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QUASILINEAR THEORY FOR MOMENTUM AND SPATIAL DIFFUSION DUE TO RADIO FREQUENCY WAVES IN NON-AXISYMMETRIC TOROIDAL PLASMAS

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Quasilinear Theory for Momentum and Spatial Diffusion due to Radio Frequency Waves in Non-Axisymmetric Toroidal Plasmas

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A theoretical description of the interaction of radio frequency (RF) waves with electrons in tokamaks requires an accounting of the toroidal magnetic field geometry. For EC waves, the description has to be relativistic so that the damping of the waves and their interaction with electrons are described correctly. In this paper we derive the quasilinear diffusion operator for the interaction of RF waves with electrons using the Lie transform perturbation technique. We use the magnetic flux coordinates of an axisymmetric toroidal plasma, and the electron motion is expressed in terms of the canonical guiding center variables. The electron motion is perturbed by RF waves and by non-axisymmetric perturbations to the confining magnetic field. The magnetic perturbations could be due to magnetic islands in a plasma.

The quasilinear action diffusion equation describes transport in momentum space and in the radial spatial direction induced by magnetic perturbations and RF waves. The diffusion tensor is non-singular and time-dependent. As a result of applying the perturbation theory to finite time intervals we avoid the presence of the usual Dirac delta functions which results in a non-vanishing diffusion tensor only on a discrete set of action surfaces which satisfy exactly a resonant condition [1]. The appearance of Dirac's delta function is a consequence of two facts. The first is related to the consideration of infinite plane waves. For the case where a finite beam size is taken into account, the dependence on the resonance condition becomes non-singular, resulting to what is known as resonance broadening effect [2]. The second is related to dynamical features of the electron motion. More specifically, singularities appear when the Markovian assumption for decorelation of the particle orbits due to perturbations, is invoked [1]. However, in many cases of interest such statistical assumptions do not hold. The underlying phase space of the system contains not only chaotic areas but also islands of "regular", quasiperiodic motion.

In a general magnetic configuration, consisting of nested toroidal magnetic surfaces, the covariant representation of the magnetic field is [3]

$$\mathbf{B} = g(\psi_p)\nabla\zeta + I(\psi_p)\nabla\theta + \delta(\psi_p,\theta)\nabla\psi_p \qquad (1)$$

where ψ_p, ζ, and θ are, respectively, the poloidal flux, the toroidal angle, and the poloidal angle. The functions g and I are related to the poloidal and toroidal currents, respectively, and δ

is related to the degree of non-orthogonality of the coordinate system. The magnetic field lines are straight lines in the (ζ, θ) plane. The guiding center Hamiltonian [3] is

$$H_{gc} = \left(m^2c^4 + m^2c^2\rho_{\|}^2 B^2 + 2mc^2\mu B\right)^{1/2} + \Phi \tag{2}$$

where $\rho_{\|} = v_{\|}/B$, $v_{\|}$ is the component of $\mathbf{v}$ along $\mathbf{B}$, m is the mass of the electron, μ is the magnetic moment, and Φ is the electrostatic potential. The two sets of canonically conjugate variables [3] are (P_θ, θ) and (P_ζ, ζ) where

$$P_\theta = \psi + \rho_{\|} I, \qquad\qquad P_\zeta = \rho_{\|} g - \psi_p \tag{3}$$

