

Gapped Domain Walls, Gapped Boundaries, and Topological Degeneracy

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Gapped domain walls, as topological line defects between $(2 + 1)$ D topologically ordered states, are examined. We provide simple criteria to determine the existence of gapped domain walls, which apply to both Abelian and non-Abelian topological orders. Our criteria also determine which $(2 + 1)$ D topological orders must have gapless edge modes, namely, which $(1 + 1)$ D *global gravitational anomalies* ensure gaplessness. Furthermore, we introduce a new mathematical object, the *tunneling matrix* \mathcal{W} , whose entries are the *fusion-space dimensions* \mathcal{W}_{ia} , to label different types of gapped domain walls. By studying many examples, we find evidence that the tunneling matrices are powerful quantities to classify different types of gapped domain walls. Since a gapped boundary is a gapped domain wall between a bulk topological order and the vacuum, regarded as the trivial topological order, our theory of gapped domain walls inclusively contains the theory of gapped boundaries. In addition, we derive a topological ground state degeneracy formula, applied to arbitrary orientable spatial 2-manifolds with gapped domain walls, including closed 2-manifolds and open 2-manifolds with gapped boundaries.

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Introduction.—The insulator has a finite energy gap, which is rather trivial at low energy. Nonetheless, domain walls, separating different symmetry-breaking insulating regions, can enrich the physics of a trivial insulator, such as some paramagnets [1]. *Topological order* [2–4], on the other hand, as a new kind of many-body quantum ordering, has a gapped bulk with exotic properties: some have (i) gapless edge modes, (ii) anyonic excitations with fractional or non-Abelian statistics [5], such as fractional quantum Hall states, and (iii) long-range entanglement [6–8]. In this Letter, we would like to investigate the gapped domain walls of topological orders, and how gapped domain walls further enrich their physics.

It was conjectured that the $(2 + 1)$ D topological orders are completely classified by the gauge connection on the moduli space of the degenerate ground states [4,9]. The non-Abelian part of the gauge connection is the non-Abelian geometric phase [10] characterized by the \mathcal{S}, \mathcal{T} matrices, which also encode the anyon statistics. The Abelian part is related to the gravitational Chern-Simons term in the effective theory and is described by the chiral central charge c_- of the edge state. Nonzero c_- implies robust gapless edge modes.

By now we understand how to label a 2D topological order by a set of “topological order parameters” $(\mathcal{S}, \mathcal{T}, c_-)$, analogous to “symmetry-breaking order parameters” for spontaneous symmetry breaking systems [11,12]. However, it is less known how different topological orders are related. To this end, it is important to investigate the following circumstance: there are several domains in the

system and each domain contains a topological order, while the whole system is gapped. In this case, different topological orders are connected by *gapped domain walls*. Our work addresses two primary questions: (Q1) Under what criteria can two topological orders be connected by a gapped domain wall, and how many different types of gapped domain walls are there? Since a gapped boundary is a gapped domain wall between a nontrivial topological order and the vacuum, we also address the question “under what criteria can topological orders allow gapped boundaries?” (Q2) When a topologically ordered system has a gapped bulk, gapped domain walls, and gapped boundaries, how can one calculate its ground state degeneracy (GSD) [2,3,13–15], on any orientable manifold?

Main result.—Consider two topological orders, phases A and B , described by $(\mathcal{S}^A, \mathcal{T}^A, c_-^A)$ and $(\mathcal{S}^B, \mathcal{T}^B, c_-^B)$. Suppose there are N and M types of anyons in phase A and phase B , then the ranks of their modular matrices are N and M , respectively. If A and B are connected by a gapped domain wall, first, their central charges must be the same $c_-^A = c_-^B$. Next we find that the domain wall can be labeled by an $M \times N$ *tunneling matrix* \mathcal{W} whose entries are *fusion-space dimensions* \mathcal{W}_{ia} satisfying the *commuting condition*, Eq. (2), and the *stable condition*, Eq. (3):

$$\mathcal{W}_{ia} \in \mathbb{N}, \quad (1)$$

$$\mathcal{S}^B \mathcal{W} = \mathcal{W} \mathcal{S}^A, \quad \mathcal{T}^B \mathcal{W} = \mathcal{W} \mathcal{T}^A, \quad (2)$$

$$\mathcal{W}_{ia} \mathcal{W}_{jb} \leq \sum_{kc} (\mathcal{N}^B)_{ij}^k \mathcal{W}_{kc} (\mathcal{N}^A)_{ab}^c. \quad (3)$$

