

RESONANT PARAMETRIC EXCITATION IN LOWER-HYBRID
HEATING OF TOKAMAK PLASMA*

E. Villalón

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Resonant parametric excitations in lower-hybrid
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E. Villalón

Plasma Fusion Center and
Research Laboratory of Electronics
Massachusetts Institute of Technology
Cambridge, Massachusetts 02139.

Abstract

Three wave parametric excitation in inhomogeneous plasmas is examined in a two-dimensional geometry relevant to supplementary rf-heating of tokamaks. The stabilization of resonant parametric excitation due to a linear mismatch in wavenumbers and to the Landau-damping rates of the decay waves is analyzed, assuming that the magnitude of the pump field is constant in time and in the spatial region where the resonant interaction takes place. Both types of temporally growing modes and spatially amplified instabilities are studied, using a WKB analysis. It is shown that either by increasing the strength of the mismatch K' or the width of the pump L , the growth rate of the fastest growing normal mode will decrease. When the excited waves are slightly damped, it is shown that there exists a finite value of the product $K'L$, such that, above it, no temporal normal modes are excited. The amount of spatial amplification is also reduced by the mismatch in wavenumbers and by the damping rates of the excited waves. Because of the finite spatial extent of the pump electric field, the amplification length is found to be smaller than or equal to L , depending on the strength of the mismatch and damping rates.

I. INTRODUCTION

At the high power levels required for supplementary rf-heating of plasmas, a variety of nonlinear phenomena are likely to occur¹, especially, in the very low density and temperature region of the plasma edge. For lower-hybrid heating, the rf-fields are principally electrostatics; they are launched into the plasma by phased waveguides array, and propagate inside the plasma along well-defined resonance cones. Among possible nonlinear effects, parametric instability is one of the most prominent. There are many different types of parametric processes that may take place during lower-hybrid heating, most of which have been discussed by several authors¹⁻³. In this paper, we consider the resonant three wave parametric instability, where the pump field is assumed to decay into two others resonant waves, such as another lower-hybrid wave and a low frequency mode. Recent experimental results indeed show the presence of such excitations⁴ near the plasma edge region, which partially motivates this work. Because the plasma density gradient near the edge is always very large, it is very important to understand how this may affect the decay process. The plasma density gradient originates a phase mismatch which may saturate the resonant excitation. The low frequency mode is very likely to be slightly damped either by electrons or ions, which also contributes to the saturation of the instability.

These two aspects have not yet been sufficiently studied, and a correct treatment of this problem is still needed. The difficulty arises from the selection of the right boundary conditions that describe the saturation of the resonant excitation due to the wavenumber mismatch. The mode coupled equations are to be analyzed in the complex plane, so to impose the right boundary conditions for the excitation of either temporally growing or spatially amplified instabilities. These boundary conditions are different from the ones found in a homogeneous plasma⁵ due to the presence of the phase mismatch. We present a thorough analysis of the stabilization of the resonant three waves parametric excitation due to both mismatch in wavenumbers and linear damping rates of the excited waves, using a WKB analysis. The correct boundary conditions are found after examining the Stokes' structure of the mode amplitudes, taking into account the

mismatch, the damping rates, and the finite spatial extent of the interaction region.

The frequency and wavevector of the three waves are assumed to be peaked around a certain (ω, \vec{k}) . The slowly varying pump amplitude is assumed constant in time and in space. The geometry of the interaction is two-dimensional, and this interaction may also evolve in time. The two-dimensional interaction has been shown^{6,7} to be equivalent to a one-dimensional one, which reduces the problem to two coupled differential equations in time and one spatial variable. We shall consider a linear mismatch gradient, and a rectangular pump profile of finite width L .

One dimensional interaction in a pump of infinite spatial extent is fairly well understood^{8,9} for the case of a linear mismatch profile. It has been shown⁹ that no temporally growing modes are possible in the linear mismatch profile, and that the amount of spatial amplification^{8,9} is always finite and may be greatly reduced by the damping rates of the excited waves¹⁰. The one dimensional interaction in a pump of finite spatial extent is the case that concerns us now, and has been analyzed by a number of authors¹¹⁻¹⁷. In spite of this work the situation is not very clear. Specifically, some authors¹¹⁻¹⁴ agree that temporally growing modes will disappear, increasing the length of the interaction (i.e., the spatial region where the pump extends). Others¹⁵⁻¹⁷ claim that temporal normal modes exist for any finite value of the length of the interaction, which does not seem to be consistent with the results in Ref. 9. To our knowledge, no satisfactory resolution of this controversy has yet been presented, and the thresholds for the excitation of normal modes, as function of the pump width, are still unknown. The amount of spatial amplification may be considerably reduced when simultaneous damping of the excited waves, mismatch in wavenumbers, and finite extent of the interaction region are considered; this has not been explored sufficiently. In this paper, we attempt to answer these questions, and give a better insight into the nature of the resonant three wave parametric instability.

The paper is organized as follows. In Sec. II, we formulate the problem and present the basic equations describing the evolution of the three waves parametric instability. Section III is divided in three subsections, in which we study the excitation of normal modes. We start by presenting the WKB solutions for the second order equation that gives the evolution of the

instability. The boundary conditions are formulated taking into account the Stokes' structure of these solutions in the complex plane. Next, we derive the dispersion relation that gives the eigenvalues of the possible normal modes. We show that the existence of these normal modes depend critically on the strength of the mismatch K' , width of the pump L , and damping rates of the excited waves. In fact, we find that either by increasing the length of the interaction L , or the strength of the mismatch K' , the growth rate of the fastest growing normal mode will decrease. Furthermore, if the excited waves are slightly damped, there exists a finite value of the product $K'L$, such that above it no temporal normal modes are possible. The thresholds for the existence of normal modes are presented in Sec. III C. In Sec. IV, we are concerned with spatially amplified instability. We calculate the spatial amplification factor, and we study how it is affected by the damping of the waves, the mismatch in wavenumbers, and the finite spatial extent of the pump field. In Sec. V, we present a brief discussion of the techniques used in the paper and a summary of our results. The Appendix contains a derivation of the coupled mode equations for the decay of the lower-hybrid pump into a sideband and a low frequency mode.

It should be noted that the analysis we present is quite general and could be applied to any three waves decay processes, even if the involved waves are not in the lower-hybrid frequency range, without major changes. The geometry of the problem has to be two-dimensional, and the pump field must have a finite spatial extent, for the application of this analysis.

II. BASIC EQUATIONS

Let us consider a two-dimensional plasma slab where x is in the direction of the plasma inhomogeneities (i.e., density, temperatures, and toroidal magnetic field inhomogeneities), and z is in the direction of the toroidal magnetic field \vec{B}_0 . An externally launched high frequency pump wave in the lower-hybrid frequency range,

$$\omega_0 = \frac{\omega_{pe}}{(1 + \omega_{pe}^2 / \Omega_e^2)^{1/2}},$$

is assumed to decay into two other resonant waves, such as another lower-hybrid wave ω_1 and a low frequency resonant mode ω_2 (e.g., ion cyclotron, ion Bernstein, and ion acoustic modes). The three waves are described by amplitudes

$$A_i = a_i(x, z, t) \exp[i(-\omega_i t + k_{iz}z + k_{iy}y) + i \int_{x_0}^x k_{ix}(x') dx'], \quad (1)$$

where $a_i(x, z, t)$ are assumed to be slowly varying in space and in time. The pump wave vector \vec{k}_0 is taken to be on the x - z plane [i.e., $\vec{k}_0 = (k_{0x}, k_{0z})$]; thus, the pump amplitude is assumed uniform in the y direction. The three waves satisfy the matching conditions: $\omega_0 = \omega_1 + \omega_2$, $k_{0z} = k_{1z} + k_{2z}$ and $k_{1y} = -k_{2y}$. The plasma inhomogeneities introduce a mismatch in the x component of the wavenumbers $\Delta k(x) = k_{0x} - k_{1x} - k_{2x}$ to be determined from the local dispersion relation for each of the waves in the inhomogeneous plasma: $K_i(k_{i\perp}, \omega_i, k_{iz}; x) = 0$, where $k_{i\perp} = (k_{ix}^2 + k_{iy}^2)^{1/2}$ and where k_{iz} and k_{iy} are constants independent of the inhomogeneity coordinate x .

We assume that the amplitude a_0 is constant in time and in space. The complex wave packet amplitudes of the excited high and low frequency waves satisfy the equations (see the Appendix):

$$\left(\frac{\partial}{\partial t} + \gamma_1 + v_{1x} \frac{\partial}{\partial x} + v_{1z} \frac{\partial}{\partial z} \right) a_1 = \gamma_0 a_0^* \exp[i \int_{x_0}^x \Delta k(x') dx'], \quad (2)$$

$$\left(\frac{\partial}{\partial t} + \gamma_2 + v_{2x} \frac{\partial}{\partial x} + v_{2z} \frac{\partial}{\partial z}\right) a_2 = \gamma_0 a_1^* \exp\left[i \int_{x_0}^x \Delta k(x') dx'\right], \quad (3)$$

where γ_1 and γ_2 are the damping rates of the excited waves, and v_{ix} , v_{iz} are their group velocity components along the x and z directions, respectively.