ψ, the toroidal flux, is given by $d\psi/d\psi_p = q(\psi_p)$ with $q(\psi_p)$ being the safety factor. Note that ψ_p and $\rho_{\|}$ are functions of P_θ and P_ζ only. The third set of canonically conjugate variables is (μ, ξ), with ξ being the gyration angle. For an axisymmetric magnetic field the three-degree of freedom system (2) has three independent conserved quantities (μ, P_ζ, W) and the particle motion is completely integrable. The Hamiltonian describes magnetically trapped particles moving in banana orbits, and passing particles circulating in the toroidal direction. An action-angle transformation can be used to eliminate θ from the Hamiltonian. A new action $\hat{P}_\theta$ where $\hat{P}_\theta = \oint P_\theta(\theta; \mu, P_\zeta, W) d\theta$ along with the canonical transformation is obtained from the generating function $S(\xi, \zeta, \theta; \hat{\mu}, \hat{P}_\zeta, \hat{P}_\theta) = \xi\hat{\mu} + \zeta\hat{P}_\zeta + \int_0^\theta P_\theta(\theta'; \hat{\mu}, \hat{P}_\zeta, \hat{P}_\theta) d\theta'$. The hatted variables are the new action-angle variables, and $\hat{\mu} = \mu$ and $\hat{P}_\zeta = P_\zeta$. We will use the new action-angle variables and drop, without leading to any confusion, the hat over this variable set.

The non-axisymmetric magnetic perturbations have the form $\tilde{\mathbf{A}} = a\mathbf{B}$ with $a(\psi_p, \theta, \zeta) = \sum_{m_1,m_2} a_{m_1,m_2}(\psi_p) e^{i(m_1\theta + m_2\zeta)}$. Such perturbations modify the parallel canonical momentum $\rho_c = \rho_{\|} + a$ [3]. The scalar and vector potentials corresponding to RF wave fields are represented in an eikonal form $\Phi_{rf}(\mathbf{x},t) = \tilde{\Phi}_{rf}(\mathbf{x}) e^{i\Psi(\mathbf{x},t)}$, $\mathbf{A}_{rf}(\mathbf{x},t) = \tilde{A}_{rf}(\mathbf{x}) e^{i\Psi(\mathbf{x},t)} \mathbf{P}_{rf}$ where $\tilde{\Phi}$ and $\tilde{A}$ are amplitudes of the scalar and vector potentials, respectively, Ψ is the phase, and $\mathbf{P}_{rf}$ is the wave polarization vector. The local wave vector $\mathbf{k}$ and the angular frequency ω of the wave fields are given by $\mathbf{k}(\mathbf{x},t) = \nabla\Psi(\mathbf{x},t)$, $\omega(\mathbf{x},t) = -\frac{\partial\Psi(\mathbf{x},t)}{\partial t}$.

To second order in the ordering parameters ε (RF wave perturbations) and $\lambda(\sim\varepsilon)$ (non-axisymmetric magnetic perturbations) $H = H_0 + \varepsilon H_1 + \varepsilon^2 H_2$, where $H_0 = mc^2\Gamma_0 + \Phi$ and

$$H_1 = -\frac{1}{\Gamma_0}\left(\rho_c B\hat{b} + (2\mu B)^{1/2}\hat{c}\right)\cdot\mathbf{A}_{rf} + \Phi_{rf} - \frac{\lambda}{\varepsilon}\frac{m}{\Gamma_0}\rho_c B^2 a \tag{4}$$

$$\begin{aligned} H_2 &= \frac{1}{2mc^2\Gamma_0}\left[c^2 A_{rf}^2 - \frac{1}{\Gamma_0^2}\left\{\left(\rho_c B\hat{b} + (2\mu B)^{1/2}\hat{c}\right)\cdot\mathbf{A}_{rf}\right\}^2\right] \\ &\quad + \frac{\lambda^2}{\varepsilon^2}\frac{B^2}{2c^2\Gamma_0^2}\left(1 - \frac{\rho_c^2 B^2}{c^2\Gamma_0^2}\right)a^2 + \frac{\lambda}{\varepsilon}\frac{aB}{\Gamma_0}\hat{b}\cdot\mathbf{A}_{rf} \end{aligned} \tag{5}$$

The unit vector $\hat{b}$ is along the axisymmetric magnetic field, $\hat{a}$ and $\hat{c}$ are perpendicular to $\hat{b}$ ($\hat{a} = \hat{b} \times \hat{c}$) gyrating with the particle and $\Gamma_0 = (1 + \rho_c^2 B^2/c^2 + 2\mu B/mc^2)^{1/2}$.

