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Finite-Larmor-Radius Kinetic Theory of a Magnetized Plasma in the Macroscopic Flow Reference Frame

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Abstract

A straightforward operator method is used to derive a form of the drift-kinetic equation for a collisionless plasma species in the moving reference frame of its macroscopic flow. This equation is valid for sonic time scales and flow velocities, with first-order finite-Larmor-radius (FLR) effects included. It applies rigorously to far-from-Maxwellian distribution functions and to general space and time variations of the magnetic field. Its velocity moments are shown to reproduce exactly the corresponding fluid equations obtained from moments of the full Vlasov equation.

I. Introduction.

The drift-kinetic equation[1−6], i.e. the dimensionally reduced kinetic equation for the gyrophase average of the distribution function of a magnetized plasma species whose Larmor gyroradius is much smaller than any other characteristic length, is a valuable and widely used research tool in theoretical plasma physics. In particular, it provides the means of closing the set of fluid equations in collisionless or low-collisionality regimes, thus allowing a consistent fluid-kinetic plasma description when the conventional high-collisionality fluid closure[7,8] does not apply. Closure of the fluid moment equations for a collisionless or low-collisionality magnetized plasma species requires the kinetic evaluation of some components of the gyrotropic or Chew-Goldberger-Low (CGL) pressure and/or heat flux tensors[9]:

$$\mathsf{P}^{CGL}_{jk} \;=\; p_\perp \delta_{jk} + (p_\| - p_\perp) b_j b_k \;=\; p\ \delta_{jk} + (p_\| - p_\perp)(b_j b_k - \delta_{jk}/3), \tag{1}$$

and

$$\mathsf{Q}^{CGL}_{jkl} \;=\; q_{T\|}(\delta_{jk} b_l + \delta_{kl} b_j + \delta_{lj} b_k) \;+\; (2q_{B\|} - 3q_{T\|}) b_j b_k b_l \ , \tag{2}$$

where b_j are the Cartesian components of the magnetic unit vector, $p_\perp$ and $p_\|$ are respectively the perpendicular and parallel pressures with the mean scalar pressure $p = (2p_\perp + p_\|)/3$; $q_{T\|}$ is the parallel flux of perpendicular heat and $q_{B\|}$ is the parallel flux of parallel heat, with the total parallel heat flux $q_\| = q_{B\|} + q_{T\|}$. In a finite-but-small-gyroradius collisionless analysis, knowledge of the gyrotropic variables is sufficient to close the fluid system because the remaining, non-gyrotropic or "perpendicular" parts of the stress and heat flux tensors can then be deduced from fluid theory alone[10,11]. On the other hand, the fluid evolution equation for any component of the CGL tensors always involves some yet unknown higher-rank gyrotropic moments. Since these are moments of the gyrophase-averaged part of the distribution function, the kinetic information needed for the fluid closure can be obtained from the drift-kinetic equation. However, for a consistent closure scheme, it is important that such drift-kinetic equation be compatible with the companion set of fluid equations, in the way that both fulfill the same validity conditions and retain the same level of accuracy in the high gyrofrequency and small gyroradius asymptotic expansion.

Recent advances in the fluid description of collisionless and low-collisionality magnetized plasmas, based on the moments of the full Vlasov-Boltzmann kinetic equation[11−13], provide improved sets of

FLR fluid equations, especially with regard their applicability to general magnetic geometries, fully electromagnetic nonlinear dynamics with large fluctuation amplitudes, arbitrary density and temperature gradients and far-from-Maxwellian distribution functions with large pressure anisotropies and parallel heat fluxes. The purpose of this work is to derive a drift-kinetic equation, compatible with such general fluid systems, that can serve as the basis for their consistent closure. The proposed approach is conceptually straightforward and affords an alternative to other recently derived FLR drift-kinetic equations[14,15]. Like in these two related works, the collisionless case will be analyzed here, leaving the collisional effects for future consideration.

The crucial observation is that the gyrotropic pressure and heat flux fluid variables are normally defined relative to the macroscopic flow velocity of the species under consideration:

$$p \;=\; (m/3)\int d^3\mathbf{v}\ |\mathbf{v}-\mathbf{u}|^2\ \bar{f}\ , \tag{3}$$

$$p_{\|} \;=\; m\int d^3\mathbf{v}\ [(\mathbf{v}-\mathbf{u})\cdot\mathbf{b}]^2\ \bar{f}\ , \tag{4}$$

$$q_{\|} \;=\; (m/2)\int d^3\mathbf{v}\ [(\mathbf{v}-\mathbf{u})\cdot\mathbf{b}]\ |\mathbf{v}-\mathbf{u}|^2\ \bar{f}\ , \tag{5}$$

$$q_{B\|} \;=\; (m/2)\int d^3\mathbf{v}\ [(\mathbf{v}-\mathbf{u})\cdot\mathbf{b}]^3\ \bar{f}\ , \tag{6}$$

where

$$\mathbf{u}(\mathbf{x},t) \;=\; n^{-1}\int d^3\mathbf{v}\ \mathbf{v}\ f(\mathbf{v},\mathbf{x},t)\ , \tag{7}$$

$$n(\mathbf{x},t) \;=\; \int d^3\mathbf{v}\ f(\mathbf{v},\mathbf{x},t)\ , \tag{8}$$

and the barred distribution function $\bar{f}$ represents its gyrophase average in the moving frame of the species macroscopic flow, which is the only part that contributes to the moments defined in Eqs.(3-6). Accordingly, it is deemed advantageous to express the drift-kinetic equation in the local reference frame of the complete macroscopic flow velocity. In fact, the more traditional approach of deriving the drift-kinetic equation either in the laboratory frame or in the frame of the electric drift velocity, makes the task of taking the moments that yield the pressure and heat flux tensors cumbersome. Another advantage of using the macroscopic flow velocity as reference is that, by guaranteeing a small electric field in the working frame, it automatically makes possible to allow for the fast flows that are becoming increasingly important in plasma research. Our use of the full macroscopic velocity to define the moving frame differs from the analyses of Refs.2,14-16 that use the frame of the electric drift velocity in order to allow for the fast flows, leading to more awkward calculations of the pressures and heat fluxes and to difficulty in reproducing the conventional fluid equations. The adoption of a reference frame tied to the magnetic field lines in Ref.3 and the method followed in Ref.17 of adding and subtracting a term in Vlasov's equation instead of performing a change of reference frames, suffer from those same drawbacks. Macroscopic flow reference frames were used in Refs.18,19 (a center of mass frame in Ref.18), but these works assume weak temporal variations of the magnetic field and linearized distribution functions near a Maxwellian, the transport analysis of Ref.19 being further limited to axisymmetric configurations. In the present work, no simplifications will be made with regard the temporal or spatial variation of the magnetic field, and the electric field will be eliminated algebraically by means of the exact momentum conservation equation. The results to be shown will apply to general nonlinear dynamical systems under collisionless conditions and, accordingly, far-from-Maxwellian distribution functions with large pressure anisotropies and parallel heat fluxes will be allowed.