\mathbb{N} denotes the set of non-negative integers. a, b, c, \dots and i, j, k, \dots are anyon indices for phases A, B . $(\mathcal{N}^A)_{ab}^c$ and $(\mathcal{N}^B)_{ij}^k$ are fusion tensors [5,16] of phases A, B .

Relations (1)–(3) comprise a set of necessary conditions a gapped domain wall must satisfy, i.e., *if there is no nonzero solution of \mathcal{W} , the domain wall must be gapless*. We conjecture that they are also sufficient for a gapped domain wall to exist. In the examples studied in the Supplemental Material [17], \mathcal{W} are in one-to-one correspondence with gapped domain walls. However, for some complicated examples [22], a \mathcal{W} matrix may correspond to more than one type of gapped domain wall. This indicates that some additional data are needed to completely classify gapped domain walls.

As a first application of our result, we give a general method to compute the GSD in the presence of gapped domain walls on any orientable 2D surface. A simple case is the GSD on a disk drilled with two holes [equivalently a sphere with three circular boundaries, see Fig. 3(c)]. The gapped boundaries are labeled by three *vectors* (one-row or one-column matrices) $\mathcal{W}^{(1)}, \mathcal{W}^{(2)}, \mathcal{W}^{(3)}$. The GSD is $\sum_{ijk} \mathcal{W}_{i1}^{(1)} \mathcal{W}_{j1}^{(2)} \mathcal{N}_{ij}^k \mathcal{W}_{1k}^{(3)}$.

For gapped boundaries, our criteria can be understood via *dimension reduction*, i.e., shrinking a 1D gapped boundary \mathcal{W} to an (composite, Ref. [23]) anyon $\mathbf{q}_{\mathcal{W}} = \bigoplus_a \mathcal{W}_{1a} a$. If the system is on a 2D surface M^2 drilled with n gapped boundaries $\mathcal{W}^{(1)}, \dots, \mathcal{W}^{(n)}$, then the GSD is the dimension of the fusion space [23] with anyons $\mathbf{q}_{\mathcal{W}^{(1)}}, \dots, \mathbf{q}_{\mathcal{W}^{(n)}}$, $\text{GSD} = \dim[\mathcal{V}(M^2, \mathbf{q}_{\mathcal{W}^{(1)}}, \dots, \mathbf{q}_{\mathcal{W}^{(n)}})]$.

Since gapped domain walls *talk to each other* through long-range entanglement, the GSD with domain walls reveals more physics than that without domain walls. We foresee its practicality in experiments, since we can read even more physics by putting the system on open surfaces with gapped domain walls. Below we shall properly introduce \mathcal{S}, \mathcal{T} and \mathcal{W} matrices.

Modular \mathcal{S}, \mathcal{T} matrices.— \mathcal{S} and \mathcal{T} are unitary matrices indexed by anyon types $\{1, a, b, c, \dots\}$. 1 labels the trivial anyon type. The anti-quasiparticle of a is denoted by a^* .

\mathcal{T} describes the self-statistics. It is diagonal $\mathcal{T}_{ab} = e^{i\theta_a} \delta_{ab}$, where $e^{i\theta_a}$ is the phase factor when exchanging two anyons a . For the trivial type, $\mathcal{T}_{11} = e^{i\theta_1} = 1$. \mathcal{S} describes the mutual statistics. \mathcal{S}_{ab} is the amplitude of the following process with proper normalization factors: first create a pair of aa^* and a pair of bb^* , then braid a around b , and finally annihilate the two pairs. \mathcal{S} is symmetric, $\mathcal{S}_{ab} = \mathcal{S}_{ba}$. If $b = 1$, the process is just creation and annihilation, and $\mathcal{S}_{a1} > 0$. \mathcal{S} and \mathcal{T} form a projective representation of the modular group: $\mathcal{S}^4 = I, (\mathcal{S}\mathcal{T})^3 = e^{2\pi ic_-/8} \mathcal{S}^2$, where I denotes the identity matrix.