The coupling coefficient γ_0 is different from zero over the region where the pump extends:

$$-\frac{w}{2} \leq z - \int_0^x \frac{v_{0z}(x')}{v_{0x}(x')} dx' \leq \frac{w}{2},$$

where v_{0z} and v_{0x} are the group velocity components of the pump wave, and w is the width of the pump resonance cone (see Fig. 1). We find (see the Appendix),

$$\gamma_0 = \frac{E_0}{2 B_0 \omega_0} \frac{\sin \phi}{\left[\left(\frac{\partial K_1}{\partial \omega}\right)_{\omega_1} \left(\frac{\partial K_2}{\partial \omega}\right)_{\omega_2}\right]^{1/2}} k_2 \chi_e(\vec{k}_2, \omega_2), \quad (4)$$

where $(\partial K_i / \partial \omega)_{\omega_i}$ is to be evaluated at the frequencies of the resonant waves, $i = 1, 2$; $\sin \phi = k_{1y} / k_{1\perp}$, and E_0 is the pump electric field. The pump amplitude a_0 is related to E_0 as $a_0 = 2 [(\partial K_0 / \partial \omega)_{\omega_0}]^{1/2} E_0$; similar expressions hold for a_1 , a_2 and E_1 , E_2 . The electron susceptibility χ_e for the resonant low frequency wave is given approximately by

$$\begin{aligned} \chi_e(\vec{k}_2, \omega_2) &\approx \frac{1}{k_2^2 \lambda_{De}^2} \text{ if } \frac{\omega_2}{|k_{2z}| v_{Te}} \ll 1, \\ &\approx -\frac{\omega_{pe}^2}{\omega_2^2} \frac{k_{2z}^2}{k_2^2} \text{ if } \frac{\omega_2}{|k_{2z}| v_{Te}} \gg 1, \end{aligned}$$

depending on whether the excited low frequency mode is an ion acoustic or an ion cyclotron mode, $\omega_2 / (|k_{2z}| v_{Te}) \ll 1$, or whether it is an ion Bernstein mode, $\omega_2 / (|k_{2z}| v_{Te}) \gg 1$. It should be noted that the group velocity components v_x and v_z for the three waves, the damping rates γ_1 , γ_2 , and the coupling coefficient γ_0 , depend on the inhomogeneity coordinate x . However, we assume that the spatial inhomogeneities are weak enough to treat them as constants over the finite region where the pump extends; we shall also find that the phase mismatch $\Delta k(x)$ further localizes the

region of resonant interaction in the inhomogeneity coordinate x .

The two-dimensional interaction described by Eqs. (2) and (3) can be reduced to an equivalent one-dimensional one by means of the transformations that follows ⁷. Let us first rotate our coordinate system to lie along and perpendicular to the pump resonance cone: $\bar{z} = (v_{0z} z + v_{0x} x)/v_0$ and $\bar{x} = (v_{0z} x - v_{0x} z)/v_0$, where $v_0 = (v_{0x}^2 + v_{0z}^2)^{1/2}$. Next, define a new coordinate system as

$$\tau = t, \quad \xi = \delta \bar{x}, \quad \theta = \bar{z} - \alpha \bar{x} - u\tau, \quad (5)$$

where $\alpha = (v_{1\bar{z}} - v_{2\bar{z}})/(v_{1\bar{x}} - v_{2\bar{x}})$, $u = v_{1\bar{z}} - \alpha v_{1\bar{x}}$ ($v_{i\bar{z}}$ and $v_{i\bar{x}}$ are the group velocity components along and perpendicular to the pump cone), and $\delta = (v_{0z} + \alpha v_{0x})/v_0$. Under these transformations Eqs. (2) and (3) become

$$\left(\frac{\partial}{\partial \tau} + v_{1\xi} \frac{\partial}{\partial \xi} + \gamma_1 \right) a_1 = \gamma_0 a_2^* \exp \left[i \int_{\xi_0}^{\xi} \Delta k(\xi' + \frac{v_{0x}}{v_0} z_f) d\xi' \right], \quad (6)$$

$$\left(\frac{\partial}{\partial \tau} + v_{2\xi} \frac{\partial}{\partial \xi} + \gamma_2 \right) a_2 = \gamma_0 a_1^* \exp \left[i \int_{\xi_0}^{\xi} \Delta k(\xi' + \frac{v_{0x}}{v_0} z_f) d\xi' \right], \quad (7)$$

where $v_{i\xi} = \delta v_{i\bar{x}}$, ($i = 1, 2$), $\xi_0 = x_0 - (v_{0x}/v_0) z_f$ and $\xi = x - (v_{0x}/v_0) z_f$. The independent variable z_f is a free parameter which defines where in the plasma the resonant interaction is taking place; the interaction region in the pump cone extends along the line, $\bar{z} - \alpha \bar{x} = z_f$, where z_f is given as an initial condition. The coupling coefficient γ_0 , the damping rates γ_1 and γ_2 , and the group velocity components $v_{i\xi}$, can now be defined at the values that they take for $\bar{z} = z_f$ and $\bar{x} = 0$. Thus, Eqs. (6) and (7) describe a one-dimensional resonant interaction in the spatial coordinate ξ , and at a fixed given distance z_f along the direction of propagation of the pump field (see Fig. 1). From now on we shall, for simplicity, omit the subscript z_f in all the equations, but we always understand that the interaction is defined for a fixed given value of z_f .

We are looking for solutions to Eqs. (6) and (7) with a temporal dependence of the form

$\exp(p\tau)$ where $p = s + iq$ is, in general, a complex number. Let us define a new set of amplitudes 7-17:

$$a_1 = \frac{\hat{a}_1}{(|v_1\xi|)^{1/2}} \exp\left[-\frac{\xi}{2}\left(\frac{p+\gamma_1}{v_1\xi} + \frac{p+\gamma_2}{v_2\xi}\right) + p\tau + \frac{i}{2}\int_{\xi_0}^{\xi} \Delta k(\xi') d\xi'\right] \quad (8)$$

$$a_2^* = \frac{\hat{a}_2}{(|v_2\xi|)^{1/2}} \exp\left[-\frac{\xi}{2}\left(\frac{p+\gamma_1}{v_1\xi} + \frac{p+\gamma_2}{v_2\xi}\right) + p\tau - \frac{i}{2}\int_{\xi_0}^{\xi} \Delta k(\xi') d\xi'\right] \quad (9)$$

Substituting these expressions into Eqs. (6) and (7) we obtain:

$$\left[\frac{1}{2}\beta(\xi) + \frac{\partial}{\partial\xi}\right]\hat{a}_1 = \text{sgn}(v_1\xi)\lambda_0\hat{a}_2, \quad (10)$$

$$\left[-\frac{1}{2}\beta(\xi) + \frac{\partial}{\partial\xi}\right]\hat{a}_2 = \text{sgn}(v_2\xi)\lambda_0^*\hat{a}_1, \quad (11)$$

where $\beta(\xi) = \eta + i[\Delta k(\xi) + q/v_1\xi - q/v_2\xi]$, $\eta = (s + \gamma_1)/v_1\xi - (s + \gamma_2)/v_2\xi$, $\lambda_0 = \gamma_0/(|v_1\xi v_2\xi|)^{1/2}$ and $\text{sgn}(v_i\xi) = v_i\xi/|v_i\xi|$. We can now further eliminate \hat{a}_2 from Eq. (10), and \hat{a}_1 from Eq. (11), to find

$$\frac{d^2\hat{a}_i}{d\xi^2} - [\text{sgn}(v_1\xi v_2\xi)|\lambda_0|^2 + \frac{1}{4}\beta^2(\xi) + \frac{(-1)^i}{2}\frac{d\Delta k}{d\xi}]\hat{a}_i = 0, \quad (12)$$

with $i = 1, 2$. In what follows we take $|\lambda_0|^2 \gg |d\Delta k/d\xi|$, allowing us to neglect $d\Delta k/d\xi$ in Eq. (12).

Then, the amplitudes \hat{a}_1 and \hat{a}_2 satisfy the same second order differential equation,

$$\frac{d^2\hat{a}_i}{d\xi^2} - [\text{sgn}(v_1\xi v_2\xi)|\lambda_0|^2 + \frac{1}{4}\beta^2(\xi)]\hat{a}_i = 0. \quad (13)$$

The phase mismatch $\Delta k(\xi)$ will always be assumed to be a linear function of ξ . If the pump extension in the ξ coordinate is

$$L = \frac{v_{0x}v_{0x} + \alpha v_{0x}^2}{v_0^2} w, \quad (14)$$

we may write

$$\Delta k(\xi) = \Delta k(0) + K' \xi, \quad (15)$$

where $K' = [\Delta k(L/2) - \Delta k(-L/2)]/L$, $\Delta k(\pm L/2)$ are the phase mismatches at both extremes of the pump cone (i.e., for $\xi = \pm L/2$) as taken along the line of pulse response, $z - \alpha \bar{x} = z_f$, and where $\Delta k(0)$ is the mismatch at the center of the pump cone. It should be noted that the trajectory of the pulse response covers a distance along the inhomogeneity coordinate x , as it goes through the pump cone, which is precisely equal to L (see Fig. 1).