II. Derivation of the FLR drift-kinetic equation.

The starting point is the Vlasov kinetic equation for the distribution function $f(\mathbf{v}, \mathbf{x}, t)$ of a collisionless plasma species,

$$\frac{\partial f}{\partial t} \;+\; \mathbf{v}\cdot\frac{\partial f}{\partial \mathbf{x}} \;+\; \frac{e}{m}\Big(\mathbf{E} + \mathbf{v}\times\mathbf{B}\Big)\cdot\frac{\partial f}{\partial \mathbf{v}} \;=\; 0 \;, \tag{9}$$

where $\mathbf{E}(\mathbf{x}, t)$ and $\mathbf{B}(\mathbf{x}, t)$ are the electric and magnetic fields, and m and e are the species mass and electric charge respectively. The present collisionless analysis applies to each species independently (without consideration of possibly small mass or charge ratios between species that might result in additional simplifications) and the species index is omitted throughout.

Carrying out the space-time dependent Galilean transformation to moving reference frames with local velocities $\mathbf{u}(\mathbf{x}, t)$ relative to the laboratory[18,19],

$$t = t \;, \qquad \mathbf{x} = \mathbf{x} \;, \qquad \mathbf{v} = \mathbf{v}' + \mathbf{u}(\mathbf{x}, t) \;, \tag{10}$$

the kinetic equation (9) becomes

$$\frac{\partial f(\mathbf{v}',\mathbf{x},t)}{\partial t} \;+\; (\mathbf{v}'+\mathbf{u})\cdot\frac{\partial f(\mathbf{v}',\mathbf{x},t)}{\partial \mathbf{x}} \;+\; \Big[\Omega_c\mathbf{v}'\times\mathbf{b} \;+\frac{\mathbf{F}}{mn} - (\mathbf{v}'\cdot\nabla)\mathbf{u}\Big]\cdot\frac{\partial f(\mathbf{v}',\mathbf{x},t)}{\partial \mathbf{v}'} \;=\; 0\;, \tag{11}$$

where $\Omega_c = eB/m$ is the cyclotron frequency, $\mathbf{b} = \mathbf{B}/B$ is the magnetic unit vector and

$$\mathbf{F}(\mathbf{x},t) \;=\; en(\mathbf{E}+\mathbf{u}\times\mathbf{B}) \;-\; mn\Big[\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u}\Big]\;. \tag{12}$$

Associated with this force density $\mathbf{F}$, is the velocity

$$\mathbf{u}_F(\mathbf{x},t) \;=\; \frac{\mathbf{F}\times\mathbf{b}}{mn\Omega_c}\;, \tag{13}$$

that will appear in the final form of our drift-kinetic equation.

As discusssed in the Introduction, our main interest is to take $\mathbf{u}(\mathbf{x},t)$ as the macroscopic flow velocity of the plasma species under consideration. Throughout this section however, $\mathbf{u}(\mathbf{x},t)$ can be taken as any velocity field provided only $F/(mn) \ll \Omega_c v_{th}$, with v_{th} the characteristic thermal speed, so that, under strong magnetization conditions, $(\Omega_c\mathbf{v}'\times\mathbf{b})\cdot\partial f/\partial\mathbf{v}'$ is the dominant term in (11) . Thus, three possible choices for $\mathbf{u}(\mathbf{x},t)$ and its associated $\mathbf{F}(\mathbf{x},t)$ and $\mathbf{u}_F(\mathbf{x},t)$ are as follows:

1.) If the flow is subsonic, hence the electric field is $E \ll v_{th}B$, one can carry out the analysis in the laboratory frame as is most traditionally done[1,4–6]. In this case one just sets $\mathbf{u}=0$, $\mathbf{F}=en\mathbf{E}$ and $\mathbf{u}_F = \mathbf{u}_E$, the electric drift velocity.

2.) One can take $\mathbf{u} = \mathbf{u}_E = \mathbf{E}\times\mathbf{b}/B$ as chosen in Refs.2,15,16 and (aside from an additional parallel component) in Ref.14. In this case $\mathbf{F} = enE_\parallel\mathbf{b} - mn[\partial\mathbf{u}_E/\partial t + (\mathbf{u}_E\cdot\nabla)\mathbf{u}_E]$ and $\mathbf{u}_F = \mathbf{u}_{polE}$, the polarization drift velocity calculated with $\mathbf{u}_E$.

3.) Our preferred choice is to take $\mathbf{u}(\mathbf{x},t)$ as the complete macroscopic flow velocity (7). In this case, by virtue of the momentum conservation equation, $\mathbf{F} = \nabla\cdot\mathsf{P}$ where $\mathsf{P} = \mathsf{P}^{CGL} + \mathsf{P}^{GV}$ is the full stress tensor made of its gyrotropic (CGL) and non-gyrotropic (gyroviscous) parts and $\mathbf{u}_F = -\mathbf{u}_{dia}$, the negative of the diamagnetic drift velocity calculated with the full $\nabla\cdot\mathsf{P}$.

The next step is to perform the change of variables to cylindrical coordinate systems in velocity space locally aligned with the magnetic field,

$$t=t\;,\qquad \mathbf{x}=\mathbf{x}\;,\qquad \mathbf{v}' = v'_\parallel\;\mathbf{b}(\mathbf{x},t) \;+\; v'_\perp\;[\cos\alpha\;\mathbf{e}_1(\mathbf{x},t) + \sin\alpha\;\mathbf{e}_2(\mathbf{x},t)]\;, \tag{14}$$

where $\mathbf{b}(\mathbf{x},t)$, $\mathbf{e}_1(\mathbf{x},t)$ and $\mathbf{e}_2(\mathbf{x},t)$ form right-handed sets of mutually orthogonal unit vectors. A possible intrinsic choice would be to take $\mathbf{e}_1(\mathbf{x},t)$ and $\mathbf{e}_2(\mathbf{x},t)$ in the directions of the principal normal and the binormal of the magnetic field. As it turns out however, the results do not depend on the choice of $\mathbf{e}_1(\mathbf{x},t)$ and $\mathbf{e}_2(\mathbf{x},t)$, the only requirement being that they be well defined and differentiable, something which is always possible if the magnetic field is sufficiently smooth locally. After carrying out this change of variables, the kinetic equation becomes of the form,

