplementary-heating methods, principally in the lower-hybrid[1] and ion-cyclotron[2] range of frequencies. Radio-frequency heating requires, for its proper implementation, a high degree of understanding of how the source couples to a nonuniform plasma. The reason is that the free-space electro-magnetic energy incident at the plasma edge undergoes a process of transformation into a normal plasma mode, which itself can successively mode-convert as the energy penetrates into the plasma. The possibilities of dissipation of the incident power are thereby greatly enhanced. In other words, the radio-frequency method of heating is based on, and relies on the existence of, mode-conversion.

The radio-frequency method of heating through the underlying mechanism of mode-conversion at the lower-hybrid frequency originated in the work of Stix[3] who showed that the cold plasma resonance is removed by adding warm plasma corrections to the dispersion relation, and that a warm ion mode is consequently born which couples to the original cold plasma mode. Adding ion-cyclotron motion will give rise to ion-Bernstein modes which couple to each other and to the already present warm ion mode. As the incident cold wave undergoes conversions, its wavevector perpendicular to the magnetic field increases, and the wave becomes more susceptible to being dissipated through collisionless processes such as perpendicular ion-Landau damping and/or ion cyclotron damping. At the heart of the theory is the kinetic Bernstein[4] dispersion relation, derived from the Vlasov-Maxwell equations by using a plane-wave assumption. The dispersion relation can be used in a nonuniform situation, and is then termed "local", if the eikonal approximation for the wavenumber can be made, that is, if the magnitude of the gradient scale-lengths of the non-uniformities greatly exceeds the local wave-length and, simultaneously, if magnitude of the local wavenumber exceeds its own gradient scale-length (WKB validity). The introduction of non-uniformities into the problem, even in this quite primitive fashion, has far-reaching consequences for a hot magneto-plasma, which is distinguished by a multiplicity of modes. In a uniform situation modes propagate independently throughout the entire spatial region. The introduction of non-uniformity makes the wavevectors vary in space and allows branches of the dispersion relation to couple. The "local" theory is very helpful in this regard as it determines the distribution of coupling points, that is, the distribution of the saddle points of the dispersion relation in the complex wavenumber k-plane and of the corresponding branch points in the complex space z-plane. Unfortunately, the local wavenumber changes very rapidly at the branch points and in consequence, the eikonal approximation breaks down there. It follows that the exchange of energy between the modes cannot be studied on the basis of the local approximation alone. The success of Stix's[3] theory, as well as of other works[5] applying his method to different situations, owes largely to the ingenious idea of attributing to the local dispersion relation a differential equation, usually one of the fourth-order, whose asymptotic solutions can be found by a suitable integral-transform method.[6] Phase-integral methods[7] have also been used[8] as they prove to be more flexible than the former technique. In either case, the method used avoids the necessity of having to cope with the branch-point singularities, and provides a technique by which one can study power flow, mode-conversion efficiency, etc. There are three limitations to the application of these methods. The first limitation is a purely technical one in that the treatment of differential equations of order greater than two with variable coefficients is difficult. The second is of a more fundamental nature and has to do with the foundation of the local method. Namely, the form of a differential equation corresponding to a local dispersion relation can be ambiguous as no *a priori* prescription is available for the commutation properties between functions and the differential operators multiplying them. The third limitation is the most restrictive one: the Vlasov-Maxwell model equations for a warm plasma are an integro-differential system whose corresponding dispersion relation contains the transcendental Plasma Dispersion Function Z. If there is Landau damping, and/or an asymptotic treatment of the Plasma Dispersion Function Z is not justified, then the dispersion relation cannot be approximated by a polynomial in the wavenumber, and consequently no direct representation in differential-equation form exists.

In order to advance toward the theoretical foundation of the local method, as well as to its applicability in situations of practical interest, it would be necessary, first, to clarify the analytical properties of the mapping represented by a local dispersion relation, second, to reduce the dispersion relation to a set of irreducible coupled-mode dispersion relations, and third, to formulate an unambiguous differential equation representation of the reduced system.

In spite of the fact that the problem of reducing a generally transcendental dispersion relation to a set of elementary dispersion relations describing the embedded coupling events is of great practical interest, it has not been, to our knowledge, previously addressed. As a matter of fact, it appears that to date not even the problem of reducing polynomial dispersion relations has been resolved to a satisfactory extent. The basic results for the polynomial problem are due to Heading.[9] He proceeds from the maxtrix formulation[10] of an n-th order linear differential equation with variable coefficients to obtain a matrix equation, whose diagonal elements give directly the WKB approximation $\sim \exp(\int q_i \, dx)$, q_i being the characteristic values of the original coefficient matrix. The WKB solutions represent waves in regions where they propagate independently, and are equivalent to the eikonal approximation, as the $q_i(x)$ can be shown to be the roots of the corresponding local dispersion relation. Departure from the local approximation occurs in regions where the $q_i(x)$ merge (typically pairwise), as certain non-diagonal elements become very large there, and, consequently, cannot be neglected. Any particular non-diagonal element will couple two modes. The equations thus coupled form Heading's approximation for the embedded coupling events. More recently, Chan *et al.*[8] observed that the WKB solutions of the fourth-order equation describing lower-hybrid to warm-plasma mode conversion permit the extraction of a complex phase factor common to both waves, thus reducing the solution to a potential which obeys a second-order turning point equation. This is a useful concept, worthy of generalization, as it embodies the fact that mode-conversion, on the one hand, will produce a backward propagating wave, like specular reflection at a cut-off, but, on the other hand, in contrast to reflection of a cut-off, will occur at a non-zero value of the wavenumber. Heading's work does, in fact, give an answer, in very general terms, to the previous question of splitting the wave-potential into common and coupling parts. We will now state Heading's basic result as it is very instructive and clarifies the concept of embedding. Let two k-roots, q_1 and q_2, of the characteristic equation (equivalent to the local dispersion relation) of the coefficient matrix merge at the point z_0. Let further $q_{1,2} = \alpha \pm \beta$ around z_0, and $\beta(z_0) = 0$. Then, the coupling of modes corresponding to the roots $q_{1,2}$ is described by the wave-potential $\Phi = F_c\varphi$, where φ is the solution of the second-order differential equation $\varphi'' - \beta^2\varphi = 0$, and $F_c = \exp(\int \alpha\, dz)$. This result exhibits very clearly the manner in which the wave-potential is made up of the complex phase factor F_c, common to both waves, and the coupling potential φ.

The next step, going beyond Heading's work is to actually determine the forms of the common wavenumber α, and of the scattering potential β^2. This is a difficult task, in general, inasmuch as the merging of roots of higher-order algebraic equations and of transcendental equations in general cannot be followed analytically. We therefore propose in the present work to extract the embedded coupling equations in a manner independent of Heading's, namely making use of the basic analytical properties of the mapping $z \to k$ represented by the dispersion relation $D(k, z) = 0$. The method is not only inherently more suitable for gaining insight into the formal and physical structure of mode-coupling, but also leads to an approximation for the coupling characteristics $\alpha(z)$ and $\beta^2(z)$. Moreover, our theory is not restricted to dispersion relations which are polynomials in the wavenumber k. Basically, we expand the dispersion relation to appropriate order in the wavenumber k around the saddle points of the mapping and along lines in the k-plane which are the maps of branch cuts from the plane of the independent spatial variable. We will refer to these lines as mode-boundaries. The resulting embedded dispersion relations are in a one-to-one correspondence with differential equations possessing turning points at the appropriate branch points. The mode-boundaries around which we expand the dispersion relation give the values of wavenumber common to the coupled modes and also determine the common phase factor. Heading's theory and ours differ not only in the analytic-theoretical aspects of viewing the problem, but on the practical side, once the analytic properties of the $z \to k$ mapping are understood, the extraction of the embedded equations is here achieved by simple operations on the dispersion relation. Moreover, we obtain the full scattering potential for a coupling event, namely in terms of the dispersion relation expanded around the appropriate mode-boundary.

It is of necessity that the mode-boundaries be maps of the branch cuts, and their determination is therefore a prime objective of this study. The concept of a mode-boundary which we have introduced here is quite natural at the more fundamental level involving the Riemann surface of the mapping. Each branch of the dispersion relation in the k-plane is represented by a Riemann sheet lying over the z-plane (which itself corresponds to the principal branch). Imagine now the mapping of the Riemann surface onto the k-plane. Each Riemann sheet in this one-to-one mapping maps onto a distinct region of the k-plane bounded by contours which are the maps of the branch cuts in the z-plane. These boundary contours are the mode-boundaries. The Riemann surface itself allows the analytic continuation of the principal branch. The order in which the Riemann sheets are stacked is determined by the algebraic pattern of power flow between the branches. In what ratio the power flux, incoming into a branch point, will actually distribute itself is given by the corresponding embedded differential equation.

We limit ourselves to the case of pair-wise coupled modes, as the coupling of three or more modes occurs exceptionally. We consider stable waves, excited by a steady source operating at a frequency ω , and we work in one spatial dimension, assuming that the plasma is uniform in the others.

II. Basic Considerations

Consider a plasma which is weakly nonuniform in one spatial dimension x, supporting waves excited at a fixed frequency ω. The local dispersion relation

$$D(k, z; \omega) = 0 \tag{1}$$

representing these waves as they propagate under the influence of nonuniformities, defines a many valued mapping of the complex spatial variable $z = x + iy$ onto the complex wavenumber plane $k = k_r + ik_i$. In Eq. (1) we have defined $k \equiv k_x$, and we have suppressed the explicit dependence on parameters such as the other two components of the wavevector, etc.; y and z are not to be confused with spatial dimensions. We will assume for now that the inverse mapping $k \to z$ is single-valued. It poses no principal difficulty to relax this assumption, and on occasion we will do so, since the case of many valued z is frequently encountered in practice.

An arbitrary contour C in the z-plane thus maps onto n branches, $n > 1$,

$$k = f(z) \tag{2}$$

and with such a mapping there are associated saddle and branch points, k_s and z_B, respectively. The branch points z_B are those for which $\frac{df}{dz}$ diverges, and they are, by definition, the only ones which map onto a single point k_s in the k-plane, the latter being the saddle points of the inverse mapping

$$z = f^{-1}(k). \tag{3}$$

It is at the saddle points where the branches intersect.

To visualize the process of branching it is helpful to draw contour-plots of $x = \mathrm{Re}\, f^{-1}(k)$. The topology of the contours around a saddle point is shown in Fig. 1, which also serves to illustrate another point, namely that the usual, intuitive concept of mode coupling associated with the process of two real modes merging to become a complex conjugate pair (Fig. 2) is just a very special case of the general situation outlined above. Recently Tang[11] defined and discussed mode-conversion points, in the lower-hybrid range, as those at which x, given by dispersion relation, has a local extremum at some real value of k. We see here that if no such extremum exists for a real value of k, there certainly must exist one for complex k along a line of steepest descent, $y = \mathrm{Im}\, z = \text{const}$. It is also easy to understand that if there are one or two saddle points, they map, respectively, onto one or two distinct branch points. If there are more than two saddle points, there cannot be less than two branch points. To see this, imagine passing from the ridge of a saddle to another, lying elsewhere at the same height x. The passage must, of course, go through a valley belonging to a saddle lying at a different height.

The general procedure of mapping $z \rightarrow k$ on the dispersion relation involves two steps. The first is to determine the saddle points k_s and branch points z_B. The second is to specify the branch lines and to construct the Riemann surface by joining the individual, properly cut, Riemann sheets.