By applying the Lie transform perturbation theory [4], the first order Lie generator w_1, obtained from the solution of the equation $\partial w_1/\partial t + [w_1, H_0] = K_1 - H_1$, by seting $K_1 = 0$, as $w_1 = -\int_{t_0}^{t} H_1(\mathbf{J}, \theta, s) ds$ where $\mathbf{J} = \left(P_\theta, P_\zeta, \mu\right)$, $\theta = (\theta, \zeta, \xi)$. The integration is along the orbits of the unperturbed, integrable, Hamiltonian H_0, $\mathbf{J}(s) =$ const. and $\theta(s) = \theta(t) + \omega_\theta(s - t)$ with $\omega_\theta = \partial H_0/\partial \mathbf{J}$. The resulting w_1 is given by

$$\begin{aligned} w_1 &= \sum_{n_1,n_2,l} G_{n_1,n_2,l}(\mathbf{J}) e^{i\mathbf{N}_{n_1,n_2,l}\cdot(\theta-\omega_\theta t)} \frac{e^{i\left(\mathbf{N}_{n_1,n_2,l}\cdot\omega_\theta-\omega\right)t} - e^{i\left(\mathbf{N}_{n_1,n_2,l}\cdot\omega_\theta-\omega\right)t_0}}{i(\mathbf{N}_{n_1,n_2,l}\cdot\omega_\theta - \omega)} \\ &+ \sum_{n_1,m_1,m_2} F_{n_1}(\mathbf{J}) a_{m_1,m_2}(\mathbf{J}) e^{i\mathbf{M}_{n_1,m_1,m_2}\cdot(\theta-\omega_\theta t)} \frac{e^{i\mathbf{M}_{n_1,m_1,m_2}\cdot\omega_\theta t} - e^{i\mathbf{M}_{n_1,m_1,m_2}\cdot\omega_\theta t_0}}{i(\mathbf{M}_{n_1,m_1,m_2}\cdot\omega_\theta)} \end{aligned} \tag{6}$$

where $\mathbf{N}_{n_1,n_2,l} = (n_1 + k_\theta, n_2 + k_\zeta, l)$ and $\mathbf{M}_{n_1,m_1,m_2} = (n_1 + m_1, m_2, 0)$. The first sum includes resonance between the RF waves and the particles and depends on the three angles. The second sum includes resonance between the magnetic perturbations and the particles and depends on the two angles θ and ζ. This form is derived by representing the terms of the first order Hamiltonian H_1 (4) in Fourier series as

$$\sum_{n_1,n_2} G_{n_1,n_2}(\mathbf{J}) e^{i(n_1\theta+n_2\zeta)} =$$

$$\left[(1/\Gamma_0)\tilde{A}_{rf}(\mathbf{X})\left(\rho_c B P_{rf\parallel} J_l + (2\mu B)^{1/2}\left(P_{rf}^{+} J_{l-1} + P_{rf}^{-} J_{l+1}\right)\right) - \tilde{\Phi}_{rf}(\mathbf{X})\right] e^{ik_{\psi_p}\psi_p}, \tag{7}$$

$$\sum_{n_1} F_{n_1}(\mathbf{J}) e^{in_1\theta} = \frac{m}{\Gamma_0}\rho_c B^2 \tag{8}$$

where the polarization vector has been analyzed to one parallel ($P_{rf\parallel}$) and two counter-rotating circular polarizations (P_{rf}^{+}, P_{rf}^{-}) and $J_l = J_l(k_\perp \rho)$ is the $l-$th order Bessel function. Both sums in eq. (6) include a functional dependence on the actions of the form

$$\mathscr{R}(\Omega; t, t_0) = \frac{e^{i\Omega t} - e^{i\Omega t_0}}{i\Omega} = \int_{t_0}^{t} e^{i\Omega s} ds \tag{9}$$

This function is smooth and localized around $\Omega = 0$ and indicates a resonance between the particle motion and the perturbations. For long times $\lim_{t\to\infty}, \mathscr{R}(\Omega; t, -t) = 2\pi\delta(\Omega)$, where $\delta(\Omega)$ is the Dirac delta function commonly appearing in quasilinear theory [1].