$$\Omega_c \, \frac{\partial f(v'_\parallel, v'_\perp, \alpha, \mathbf{x}, t)}{\partial \alpha} \;=\; \sum_{l=-2}^{2} e^{il\alpha} \left[\Lambda_l \; f \;+\; \lambda_l \, \frac{\partial f}{\partial \alpha} \right] , \tag{15}$$

where $\Lambda_l(\partial/\partial v'_\parallel, \partial/\partial v'_\perp, \partial/\partial \mathbf{x}, \partial/\partial t, v'_\parallel, v'_\perp, \mathbf{x}, t) = \Lambda^*_{-l}$ are gyrophase-independent operators and $\lambda_l(v'_\parallel, v'_\perp, \mathbf{x}, t) = \lambda^*_{-l}$ are gyrophase-independent functions. Specifically,

$$\begin{aligned} \Lambda_0 \;=\; & \frac{\partial}{\partial t} \;+\; (\mathbf{u} + v'_\parallel \mathbf{b}) \cdot \frac{\partial}{\partial \mathbf{x}} \;+\; \left\{ \frac{\mathbf{b}\cdot\mathbf{F}}{mn} - v'_\parallel \mathbf{b}\cdot[(\mathbf{b}\cdot\nabla)\mathbf{u}] + \frac{v'^2_\perp}{2} \nabla\cdot\mathbf{b} \right\} \frac{\partial}{\partial v'_\parallel} \;+ \\ & +\; \frac{v'_\perp}{2} \Big\{ \mathbf{b}\cdot[(\mathbf{b}\cdot\nabla)\mathbf{u}] - \nabla\cdot\mathbf{u} - v'_\parallel \nabla\cdot\mathbf{b} \Big\} \frac{\partial}{\partial v'_\perp} \;, \end{aligned} \tag{16}$$

$$\begin{aligned} \Lambda_1 \;=\; & \frac{v'_\perp}{2} (\mathbf{e}_1 - i\mathbf{e}_2) \cdot \frac{\partial}{\partial \mathbf{x}} \;+\; \frac{v'_\perp}{2} (\mathbf{e}_1 - i\mathbf{e}_2) \cdot \left[\frac{\partial \mathbf{b}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{b} - (\mathbf{b}\cdot\nabla)\mathbf{u} - \mathbf{b}\times\boldsymbol{\omega} + v'_\parallel \boldsymbol{\kappa} \right] \frac{\partial}{\partial v'_\parallel} \;+ \\ & +\; \frac{1}{2} (\mathbf{e}_1 - i\mathbf{e}_2) \cdot \left\{ \frac{\mathbf{F}}{mn} - v'_\parallel \left[\frac{\partial \mathbf{b}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{b} + (\mathbf{b}\cdot\nabla)\mathbf{u} + v'_\parallel \boldsymbol{\kappa} \right] \right\} \frac{\partial}{\partial v'_\perp} \;, \end{aligned} \tag{17}$$

$$\begin{aligned} \Lambda_2 \;=\; & \frac{i v'^2_\perp}{4} (\mathbf{e}_1 - i\mathbf{e}_2) \cdot \Big[\nabla \times (\mathbf{e}_1 - i\mathbf{e}_2) \Big] \frac{\partial}{\partial v'_\parallel} \;- \\ & -\; \frac{v'_\perp}{4} (\mathbf{e}_1 - i\mathbf{e}_2) \cdot \Big\{ [(\mathbf{e}_1 - i\mathbf{e}_2)\cdot\nabla]\mathbf{u} + i v'_\parallel \; \nabla \times (\mathbf{e}_1 - i\mathbf{e}_2) \Big\} \frac{\partial}{\partial v'_\perp} \;, \end{aligned} \tag{18}$$

$$\begin{aligned} \lambda_0 \;=\; & \frac{1}{2} \Big\{ \mathbf{e}_1 \cdot \left[\frac{\partial \mathbf{e}_2}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{e}_2 + (\mathbf{e}_2\cdot\nabla)\mathbf{u} + v'_\parallel (\mathbf{b}\cdot\nabla)\mathbf{e}_2 \right] - \\ & -\; \mathbf{e}_2 \cdot \left[\frac{\partial \mathbf{e}_1}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{e}_1 + (\mathbf{e}_1\cdot\nabla)\mathbf{u} + v'_\parallel (\mathbf{b}\cdot\nabla)\mathbf{e}_1 \right] \;-\; v'_\parallel \mathbf{b}\cdot(\nabla\times\mathbf{b}) \Big\} \;, \end{aligned} \tag{19}$$

$$\lambda_1 \;=\; \frac{i}{2 v'_\perp} (\mathbf{e}_1 - i\mathbf{e}_2) \cdot \left\{ \frac{\mathbf{F}}{mn} - v'_\parallel \left[\frac{\partial \mathbf{b}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{b} + (\mathbf{b}\cdot\nabla)\mathbf{u} + v'_\parallel \boldsymbol{\kappa} \right] \right\} \;-\; \frac{v'_\perp}{2} \mathbf{b} \cdot \Big[\nabla \times (\mathbf{e}_1 - i\mathbf{e}_2) \Big] \tag{20}$$

and

$$\lambda_2 \ = \ -\frac{i}{4}(\mathbf{e}_1 - i\mathbf{e}_2)\cdot\Big\{[(\mathbf{e}_1 - i\mathbf{e}_2)\cdot\nabla]\mathbf{u} + iv'_\parallel \ \nabla\times(\mathbf{e}_1 - i\mathbf{e}_2)\Big\} \ , \tag{21}$$

with $\boldsymbol{\omega} = \nabla\times\mathbf{u}$ and $\boldsymbol{\kappa} = (\mathbf{b}\cdot\nabla)\mathbf{b}$. In (17) and (20), the time derivative of the magnetic unit vector $\partial\mathbf{b}/\partial t$ is evaluated using Faraday's law, with the electric field eliminated algebraically in favor of $\mathbf{F}$ after (12). The resulting expressions for Λ_1 and λ_1 are given in Appendix A.