The saddle and branch points are inherent to the mapping. Each saddle point defines a distinct coupling event between the branches; we will assume throughout that the branches couple pair-wise, which occurs when

$$\frac{df^{-1}}{dk} = 0, \qquad \frac{d^2 f^{-1}}{dk^2} \neq 0. \tag{4}$$

If the mapping $k \rightarrow z$ is single-valued, then each saddle point corresponds to just one branch point

$$z_B = f^{-1}(k_s). \tag{5}$$

The problem of locating the saddle points is of a purely technical nature. What can be said generally is that the number of saddle points is correlated with the pattern of intersection between the branches, which can become rather complicated when the mapping is many valued in both directions. It is a useful excercise to consider, in this regard, polynomial dispersion relations in the variable k,

$$D \equiv \sum_{l=0}^{L} P_l(z) k^l, \tag{6}$$

having polynomial coefficients in z, all of the same order N. In the simplest case where just one coefficient, say $P_r(z)$, is not a constant, the number of saddle points is $L - 1$ when $r = 0$, and L for any other value of r. This particular case is instructive in that it also provides a picture of what happens in the most simple case of many valued $k \to z$ maps. Namely, if $w = P_r(z)$ has N branches $z = P_r^{-1}(w)$, then each saddle point of the map $k \to w$ will correspond to one branch point, but the composite map $k \to w \to z$ will give N branch points for each saddle point. A many valued map $k \to z$ is therefore interpreted as follows: each coupling event between two particular branches in the k-plane takes place at more than one location in the z-plane. The practical significance of this will become clearer in Section IV. where we discuss differential equations describing the coupling between two branches.

The second step in the analysis of the dispersion relation regarding the construction of the Riemann surface is conceptually — if not technically — the crucial one. On the one hand, the branch cuts define the process of mode coupling and the process of analytic continuation of a branch, but on the other, the dispersion relation alone does not provide any ready-made information on how to specify the branch lines. This is, of course, a source of major concern since mode coupling occurs, by definition, every time the wave-propagation contour C (the real axis $x = \mathrm{Re}\, z$) crosses a branch cut. The intuitive reaction in this situation would be to trace the branch cuts in some arbitrary manner away from the real axis, to avoid crossings other than through the branch points themselves. But what if all branch points lie away from the real axis? Does this imply that now no mode coupling takes place? The answer to these questions, and to the problem of constructing the Riemann surface, lies in the proper definition of mode-coupling and of a mode itself. The relevant question to ask is which are the physical conditions for coupling and find their analytical counterparts.

Before we pass to the discussion of this problem in Section III., it will be useful to familiarize ourselves with the technical side of the $z \to k$ mappings, as well as with the technique of constructing the Riemann surface. Consider, for example, a triple-valued mapping possessing two distinct saddle points and two corresponding branch points. We will now remove the restriction of the propagation contour C having to coincide with the real axis, as all that matters, as far as coupling of the branches is concerned, is the mutual configuration of C and the branch cuts. The mapping we are considering represents pair-wise coupling of the branches. The Riemann surface of such a mapping is shown in Fig. 3. To start a process of analytic continuation we first determine the principal branch: it is the branch corresponding to the physical boundary condition in the problem. For plasma heating at the lower-hybrid frequency, for example, the principal branch is the cold lower-hybrid wave, as it is that which carries energy into the system. Let the principal branch corresponds to the first Riemann sheet, RS(I). The process of analytic continuation can be viewed as follows: when the contour C passes through the first branch point z_{B1}, coupling between the first two branches occurs in a manner shown in Fig. 4 for the contour C. The contour is then continued on the second Riemann sheet which, in turn, must be cut twice, since z_{B2} couples the branch corresponding to this sheet with the one represented by the third sheet. For a more general pair-wise coupled system, the situation would be similar: the sheets are stacked in that order in which the branches intersect, and are cut in a manner as to allow access only to the nearest neighbors. We stress that the ordering has nothing to do with the order in which branch points are encountered as the propagation contour C is traced in one direction.

We now examine what happens if the contour C does not pass through a branch point. The two possibilities are shown in Figure 4. For a contour $\tilde{C}$ not crossing the branch cut, C_b, the branches $f(\tilde{C})$ must lie, unconnected, on either side of the map, $f(C_b)$, of the cut. A branch cut itself maps onto the k-plane as extending from the saddle point k_s into two directions, and represents the boundary between the maps of the two Riemann sheets it connects. In the other case, of C' crossing the branch cut, the branches connect through the boundary $f(C_b)$ at two points signifying mode-coupling events. Two passages through the same cut will bring us back onto the original sheet as shown in Fig. 5.

It is useful to note, at this point, that our assumption of pair-wise coupling has reduced the problem of the n-valued mapping into a series of double-valued maps, each of which are easily resolved. If the mapping appears as in Fig. 6, for example, then since A and A' represent the same point in the z-plane, we will remember that the branches colliding with oppositely oriented arrowheads describe waves propagating in the same regions of space.

With the elementary operations described in Figs. 4-6 we can interpret any mapping which describes pair-wise coupled modes. Consider, for example, a triple-valued mapping which posesses three saddle points, and which gives a pattern of branches and boundaries as in Fig. 7. If power is incident along the first branch, it can mode-convert at k_{s1}; if the mode-conversion is not complete, a fraction of it will continue to flow along the first branch to possibly mode-convert again at k_{s2}. No power flows from k_{s2} to k_{s3}. The mode-converted power from k_{s1} is partially channeled into the third branch at k_{s3}, but a fraction of it will flow into k_{s2}. From there, the power which does not flow away along the second branch mode converts to the first branch. Each saddle point thus represents a coupling event occurring as many times as there is an influx of power into it. Whether a possible mode-conversion event actually takes place, and in what ratio the power incident at a coupling point will distribute itself between the branches, depends on the spatially asymptotic states of the coupled branches. The principal selection rule is that in a passive system driven at a real frequency, a branch corresponding to a spatially amplified mode cannot be excited.

It is instructive to consider some simple examples at a more definite level. Most cases of physical interest will be described by dispersion relations which have an even number of branches k. This embodies the fact that a physical mode itself should be represented by a pair of branches, allowing for propagation, in principle, along both orientations of the x-axis. A familiar example would be the lower-hybrid dispersion relation considered by Stix[3]

$$k^4 - 2zk^2 + b = 0 \tag{7}$$

where b is a positive constant corresponding to the parallel component $K_{||}$ of the dielectric tensor, and where the perpendicular component $K_{\perp}$ has been linearized around the cold-plasma resonant point $z = 0$. Another example is the dispersion relation corresponding to differential equations considered by Swanson,[12] Ngan and Swanson,[5] and Faulconer[13]

$$k^4 - 2zk^2 + 2z - a = 0, \tag{8}$$

describing the mode-conversion and tunnelling near the two-ion hybrid frequency as well as mode-conversion below the second electron cyclotron harmonic and in the neighborhood of the first harmonic of the ion-cyclotron resonance. The mappings of the real axis $x = \mathrm{Re}\, z$, given by Eqs. (7) and (8) are shown in Figs. 8, 9 and 10, respectively. Fig. 9 corresponds to $a < 0$, Fig. 10 to $a > 0$. The solid and dashed lines distinguish between different physical modes. Let us consider in some detail the example of Eq. (7), Fig. 8. Let the boundary condition specify that power flows into the system along a lower-hybrid branch in the direction from $x \to +\infty$ toward $x = z_{B1}$. The selection of $k > 0$ for this branch specifies the principal branch as the dashed branch lying in the fourth quadrant. As the wave propagates in, we follow this branch along the real axis toward $k_{s1} = +b^{1/4}$. Once the wave arrives at $z_{B1} = +b^{1/2}$ it mode-converts into a warm-plasma wave represented by the solid line going away from k_{s1} along the real axis, corresponding to a mode propagating backward from z_{B1}. However, the branch point z_{B1} has another saddle point asociated with it, namely $k_{s3} = -b^{1/4}$. Whatever coupling materializes at k_{s3} is determined by the continuation of the principal branch. The first analytic continuation is the warm-plasma branch which goes away from k_{s1} along the real axis; that branch does not mode-convert again. We now follow the principal branch along the dashed line toward $k_{s4} = -ib^{1/4}$, corresponding to an evanescent lower-hybrid branch. At k_{s4} this lower-hybrid branch can be analytically continued into the third quadrant, corresponding to warm plasma branch evanescent in the direction from z_{B1} toward $x = -\infty$. The continuation must terminate at k_{s3}, since the branch in the second quadrant corresponds to a growing mode, and the warm-plasma branch going away from k_{s3} along the real axis designates a warm wave incoming toward z_{B1} from $x = +\infty$. We conclude that a lower-hybrid wave incident at z_{B1} from $x = +\infty$ generates a backward-propagating warm-plasma wave, and evanescent lower-hybrid and warm-plasma modes in the direction from z_{B1} toward $x = -\infty$. The coupling diagram of Fig. 9 can be analyzed in the same fashion. The main difference we see is that specular reflection is now possible since two branches of the same physical mode can interact, namely at the saddle point $k_s = 0$. The mapping of Fig. 10 also produces specular reflection, but no mode-coupling appears to occur. Another fact which can be seen from these mappings is that the location of the cold plasma resonance, $x = 0$, now becomes a point of no significance to the coupling. There appears to be some confusion in the literature on this matter as, for example, Ngan and Swanson[5] refer to $x = 0$ as a turning point. It should be stressed that asymptotic regions of wave-propagation are determined with respect to the positions only of the branch points.

The example of Fig. 10 brings to our attention the following problem. The mapping of the real axis does not indicate coupling, while asymptotic solutions[13] of the corresponding differential equation do exhibit some degree of mode-conversion. In order to see how the coupling materializes in such situations, the real axis $x = \mathrm{Re}\, z$ must be deformed and traced through the branch points. We will elaborate more on this matter in the next section.

III. Coupling Points and Branch Cuts

We will now return to the discussion of the role of the branch cuts. Branches couple, by definition, whenever the contour C in the z-plane crosses a branch cut. The usual definition of a mode, in terms of a branch of the dispersion relation, is thus not complete without associating with it a properly cut Riemann sheet. The Riemann sheet then maps onto a simply connected region of the k-plane, whose boundaries (given by the branch cuts) limit the extent of wavenumbers accessible to the branch under deformations of the x-axis. We will show that the branch cuts C_b and the mode-boundaries k_c are given by the mappings

$$D(k_c, C_b) = 0, \qquad \frac{\partial D(k_c, z)}{\partial k} = 0, \qquad z \in C, \tag{9}$$

where C is required to pass through the branch points and, therefore, might occasionally deviate from the real x-axis. The only coupling events which materialize under these conditions are those defined by the branch (and saddle) points of the dispersion relation. We emphasize that it is actually through the mapping $\frac{\partial D}{\partial k} = 0$ that we make contact with the physical requirements for mode-conversion.

Our argument is as follows. Consider a wave $A(x)\exp\left[-i\omega t + ik_r(x)x\right]$ propagating along the x-direction, and characterized by a slowly varying amplitude A (incorporating the effect of $k_i(x)$) and wavenumber $k_r(x)$. The wave has associated with it an averaged energy flux proportional to $v_g AA^*$, where A^* is the conjugate to A, and v_g is the local group velocity

$$v_g = \frac{d\omega}{dk} = -\frac{\frac{\partial D}{\partial k}}{\frac{\partial D}{\partial \omega}}, \tag{10}$$

evaluated implicitly from the dispersion relation $D(k, z; \omega) = 0$. The energy flux thus defined depends on position not only through the amplitude A, but also via v_g. Under these conditions, energy ceases to flow in the form of a particular mode where either $A \to 0$ or $v_g \to 0$. The former case is indicative of dissipative processes and A vanishes gradually. We are concerned with the latter case as that is the one which is indicative of mode-conversion, at least as far as the coupling of energy between different wave-types is concerned. This is because when $v_g \to 0$ without A vanishing simultaneously, the global conservation of energy flux requires that the energy be either transformed into a different wave form, or the amplitude diverges and we have a resonance. Cut-offs ($k \to 0$) and cold-plasma resonances ($k \to \infty$) are special cases of the singular points we are considering here. The preceding argument for the coupling to occur at a branch point rather than at any other point on a branch cut can be reinforced by inspection of the coupling event in the k-plane, Fig. 4. We see that the coupling caused by the contour C' crossing the branch cut will separate the coupled wavenumbers, thus introducing a phase-mismatch between the coupled waves. We may also say that coupling across a branch cut away from a branch point does not conserve momentum associated with the coupled waves.