The anti-quasiparticle can be read from \mathcal{S}^2 , $(\mathcal{S}^2)_{ab} = \delta_{a^*b}$. The fusion tensor \mathcal{N}_{ab}^c can be calculated via the *Verlinde formula* [16]:

$$\mathcal{N}_{ab}^c = \sum_m \frac{\mathcal{S}_{am} \mathcal{S}_{bm} \overline{\mathcal{S}_{cm}}}{\mathcal{S}_{1m}} \in \mathbb{N}. \quad (4)$$

Gapped domain walls.—Below we demonstrate the physical meanings of the gapped domain wall conditions, Eqs. (1)–(3). First we put phase A and phase B on a sphere S^2 , separated by a gapped domain wall. Note that there can be many types of domain walls separating the same pair of phases A and B . What data characterize those different types of domain walls? We fix the domain-wall type, labeled by W , and trap [23] an anyon a^* in phase A , and an anyon i in phase B . This configuration is denoted by (S^2, i, W, a^*) . The states with such a configuration may be degenerate and the degenerate subspace is the fusion space $\mathcal{V}(S^2, i, W, a^*)$. Here we propose using the *fusion-space dimensions* $\mathcal{W}_{ia} \equiv \dim[\mathcal{V}(S^2, i, W, a^*)] \in \mathbb{N}$ to characterize the gapped domain wall W .

There are nonlocal operators O_{W,ia^*} that create a pair aa^* in phase A , and then tunnel a through the domain wall to an anyon i in phase B , $O_{W,ia^*} |\psi_{S^2,W}\rangle \in \mathcal{V}(S^2, i, W, a^*)$, where $|\psi_{S^2,W}\rangle$ is the ground state. Since we care about the fusion states rather than the operators themselves, we would take the equivalent class $[O_{W,ia^*}] = \{U_{W,ia^*} (O_{W,ia^*} - U_{W,ia^*}) |\psi_{S^2,W}\rangle = 0\}$. We refer to $[O_{W,ia^*}]$ as *tunneling channels*, which correspond to fusion states in $\mathcal{V}(S^2, i, W, a^*)$. Therefore, the fusion space dimension \mathcal{W}_{ia} is the number of linearly independent tunneling channels. So we also refer to \mathcal{W} as the “tunneling matrix.”

The *commuting condition* Eq. (2) dictates the consistency of anyon statistics in the presence of gapped domain walls. Since modular \mathcal{S}, \mathcal{T} matrices encode the anyon statistics, we require that \mathcal{W} should commute with them as Eq. (2): $\mathcal{S}^B \mathcal{W} = \mathcal{W} \mathcal{S}^A, \mathcal{T}^B \mathcal{W} = \mathcal{W} \mathcal{T}^A$.

We may as well create a pair ii^* in phase B and tunnel i^* to a^* . \mathcal{W}^\dagger describes such tunneling in the opposite direction (i.e., $\mathcal{W}: A \rightarrow B, \mathcal{W}^\dagger: B \rightarrow A$). \mathcal{W}^\dagger and \mathcal{W} contains the same physical data. To be consistent, tunneling i^* to a^* should give the same fusion-space dimension, $(\mathcal{W}^\dagger)_{a^*i^*} = \mathcal{W}_{i^*a^*} = \mathcal{W}_{ia}$. This is guaranteed by $\mathcal{W}(\mathcal{S}^A)^2 = (\mathcal{S}^B)^2 \mathcal{W}$ and $(\mathcal{S}^2)_{ab} = \delta_{a^*b}$.