A formal solution of Eq.(15) can be written as a Fourier series in harmonics of the gyrophase,

$$f(v'_\parallel, v'_\perp, \alpha, \mathbf{x}, t) \ = \ \sum_{l=-\infty}^{\infty} e^{il\alpha} \ f_l(v'_\parallel, v'_\perp, \mathbf{x}, t) \ , \tag{22}$$

where the Fourier coefficients are determined by the following coupled system:

$$f_l \ = \ \frac{1}{il\Omega_c}\sum_{l'=-2}^{2}\Big[\Lambda_{l'} f_{l-l'} \ + \ i(l-l')\lambda_{l'} f_{l-l'}\Big] \qquad\quad for \qquad l\neq 0 \tag{23}$$

and

$$\sum_{l=-2}^{2}\Big(\Lambda_l f_{-l} \ - \ il\lambda_l f_{-l}\Big) \ = \ 0 \ . \tag{24}$$

The $l = 0$ Fourier coefficient, i.e. the gyrophase average of the distribution function, will also be denoted as $f_0 = \bar{f}$. All the expressions given until now are exact, no approximations having been made yet.

At this point we introduce the drift-kinetic asymptotic expansion for strong magnetization, assuming small ratios between the Larmor gyration period and any other characteristic time T, and between the Larmor gyration radius ρ and any other characteristic length L. We will consider here dynamical evolution on the sonic time scale, so that those two ratios can be taken as comparable and define one basic expansion parameter:

$$\delta \sim \frac{1}{\Omega_c T} \sim \frac{\rho}{L} = \frac{v_{th}}{\Omega_c L} \ll 1 \ . \tag{25}$$

Macroscopic flows as fast as sonic will be allowed and the analysis will be carried to the first FLR order in δ, so that the dynamical effects on the diamagnetic scale are also included: $\partial/\partial t = O(\delta\Omega_c) + O(\delta^2\Omega_c)$

and $u = O(v_{th}) + O(\delta v_{th})$. Under these orderings, a recursive asymptotic solution of (23,24) can be constructed as

$$f_0 = f_0^{(0)} + f_0^{(1)} + ... \;, \qquad f_{\pm 1} = f_{\pm 1}^{(1)} + ... \;, \qquad f_{\pm 2} = f_{\pm 2}^{(1)} + ... \;, \qquad ... \;, \tag{26}$$

with $f_l^{(n)} = O(\delta^n f_0^{(0)})$. In its lowest order, Eq.(24) yields

$$\Lambda_0 f_0^{(0)} = 0 \;, \tag{27}$$

which is the zero-Larmor-radius drift-kinetic equation. Then, Eq.(23) yields the first-order solutions

$$f_1^{(1)} = f_{-1}^{(1)*} = \frac{1}{i\Omega_c} \Lambda_1 f_0^{(0)} \tag{28}$$

and

$$f_2^{(1)} = f_{-2}^{(1)*} = \frac{1}{2i\Omega_c} \Lambda_2 f_0^{(0)} \;. \tag{29}$$

The first-order correction to the gyrophase-independent Fourier component, $f_0^{(1)}$, is determined by Eq.(24) in its first order:

$$(\Lambda_{-2} + 2i\lambda_{-2}) f_2^{(1)} + (\Lambda_{-1} + i\lambda_{-1}) f_1^{(1)} + \Lambda_0 f_0^{(1)} + (\Lambda_1 - i\lambda_1) f_{-1}^{(1)} + (\Lambda_2 - 2i\lambda_2) f_{-2}^{(1)} = 0 \tag{30}$$

which, after substituting the solutions (28,29) for $f_{\pm 1}^{(1)}$ and $f_{\pm 2}^{(1)}$, becomes:

$$\Lambda_0 f_0^{(1)} \;-\; \frac{1}{i\Omega_c} \Big\{ \Big[(\Lambda_1 - i\lambda_1)\Lambda_{-1} - (\Lambda_{-1} + i\lambda_{-1})\Lambda_1 \Big] \;+\; \Big[(\frac{1}{2}\Lambda_2 - i\lambda_2)\Lambda_{-2} - (\frac{1}{2}\Lambda_{-2} + i\lambda_{-2})\Lambda_2 \Big] \;-$$
$$- \;\frac{v'_\perp}{2} \Big[(\mathbf{e}_1 - i\mathbf{e}_2) \cdot \nabla \ln B \;\Lambda_{-1} - (\mathbf{e}_1 + i\mathbf{e}_2) \cdot \nabla \ln B \;\Lambda_1 \Big] \Big\} f_0^{(0)} \;=\; 0 \;. \tag{31}$$

This is the first-order contribution to our sought after FLR drift-kinetic equation. Here, only the operator commutators defined by the three terms inside square brackets need to be evaluated. The results are given in Appendix B and do not depend on $(\mathbf{e}_1, \mathbf{e}_2)$, involving only the intrinsic geometry of the magnetic field along with the velocity field $\mathbf{u}$ and the force density $\mathbf{F}$. Adding (27) and (31) and calling $\bar{f} = f_0^{(0)} + f_0^{(1)}$, collecting terms and using some standard vector identities, we obtain our final drift-kinetic equation, accurate to the first order in the FLR asymptotic expansion:

$$\frac{\partial \bar{f}(v'_\parallel, v'_\perp, \mathbf{x}, t)}{\partial t} \;+\; \dot{\mathbf{x}} \cdot \frac{\partial \bar{f}}{\partial \mathbf{x}} \;+\; \dot{v}'_\parallel \, \frac{\partial \bar{f}}{\partial v'_\parallel} \;+\; \dot{v}'_\perp \, \frac{\partial \bar{f}}{\partial v'_\perp} \;=\; 0 \;, \tag{32}$$

with the coefficient functions:

$$\dot{\mathbf{x}} \;=\; \mathbf{u} + \mathbf{u}_F \;+\; v'_\parallel \mathbf{b} \;+\; \frac{v'^2_\perp}{2} \nabla \times \Big(\frac{\mathbf{b}}{\Omega_c} \Big) \;+\; \frac{\mathbf{b}}{\Omega_c} \times \Big[2v'_\parallel (\mathbf{b} \cdot \nabla) \mathbf{u} + \Big(v'^2_\parallel - \frac{v'^2_\perp}{2} \Big) \boldsymbol{\kappa} \Big] \;, \tag{33}$$

$$\dot{v}'_\parallel \;=\; \frac{\mathbf{b}\cdot\mathbf{F}}{mn} \;-\; v'_\parallel \mathbf{b}\cdot\Big[(\mathbf{b}\cdot\nabla)(\mathbf{u}+\mathbf{u}_F)\Big] \;-\; \frac{v'^2_\perp}{2}\mathbf{b}\cdot\nabla\ln B \;+\; \frac{v'^2_\perp}{2}\nabla\cdot\Big[\frac{\mathbf{b}}{\Omega_c}\times(\boldsymbol{\omega}\times\mathbf{b}+v'_\parallel\boldsymbol{\kappa})\Big] \;+$$