Let us now identify the contours k_c in the k-plane where $\frac{\partial D}{\partial k}$ vanishes for a contour C of the z-plane. It is obvious, from the definition (10), that the $k_c(z)$ are the loci of saddle points of the mapping $k \to \omega$, or, equivalently, of the mapping $w = \mathfrak{D}(k) \equiv D(k, z)$, where z is swept through C. It follows that the contours C_b given by Eqs. (9) are indeed the required branch cuts. Recalling now that, for any particular ω the saddle and branch points of the mapping (1) are given by the system

$$D(k,z)=0, \qquad \frac{\partial D(k,z)}{\partial k}=0, \tag{11}$$

we see that the mapping of the contour C

$$\begin{aligned} D(k,z)=0; \qquad z\in C \\ k=f(C) \end{aligned} \tag{12}$$

has no common points with the mapping

$$\begin{aligned} \frac{\partial D(k,z)}{\partial k}=0; \qquad z\in C \\ k=g(C) \end{aligned} \tag{13}$$

except for the saddle and branch points which are therefore the only coupling points. Also, the lines $g(C)$ form the natural boundaries in the k-plane for the branches $f(C)$ of the dispersion relation, since they never cross the branches except at the saddle points. As long as all the branch points lie on the contour C, we cut the z-plane along the lines

$$z=f^{-1}[g(C)]. \tag{14}$$

As a result, the modes will only couple at the branch points as other coupling events can not materialize with the branch cuts so defined. However, a difficulty arises if a branch point is not on C as, for instance, in the example of Fig. 10. Then also the corresponding saddle point is not on $f(C)$ and $g(C)$, with the result that the line (14) can no longer be a branch cut. As a matter of fact, this line now loops around the branch point, as illustrated in Fig. 11. Apparently, no coupling now occurs. This is, however, in contradiction with the fact that differential equations possessing generally complex turning points do exhibit coupling. To remedy this shortcoming of local dispersion relations we have to allow the contour to pass through the branch point. It is worth mentioning that when the mapping between k and z is real-analytic (mappings (7) and (8), for example), then the maps $f(x)$ and $g(x)$, of course, coincide. This degeneracy can be easily removed, if we so desire, by adding to x an infinitesimal imaginary part.

We have shown, essentially, that local dispersion relations are consistent with the underlying physics of mode-conversion only if the z-plane is cut along contours given by Eq. (9). In the next section we will show how the branch points and branch cuts are instrumental in obtaining a representation of the dispersion relation in the form of a system of second-order differential equations.

IV. Coupled-Mode Equations

The principal practical application of the analysis of the previous sections is that it lays the foundation for relating to the dispersion relation a system of second-order differential equations describing the elementary pair-wise coupling events. Regarding the uniqueness of the differential-equation representation, we require that the branch points of the dispersion relation become turning points of the differential equations. We will return to this problem later.

The simplest and straightforward differential-equation representation of a pair of modes which couple at the saddle and branch points k_s and z_B, is obtained by expanding the dispersion relation to second-order in k and to first order in z, and substituting thereafter $i\frac{d}{dz}$ for k. The equation thus obtained has a turning point at $z = z_B$, but it is an unsatisfactory representation for a number of reasons. First, the approximation is only valid in a close neighborhood of the coupling points, so that if the branch point z_B is complex, it is not clear whether and how the equation can be continued to its physical domain: the real axis $x = \mathrm{Re}\, z$. Second, if the mapping $k \rightarrow z$ is itself many-valued as, for example, when $D(k, z)$ is a quadratic function of z, then every saddle point corresponds to more than one branch point (to two in the quadratic case). Each such branch point is represented by a separate turning-point equation. This description of the coupling is generally incorrect as it is well known that certain configuations of turning points have to be treated as groups, rather than separately, to obtain the correct wave-propagation characteristics. Examples of such groups are turning points corresponding to potential barriers or wells, which are the scattering potentials corresponding to double-valued $k \rightarrow z$ maps. This particular shortcoming of the straight-forward expansion technique appears to be the most serious limitation of the representation in terms of its applicability to situations of practical interest. To remove the above restrictions, we will expand the dispersion relation around the mode-boundaries $k = g(z)$, defined by Eq. (13), rather than around the coupling points themselves. In doing so we can follow, if necessary, the approximation from a complex branch point all the way to the real axis. Moreover, we obtain the explicit form of the appropriate scattering potential giving all the turning points relevant to the coupling of two particular modes. Finally, we obtain, at no extra cost, the common part of the wave-potential.

To proceed, let us define the function

$$\mathfrak{I}(k) \equiv D(k, z) \tag{15}$$

of the independant variable k, with z as a parameter. To obtain an approximation for the two roots k of the dispersion relation which couple at $k = k_s$, we expand $\mathfrak{I}(k)$ around that mode-boundary $k_c(z)$,

$$k_c(z) \equiv g(C), \tag{16}$$

defined by the mapping (13), for which $k_c(z_B) = k_s$. Since $k_c(z)$ is a saddle point of the function $\mathfrak{D}(k)$, we thus obtain to second order in k for all z:

$$\mathfrak{D}(k) = \mathfrak{D}(k_c) + \frac{1}{2}(k - k_c)^2 \mathfrak{D}''(k_c). \tag{17}$$

This approximation for $\mathfrak{D}(k)$ is excellent near the coupling points k_s and z_B, by virtue of the definition of $k_c(z)$, providing that $\mathfrak{D}''(k_s) \neq 0$. Moreover, it is a good approximation away from k_s and z_B, as long as $\mathfrak{D}''(k_c)$ does not vanish in the region of interest. The latter condition is particularly important when the branch point is not real, since then the approximation must be extended all the way from z_B to the real axis, which is where the waves actually propagate. It follows that the particular path in the z-plane must avoid passing through a branch point of the mapping (13), since $\mathfrak{D}''(k_c)$ vanishes there and $k_c(z)$ joins onto a different branch. Except for this restriction, the path can be quite arbitrary, and the transition to the real axis can be therefore performed by substituting x for z and making sure that $k_c(x)$ belongs to the correct branch as specified by the appropriate saddle point $k_c(z_B) = k_s$. The dispersion relation $\mathfrak{D}(k) = 0$ on the real axis Re z is then approximated by

$$\mathfrak{D}[k_c(x)] + \frac{1}{2}[k - k_c(x)]^2 \mathfrak{D}''[k_c(x)] = 0, \tag{18}$$

which can also be written as

$$k^2 - 2k_c(x)k + k_c^2(x) - Q(x) = 0, \tag{19}$$

where

$$Q(x) = -2\,\mathfrak{D}(k)/\mathfrak{D}''(k)\big|_{k_c(x)}. \tag{20}$$

is the scattering potential. We now note that Eq. (19) contains the ambiguous combination $k_c k$ which permits the interpretations ($k \rightarrow i\frac{d}{dx}$)

$$ik_c\frac{d\Phi}{dx}, \qquad i\frac{d}{dx}(k_c\Phi) \tag{21}$$

or some linear combination of these terms, where Φ is the wave-potential. The problem of uniqueness of the representation thus arises. This leads us to examine more closely the nature of the branch points of the dispersion relation.

We will show, in particular, that the branch points of the dispersion relation are in fact the regular (as opposed to singular) turning points of the corresponding wave-propagation problem. We have already shown that the individual branches, $f(C)$, of the dispersion relation $D = 0$ lie within the interior of the non-intersecting regions bounded by lines $g(C)$ on which $\frac{\partial D}{\partial k}$ vanishes. Thus, the lines $g(C)$ are the contours on which both the real and imaginary parts, u and v respectively, of $\frac{\partial D}{\partial k}$ vanish. We now exploit a basic property of harmonic functions, namely, that a sufficiently small surface of u centered around a point k_0 lying on a contour $u(k_0) = u_0$ is divided by the contour into an even number of sectors, 2n, in which u is alternately smaller and larger than u_0 where n is the multiplicity of the zero at $u - u_0$. Since along the boundaries $\frac{\partial^2 D}{\partial k^2}$ can vanish only exceptionally, namely at points where the boundaries intersect, we have n = 1, and hence u changes sign when we cross the boundary $g(C)$. By the principle of analytic continuation, it follows that u must preserve its sign throughout the entire bounded region corresponding to one Riemann sheet. Thus, each branch of the dispersion relation has a group-velocity of definite sign associated with it. Moreover, branches which correspond to neighboring Riemann sheets have oppositely directed group velocities, and since the group velocity vanishes as the branch approaches a branch point, we conclude that these branch points are indeed regular turning points. Any differential equation describing the coupled modes must respect this correspondence principle.

The criterion for deciding the existence of a unique combination of the terms (21) is that the resulting equation must have regular turning points where $Q = 0$, as this is where, by definition, the mapping represented by the dispersion relation has branch points. Such a combination does indeed exist, and the differential equation is

$$\Phi'' + ik_c\Phi' + i(k_c\Phi)' + (Q - k_c^2)\Phi = 0 \tag{22}$$

where the prime now denotes differentiation with respect to x. To prove that Eq. (22) has the desired turning points it suffices to write Φ as

$$\Phi = \varphi \exp\left(-i \int k_c(x)\, dx\right) \equiv \varphi F_c \tag{23}$$

which reduces Eq. (22) to the form

$$\varphi'' + Q(x)\varphi = 0. \tag{24}$$

Since Eq. (24) is derived under the assumption of vanishing $\frac{\partial D}{\partial k}\big|_{k_c}$ we see that its regular turning points coincide with the branch points of the mapping (10). Moreover, we see how the wave-potential Φ factorizes into a common phase factor F_c and a coupling potential φ.

The system, composed of the elementary local dispersion relations (19), together with their corresponding differential equations (24) can be appropriately termed as the "coupling approximation" of the dispersion relation (1). It reduces the study of the dispersion relation to the analysis of individual coupling events as they appear around the coupling points and along the mode-boundaries. The approximation deteriorates as the branches of the dispersion relation depart from the boundaries $g(C)$ of their regions of definition. Nevertheless, the causality built into the system in the form of coupling, together with the requirement of continuity of a branch away from the coupling points, permits us to extend the range of validity of the approximation. Let us consider how the system of equations (24), with $k_c(z)$ running in succession through the boundary regions $g(C)$, can be used to follow the energy flow through the branches. In this context, we recall that any particular branch can eventually interact with only two branches which are its nearest neighbors as determined by the boundaries $g(C)$. The coupled branches have associated with them oppositely directed group-velocities, so that the power flowing into a coupling point will be distributed as follows. The transmission coefficient $|T|^2$ associated with Eq. (24) determines the fraction of power flowing away along the branch which was incident, and the reflection coefficient $|R|^2$ determines the fraction of power transferred to the other branch. If $|R|^2 + |T|^2 < 1$, then some power is dissipated in the coupling region. The global picture of power flow, and of power deposition, is thus determined by the reflection and transmission coefficients of the coupling approximation.