To determine the phase shift q let us substitute the mismatch $\Delta k(\xi)$ in Eq. (15) into the definition of $\beta(\xi)$ after Eq.(11). We find, $\beta(\xi) = \eta + i(K' \xi + Q)$ where $Q = \Delta k(0) + q/v_1 \xi - q/v_2 \xi$. Note that the zero-mismatch $\Delta k(0)$ is independent of K' and L . It comes from the fact that in a homogeneous plasma (i.e., for $K' = 0$), the k -vectors of the three waves are not necessarily matched since they are independently obtained solving for their corresponding dispersion relations. This zero-mismatch originates a frequency shift q which is given by setting $Q = 0$, as

$$q = - \frac{\Delta k(0)}{1/v_1 \xi - 1/v_2 \xi}. \quad (16)$$

The complex function $\beta(\xi)$ becomes now only a function of K' , and independent of the zero-mismatch $\Delta k(0)$:

$$\beta(\xi) = \eta + iK' \xi, \quad (17)$$

where η is real and is defined after Eq.(11); from now on, we shall simply call mismatch to the function $K' \xi$.

III. NORMAL MODES AND TEMPORAL GROWTH RATES

We wish to examine Eqs. (10) and (11) for temporally growing modes (i.e., for $s > 0$), using a WKB analysis. Such time growing modes can only exist if $v_1 \xi v_2 \xi < 0^{7-17}$.

A. WKB solutions and boundary conditions

Let us first start defining a new complex variable z as

$$z = z_0 + i \frac{K' \xi}{2|\lambda_0|}, \quad (18)$$

where $z_0 = \eta/2|\lambda_0|$, and η and λ_0 are defined after Eq. (11) (note that η depends on the eigenvalue s of the possible normal mode). The pump boundary limits $\xi = \pm L/2$, in the new variable z are c and c^* (the complex conjugate of c), where

$$c = z_0 + i \frac{K' L}{4|\lambda_0|}. \quad (19)$$

In the WKB approximation, two independent solutions of Eq. (13) are

$$\psi_{\pm}(z) = \frac{1}{[\zeta(z)]^{1/2}} \exp\left[\pm \frac{2|\lambda_0|^2}{K'} \int_z^1 (1 - z'^2)^{1/2} dz'\right], \quad (20)$$

where $\zeta(z) = |\lambda_0|(1 - z^2)^{1/2}$. The turning points are $z = \pm 1$, and $\zeta(z)$ is real analytic in the cut complex plane of Fig. 2. Any linear combination of the form $C_1 \psi_+(z) + C_2 \psi_-(z)$, with C_1 different from zero, is only defined in one semiplane of the complex plane [i.e., either for $\text{Im}(z)$ greater or smaller than zero]. If z crosses the real axis intersecting the Stokes' line,

$$\text{Im}\left[\int_z^1 (1 - z'^2)^{1/2} dz'\right] = 0, \quad (21)$$

(Im denotes the imaginary part of the expression between brackets) lying along the real axis between $z = 1$ and $z = -1$ (see Fig. 2), such linear combination should be modified according to the connection formulae given, for instance, in Ref. 18.

Let us denote by r and r^* the points of intersection of the line $\text{Re}(z) = z_0$ with the anti-Stokes' lines

$$\text{Re}\left[\int_z^1 (1 - z'^2)^{1/2} dz'\right] = 0. \quad (22)$$

(Re denotes the real part of the expression between brackets). The rate of change of the magnitudes of $\psi_{\pm}(z)$ with respect to z , is given by

$$A_{\pm}(z) = \exp\left\{\pm \frac{2|\lambda_0|^2}{K} \text{Re}\left[\int_z^1 (1 - z'^2)^{1/2} dz'\right]\right\},$$

$A_+(z)$ is exponentially large for $|\text{Im}(z)| < \text{Im}(r)$, and becomes exponentially small for $|\text{Im}(z)| > \text{Im}(r)$. When $z = z_0$, the mismatch $K'\xi$ is zero, and then $A_{\pm}(z_0)$ are maximal and minimal, respectively. Thus, the WKB solution, $\psi_+(z)$, decays in magnitude away from the zero-matching point $z = z_0$ as $|\text{Im}(z)|$ increases; the other WKB solution, $\psi_-(z)$, reaches its minimum value at the zero-matching point, and grows continuously in magnitude as $|\text{Im}(z)|$ increases (i.e., as the mismatch $K'\xi$ increases).

The mode amplitude, $\hat{a}_i(z)$, ($i = 1, 2$), is given as two different linear combinations of $\psi_{\pm}(z)$, each of them defined for each of the semiplanes, $\text{Im}(z) > (<) 0$. The complex variable z is now restricted to constant $\text{Re}(z) = z_0$ and to $|\text{Im}(z)| \leq \text{Im}(c)$. The right linear combinations that defines $\hat{a}_i(z)$ are dictated by the boundary conditions, which depend on the location of c and c^* in the complex plane. We distinguish between the two cases: (a) when c and c^* are between the two anti-Stokes' lines defined in Eq. (22); and (b) when c and c^* are outside the region delimited by these anti-Stokes' lines and the imaginary axis. In case (b) [i.e., for $\text{Im}(c) > \text{Im}(r)$], the magnitude of $\hat{a}_i(z)$ must decay exponentially as $|\text{Im}(z)|$ becomes larger than $\text{Im}(r)$. This means that $\hat{a}_i(z)$ must be proportional to $\psi_+(z)$ for both $\text{Im}(z)$ greater and smaller than zero. However, $\psi_+(z)$ cannot be simultaneously defined as an approximate solution of Eq. (13), in both semiplanes of the complex plane. This implies that the eigenvalue problem cannot be solved under the conditions of case (b).

We shall discuss in Sec. III C the restrictions that this imposes on the eigenvalue s of the possible normal mode.

In case (a), both solutions $\psi_{\pm}(z)$ are physically acceptable, since even if $\psi_{-}(z)$ grows as $|\text{Im}(z)|$ increases, its contribution to $\hat{a}_i(z)$ is always exponentially small; the dominant contribution is given by the well-behaved solution $\psi_{+}(z)$. The mode amplitude $\hat{a}_i(z)$ is given by two different linear combinations of $\psi_{\pm}(z)$, for each of the semiplanes, $\text{Im}(z) > (<) 0$; these combinations are connected along the real axis [i.e., along the Stokes' line defined in Eq. (21)]. Next, we present these solutions and derive the dispersion relation that defines the eigenvalue s of the possible normal mode.

B. The dispersion relation

Assuming always that $\text{Im}(c) < \text{Im}(r)$, the appropriate boundary conditions for the excitation of normal modes are $\hat{a}_1(c^*) = \hat{a}_2(c) = 0$. These boundary conditions give

$$\hat{a}_1(z) = \frac{A}{[\zeta(z)]^{1/2}} \sin[-i \frac{2|\lambda_0|^2}{K'} \int_{c^*}^z (1-z'^2)^{1/2} dz'], \quad (23)$$

$$\hat{a}_2(z) = \frac{B}{[\zeta(z)]^{1/2}} \sin[-i \frac{2|\lambda_0|^2}{K'} \int_c^z (1-z'^2)^{1/2} dz'], \quad (24)$$

where A and B are two integration constants which are related to each other in the way we shall specify. The solutions in Eqs. (23) and (24) are defined for $\text{Im}(z) < 0$ and for $\text{Im}(z) > 0$, respectively, and they have a common boundary along the real axis. The integration paths are now straight lines parallel to the imaginary axis, and are displaced from it a distance z_0 , which depends on the eigenvalue s . The system of equations (10) and (11) must be exactly satisfied at $z = z_0$, where the common boundary and the integration paths intersect:

$$\frac{\eta}{2} \hat{a}_1(z_0) + \frac{iK'}{2|\lambda_0|} \hat{a}_1'(z_0) = \lambda_0 \hat{a}_2(z_0), \quad (25)$$

$$-\frac{\eta}{2} \hat{a}_2(z_0) + \frac{iK'}{2|\lambda_0|} \hat{a}_2'(z_0) = -\lambda_0^* \hat{a}_1(z_0); \quad (26)$$

$\hat{a}'(z_0)$ denotes differentiation with respect to z evaluated at $z = z_0$. Equations (25) and (26) together with Eqs. (23) and (24), yield a system of algebraic equations to be solved for A and B . The compatibility of this system gives the following dispersion relation for the eigenvalue s of the temporal growing mode,

$$\tan\left[-i \frac{2|\lambda_0|^2}{K'} \int_{c^*}^c (1-z^2)^{1/2} dz\right] = -\left(\frac{|\lambda_0|^2}{\eta^2} - \frac{1}{4}\right)^{1/2}. \quad (27)$$

To solve for a real s , we have to require that $|\lambda_0| \geq \eta/2$. This implies that, for any given value of K' and L , the eigenvalue s is always smaller than or equal to s_0 as given by,

$$s_0 = \frac{2|\lambda_0| - \gamma_1/v_1\xi + \gamma_2/v_2\xi}{1/v_1\xi - 1/v_2\xi}. \quad (28)$$

Equations (25) and (26) also give the following relation between the two integration constants A and B :

$$\frac{B}{A} = \frac{-\eta/2 \sin\psi^* + (|\lambda_0|^2 - \eta^2/4)^{1/2} \cos\psi^*}{\lambda_0 \sin\psi}. \quad (29)$$

where $\psi = -i \frac{2|\lambda_0|^2}{K'} \int_c^{z_0} (1-z^2)^{1/2} dz$.