$$+\;\Big[\frac{\mathbf{b}}{\Omega_c}\times(\boldsymbol{\omega}\times\mathbf{b})\Big]\cdot\Big[\frac{\mathbf{F}}{mn}-2v'_\parallel(\mathbf{b}\cdot\nabla)\mathbf{u}-v'^2_\parallel\boldsymbol{\kappa}\Big] \;-\; 2v'^2_\parallel\Big(\frac{\mathbf{b}}{\Omega_c}\times\boldsymbol{\kappa}\Big)\cdot[(\mathbf{b}\cdot\nabla)\mathbf{u}] \;-\; \frac{v'^2_\perp}{2}\sigma\;, \qquad (34)$$

where $\sigma(\mathbf{x},t)$ is the scalar defined in (B.4), and

$$\dot{v}'_\perp = \frac{v'_\perp}{2}\;\Big\{\;\mathbf{b}\cdot\Big[(\mathbf{b}\cdot\nabla)(\mathbf{u}+\mathbf{u}_F)\Big]-\nabla\cdot(\mathbf{u}+\mathbf{u}_F)+v'_\parallel\mathbf{b}\cdot\nabla\ln B \;-\; v'_\parallel\nabla\cdot\Big[\frac{\mathbf{b}}{\Omega_c}\times\Big(2(\mathbf{b}\cdot\nabla)\mathbf{u}+v'_\parallel\boldsymbol{\kappa}\Big)\Big]+$$

$$+2\Big[\frac{\mathbf{b}}{\Omega_c}\times(\boldsymbol{\omega}\times\mathbf{b})\Big]\cdot\Big[(\mathbf{b}\cdot\nabla)\mathbf{u}+v'_\parallel\boldsymbol{\kappa}\Big]+4v'_\parallel\Big(\frac{\mathbf{b}}{\Omega_c}\times\boldsymbol{\kappa}\Big)\cdot[(\mathbf{b}\cdot\nabla)\mathbf{u}]\;\Big\}\;. \qquad (35)$$

This compact form of the FLR drift-kinetic coefficient functions exhibits clearly the phase-space conservation property:

$$\nabla\cdot\dot{\mathbf{x}} \;+\; \frac{\partial\dot{v}'_\parallel}{\partial v'_\parallel} \;+\; \frac{1}{v'_\perp}\frac{\partial(v'_\perp\dot{v}'_\perp)}{\partial v'_\perp} \;=0\;, \qquad (36)$$

where $\nabla\cdot$ represents the divergence operator in 3-dimensional $\mathbf{x}$-space. According to our derivation, Eqs.(32-35) are valid for any velocity field $\mathbf{u}(\mathbf{x},t)$, upon substitution of the corresponding $\mathbf{F}(\mathbf{x},t)$ and $\mathbf{u}_F(\mathbf{x},t)$ defined in (12,13), provided $F/(mn) \ll \Omega_c v_{th}$. In particular, the well known result[6] that applies to the case of subsonic flows with $E \ll v_{th}B$ can be recovered by setting $\mathbf{u}=0$, $\mathbf{F}=en\mathbf{E}$ and $\mathbf{u}_F=\mathbf{u}_E$ as shown in Appendix C. For our present goal of calculating fluid moments including the possibility of sonic macroscopic flows, we will choose to set $\mathbf{u}$ equal to the complete macroscopic flow velocity.

III. Fluid moments of the FLR drift-kinetic equation.

In order to derive the fluid equations for the gyrotropic variables from the velocity moments of the drift-kinetic equation, we find it most advantageous to work in the reference frame of the macroscopic flow. So, in this section we will use $\mathbf{u}=n^{-1}\int d^3\mathbf{v}\,\mathbf{v}\,f$, $\mathbf{F}=\nabla\cdot\mathbf{P}$ and $\mathbf{u}_F=(\nabla\cdot\mathbf{P})\times\mathbf{b}/(mn\Omega_c)=-\mathbf{u}_{dia}$. The perpendicular components of $\mathbf{F}$ appear only in the first-order terms of the coefficient functions (33-35), always divided by Ω_c. Therefore, only the zero-Larmor-radius CGL part needs to be retained in the perpendicular components of $\nabla \cdot \mathsf{P}$:

$$\frac{\mathbf{b} \times \mathbf{F}}{\Omega_c} \;=\; \frac{\mathbf{b} \times (\nabla \cdot \mathsf{P}^{CGL})}{\Omega_c} \;=\; \frac{1}{\Omega_c} \left[\mathbf{b} \times \nabla p_\perp + (p_\parallel - p_\perp)(\mathbf{b} \times \boldsymbol{\kappa}) \right] . \tag{37}$$

On the other hand, the term involving the parallel component of $\mathbf{F}$ in (34) does not have an inverse gyrofrequency factor, therefore the first-order gyroviscous part of $\nabla \cdot \mathsf{P}$ must be included there. This parallel component of the gyroviscous force can be obtained from "perpendicular" (non-gyrotropic) fluid theory alone, so we can use the result[11,12]:

$$\mathbf{b} \cdot \mathbf{F} \;=\; \mathbf{b} \cdot [\nabla \cdot (\mathsf{P}^{CGL} + \mathsf{P}^{GV})] \;=\; \mathbf{b} \cdot \nabla p_\parallel - (p_\parallel - p_\perp)\mathbf{b} \cdot \nabla \ln B \; +$$

$$+ \; \nabla \cdot \left\{ \frac{\mathbf{b}}{\Omega_c} \times \left[2p_\parallel (\mathbf{b} \cdot \nabla)\mathbf{u} + p_\perp \mathbf{b} \times \boldsymbol{\omega} + \nabla q_{T\parallel} + 2(q_{B\parallel} - q_{T\parallel})\boldsymbol{\kappa} \right] \right\} \; +$$

$$+ \; \frac{\mathbf{b} \times \boldsymbol{\kappa}}{\Omega_c} \cdot \left[2p_\parallel (\mathbf{b} \cdot \nabla)\mathbf{u} + p_\perp \mathbf{b} \times \boldsymbol{\omega} + \nabla q_{T\parallel} \right] \; + \; p_\perp \sigma \; . \tag{38}$$