V. Summary and Concluding Remarks

A local dispersion relation $D(k, z) = 0$ is a generally many valued mapping $z \rightarrow k$ of the wave-propagation contour C (the real x-axis) onto the complex plane of the wavenumber k. The wave-modes described by the dispersion relation are given by the branches of the mapping $z \rightarrow k$. The branches, in turn, are defined by associating with them a system of Riemann sheets, which, when joined along branch cuts, form the Riemann surface of the mapping. The Riemann surface restores singlevaluedness of the mapping, and it permits the analytic continuation of a branch to be performed. This point is crucial to our analysis, since coupling between modes is nothing other than the analytic continuation of a branch from its own region of definition – the respective Riemann sheet – to another. An analytic continuation materializes, in principle, whenever the x-

axis crosses a branch cut. However, all such continuations do not signify a mode-transformation event. The only point along a branch cut which does actually satisfy the physical requirement for mode-coupling is the branch point itself. The argument supporting that statement is as follows. First, the only points in the k-plane at which coupled wavenumbers match perfectly, thereby conserving momentum in the coupling process, are the saddle points – the maps of the branch points under the mapping $z \rightarrow k$. Second, the wave group-velocity, and with it the energy flux associated with the wave, vanishes at the branch point. It follows, that in order for energy flux to be globally conserved, the mode must undergo a process of transformation at the branch point.

As regards the branch cuts themselves, they can not be traced in an arbitrary fashion away from the x-axis, but rather are specified as the contours C_b given implicitly by the mappings $D(k_c, C_b) = 0$, $\frac{\partial D(k_c,z)}{\partial k} = 0$, where z is made to trace the x-axis deformed to pass through the branch points. Such branch cuts do not have common points with the contour which generates them via this mapping, except, of course, for the branch points. We actually never make use of the branch cuts themselves in a technical sense, they merely serve as a conceptual tool to arrive at the definition of a mode. What we do use, however, are the related contours $k_c(z)$, which are, by definition, the loci of saddle points of the mapping $w = \Im(k) \equiv \Im(k, z)$, but also are the maps of the branch cuts C_b. Hence, the k_c are the boundaries of the maps of the Riemann sheets, and so deserve to be called mode-boundaries.

With the assumption that modes couple pairwise (other coupling events being exceptional), we expand the dispersion relation to second order in k around the saddle points and along the boundaries k_c to obtain the embedded dispersion relations (19) describing the coupled modes. Differential equations which are in a one-to-one correspondence with the embedded dispersion relations are then obtained through Fourier inversion ($k \rightarrow i\frac{d}{dz}$) and by requiring that the resulting differential equation have turning points at the appropriate branch points of the dispersion relation. The latter condition removes the ambiguity inherent in the inversion of products of k with functions of z.

Once the mapping $z \rightarrow k$ has been performed, we can follow the algebraic pattern of powerflow through the branches by analytically continuing the principal branch (determined by the boundary condition in the problem) via the saddle points. The dynamics of powerflow is given by the system of differential equations (24). The power incident at a saddle point will distribute itself between the incident and the other branch in a ratio which is equal to the ratio of the transmission and reflection coefficients of the turning point problem given by corresponding the differential equation (24). The number of turning points associated with a saddle point depends on the many valuedness of the mapping $k \rightarrow z$. If, for example, all non-uniformities are linearized, then each saddle point has associated with it one turning point; if the non-uniformities are quadratic in nature, there will be two turning points, etc.

The implementation of the method requires, in practical situations, numerical work, both on the algebraic and differential equation aspects of the problem. The approximation (19) for k should be a reliable one. Some indication of the quality of the approximation can be obtained by applying the procedure to the most simple examples of dispersion relations. For quadratic (in k) forms the approximation coincides with the exact form. For biquadratic forms such as Eqs. (7) and (8), the approximate $k(x)$ coincides with the exact form around the saddle points and is therefore expected to represent a good approximation on a sufficiently large portion of the real axis. We obtain good agreement in much more complicated cases. We have recently reported results of an analysis of the effect of ion-cyclotron harmonics on the mode-conversion of the lower-hybrid to a warm-ion wave under Alcator-A conditions[14,15]. The wavenumber $k(x)$ computed from the approximate [Eq. (19)], and the exact[16] dispersion relations are shown, respectively, in Figs. 12a and 12b. The approximation is seen to be adequate and, therefore, the appropriate Eq. (24) can be utilized to calculate power flow and dissipation. A detailed account of this investigation, which is now being carried out for an entire n_z spectrum of incident lower-hybrid power, will be presented elsewhere.

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Figure Captions

Fig. 1. Contour-plot of $x = \mathrm{Re}\, f^{-1}(k)$, Eq. (3). The contours of constant x form an orthogonal net with the contours of constant $y = \mathrm{Im}\, f^{-1}(k)$. The two y-contours of steepest descent marked $++$ and $--$ correspond to the branches of some particular line parallel to the x-axis. Other such lines produce branches which break up around the saddle point and do not couple.

Fig. 2. The event of two real modes merging at x_B to become a complex-conjugate pair produces a saddle point k_s in the k-plane.

Fig. 3. The Riemann surface of a triple-valued mapping $z \rightarrow k$, with branches coupled pair-wise. The Riemann sheet RS(1) corresponds to the principal branch which can be analytically continued through the cuts C_{b1} and C_{b2}.

Fig. 4. An illustration of how the double-valued mapping of a contour depends on the position of the branch cut.

Fig. 5. Two crossings of one branch cut in a pair-wise coupled system will result in returning to the original Riemann sheet.

Fig. 6. The mapping of the real axis $x = \mathrm{Re}\, z$ onto the k-plane, and in the $k = k(x)$ representation. If the branch point z_B does not lie on the real axis, the branches do not intersect.

Fig. 7. The algebraic power-flow pattern in a triple-valued mapping possessing three saddle points.

Fig. 8. The mapping of the real x-axis onto the k-plane, as determined by the dispersion relation (7).

Fig. 9. The mapping of the real x-axis onto the k-plane, as determined by the dispersion relation (8); $a < 0$.

Fig. 10. The mapping of the real x-axis onto the k-plane, as determined by the dispersion relation (8); $a > 0$.

Fig. 11. When a branch point z_B does not lie on the real x-axis, the contour must be deformed to pass through z_B. This recovers the coupling property embodied in a local dispersion relation.

Fig. 12. Coupling of an incident lower–hybrid wave to a warm–plasma wave under Alcator–A conditions: the perpendicular component k_x of the wave–vector evaluated as a function of the spatial direction x perpendicular to the magnetic field. a) Exact dispersion relation. b) Embedded dispersion relation Eq. (19).