The fusion spaces with four anyons further provide us consistence conditions of \mathcal{W} . To see this, first notice that there are *generalized* tunneling channels, $[O_{W,ia^*,x}]$, which, in addition to tunneling a to i , also create the quasiparticle x on the domain wall. If we combine the tunneling channels $[O_{W,ia^*,x}]$ and $[O_{W,jb^*,x^*}]$, we can create fusion states with a domain wall W and four anyons i, j, a^*, b^* , as in Fig. 1(a). In other words, $[O_{W,ia^*,x} O_{W,jb^*,x^*}]$ form a basis of the fusion space $\mathcal{V}(S^2, i, j, W, a^*, b^*)$. Let \mathcal{K}_{ia}^x denote the number of tunneling channels $[O_{W,ia^*,x}]$, and we know that $\dim \mathcal{V}(S^2, i, j, W, a^*, b^*) = \sum_x \mathcal{K}_{ia}^x \mathcal{K}_{jb}^{x^*}$. However, the tunneling process as shown in Fig. 1(b), i.e., fusing a, b to c , using $[O_{W,kc^*}]$ to tunnel c to k and splitting k to i, j , forms

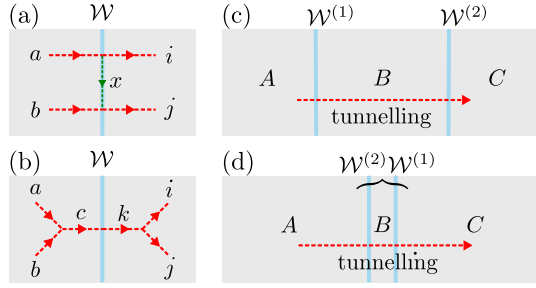


FIG. 1 (color online). (a),(b) Tunneling channels. (c) Separated domain walls $\mathcal{W}^{(1)}$ and $\mathcal{W}^{(2)}$. (d) Composite domain wall $\mathcal{W}^{(2)}\mathcal{W}^{(1)}$.

another basis of the fusion space. The number of such fusion, tunneling, or splitting channels is $\sum_{kc} (\mathcal{N}^B)^k_{ij} \mathcal{W}_{kc} (\mathcal{N}^A)^c_{ab}$. Therefore, we must have

$$\sum_x \mathcal{K}_{ia}^x \mathcal{K}_{jb}^{x*} = \sum_{kc} (\mathcal{N}^B)^k_{ij} \mathcal{W}_{kc} (\mathcal{N}^A)^c_{ab}. \quad (5)$$

We are interested in classifying *stable* gapped domain walls, i.e., the GSD cannot be reduced no matter what small perturbations are added near the domain wall. For stable gapped domain walls we have $\mathcal{W}_{ia} = \mathcal{K}_{ia}^1$. Unstable gapped domain walls \mathcal{U} split as the sum of stable ones $\mathcal{W}^{(1)}, \mathcal{W}^{(2)}, \dots, \mathcal{W}^{(N)}$, and $\mathcal{U}_{ia} = \sum_{n=1}^N \mathcal{W}_{ia}^{(n)}$, for $N \geq 2$.

Now if a gapped domain wall \mathcal{W} is stable, Eq. (5) becomes $\sum_{kc} (\mathcal{N}^B)^k_{ij} \mathcal{W}_{kc} (\mathcal{N}^A)^c_{ab} = \mathcal{W}_{ia} \mathcal{W}_{jb} + \sum_{x \neq 1} \mathcal{K}_{ia}^x \mathcal{K}_{jb}^{x*} \geq \mathcal{W}_{ia} \mathcal{W}_{jb}$. We know that Eq. (3) is necessary for a gapped domain wall to be stable. Furthermore, setting $i = j = a = b = 1$ we know that $\mathcal{W}_{11} \geq \mathcal{W}_{11}^2$ and Eq. (2) requires that $\mathcal{W}_{11} > 0$, thus $\mathcal{W}_{11} = 1$ and \mathcal{W} cannot be the sum of more than one stable tunneling matrix; it must be stable itself. Therefore Eq. (3) with Eq. (2) is also sufficient for a gapped domain wall to be stable.