Let $M^{\alpha\beta}$ (with α a non-negative integer and β a non-negative even integer) denote a generic gyrotropic moment of the distribution function,

$$M^{\alpha\beta}(\mathbf{x}, t) \;=\; 2\pi \int dv'_\parallel \; dv'_\perp \; v'_\perp \; {v'_\parallel}^{\alpha} \; {v'_\perp}^{\beta} \; \bar{f}(v'_\parallel, v'_\perp, \mathbf{x}, t) \; , \tag{39}$$

so that $M^{00} = n$, $M^{10} = 0$, $p_\parallel = mM^{20}$, $p_\perp = mM^{02}/2$, $q_{B\parallel} = mM^{30}/2$ and $q_{T\parallel} = mM^{12}/2$. We also define the following higher-rank moments:$r_{B\parallel} = m^2 M^{40}/4$, $r_{B\perp} = m^2 M^{22}/4$, $r_{T\perp} = m^2 M^{04}/4$, $s_{B\parallel} = m^2 M^{50}/4$, $s_{B\perp} = m^2 M^{32}/4$ and $s_{T\perp} = m^2 M^{14}/4$. Taking the ${v'_\parallel}^{\alpha} {v'_\perp}^{\beta}$ moment of the drift-kinetic equation (32), we obtain after integration by parts:

$$\frac{\partial M^{\alpha\beta}}{\partial t} \; + \; \nabla \cdot \left(2\pi \int dv'_\parallel \; dv'_\perp \; v'_\perp \; {v'_\parallel}^{\alpha} \; {v'_\perp}^{\beta} \; \dot{\mathbf{x}} \; \bar{f} \right) \; -$$

$$- \; 2\pi \int dv'_\parallel \; dv'_\perp \; v'_\perp \; {v'_\parallel}^{\alpha} \; {v'_\perp}^{\beta} \left[\nabla \cdot \dot{\mathbf{x}} + \frac{\partial \dot{v}'_\parallel}{\partial v'_\parallel} + \frac{\partial \dot{v}'_\perp}{\partial v'_\perp} + \alpha \frac{\dot{v}'_\parallel}{v'_\parallel} + (\beta + 1) \frac{\dot{v}'_\perp}{v'_\perp} \right] \; \bar{f} \; = 0 \tag{40}$$

which, using the phase-space conservation property (36), reduces to:

$$\frac{\partial M^{\alpha\beta}}{\partial t} \; + \nabla \cdot \left(2\pi \int dv'_\parallel \; dv'_\perp \; v'_\perp \; {v'_\parallel}^{\alpha} \; {v'_\perp}^{\beta} \; \dot{\mathbf{x}} \; \bar{f} \right) \; - \; 2\pi \int dv'_\parallel \; dv'_\perp \; v'_\perp \; {v'_\parallel}^{\alpha} \; {v'_\perp}^{\beta} \left(\alpha \frac{\dot{v}'_\parallel}{v'_\parallel} + \beta \frac{\dot{v}'_\perp}{v'_\perp} \right) \; \bar{f} \; = 0 \; . \tag{41}$$

This is the generic evolution equation for the gyrotropic moments. After substituting the coefficient functions $\dot{\mathbf{x}}$, $\dot{v}'_\parallel$ and $\dot{v}'_\perp$ (33-35), assigning specific values to the exponents α and β, and recalling the definitions given after (39), it reproduces the corresponding macroscopic equations in terms of conventional fluid variables, as will be shown next for the first six moments.

i) Density moment and continuity equation.

Setting $\alpha = \beta = 0$, Eq.(41) becomes

$$\frac{\partial n}{\partial t} \;+\; \nabla\cdot\Big(2\pi\int dv'_\parallel\; dv'_\perp\; v'_\perp\; \dot{\mathbf{x}}\;\bar{f}\Big) \;=0 \tag{42}$$

where, bringing in our expression (33) for $\dot{\mathbf{x}}$, the velocity-space integral can be written in terms of the gyrotropic moments as

$$2\pi\int dv'_\parallel\; dv'_\perp\; v'_\perp\; \dot{\mathbf{x}}\;\bar{f} \;=\; n(\mathbf{u}+\mathbf{u}_F) \;+\; \frac{p_\perp}{m}\nabla\times\Big(\frac{\mathbf{b}}{\Omega_c}\Big) \;+\; \frac{(p_\parallel - p_\perp)(\mathbf{b}\times\boldsymbol{\kappa})}{m\Omega_c}\;. \tag{43}$$

From (13) and (37) we get

$$\mathbf{u}_F \;=\; -\,\frac{1}{mn\Omega_c}\,\Big[\mathbf{b}\times\nabla p_\perp + (p_\parallel - p_\perp)(\mathbf{b}\times\boldsymbol{\kappa})\Big]\;, \tag{44}$$

so that

$$2\pi\int dv'_\parallel\; dv'_\perp\; v'_\perp\; \dot{\mathbf{x}}\;\bar{f} \;=\; n\;\mathbf{u} \;+\; \nabla\times\Big(\frac{p_\perp\mathbf{b}}{m\Omega_c}\Big) \tag{45}$$

and (42) becomes identical to the continuity equation

$$\frac{\partial n}{\partial t} \;+\; \nabla\cdot(n\;\mathbf{u}) \;=0\;. \tag{46}$$

ii) Parallel relative velocity moment.

Setting $\alpha = 1$ and $\beta = 0$, and multiplying by the mass, Eq.(41) becomes

$$\nabla\cdot\Big(2\pi m\int dv'_\parallel\; dv'_\perp\; v'_\perp\; v'_\parallel\; \dot{\mathbf{x}}\;\bar{f}\Big) \;-\; 2\pi m\int dv'_\parallel\; dv'_\perp\; v'_\perp\; \dot{v}'_\parallel\;\bar{f} \;=\; 0\;, \tag{47}$$

where, recalling (33,34), the velocity-space integrals are again written in terms of the gyrotropic moments as

$$2\pi m\int dv'_\parallel\; dv'_\perp\; v'_\perp\; v'_\parallel\; \dot{\mathbf{x}}\;\bar{f} \;=\; p_\parallel\mathbf{b} \;+\; q_{T\parallel}\nabla\times\Big(\frac{\mathbf{b}}{\Omega_c}\Big) \;+\; \frac{\mathbf{b}}{\Omega_c}\times\Big[2p_\parallel(\mathbf{b}\cdot\nabla)\mathbf{u} + (2q_{B\parallel} - q_{T\parallel})\boldsymbol{\kappa}\Big] \tag{48}$$

and

$$2\pi m \int dv'_\parallel \; dv'_\perp \; v'_\perp \; \dot{v}'_\parallel \; \bar{f} \; = \mathbf{b}\cdot\mathbf{F} \; - \; p_\perp \mathbf{b}\cdot\nabla \ln B \; + \; p_\perp \nabla\cdot\Big[\frac{\mathbf{b}}{\Omega_c}\times(\boldsymbol{\omega}\times\mathbf{b})\Big] \; +$$

$$+ \; q_{T\parallel}\nabla\cdot\Big(\frac{\mathbf{b}\times\boldsymbol{\kappa}}{\Omega_c}\Big) \; + \; \Big[\frac{\mathbf{b}}{\Omega_c}\times(\boldsymbol{\omega}\times\mathbf{b})\Big]\cdot(\mathbf{F}-p_\parallel\boldsymbol{\kappa}) \; - \; 2p_\parallel\Big(\frac{\mathbf{b}\times\boldsymbol{\kappa}}{\Omega_c}\Big)\cdot[(\mathbf{b}\cdot\nabla)\mathbf{u}] \; - \; p_\perp\sigma \; . \qquad (49)$$