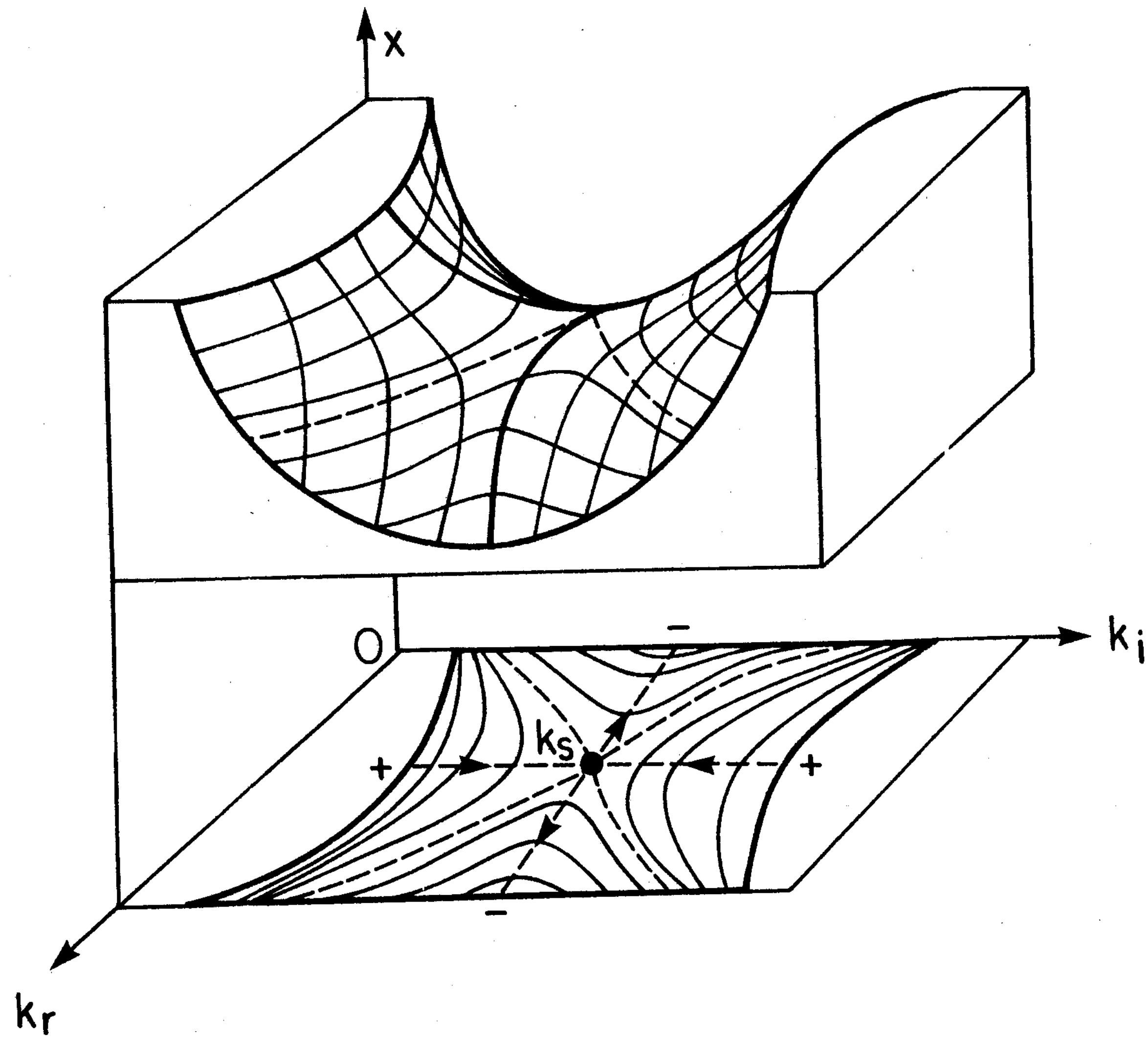


Figure 1

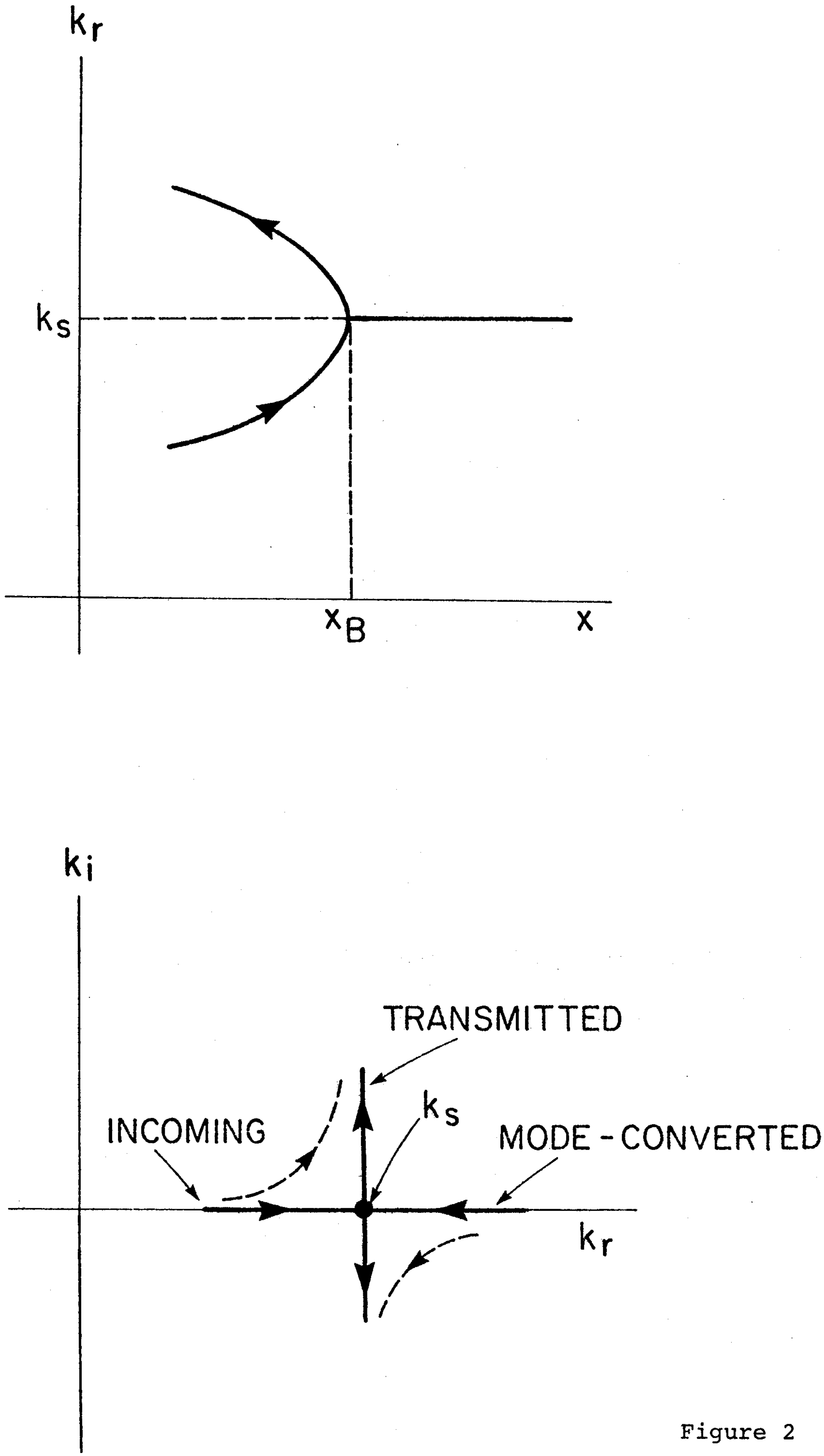


Figure 2

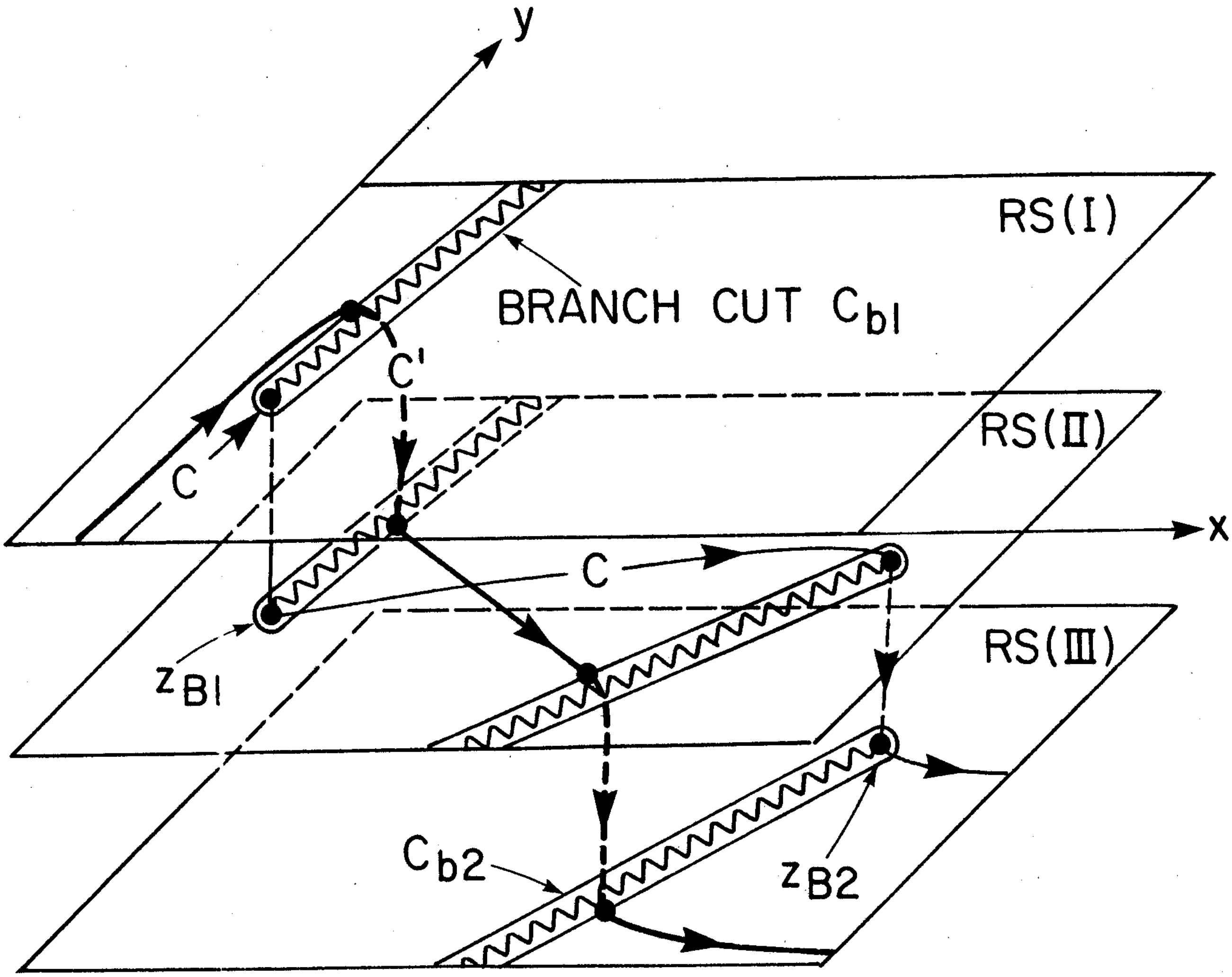
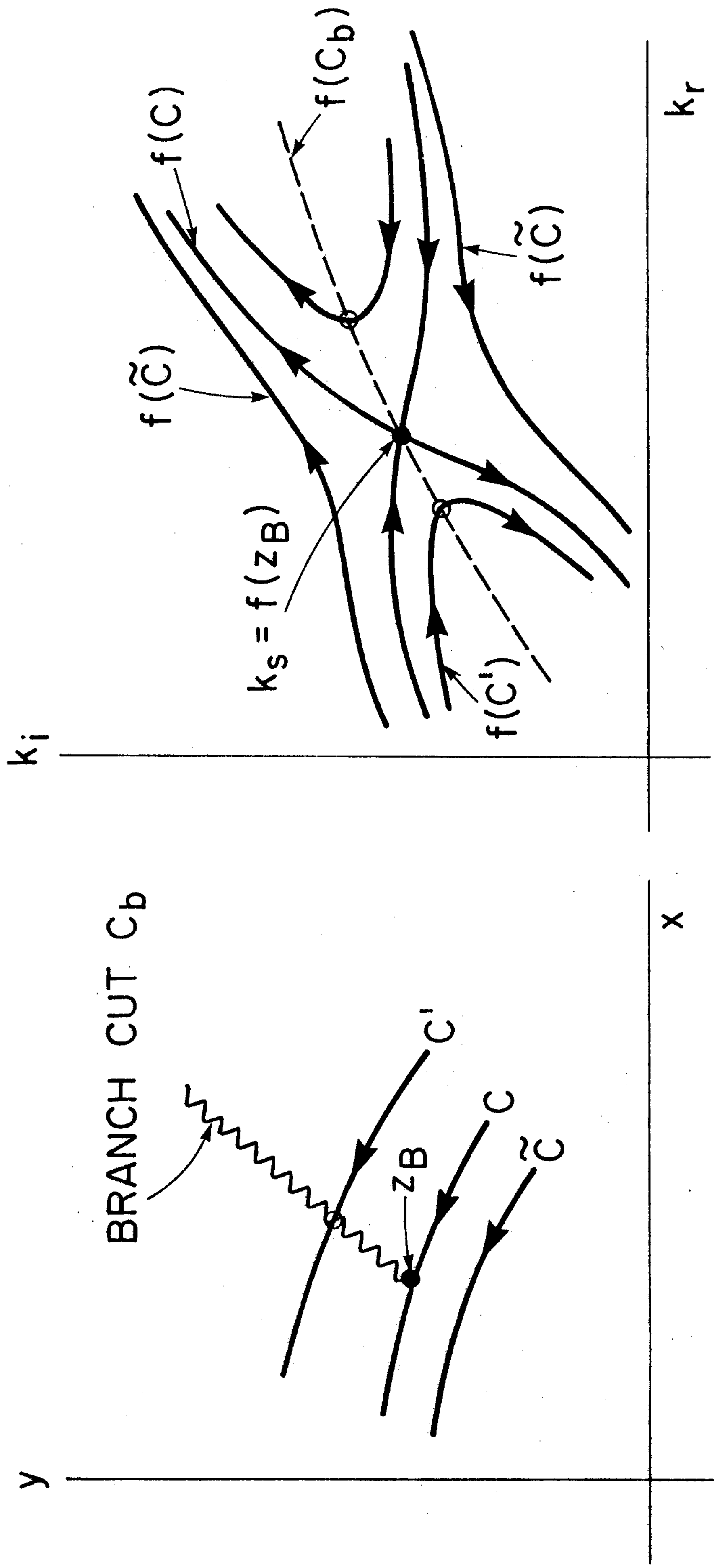


Figure 3



BRANCH CUT C_b
z_B
C'
C
$\tilde{C}$
x
y
k_i
k_r
$f(C_b)$
$f(C)$
$f(\tilde{C})$
$f(\tilde{C})$
$k_s = f(z_B)$
$f(C')$