Let us now consider,

$$\int_{c^*}^c (1-z^2)^{1/2} dz = \frac{1}{2} [c(1-c^2)^{1/2} - c^*(1-c^{*2})^{1/2}] + \frac{1}{2} \arcsin [c(1-c^2)^{1/2} - c^*(1-c^{*2})^{1/2}].$$

If K' is assumed very small we can approximate $(1-c^2)^{1/2}$ by $[1 - (\eta/2|\lambda_0|)^2]^{1/2}$. Equation (27) now becomes:

$$\tan\left\{|\lambda_0| \frac{L}{2} \left(1 - \frac{\eta^2}{4|\lambda_0|^2}\right)^{1/2} + \frac{|\lambda_0|^2}{K'} \operatorname{arcsinh}\left[\frac{K'}{|\lambda_0|} \frac{L}{2} \left(1 - \frac{\eta^2}{4|\lambda_0|^2}\right)^{1/2}\right]\right\} = -\left(\frac{|\lambda_0|^2}{\eta^2} - \frac{1}{4}\right)^{1/2}. \quad (30)$$

The dispersion relation that gives the growth rates in a homogeneous plasma may now be recovered by letting $K' \rightarrow 0$ in Eq. (30), and it is, as presented in Ref. 5,

$$\tan[|\lambda_0|L (1 - \frac{\eta^2}{4|\lambda_0|^2})^{1/2}] = -(\frac{|\lambda_0|^2}{\eta^2} - \frac{1}{4})^{1/2}.$$

The minimum thresholds for the excitation of normal modes are given by setting $s = 0$ in Eq.(27). If we assume that the damping rates are small, we find

$$|\lambda_0| \frac{L}{2} [1 + (\frac{K'L}{4|\lambda_0|})^2]^{1/2} + \frac{|\lambda_0|^2}{K'} \operatorname{arcsinh}\{\frac{K'L}{2|\lambda_0|} [1 + (\frac{K'L}{4|\lambda_0|})^2]^{1/2}\} = (\pi + \frac{1}{2}) \pi, \quad (31)$$

where π is a positive integer, which classifies the unstable modes. Equation (31) determines, for given K' and L , the minimum value of $|\lambda_0|$ which is needed for the excitation of the normal mode of order π . Note that as $K'L$ increases these thresholds decrease, and eventually, as $K'L \rightarrow \infty$, there will be an infinite numerable of excited normal modes whose eigenvalues accumulate around $s = 0$. Nevertheless, we remark that these thresholds are only necessary but not sufficient conditions for the appearance of the unstable modes, since for $K'L = \infty$ no solutions with non-zero s exist for any finite value of $|\lambda_0|$. This will be easily understood in the discussion that follows next. It should also be noted that by letting $K' \rightarrow 0$ in Eq. (31), we recover the familiar thresholds for the excitation of normal modes in a homogeneous plasma: $|\lambda_0|L = (\pi + 1/2) \pi$, as calculated in Ref. 5.

C. Instability thresholds

The zero-matching point between the pump and excited waves [i.e., the point where the coupling is as in a homogeneous plasma, see Eqs. (13) and (17)] is $\xi = 0$, or in the z variable, $z = z_0$. This means that the strongest coupling between the three waves occurs half way across the pump cone along the trajectory $\bar{z} - \alpha\bar{x} = z_f$ for a given constant value of the free parameter z_f . This interaction will get weaker as the excited waves move through the pump cone, away from the zero-matching point, and the rate of decoupling is given by the product $K'L$. By increasing $K'L$ and keeping the eigenvalue s fixed, c and c^* will go beyond the anti-Stokes' lines departing from $z = 1$, and then that particular mode s or any one growing fastest cannot be excited. As explained in

Sec. III A, this happens because the magnitude of $\hat{a}_i(z)$ must decrease exponentially as $\text{Im}(z)$ moves away from the anti-Stokes' lines. Since no solution to Eq. (13) exists satisfying this requirement, we conclude that normal modes can only be excited if c and c^* are inside the region delimited by these anti-Stokes' lines and the imaginary axis.

These considerations allow us to find the maximum value, s_m , that the eigenvalue s can take for given value $|\lambda_0|$ and for fixed decoupling rate $K'L$. If we define

$$z_m = \frac{1}{2|\lambda_0|} \left(\frac{s_m + \gamma_1}{v_1\xi} - \frac{s_m + \gamma_2}{v_2\xi} + \frac{i}{2} K'L \right), \quad (32)$$

the fastest growing mode has a growth rate smaller than or equal to s_m as calculated from

$$\text{Re} \left[\int_1^{z_m} (1 - z^2)^{1/2} dz \right] = 0. \quad (33)$$

The upper bound s_m is always smaller than s_0 , which is defined in Eq. (28), and as $K'L \rightarrow 0$, $s_m \rightarrow s_0$. By increasing the decoupling rate $K'L$ (i.e., by increasing either the length of the interaction L or the strength of the mismatch K'), s_m will become smaller. If $K'L$ is greater than the inverse of a certain critical length, L_c , no temporal normal modes are possible. This critical length L_c depends on the linear damping rates γ_1 and γ_2 , and is defined solving for

$$\text{Re} \left[\int_1^{z_c} (1 - z^2)^{1/2} dz \right] = 0, \quad (34)$$

where

$$z_c = \frac{1}{2|\lambda_0|} \left(\frac{\gamma_1}{v_1\xi} - \frac{\gamma_2}{v_2\xi} + \frac{i}{2L_c} \right). \quad (35)$$

As $(\gamma_1/v_1\xi - \gamma_2/v_2\xi)$ goes to zero, the critical length L_c becomes smaller; when the damping rates become larger, so does also L_c . By letting the length of the interaction L go to infinity, we find that $K'L$ is always larger than $1/L_c$, implying that, in a pump of infinite extent, no purely growing modes are possible⁹. The thresholds defined through Eqs. (32), (33) and (34), (35), are depicted in Fig. 2.

Let us now assume that $K'L$ is smaller than $1/L_c$. By increasing the damping rates γ_1 and γ_2 , the upper bound s_m , defined through Eqs. (32) and (33), becomes smaller. If the damping rates are large enough so that $(\gamma_1/v_{1\xi} - \gamma_2/v_{2\xi}) \geq 2|\gamma_0|$, that is, for

$$\gamma_2 \left| \frac{v_{1\xi}}{v_{2\xi}} \right|^{1/2} + \gamma_1 \left| \frac{v_{2\xi}}{v_{1\xi}} \right|^{1/2} \geq 2|\gamma_0|, \quad (36)$$

we find that no temporal growing modes are possible for any value of $K'L$.

It should be noted that in the former discussions we have been assuming that the coupling coefficient γ_0 is constant. However, for fixed values of $K'L$, $\gamma_1/v_{1\xi}$ and $\gamma_2/v_{2\xi}$, the growth rate of the fastest growing mode increases with increasing γ_0 , and tends to s_0 , as defined in Eq. (28), when $|\gamma_0| \rightarrow \infty$. The critical length L_c decreases with increasing γ_0 , and as $|\gamma_0| \rightarrow \infty$, $L_c \rightarrow 0$.

IV. SPATIAL AMPLIFICATION

For spatial amplification we require the eigenvalue p to be purely imaginary [i.e., $p = iq$, with q as defined in Eq. (16)], so that the instability may only grow in space for a certain finite plasma length smaller than or equal to the width of the pump L . These excitations grow from initial thermal fluctuations, which develop at a certain point in the interaction region L . The problem poses now as a boundary value problem with boundary conditions, which are defined at the point where the thermal source interacts with the pump field. Spatial amplification can occur in both cases $v_1\xi v_2\xi > 0$, and $v_1\xi v_2\xi < 0$. We first study the case $v_1\xi v_2\xi > 0$, for which we know that no temporal growing modes are possible.

A. Group velocities in the same direction

Let us consider the complex variable z as defined in Eq. (18), where now $z_0 = \lambda_D/2|\lambda_0|$ and $\lambda_D = \gamma_1/v_1\xi - \gamma_2/v_2\xi$. The pump boundary limits $\xi = \pm L/2$ in the variable z are c and c^* , with $c = (\lambda_D + iK'L/2)/2|\lambda_0|$.