Substituting the expressions (37,38) for the perpendicular and parallel components of $\mathbf{F}$, and using some vector identities, (49) reduces to

$$2\pi m \int dv'_\parallel \; dv'_\perp \; v'_\perp \; \dot{v}'_\parallel \; \bar{f} \; = \; \nabla\cdot\Big\{p_\parallel\mathbf{b} \; + \; \frac{\mathbf{b}}{\Omega_c}\times\Big[\nabla q_{T\parallel} + 2p_\parallel(\mathbf{b}\cdot\nabla)\mathbf{u} + (2q_{B\parallel}-q_{T\parallel})\boldsymbol{\kappa}\Big]\Big\} \qquad (50)$$

and, combining (50) with the divergence of (48), we verify that (47) is satisfied identically. Thus our drift-kinetic equation is compatible with the required condition that the parallel relative velocity moment of $\bar{f}$, i.e. M^{10}, be equal to zero.

iii) Parallel pressure.

Setting $\alpha = 2$ and $\beta = 0$, multiplying by half the mass and following a procedure analogous to the previous two cases, we get

$$\frac{1}{2}\frac{\partial p_\parallel}{\partial t} \; + \; \nabla\cdot\Big\{\frac{p_\parallel}{2}(\mathbf{u}+\mathbf{u}_F) \; + \; q_{B\parallel}\mathbf{b} \; + \; \frac{r_{B\perp}}{m}\nabla\times\Big(\frac{\mathbf{b}}{\Omega_c}\Big) \; + \; \frac{\mathbf{b}}{\Omega_c}\times\Big[2q_{B\parallel}(\mathbf{b}\cdot\nabla)\mathbf{u} + \frac{(2r_{B\parallel}-r_{B\perp})}{m}\boldsymbol{\kappa}\Big]\Big\} \; +$$

$$+ \; p_\parallel\mathbf{b}\cdot\Big[(\mathbf{b}\cdot\nabla)(\mathbf{u}+\mathbf{u}_F)\Big] \; + \; q_{T\parallel}\mathbf{b}\cdot\nabla\ln B \; - \; q_{T\parallel}\nabla\cdot\Big[\frac{\mathbf{b}}{\Omega_c}\times(\boldsymbol{\omega}\times\mathbf{b})\Big] \; - \frac{2r_{B\perp}}{m}\nabla\cdot\Big(\frac{\mathbf{b}\times\boldsymbol{\kappa}}{\Omega_c}\Big) \; +$$

$$+ \; 2\Big[\frac{\mathbf{b}}{\Omega_c}\times(\boldsymbol{\omega}\times\mathbf{b})\Big]\cdot[p_\parallel(\mathbf{b}\cdot\nabla)\mathbf{u} + q_{B\parallel}\boldsymbol{\kappa}] \; + \; 4q_{B\parallel}\Big(\frac{\mathbf{b}\times\boldsymbol{\kappa}}{\Omega_c}\Big)\cdot[(\mathbf{b}\cdot\nabla)\mathbf{u}] \; + \; q_{T\parallel}\sigma \; = \; 0 \qquad (51)$$

which, substituting for $\mathbf{u}_F$ (44) and using vector identities, becomes

$$\frac{1}{2}\Big[\frac{\partial p_\parallel}{\partial t} + \nabla\cdot(p_\parallel\mathbf{u})\Big] \; + \; p_\parallel\mathbf{b}\cdot[(\mathbf{b}\cdot\nabla)\mathbf{u}] \; + \; \nabla\cdot(q_{B\parallel}\mathbf{b}) \; + \; q_{T\parallel}(\mathbf{b}\cdot\nabla\ln B + \sigma) \; +$$

$$+\nabla\cdot\Big\{\frac{\mathbf{b}}{m\Omega_c}\times\Big[-\frac{p_\parallel}{2n}\nabla p_\perp - \frac{p_\parallel(p_\parallel-p_\perp)}{2n}\boldsymbol{\kappa} + 2mq_{B\parallel}(\mathbf{b}\cdot\nabla)\mathbf{u} + mq_{T\parallel}\mathbf{b}\times\boldsymbol{\omega} + \nabla r_{B\perp} + (2r_{B\parallel}-3r_{B\perp})\boldsymbol{\kappa}\Big]\Big\}+$$

$$+2\Big(\frac{\mathbf{b}\times\boldsymbol{\kappa}}{m\Omega_c}\Big)\cdot\Big\{-\frac{p_\parallel}{2n}\nabla p_\perp + mq_{B\parallel}\Big[2(\mathbf{b}\cdot\nabla)\mathbf{u}+\mathbf{b}\times\boldsymbol{\omega}\Big] + \nabla r_{B\perp}\Big\} - \Big[\frac{\mathbf{b}}{\Omega_c}\times(\mathbf{b}\times\boldsymbol{\omega})\Big]\cdot[2p_\parallel(\mathbf{b}\cdot\nabla)\mathbf{u} + \nabla q_{T\parallel}] = 0. \qquad (52)$$

It is useful to make explicit the contribution of a two-temperature Maxwellian $f_{2M}(v'_\parallel, v'_\perp, \mathbf{x}, t)$ to the fourth-rank gyrotropic moments, separating it from the contribution of the difference between the actual distribution function and the two-temperature Maxwellian:

$$r_{B\parallel} = \frac{3p_\parallel^2}{4n} + \tilde{r}_{B\parallel} \ , \tag{53}$$

$$r_{B\perp} = \frac{p_\parallel p_\perp}{2n} + \tilde{r}_{B\perp} \tag{54}$$

and

$$r_{T\perp} = \frac{2p_\perp^2}{n} + \tilde{r}_{T\perp} \ , \tag{55}$$

where $\tilde{r}_{B\parallel}$, $\tilde{r}_{B\perp}$ and $\tilde{r}_{T\perp}$ are the corresponding moments of $(\bar{f} - f_{2M})$ and

$$f_{2M}(v'_\parallel, v'_\perp, \mathbf{x}, t) = \Big(\frac{m}{2\pi}\Big)^{3/2} \frac{n^{5/2}}{p_\parallel^{1/2} p_\perp} \exp\Big[\frac{mn}{2}\Big(\frac{v'^2_\parallel}{p_\parallel} + \frac{v'^2_\perp}{p_\perp}\Big)\Big] \ . \tag{56}$$