Figure 4

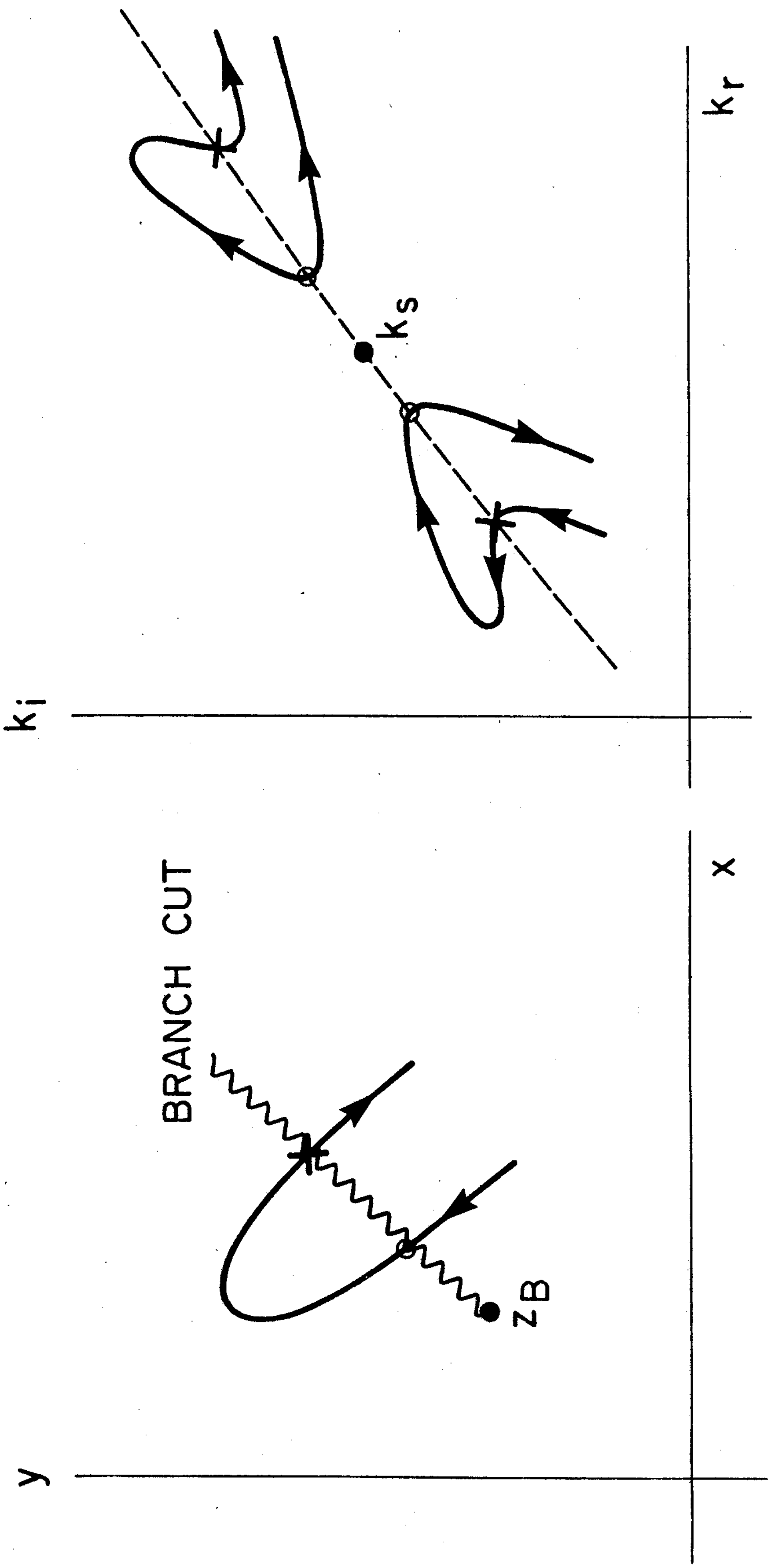


Figure 5

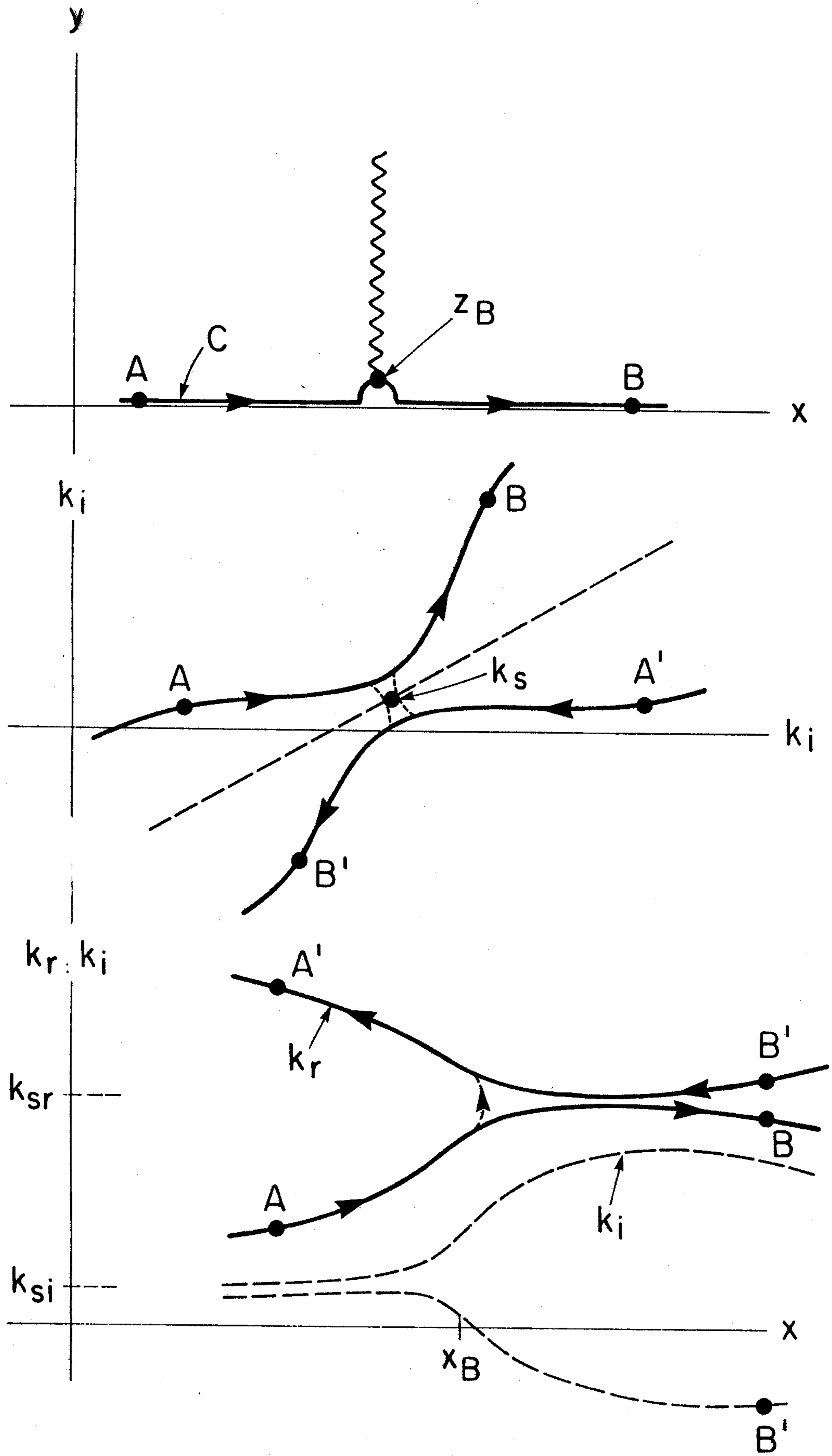


Figure 6

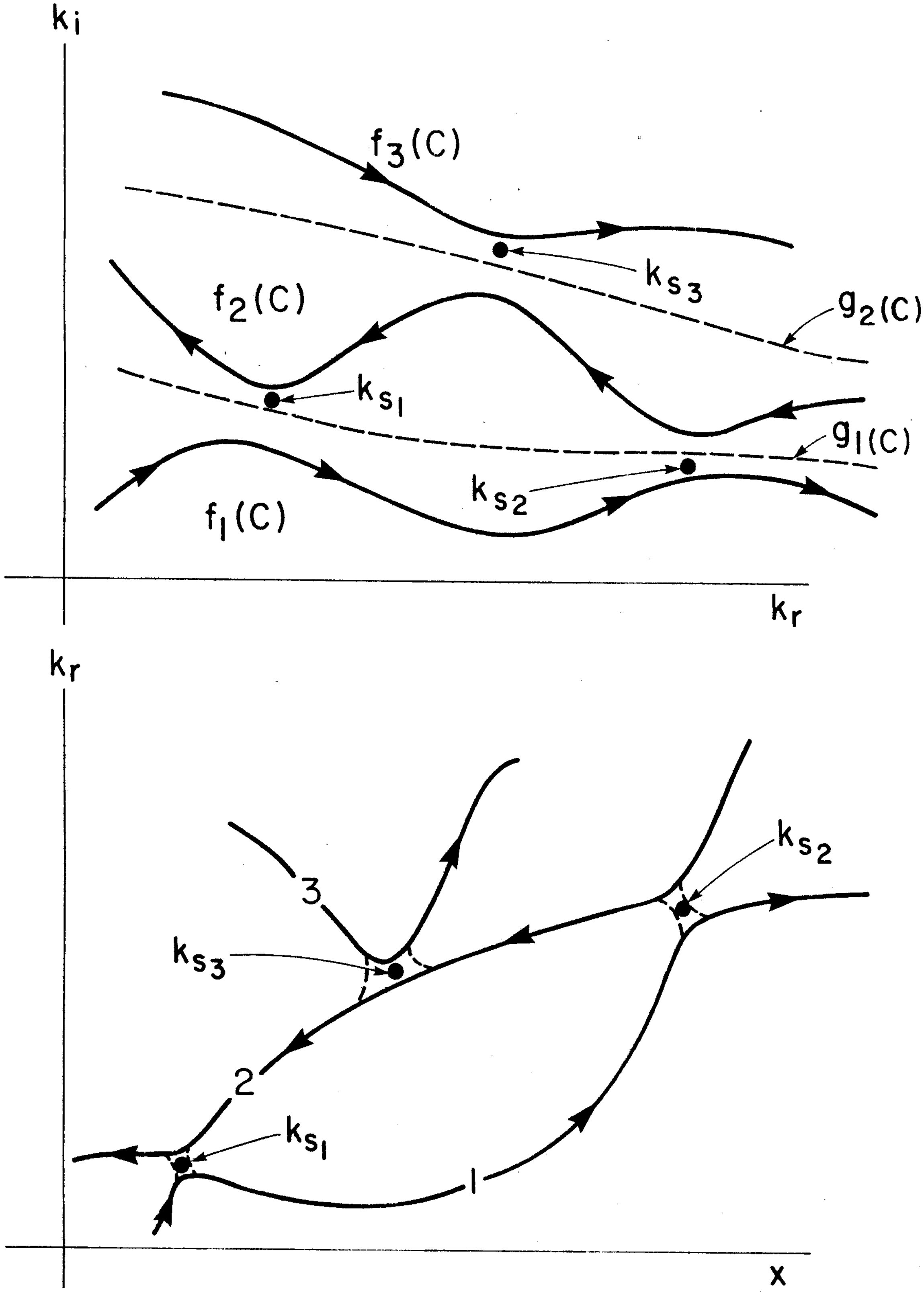
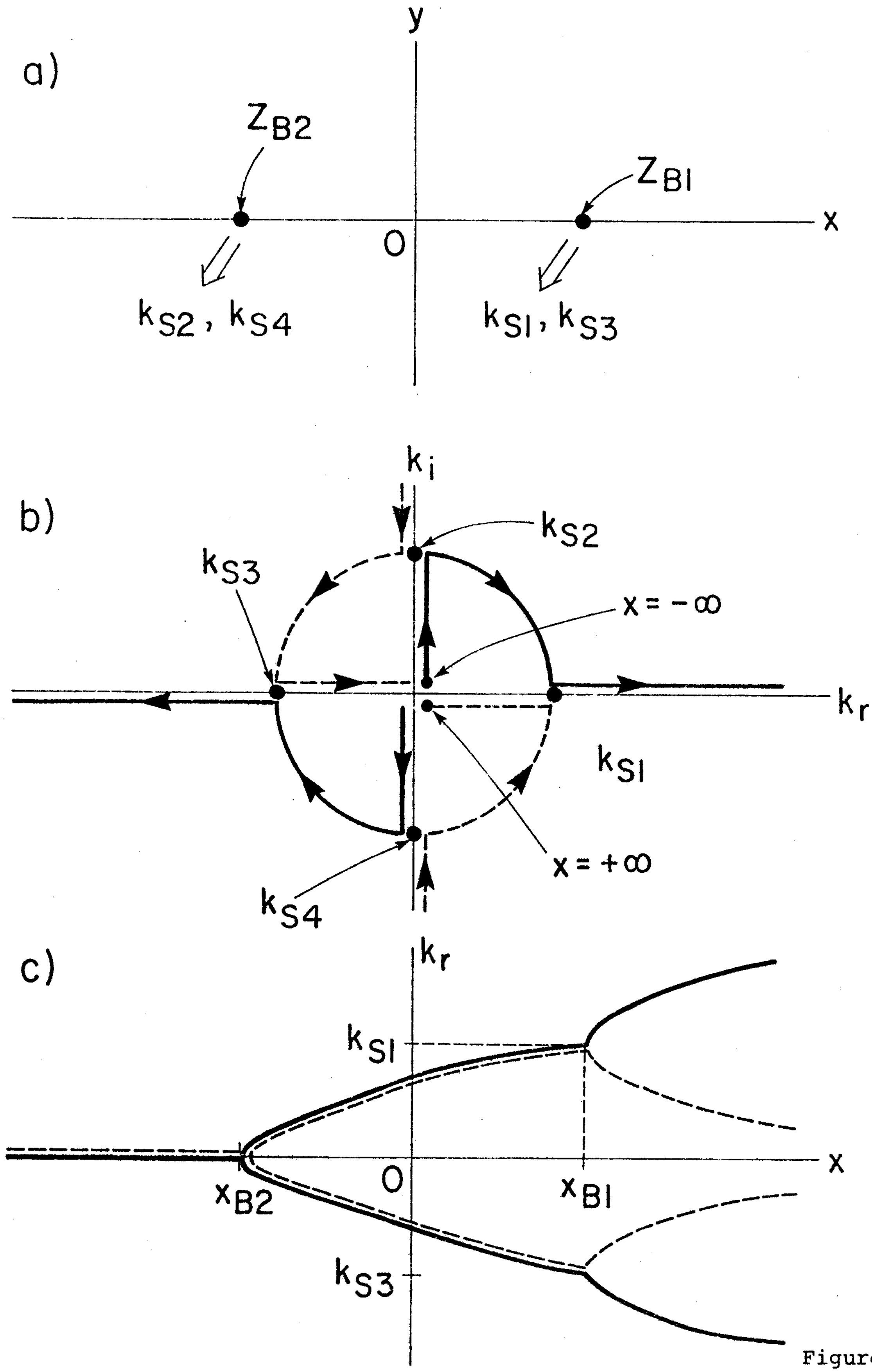
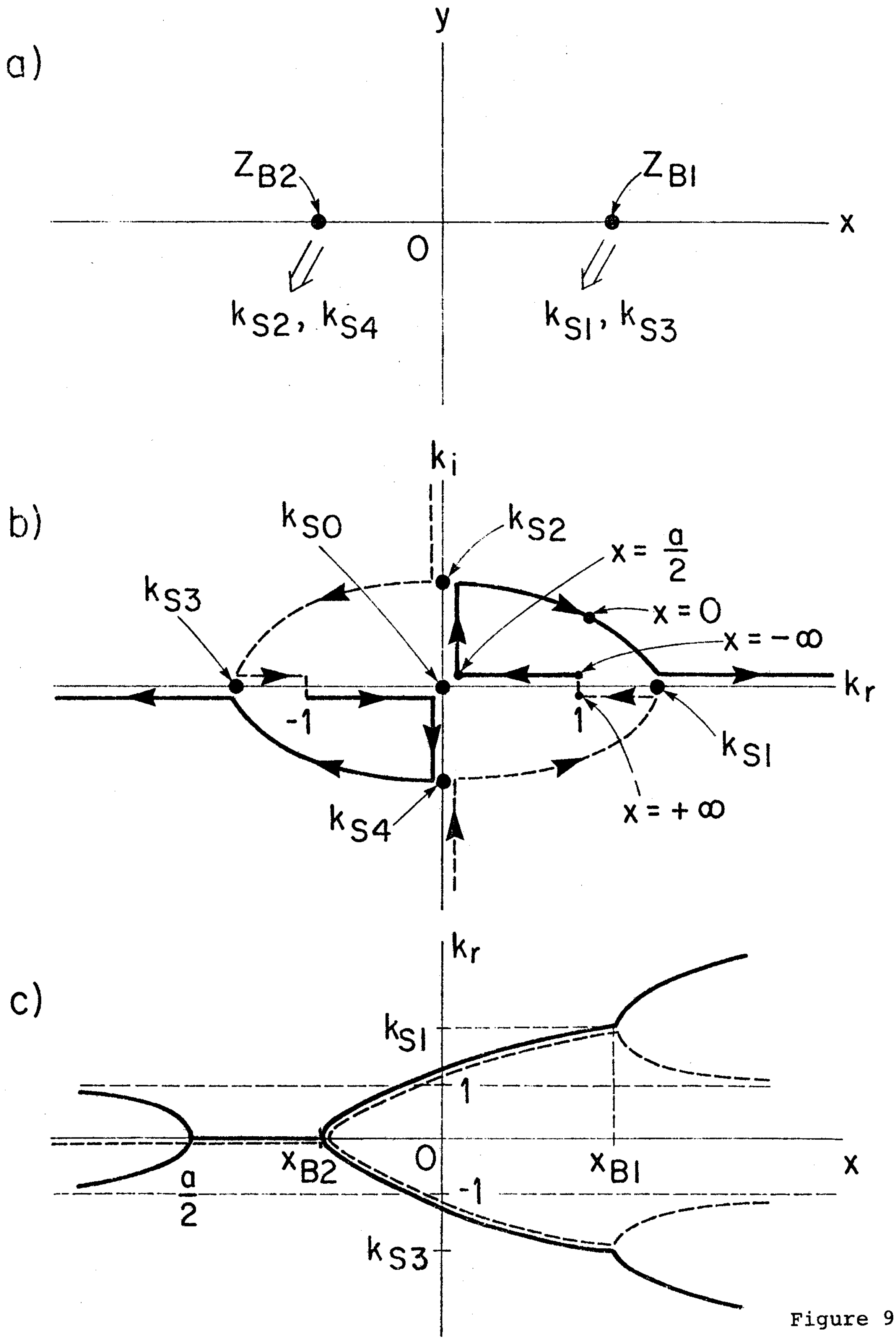