In the WKB approximation, two independent solutions of Eq. (13) are:

$$\psi_{\pm}(z) = \frac{1}{[\zeta(z)]^{1/2}} \exp[\pm i \frac{2|\lambda_0|^2}{K'} \int_{\pm i}^z (1+z'^2)^{1/2} dz'] \quad (37)$$

The turning points are $z = \pm i$, and $\zeta(z) = |\lambda_0|(1+z^2)^{1/2}$ is real analytic in the cut complex plane of Fig. 3. Let us call r and r^* the points where the line $\text{Re}(z) = z_0$ intersect the anti-Stokes' lines,

$$\text{Im} \left[\int_{\pm i}^z (1+z'^2)^{1/2} dz' \right] = 0. \quad (38)$$

The mode amplitudes $\hat{a}_i(z)$ ($i = 1, 2$) are given as linear combinations of $\psi_{\pm}(z)$; these combinations are dictated by the boundary conditions as follows. We distinguish between the two cases: (a) when $K'L$ is small enough so that c and c^* lie in between the two anti-Stokes' lines; and (b) when

$K'L$ is such that c and c^* are outside the region delimited by these anti-Stokes' lines and the imaginary axis.

In case (a), the boundary conditions are $\hat{a}_1(c^*) = \hat{a}_2(c^*) = 0$ and $(d\hat{a}_1(z)/dz)_{c^*} = b$, where b is the level of the thermal fluctuations. The amplification factor Γ , that gives the number of e foldings of the mode amplitudes \hat{a}_i , is found to be

$$\Gamma = -i \frac{2|\lambda_0|^2}{K'} \int_{c^*}^c (1+z^2)^{1/2} dz. \quad (39)$$

The path of integration is a straight line parallel to the imaginary axis which is displaced from it a distance z_0 , which depends on the damping rates of the excited waves. Let us next consider,

$$\int_{c^*}^c (1+z^2)^{1/2} dz = \frac{1}{2} \{c(1+c^2)^{1/2} - c^*(1+c^{*2})^{1/2} + \operatorname{arcsinh}[c(1+c^2)^{1/2} - c^*(1+c^{*2})^{1/2}]\}.$$

Under the limit $\lambda_D \rightarrow 0$ Γ becomes

$$\Gamma = \frac{|\lambda_0|L}{2} [1 - (\frac{K'L}{4|\lambda_0|})^2]^{1/2} + \frac{|\lambda_0|^2}{K'} \operatorname{arc} \sin \{ \frac{K'L}{2|\lambda_0|} [1 - (\frac{K'L}{4|\lambda_0|})^2]^{1/2} \}. \quad (40)$$

If we now let $K' \rightarrow 0$, we recover the homogeneous limit

$$\Gamma = \frac{|\gamma_0|L}{(|v_1\xi v_2\xi|)^{1/2}},$$

as calculated in Ref. 5. The amplification factor for the mode amplitudes a_1 and a_2 [see Eqs. (8) and (9)] should be calculated as $\Gamma = (\gamma_1/v_1\xi + \gamma_2/v_2\xi) L/2$.

In case (b) [i.e., for $|\operatorname{Im}(c)| > \operatorname{Im}(r)$], the thermal source is located at, say, $z = r^*$. The mode amplitudes must decrease exponentially as $|\operatorname{Im}(z)|$ becomes larger than $\operatorname{Im}(r)$. This implies that $\hat{a}_i(z)$ must be proportional to $\psi_+(z)$ for $\operatorname{Im}(z) > \operatorname{Im}(r^*)$. For $\operatorname{Im}(z) < \operatorname{Im}(r^*)$, $\hat{a}_i(z)$ is proportional to $\psi_-(z)$. The solutions in the two different regions should be matched at $z = r^*$. The modes $\hat{a}_i(z)$ are continuous functions of z , but their first derivatives at $z = r^*$ are not; the discontinuity in the derivatives is due to the presence of the thermal fluctuations, and the amount of discontinuity is

given by the level of the fluctuations. This discontinuity can be represented by introducing an inhomogeneous term, proportional to the Dirac-delta function, on the right-hand side of Eq. (13). Then, because of the presence of the thermal source, we can find solutions to the boundary value problem, such that they are well-behaved for $|\text{Im}(z)| > \text{Im}(r)$, and amplify in the region between the anti-Stokes' lines.

The amplification factor Γ for the modes $\hat{a}_i(z)$ is now⁹

$$\Gamma = \pi \frac{|\gamma_0|^2}{K' |v_{1\xi} v_{2\xi}|}. \quad (41)$$

The amplification factor for the modes a_1 and a_2 , as given in Eqs. (8) and (9), is calculated as: $\Gamma = (\gamma_1/v_{1\xi} + \gamma/v_{2\xi}) L_{eff}/2$, where L_{eff} is the amplification length which is given by

$$r = \frac{1}{2|\lambda_0|} (\lambda_D + \frac{i}{2} K' L_{eff}), \quad (42)$$

L_{eff} is the range for which the excited waves amplify inside the interaction region.

B. Group velocities in opposite directions

We next study the case $v_{1\xi} v_{2\xi} < 0$. The WKB-solutions, $\psi_{\pm}(z)$, are defined in Eq. (20), where z_0 is now equal to $\lambda_D/2|\lambda_0|$; the modes $\hat{a}_i(z)$ are given as certain linear combinations of $\psi_{\pm}(z)$. The Stokes structure of the complex z -plane has also been discussed in Sec. III A, and it is as presented in Fig. 2. We again have to distinguish between the two cases (a) and (b), depending on whether c and c^* are between the two anti-Stokes' lines in the semiplane $\text{Re } z \geq 0$, or whether c and c^* are beyond these anti-Stokes' lines, respectively. Case (a) has no interest here since if the instability grows [i.e., if $|\lambda_0|$ is above the thresholds given in Eq. (31)], it will build up in time in the way explained in Sec. III B.

In case (b) the instability can never build up in time because $K' L$ is always greater than the inverse of the critical length L_c , as defined in Eq. (34) and (35). The thermal source is located at, say, $z = r^*$, and the first derivative of $\hat{a}_i(z)$ is discontinuous at this point. The amplitudes of the

modes must be such that they decrease in magnitude exponentially as $|\text{Im}(z)|$ becomes larger than $\text{Im}(r)$. The amplification factor is found to be:

$$\Gamma = \frac{2|\lambda_0|^2}{K'} \left[\int_{r^*}^{z_0} (1-z^2)^{1/2} dz + \int_r^{z_0} (1-z^2)^{1/2} dz \right] \quad (43)$$

Equation (43) can also be rewritten in the more appropriate form,

$$\Gamma = \frac{4|\lambda_0|^2}{K'} \int_{z_0}^1 (1-z^2)^{1/2} dz, \quad (44)$$

within the limit $\lambda_D \rightarrow 0$ (i.e., for $z_0 = 0$) we find that Γ is as given in Eq.(41). The spatial amplification for a_1 and a_2 is calculated as $\Gamma - (\gamma_1/v_1\xi + \gamma_2/v_2\xi) L_{eff}/2$, where L_{eff} is the amplification length, which has been defined in Eq. (42).

V. SUMMARY AND CONCLUSION

Three wave parametric excitations have been analyzed in this paper, assuming a pump electric field of finite spatial extent and of constant magnitude in space and in time. We have taken into account the wavenumber mismatch due to the plasma inhomogeneities, and the Landau-damping rates of the excited waves; the wavenumber mismatch has been assumed to obey a linear profile. The resonant excitation can either grow in time or in space, and both types of temporal normal modes and spatially amplified solutions have been studied using a WKB-type of analysis.

Temporal normal modes may be excited if the group velocities of the excited waves have different directions across the pump propagation cone. We have derived an implicit transcendental equation, Eq. (27), that gives the growth rates of the excited modes; the dispersion relation is shown to allow transition to the homogeneous limit. The growth rate s of the fastest growing mode is shown to be smaller than or equal to s_m , as defined in Eqs. (32) and (33). This upper bound decreases upon increasing either the strength of the mismatch K' or the spatial width of the pump L ; if $K'L$ goes to infinity, s_m goes to zero. Moreover, if one allows some small damping for the excited waves, it is shown that there is a finite critical value, L_c^{-1} , of the product $K'L$, such that above it no normal modes are possible; this critical value is given by Eqs. (34) and (35). By increasing the damping rates of the excited waves, L_c^{-1} becomes smaller and so also do the growth rates of the excited normal modes. If the damping rates are such that Eq. (36) is fully satisfied, then no normal modes are possible for any value of $K'L$.