The evolution of any function $f(\mathbf{z})$ of the phase space variables over an infinitesimal time interval $[t_0, t_0 + \Delta t]$ is $f(\mathbf{z}(t_0 + \Delta t; t_0), t_0 + \Delta t) = T(\mathbf{z}_0, t_0) S_K(t_0 + \Delta t; t_0) T^{-1}(\mathbf{z}_0, t_0) f(\mathbf{z}_0, t_0)$, where $T = e^{-L}$, $Lf = [w, f]$. As a result of applying the canonical perturbation theory in finite time intervals $[t_0, t]$, one can easily show that $w_n(\mathbf{z}_0, t_0) = 0$. Thus, $T(\mathbf{z}_0, t_0) = I$. Furthermore, we have chosen $K_n = 0$ for $n = 1, 2$. Then the time evolution of S_K is given by the H_0, i.e., by integrating along unperturbed orbits $S_K = S_{K_0} = S_{H_0}$. Upon taking the limit $\Delta t \to 0$ we obtain $\partial f(\mathbf{z},t)/\partial t = \partial \left[T^{-1} - I\right](\mathbf{z},t)/\partial t\, f(\mathbf{z},t)$. For the case where $f(\mathbf{z})$ is the distribution function this equation is an approximation, up to the same order as T^{-1}, to the original Vlasov (Liouville) equation. For a function $F(\mathbf{J})$ which is an average of f over the angles, $F(\mathbf{J}) = \langle f(\theta,\mathbf{J})\rangle_\theta$ we have

$$\frac{\partial F(\mathbf{J},t)}{\partial t} = \frac{\partial \left\langle [T^{-1} - I](\mathbf{z},t)\right\rangle_\theta}{\partial t} F(\mathbf{J},t). \tag{10}$$

Up to second order in ε we have $T^{-1} - I = L_1 + (1/2)L_2 + (1/2)L_1^2$ with $L_n F = [w_n, F]$ [4]. Upon integration by parts and using the fact that the dependence on all the angles is periodic, we find that $< L_n F(\mathbf{J}) >_\theta = 0$ for $n = 1,2$ and $< L_1^2 F(\mathbf{J}) >_\theta = \nabla_\mathbf{J} \cdot [< (\nabla_\theta w_1)^2 >_\theta \cdot \nabla_\mathbf{J} F(\mathbf{J})]$. An important point emerges from these equations.The angle-averaged operators that are needed in the evolution equation (10) can be calculated up to second order in the perturbation parameter using results from first order perturbation theory, namely w_1 [5]. The evolution equation (10) then becomes

$$\frac{\partial F(\mathbf{J},t)}{\partial t} = \nabla_\mathbf{J} \cdot [\mathbf{D}(\mathbf{J},t) \cdot \nabla_\mathbf{J} F(\mathbf{J},t)], \qquad \text{where} \qquad \mathbf{D}(\mathbf{J},t) = \frac{1}{2} \frac{\partial \left\langle (\nabla_\theta w_1)^2 \right\rangle_\theta}{\partial t} \tag{11}$$

is the generalized quasilinear tensor. It can be shown that the first order momentum variation can be written as $< (\Delta \mathbf{J})^2 >_\theta = < (\nabla_\theta w_1)^2 >_\theta$, from which we can see that $D(\mathbf{J},t) = \lim_{\Delta t \to 0} < (\Delta \mathbf{J})^2 >_\theta / 2\Delta t$ corresponding to the common definition of the quasilinear diffusion tensor. The evolution equation (11), can be transformed to the physical variables $\mathbf{P} = (\psi_p, v_\parallel, v_\perp)$ describing particle transport, heating, and current drive through the variation of the distribution function with respect to (ψ_p, θ, ζ), $v_\perp$, and $v_\parallel$, respectively.

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