Figure 7



a)
y
Z_{B2}
Z_{B1}
x
O
k_{S2}, k_{S4}
k_{S1}, k_{S3}
b)
k_i
k_{S2}
k_{S3}
$x = -\infty$
k_r
k_{S1}
$x = +\infty$
k_{S4}
c)
k_r
k_{S1}
x
x_{B2}
O
x_{B1}
k_{S3}

Figure 8



a)
y
Z_{B2}
Z_{B1}
x
0
k_{S2}, k_{S4}
k_{S1}, k_{S3}
b)
k_i
k_{S0}
k_{S2}
$x = \frac{a}{2}$
k_{S3}
x = 0
$x = -\infty$
k_r
-1
1
k_{S1}
$x = +\infty$
k_{S4}
c)
k_r
k_{S1}
1
x_{B2}
0
x_{B1}
x
$\frac{a}{2}$
-1
k_{S3}

Figure 9

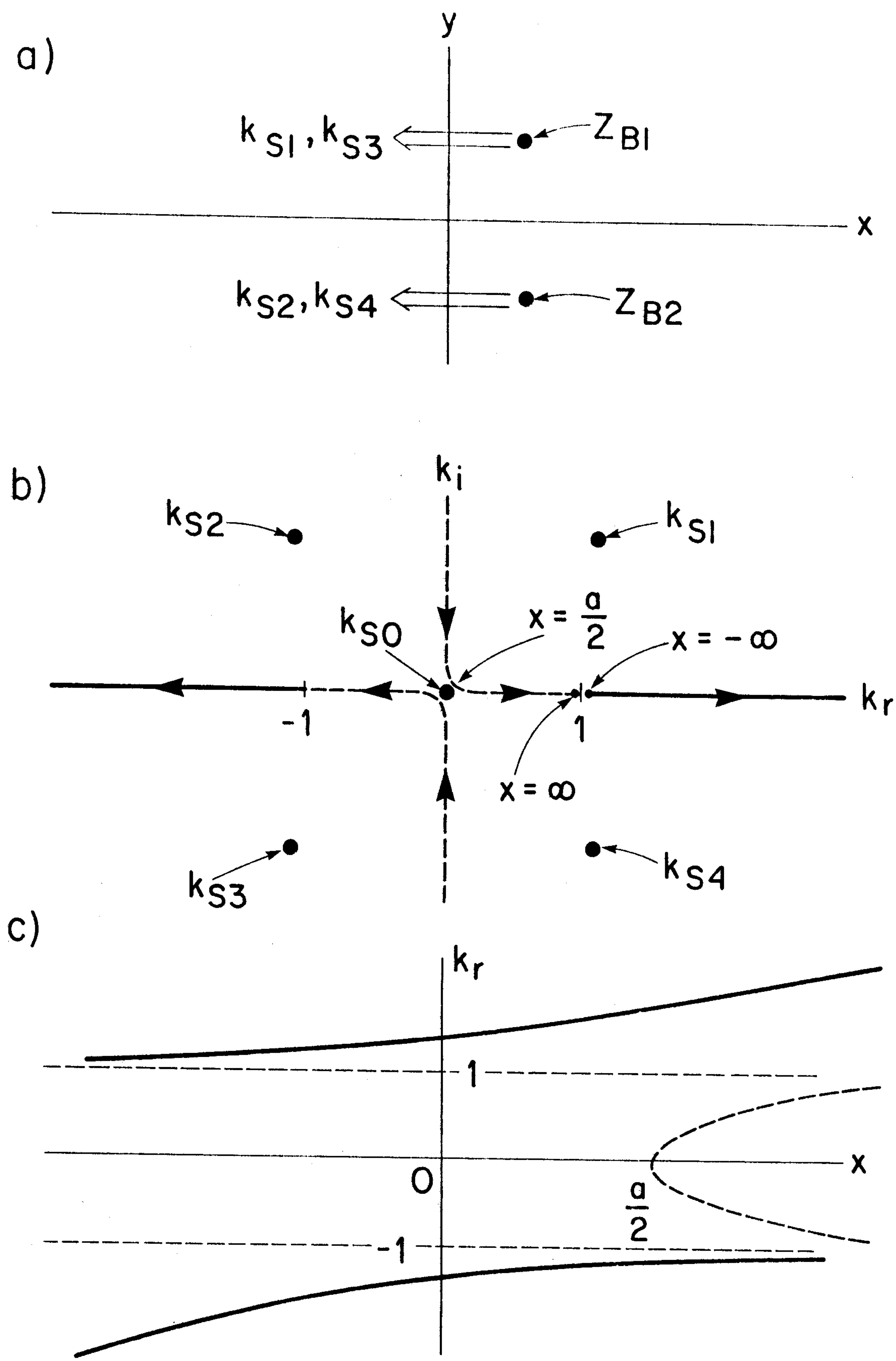
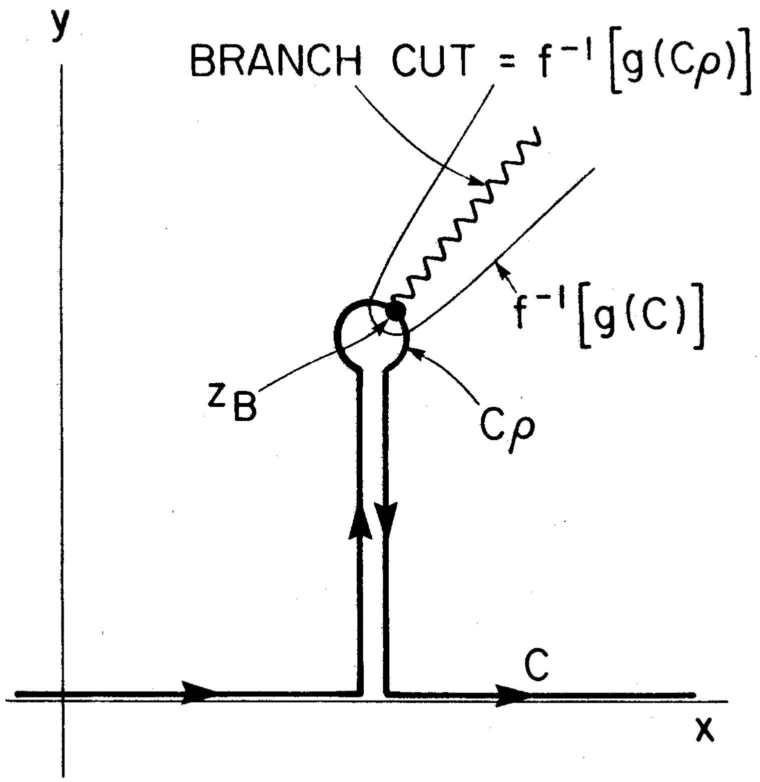

a)
y
k_{S1}, k_{S3}
Z_{B1}
x
k_{S2}, k_{S4}
Z_{B2}
b)
k_i
k_{S2}
k_{S1}
k_{S0}
$x = \frac{a}{2}$
$x = -\infty$
k_r
-1
1
$x = \infty$
k_{S3}
k_{S4}
c)
k_r
1
x
0
$\frac{a}{2}$
-1