To obtain these results we have carried out a detailed examination of the Stokes' structure of the mode amplitudes in the complex plane. This allows us to select the correct boundary conditions, which are different from the ones found in a homogeneous plasma due to the existence of the wavenumber mismatch. The mismatch introduces complex turning points that, even

if they do not lie on the real space, are positively affecting the rate of growth of the instability. The spatial dependences of the mode amplitudes may change its direction of amplification at certain points ("turning points") inside the interaction region if K' or L becomes sufficiently large. When this happens that eigenmode, or anyone growing fastest, cannot be excited. This is because we cannot find solutions that are evanescent beyond the "turning points". In Refs. 15-17, the boundary conditions that were assumed, for any K' and L , are $a_1(-L/2) = a_2(L/2) = 0$; these are not always correct in inhomogeneous plasmas. The WKB approximation is an useful technique to approximate the values of the "turning points" r and r^* , as given by Eq. (22). This approximation is restricted to small values of K' as compared with $|\lambda_c|^2$. To apply the WKB approximation, we have to be careful in writing the solutions for the two mode amplitudes within this approximation. In Ref. 15, this was not correctly done, since by requiring that the boundary conditions, at both extremes of the pump cone, to be imposed only on one of the modes (e.g., a_1) the fast and slow spatial variations were not appropriately treated. This leads to an ambiguity in the dispersion relation, as it becomes different upon imposing the boundary conditions on the other mode a_2 .

If the group velocities of the excited waves have the same direction across the pump cone, or if they have different directions but $K'L$ is such that it is above L_c^{-1} , the instability can only grow in space. The amplification factors have been obtained, taking into account the damping rates of the excited modes and the mismatch in wavenumbers. We have distinguished between two different cases: (a) when the strength of the mismatch and damping rates are small enough so that the pulse response amplifies along all its trajectory in the pump cone; and (b) when they are large enough so that, owing to the presence of complex turning points, the pulse response cannot amplify along all its trajectory in the pump cone. In case (a), the amplification length is equal to the pump width L ; the amplification factor is given by Eq. (39). In case (b) the amplification length is always smaller than L , and it is defined in Eq. (42); the amplification factor is given by Eqs. (41) and (44), depending on the directions of the group velocities.

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APPENDIX

The weak coupled equations, Eqs. (2) and (3), have been derived in different physical situations by a number of authors^{6,14,19}. The derivation we are presenting is slightly different from what is usually found in the literature, and is particularized to the coupling of lower-hybrid waves.

We start writing Poisson's equation for the high frequency lower-hybrid waves

$$\nabla \cdot [\underline{K} (n_{nl}) \nabla \phi_{\ell h}(\vec{x}, t)] = 0, \quad (\text{A1})$$

where $\underline{K} (n_{nl})$ is the dielectric tensor evaluated at the nonlinear density n_{nl} . We assume that there are no free charges in the plasma, i.e., that all the charges are polarized and are contained in the nonlinear density n_{nl} . The dielectric tensor \underline{K} is an operator which operates on both space and time variables; it has the form:

$$\underline{K} (n_{nl}) = \begin{pmatrix} K_{\perp}(n_{nl}) & +i K_x(n_{nl}) & 0 \\ -i K_x(n_{nl}) & K_{\perp}(n_{nl}) & 0 \\ 0 & 0 & K_{\parallel}(n_{nl}) \end{pmatrix}. \quad (\text{A2})$$

These components acting on a wave (ω, \vec{k}) give:

$$K_{\parallel}(n_{nl}) = 1 - \frac{\omega_{pe}^2(n_{nl})}{\omega^2}, \quad (\text{A3a})$$

$$K_{\perp}(n_{nl}) = 1 + \frac{\omega_{pe}^2(n_{nl})}{\Omega_e^2} - \frac{\omega_{pi}^2(n_{nl})}{\omega^2}, \quad (\text{A3b})$$

$$K_x(n_{nl}) = \frac{\omega_{pe}^2(n_{nl})}{\omega \Omega_e}. \quad (\text{A3c})$$

We shall simply write K_{\parallel} , K_{\perp} , and K_x when they are evaluated at the unperturbed density of the plasma n_0 , which is assumed to be obtained from a Maxwellian distribution function.

For the decay of a lower-hybrid pump wave (ω_0, \vec{k}_0) into another lower-hybrid wave (ω_1, \vec{k}_1) plus a low frequency mode (ω_2, \vec{k}_2) , we simply have:

$$\phi_{\ell h} = \phi_0 + \phi_1, \quad (\text{A4a})$$

$$\phi_i = \frac{1}{2} \exp[i\psi_i(\vec{x})] \phi_{is}(x, z, t) + c.c., \quad (i = 0, 1), \quad (\text{A4b})$$

$$n_{\ell} = n_0 + n_1 \{ \exp[i\psi_2(\vec{x})] \eta_s(x, z, t) + c.c. \}, \quad (\text{A4c})$$

where

$$\psi_i(\vec{x}) = k_{ix}x + k_{iy}y + \int_{x_0}^x k_{ix}(x') dx', \quad (i = 0, 1, 2),$$

and where ϕ_{is} and η_s are slowly varying complex amplitudes. Upon substituting Eqs. (A2) and (A4) into Eq. (A1), we find, up to first order in the spatial derivatives of ϕ_{0s} and ϕ_{1s} :

$$L_0 \phi_{0s} = C_{01} \phi_{1s} \eta_s \exp[-i \int_{x_0}^x \Delta k(x') dx'], \quad (\text{A5a})$$

$$L_1 \phi_{1s} = C_{10}^* \phi_{0s} \eta_s^* \exp[i \int_{x_0}^x \Delta k(x') dx'], \quad (\text{A5b})$$

$$L_i = -k_{iz}^2 K_{ii}(\frac{\partial}{\partial t}) - k_{i\perp}^2 K_{i\perp}(\frac{\partial}{\partial t}) + 2i (K_{i\perp} k_{ix} \frac{\partial}{\partial x} + K_{ii} k_{iz} \frac{\partial}{\partial z}), \quad (\text{A5c})$$

$$C_{01} = (K_{1\perp} - 1) k_{0\perp} k_{1\perp} \cos \phi + (K_{1\parallel} - 1) k_{0z} k_{1z} + i K_{1x} k_{0\perp} k_{1\perp} \sin \phi, \quad (\text{A5d})$$

where $\Delta k(x) = k_{0x} - k_{1x} - k_{2x}$, and $\sin \phi = k_{1y}/k_{1\perp}$. If we assume $K_{\perp} \approx 1$ and if $K_{1x} \sin \phi \gg 1$, one can simplify Eqs. (A5a) and (A5b) by neglecting the first two terms in Eq. (A5d); these approximations are called $E \times B$ approximation.

The operator L_i ($i = 0, 1$) can be rewritten in a more convenient form by considering,

$$\frac{k_z^2}{k_{\perp}^2} K_{ii}(\frac{\partial}{\partial t}) + K_{i\perp}(\frac{\partial}{\partial t}) \approx i(\frac{\partial K}{\partial \omega})(\gamma + \frac{\partial}{\partial t}), \quad (\text{A6a})$$

$$K_{\perp} k_x \frac{\partial}{\partial x} + K_{ii} k_z \frac{\partial}{\partial z} = -\frac{k_{\perp}^2}{2} (\frac{\partial K}{\partial \omega})(v_x \frac{\partial}{\partial x} + v_z \frac{\partial}{\partial z}), \quad (\text{A6b})$$

where $\partial K/\partial \omega = 2/\omega (1 + \omega_{pe}^2/\Omega_e^2)$, (ω, \vec{k}) stands for either (ω_0, \vec{k}_0) or (ω_1, \vec{k}_1) , γ is the linear damping rate and v_x, v_z are the group velocity components. Combining Eqs. (A5) and (A6) and using the $E \times B$ approximation, we find:

$$(\gamma_p + \frac{\partial}{\partial t} + v_{0z} \frac{\partial}{\partial z} + v_{0x} \frac{\partial}{\partial x}) \phi_{0s} = - \frac{K_{1x} k_{1\perp} \sin \phi}{(\partial K_0 / \partial \omega)_{\omega_0} k_{0\perp}} \phi_{1s} \eta_s \exp[-i \int_{x_0}^x \Delta k(x') dx'], \quad (A7a)$$

$$(\gamma_1 + \frac{\partial}{\partial t} + v_{1z} \frac{\partial}{\partial z} + v_{1x} \frac{\partial}{\partial x}) \phi_{1s} = + \frac{K_{0x} k_{0\perp} \sin \phi}{(\partial K_1 / \partial \omega)_{\omega_1} k_{1\perp}} \phi_{0s} \eta_s^* \exp[i \int_{x_0}^x \Delta k(x') dx'], \quad (A7b)$$

where γ_p and γ_1 are the pump and sideband linear damping rates, respectively.