Stability of composite domain walls.—Let us consider two stable domain walls, $\mathcal{W}^{(1)}$ between phases A and B , and $\mathcal{W}^{(2)}$ between phases B and C , as in Fig. 1(c). When the two domain walls are far separated, they are both stable. Any small perturbations added near $\mathcal{W}^{(1)}$, or near $\mathcal{W}^{(2)}$, cannot reduce the GSD.

We then shrink the size of the middle phase B , such that the two domain walls are near enough to be regarded as a single domain wall. This way we obtain a composite domain wall, whose tunneling matrix is the composition $\mathcal{W}^{(2)}\mathcal{W}^{(1)}$, as in Fig. 1(d). However, this composite domain wall $\mathcal{W}^{(2)}\mathcal{W}^{(1)}$ may no longer be stable. Unless phase B is a vacuum, we allow more perturbations to $\mathcal{W}^{(2)}\mathcal{W}^{(1)}$ than when $\mathcal{W}^{(1)}$ and $\mathcal{W}^{(2)}$ are far separated. Some operators simultaneously acting on both $\mathcal{W}^{(1)}$ and $\mathcal{W}^{(2)}$ may reduce the GSD, in which case, the composite domain wall $\mathcal{W}^{(2)}\mathcal{W}^{(1)}$ is not stable.

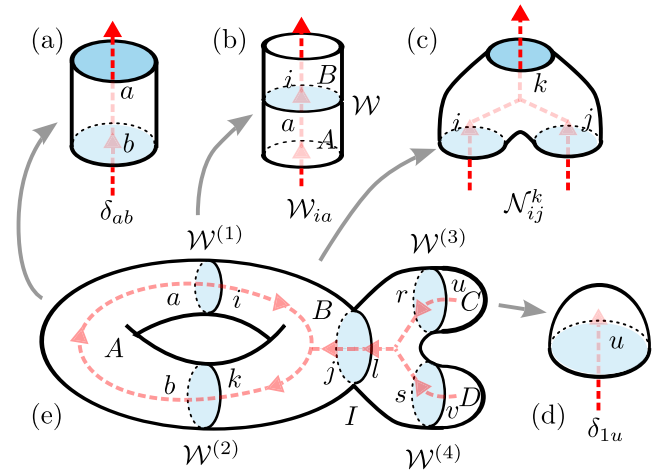


FIG. 2 (color online). Computing GSD by tensor contraction: Cut a complicated manifold (e) into simple segments, add oriented skeletons and anyon indices. Associate the segments with: (a) a cylinder with δ_{ab} , (b) a domain wall with its tunneling matrix \mathcal{W}_{ia} , (c) a pair of pants with the fusion tensor \mathcal{N}_{ij}^k , and (d) a cap with δ_{1u} . Finally, contract all the tensors.

In the special case when phase B is a vacuum, the composite $\mathcal{W}^{(2)}\mathcal{W}^{(1)}$ remains stable. One can explicitly check this with Eq. (3).

GSD in the presence of gapped domain walls.—Below we derive the GSD, for a 2D system containing several topological orders separated by looplike gapped domain walls. Domain walls cut a whole 2D system into several segments. Without losing generality, let us consider an example in Fig. 2 with topological orders, phases A, B, C, D , and four nontrivial domain walls, $\mathcal{W}^{(1)}, \mathcal{W}^{(2)}, \mathcal{W}^{(3)}, \mathcal{W}^{(4)}$, on a manifold, Fig. 2(e). We first add extra trivial domain walls $\mathcal{W} = I$, so that all segments between domain walls are reduced to simpler topologies: caps, cylinders, or pants. We also add oriented skeletons to the manifold, and put anyon indices on both sides of the domain walls, as shown in Fig. 2(e). Next, see Figs. 2(a)–2(d), for the segments with oriented skeletons and anyon indices, we associate certain tensors: caps with δ_{1u} , cylinders with δ_{ab} , pants with \mathcal{N}_{ij}^k in the corresponding topological order, and domain walls with their tunneling matrices \mathcal{W}_{ia} . We may reverse the orientation and at the same time replace the index