Figure 10



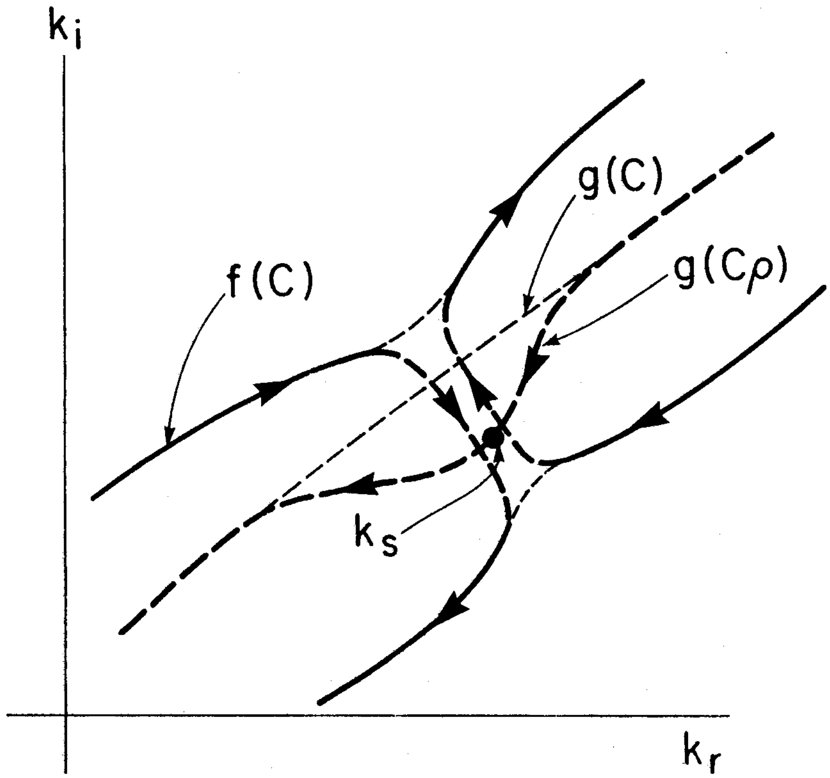


Figure 11

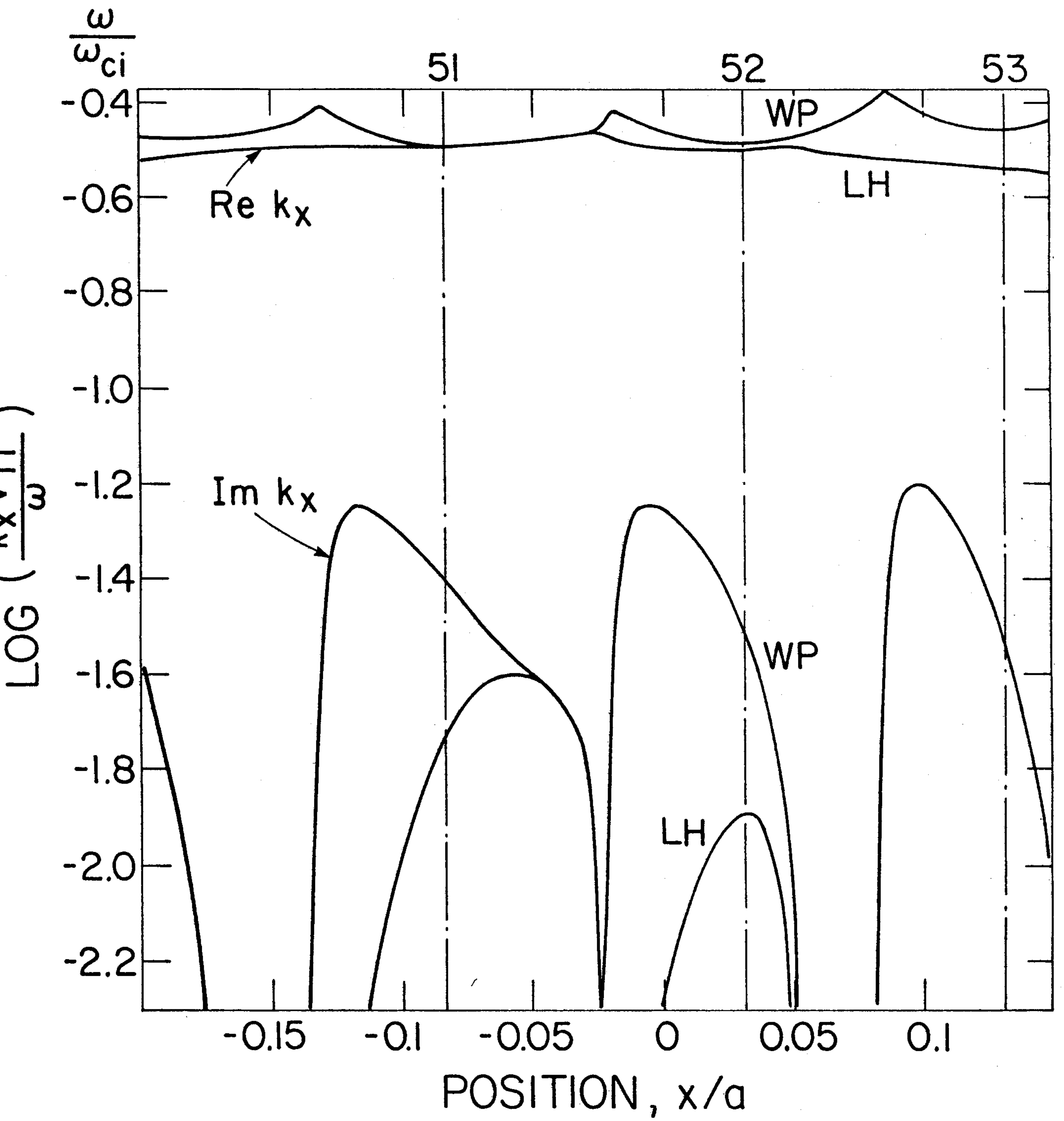

$\frac{\omega}{\omega_{ci}}$
51
52
53
-0.4
-0.6
-0.8
-1.0
-1.2
-1.4
-1.6
-1.8
-2.0
-2.2
WP
LH
Re k_x
Im k_x
WP
LH
LOG $\left(\frac{k_x v_{Ti}}{\omega}\right)$
-0.15
-0.1
-0.05
0
0.05
0.1
POSITION, x/a

Figure 12a

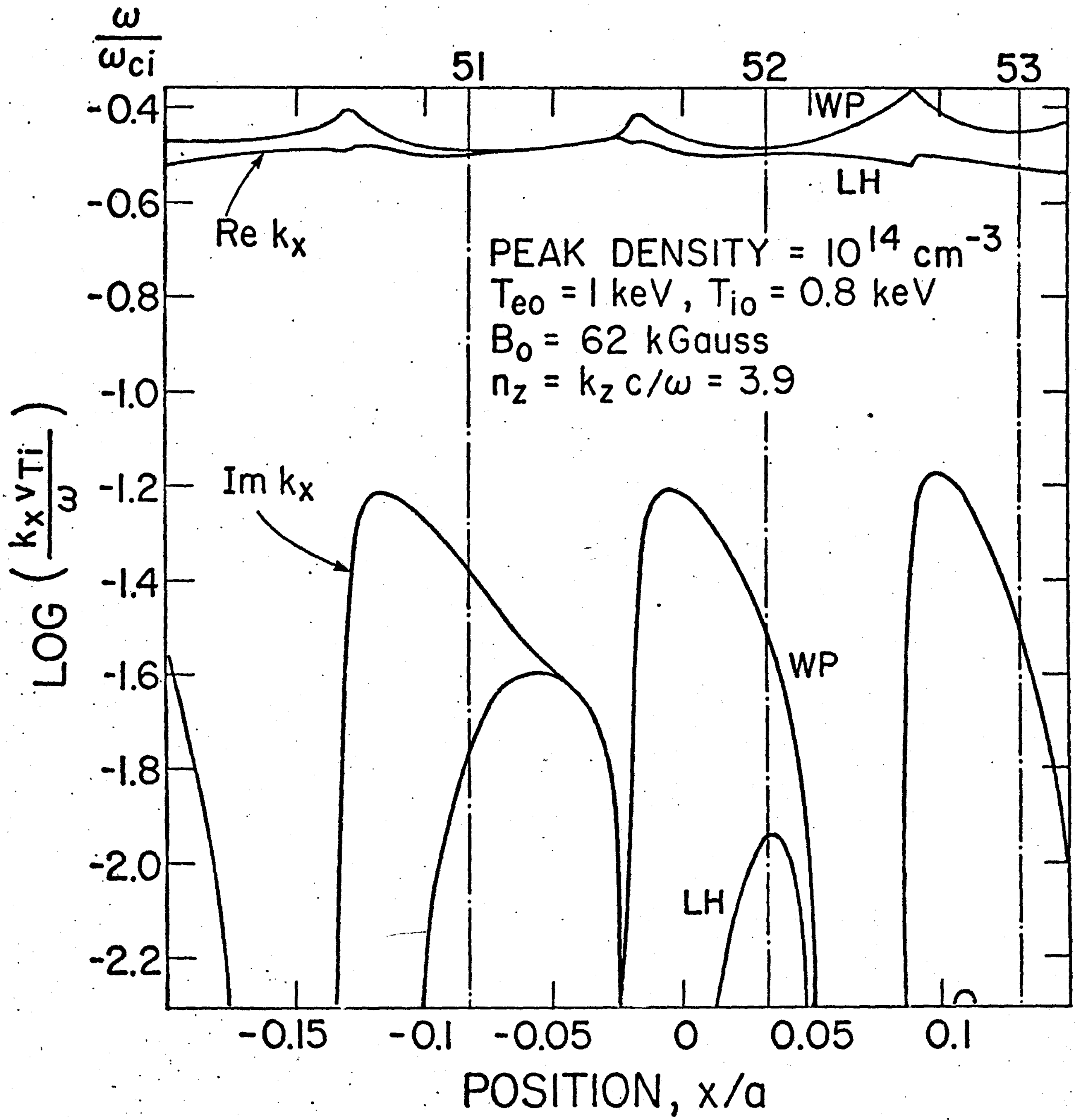

ω/ω_ci
51
52
53
IWP
LH
Re k_x
PEAK DENSITY = 10^14 cm^-3
T_eo = 1 keV, T_io = 0.8 keV
B_o = 62 kGauss
n_z = k_z c/ω = 3.9
Im k_x
WP
LH
LOG (k_x v_Ti / ω)
-0.4
-0.6
-0.8
-1.0
-1.2
-1.4
-1.6
-1.8
-2.0
-2.2
-0.15
-0.1
-0.05
0
0.05
0.1
POSITION, x/a

Figure 12 b