The nonlinear density η is due to the parallel ponderomotive force produced by ϕ_0 and ϕ_1 which acts on the electrons along \vec{B}_0 . The ponderomotive potential is given by $\phi_p = -e/m_e [(\vec{v}_0 \cdot \nabla) v_{1z}^* + (\vec{v}_1 \cdot \nabla) v_{0z}^* + c.c.]$, where \vec{v}_0 and \vec{v}_1 are the fluid velocities of the pump and sideband waves which are calculated from the fluid equation: $\partial \vec{v}_i / \partial t = -e/m_e (\vec{E}_i + \vec{v}_i \times \vec{B}_0)$, $i = 0, 1$. Hence, we find

$$\phi_{ps} = \frac{1}{4} \left(-\frac{e}{m_e} \frac{k_{0z} k_{1z}}{\omega_0 \omega_1} + \frac{i}{B_0 \omega_0} k_{0\perp} k_{1\perp} \sin \phi \right) \phi_{0s} \phi_{1s}^* \quad (A8)$$

the $E \times B$ approximation consists in neglecting the first term of the right-hand side of Eq. (A8).

The perturbed electron density can be found solving for

$$\left(\frac{\partial}{\partial t} + v_x \frac{\partial}{\partial z} \right) f_e = -\frac{e}{m_e} \frac{df_{Me}}{dv_z} \frac{\partial}{\partial z} (\phi_p + \phi_{sc}), \quad (A9)$$

where f_{Me} stands for a Maxwellian distribution function, and ϕ_{sc} is the self-consistent potential to be determined from Poisson's equation. From Eq. (A9) one gets,

$$\eta = \frac{\epsilon_0}{en_0} k_2^2 \chi_e \phi_2 \quad (A10)$$

The electron susceptibility χ_e is an operator which acts on the low frequency electric potential ϕ_2 , where $\phi_2 = \phi_p + \phi_{sc}$. For the ions the ponderomotive force is negligible and one finds: $n_i = -(\epsilon_0/en_0) k_2^2 \chi_i \phi_{sc}$. Putting together Eq. (A10) and n_i into Poisson equation we get

$$K_2 \left(\nabla \cdot \frac{\partial}{\partial t} \right) \phi_2 = [1 + \chi_i(\vec{k}_2, \omega_2)] \phi_p, \quad (A11)$$

where K_2 is the dispersion relation operator for the low frequency wave.

If the low frequency wave is nonresonant [i.e., $K_2(\vec{k}_2, \omega_2) \neq 0$], we may simply write $\phi_2 =$

$[1 + \chi_i(\vec{k}_2, \omega_2)]/K_2(\vec{k}_2, \omega_2) \phi_p$; putting this together with Eqs. (A7), (A8), and (A10), we could obtain the quasi-mode coupled equations as presented in Ref. 3. However, in our case, the low frequency wave is resonant [i.e., $K_2(\vec{k}_2, \omega_2) \approx 0$]; we may then write, in lowest order,

$$K_2(\nabla \cdot \frac{\partial}{\partial t})\phi_{2r} \approx i(\frac{\partial K_2}{\partial \omega})_{\omega_2} (\gamma_2 + \frac{\partial}{\partial t} + v_{2x} \frac{\partial}{\partial x} + v_{2z} \frac{\partial}{\partial z})\phi_{2r} \quad (\text{A12})$$

where γ_2 is the linear damping rate, and v_{2x}, v_{2z} are the group velocity components for the low frequency mode.

Let us next define the wave packet amplitudes a_i as

$$a_i = [(\frac{\partial K_i}{\partial \omega})_{\omega_i}]^{1/2} k_i \phi_{ir} \quad (\text{A13})$$

where $i = 0, 1, 2$, and $E_i = 1/2 k_i \phi_{ir}$ is the magnitude of the electric field. By combining Eqs. (A7) through (A13), we finally arrive at the coupled equations describing the resonant interaction between the three waves:

$$(\gamma_0 + \frac{\partial}{\partial t} + \vec{v}_0 \cdot \nabla)a_0 = -C a_1 a_2, \quad (\text{A14a})$$

$$(\gamma_1 + \frac{\partial}{\partial t} + \vec{v}_1 \cdot \nabla)a_1 = C a_0 a_2^*, \quad (\text{A14b})$$

$$(\gamma_2 + \frac{\partial}{\partial t} + \vec{v}_2 \cdot \nabla)a_2 = C a_0 a_1^*. \quad (\text{A14c})$$

We have taken $\omega_0 \approx \omega_1$, where

$$C = \frac{1}{4} \frac{1}{B_0 \omega_0} \frac{\sin \phi}{[(\frac{\partial K_0}{\partial \omega})_{\omega_0} (\frac{\partial K_1}{\partial \omega})_{\omega_1} (\frac{\partial K_2}{\partial \omega})_{\omega_2}]^{1/2}} k_2 \chi_e(\vec{k}_2, \omega_2). \quad (\text{A15})$$

If the pump is assumed constant, and if $\gamma_0 = C a_0$, we obtain the system of coupled equations (2) and (3).

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FIGURE CAPTIONS

Fig. 1. Pump propagation cone and trajectory of the pulse response. The coordinates (x, z) lie along the direction of the plasma inhomogeneities and toroidal magnetic field, respectively. The coordinates (\bar{x}, \bar{z}) lie along and perpendicular to the pump propagation cone. The line of the pulse response is $\bar{z} - \alpha\bar{x} = z_f$, with z_f a free parameter defining where in the plasma the resonant interaction is taking place.

Fig. 2. The complex z -plane is cut at the turning points, $z = \pm 1$. These cuts run from $z = \pm 1$ to $z = \pm\infty$. The anti-Stokes' lines are full lines departing from $z = 1$. The Stokes' line is the broken line along the real axis. Normal modes may be excited for $K'L < L_c^{-1}$. The interaction range extends along the line that joins c and c^* , where $z_0 = [\lambda_D + s(1/v_1\xi - 1/v_2\xi)]/2|\lambda_0|$, s is the growth rate of the possible normal mode, and $c = z_0 + K'L/4|\lambda_0|$. The growth rate of the fastest growing mode is smaller than or equal to s_m , as given by, $z_m = [\lambda_D + s_m(1/v_1\xi - 1/v_2\xi) + i/2 K'L]/2|\lambda_0|$.

Fig. 3. The complex z -plane is cut at the turning points $z = \pm i$. These cuts run from $z = \pm i$ to $z = \pm i\infty$. The anti-Stokes' lines are full lines departing from $z = \pm i$. The Stokes' line is the broken line along the imaginary axis. We distinguish between two cases (a) and (b), depending on whether the pump boundary limits c and c^* , where $c = (\lambda_D + i/2 K'L)/2|\lambda_0|$, lie between the two anti-Stokes' lines or not. In case (a) the amplification length is equal to the pump width L . In case (b), it is L_{eff} defined through $\tau = (\lambda_D + i/2 K'L_{eff})/2|\lambda_0|$.