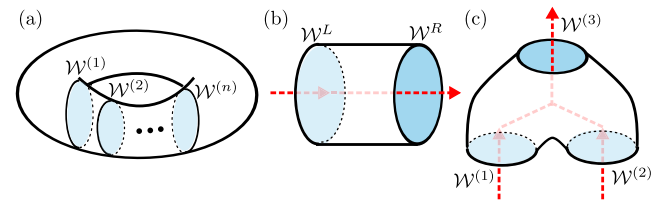


FIG. 3 (color online). Some 2-manifolds with gapped domain walls. (a) Several domain walls on the torus. (b) A cylinder with two gapped boundaries. (c) A pair of pants with three gapped boundaries.

TABLE I. The number of different gapped domain-wall types (“No. gapped DW” for short) sandwiched by two topological orders (one from the first column and the other from the first row). $D^{\omega_3}(G)$ stands for the twisted quantum double model of gauge group G with a three-cocycle twist ω_3 .

No. gapped DW	Vacuum	Toric code
Toric code	2	6
Double semion	1	2
Doubled Fibonacci	1	2
Doubled Ising	1	3
$D(S_3)$	4	12

No. gapped DW	Vacuum
$D(D_4)$	11
$D(Q_8)$	6
$D^{\omega_3[3d]}(Z_2^3)$	5
$D^{\omega_3[5]}(Z_2^3)$	3
$D^{\omega_3[7]}(Z_2^3)$	1

a with a^* . Finally, we multiply these tensors together and contract all the anyon indices. Physically such tensor contraction computes the total number of winding channels of anyons, which exactly counts the number of ground states, thus the GSD.

Systems with *gapped boundaries* are included in our method; just imagine that there are vacuums on caps connected to the boundaries, e.g., phases C, D in Fig. 2(e) can be a vacuum. Dimensions of generic fusion spaces can also be calculated, by putting the anyon a on the cap and associating the tensor δ_{au} instead of δ_{1u} .

We derive the GSD on exemplary manifolds: 1. A stable domain wall \mathcal{W} on the sphere: $\text{GSD} = \mathcal{W}_{11} = 1$. 2. A domain wall \mathcal{W} on the torus: $\text{GSD} = \text{Tr}(\mathcal{W})$. Several domain walls $\mathcal{W}^{(1)}, \dots, \mathcal{W}^{(n)}$ on the torus, in Fig. 3(a): $\text{GSD} = \text{Tr}(\mathcal{W}^{(1)} \dots \mathcal{W}^{(n)})$. In particular, $\text{Tr}[\mathcal{W}^{(1)}(\mathcal{W}^{(2)})^\dagger]$ counts the types of $0D$ defects between 1D gapped domain walls $\mathcal{W}^{(1)}, \mathcal{W}^{(2)}$. 3. A sphere with punctures: A cylinder with two gapped boundaries \mathcal{W}^L and \mathcal{W}^R , in Fig. 3(b):

TABLE II. GSD of a single topological order (the first column) on a sphere with a number of punctures (the first row). Each puncture has a gapped boundary. The last three orders allow only one type of gapped boundary, so its GSD is unique for a given topology. Toric code allows two types of gapped boundaries, and its GSD varies, which depends on boundary types associated to each puncture. This agrees with Refs. [13,24].

GSD (No. of punctures)	1	2	3	4
Toric code	1	1, 2	2, 4	2, 4, 8
Double semion	1	2	4	8
Doubled Fibonacci	1	2	5	15
Doubled Ising	1	3	10	36

$\text{GSD} = \sum_a \mathcal{W}_{a1}^L \mathcal{W}_{1a}^R$. A pair of pants with three gapped boundaries $\mathcal{W}^{(1)}, \mathcal{W}^{(2)}$ and $\mathcal{W}^{(3)}$, in Fig. 3(c): $\text{GSD} = \sum_{ijk} \mathcal{W}_{i1}^{(1)} \mathcal{W}_{j1}^{(2)} \mathcal{N}_{ij}^k \mathcal{W}_{1k}^{(3)}$. 4. The rocket graph in Fig. 2(e): $\text{GSD} = \sum_{a,i,j,k,r,s} \mathcal{W}_{ia}^{(1)} \mathcal{W}_{ak}^{(2)} (\mathcal{N}^B)_{ij}^k (\mathcal{N}^B)_{rs}^j \mathcal{W}_{r1}^{(3)} \mathcal{W}_{s1}^{(4)}$. We apply our formalism to several topological orders. Details of our examples are organized in the Supplemental Material [17]. Part of our result is listed in Table I (the number of gapped domain-wall types) and Table II (GSD).