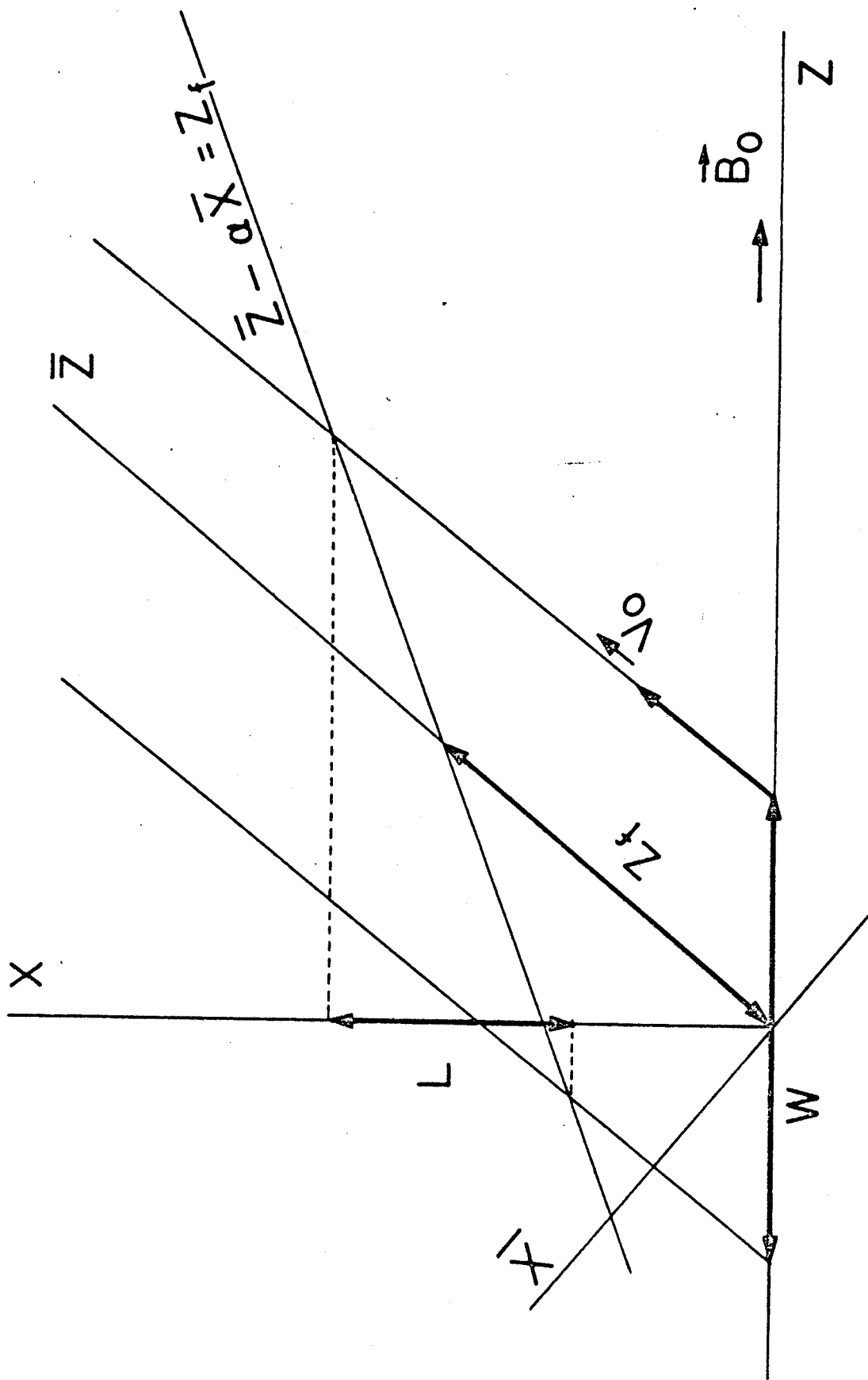


FIGURE 1

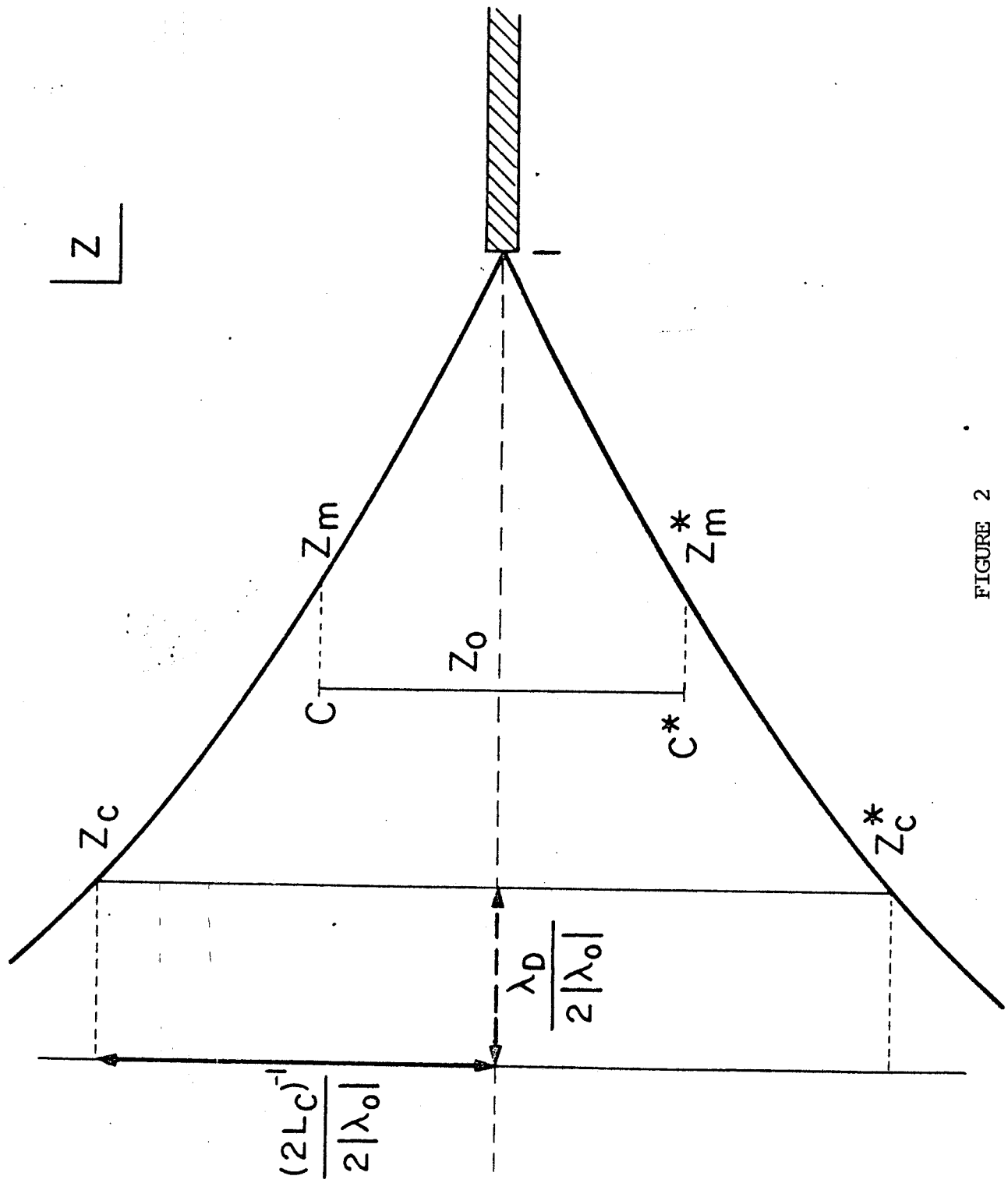


FIGURE 2

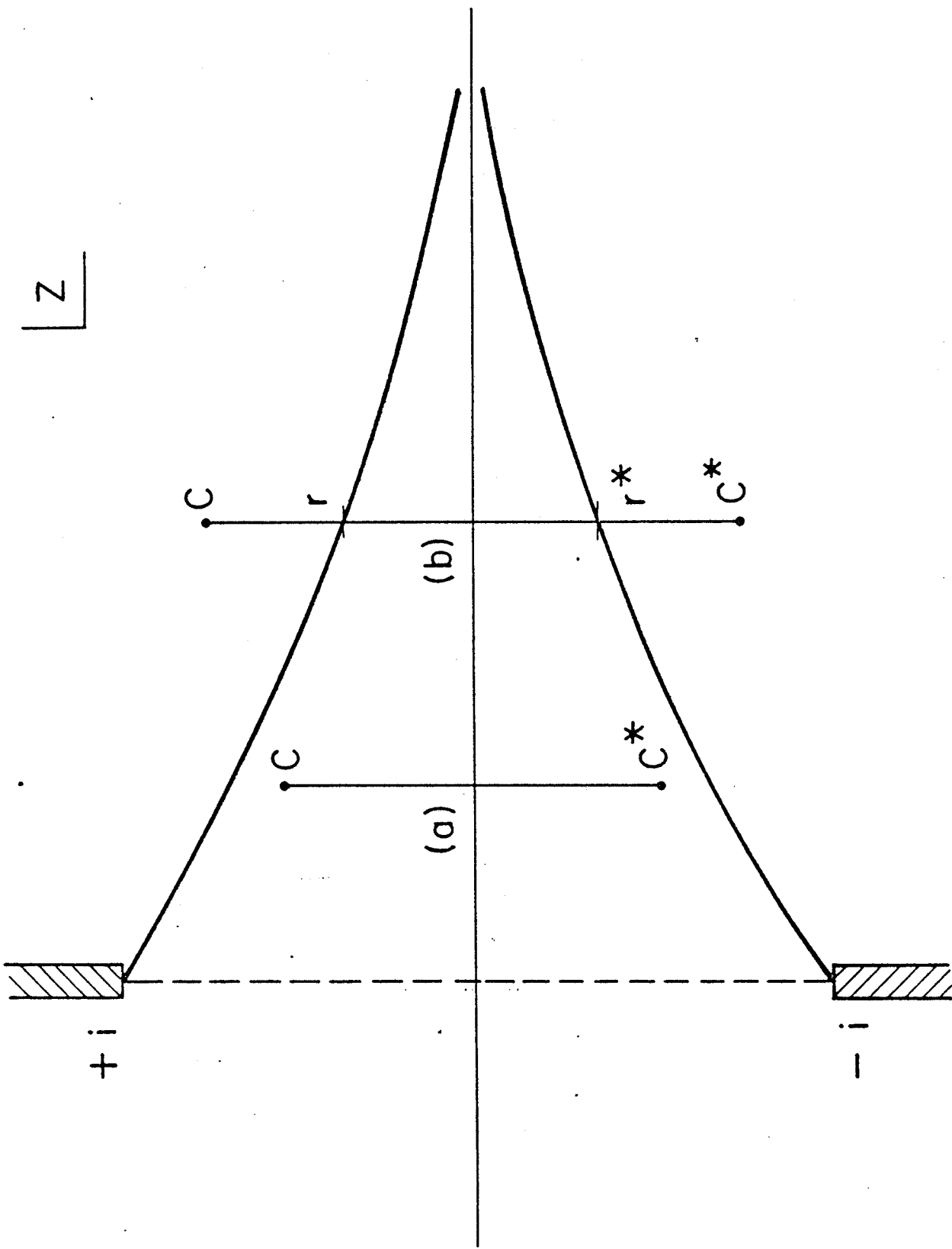


FIGURE 3