Conclusion.—Given \mathcal{S}, \mathcal{T} matrices of topological orders with the same central charge, we have provided simple criteria, Eqs. (1)–(3), to check the existence of gapped domain walls. We want to mention that, a gapped domain wall can be related to a gapped boundary by the *folding trick* [25]. By studying gapped boundaries, we can also obtain all the information of gapped domain walls. But, to compute the GSD, gapped domain walls allow more configurations on 2D surfaces than gapped boundaries.

The gapped domain walls and boundaries can be explicitly realized in lattice models [25–27]. Levin-Wen string-net models [28] are exactly solvable models for topological orders. Recently, it was found that a topological order can be realized by a Levin-Wen model if and only if it has gapped boundaries [25,27]. Thus our work provides the criteria of whether a topological order has a Levin-Wen realization.

2D Abelian topological orders can be described by Chern-Simons field theories. The boundary of a Chern-Simons theory is gappable, if and only if there exists a *Lagrangian subgroup* [13,14,29–32]. Our tunneling matrix criteria, Eqs. (1)–(3), are equivalent to the Lagrangian subgroup criteria for Abelian topological orders (a detailed proof is given in the Supplemental Material [17]), but are more general and also apply to non-Abelian topological orders.

One can also use the *anyon condensation* approach [33–39] to determine the gapped boundaries of (non-Abelian) topological orders, by searching for the *Lagrangian condensable anyons* (mathematically, *Lagrangian algebras* [35,36]), whose condensation will break the topological order to vacuum. However, we use only an integer vector \mathcal{W}_{1a} to determine the anyon $\mathbf{q}_{\mathcal{W}}$, while in the anyon condensation approach, besides the multiplicity \mathcal{W}_{1a} , there are many additional data satisfying a series of formulas. These formulas put certain constraints on the condensable anyon, but not in a simple and explicit manner. Our claim that Eqs. (1)–(3) are necessary and sufficient for a gapped domain wall to exist means that, Lagrangian condensable anyons must satisfy Eqs. (1)–(3), and, for the anyon $\mathbf{q}_{\mathcal{W}}$ satisfying Eqs. (1)–(3), there must exist solutions to the additional data in the anyon condensation approach.

We know that the effective $(1+1)$ D edge theory of a $(2+1)$ D topological order has a gravitational anomaly.

The gravitational anomalies are classified by the bulk topological order $(\mathcal{S}, \mathcal{T}, c_-)$ [40,41]. When $c_- \neq 0$, the edge effective theory has a perturbative gravitational anomaly which leads to a topological gapless edge (i.e., the gaplessness of the edge is robust against any change of the edge Hamiltonian). Even in the absence of a perturbative gravitational anomaly, $c_- = 0$, certain global gravitational anomalies [42] [characterized by $(\mathcal{S}, \mathcal{T}, 0)$] can also lead to a topological gapless edge [13,30]. Our work points out that such global gravitational anomalies are described by \mathcal{S}, \mathcal{T} which do not allow any nonzero solution \mathcal{W} of Eqs. (1)–(3). The corresponding 2D topological order $(\mathcal{S}, \mathcal{T}, 0)$ will have a topological gapless edge.

Since a domain wall sits on the border between two topological orders, our study on domain walls can also guide us to better understand the *phase transitions* of topological orders.

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Note added.—We recently became aware that related work, Ref. [39], has independently obtained part of our results using a different approach: anyon condensation. The comparison between our new approach and anyon condensation is explained in the *Conclusion* section.

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