In terms of these, (52) is rewritten as:

$$\frac{1}{2}\Big[\frac{\partial p_\parallel}{\partial t} + \nabla\cdot(p_\parallel \mathbf{u})\Big] \ + \ p_\parallel \mathbf{b}\cdot[(\mathbf{b}\cdot\nabla)\mathbf{u}] \ + \ \nabla\cdot(q_{B\parallel}\mathbf{b}) \ + \ q_{T\parallel}(\mathbf{b}\cdot\nabla\ln B + \sigma) \ +$$

$$+\nabla\cdot\Big\{\frac{\mathbf{b}}{m\Omega_c}\times\Big[\frac{p_\perp}{2}\nabla\Big(\frac{p_\parallel}{n}\Big) + \frac{p_\parallel(p_\parallel - p_\perp)}{n}\boldsymbol{\kappa} + 2mq_{B\parallel}(\mathbf{b}\cdot\nabla)\mathbf{u} + mq_{T\parallel}\mathbf{b}\times\boldsymbol{\omega} + \nabla\tilde{r}_{B\perp} + (2\tilde{r}_{B\parallel} - 3\tilde{r}_{B\perp})\boldsymbol{\kappa}\Big]\Big\}+$$

$$+2\Big(\frac{\mathbf{b}\times\boldsymbol{\kappa}}{m\Omega_c}\Big)\cdot\Big\{\frac{p_\perp}{2}\nabla\Big(\frac{p_\parallel}{n}\Big) + mq_{B\parallel}\Big[2(\mathbf{b}\cdot\nabla)\mathbf{u} + \mathbf{b}\times\boldsymbol{\omega}\Big] + \nabla\tilde{r}_{B\perp}\Big\} - \Big[\frac{\mathbf{b}}{\Omega_c}\times(\mathbf{b}\times\boldsymbol{\omega})\Big]\cdot[2p_\parallel(\mathbf{b}\cdot\nabla)\mathbf{u} + \nabla q_{T\parallel}] = 0. \tag{57}$$

From "perpendicular" (non-gyrotropic) fluid theory[11], the perpendicular flux of parallel heat is known to be

$$\mathbf{q}_{B\perp} = \frac{\mathbf{b}}{m\Omega_c}\times\Big[\frac{p_\perp}{2}\nabla\Big(\frac{p_\parallel}{n}\Big) + \frac{p_\parallel(p_\parallel - p_\perp)}{n}\boldsymbol{\kappa} + 2mq_{B\parallel}(\mathbf{b}\cdot\nabla)\mathbf{u} + mq_{T\parallel}\mathbf{b}\times\boldsymbol{\omega} + \nabla\tilde{r}_{B\perp} + (2\tilde{r}_{B\parallel} - 3\tilde{r}_{B\perp})\boldsymbol{\kappa}\Big] \tag{58}$$

and also

$$\mathbf{b}\cdot\mathsf{P}^{GV} = \frac{\mathbf{b}}{\Omega_c}\times\Big[2p_\parallel(\mathbf{b}\cdot\nabla)\mathbf{u} + p_\perp\mathbf{b}\times\boldsymbol{\omega} + \nabla q_{T\parallel} + 2(q_{B\parallel} - q_{T\parallel)}\boldsymbol{\kappa}\Big] \ . \tag{59}$$

Using these expressions, the compact form of the parallel pressure evolution equation (57) is

$$\frac{1}{2}\Big[\frac{\partial p_\parallel}{\partial t} + \nabla\cdot(p_\parallel \mathbf{u})\Big] \ + \ p_\parallel \mathbf{b}\cdot[(\mathbf{b}\cdot\nabla)\mathbf{u}] \ + \ \nabla\cdot(q_{B\parallel}\mathbf{b} + \mathbf{q}_{B\perp}) \ + \ q_{T\parallel}(\mathbf{b}\cdot\nabla\ln B + \sigma) \ +$$

$$+ \ \mathbf{b}\cdot\mathsf{P}^{GV}\cdot(\mathbf{b}\times\boldsymbol{\omega}) \ - \ 2\mathbf{q}_{B\perp}\cdot\boldsymbol{\kappa} \ = \ 0 \ , \tag{60}$$

identical to the result derived from moments of the Vlasov equation[10,11].

iv) Perpendicular pressure and energy conservation.

Setting $\alpha = 0$ and $\beta = 2$, and multiplying by half the mass in Eq.(41), we get

$$\frac{\partial p_\perp}{\partial t} + \nabla \cdot \Big\{ p_\perp(\mathbf{u}+\mathbf{u}_F) + q_{T\|}\mathbf{b} + \frac{r_{T\perp}}{m}\nabla\times\Big(\frac{\mathbf{b}}{\Omega_c}\Big) + \frac{\mathbf{b}}{\Omega_c}\times\Big[2q_{T\|}(\mathbf{b}\cdot\nabla)\mathbf{u} + \frac{(2r_{B\perp}-r_{T\perp})}{m}\boldsymbol{\kappa}\Big]\Big\} +$$

$$+ p_\perp\Big\{\nabla\cdot(\mathbf{u}+\mathbf{u}_F) - \mathbf{b}\cdot[(\mathbf{b}\cdot\nabla)(\mathbf{u}+\mathbf{u}_F)]\Big\} - q_{T\|}\mathbf{b}\cdot\nabla\ln B + 2q_{T\|}\nabla\cdot\Big\{\frac{\mathbf{b}}{\Omega_c}\times\Big[(\mathbf{b}\cdot\nabla)\mathbf{u}\Big]\Big\} +$$

$$+ \frac{2r_{B\perp}}{m}\nabla\cdot\Big(\frac{\mathbf{b}\times\boldsymbol{\kappa}}{\Omega_c}\Big) - 2\Big[\frac{\mathbf{b}}{\Omega_c}\times(\boldsymbol{\omega}\times\mathbf{b})\Big]\cdot[p_\perp(\mathbf{b}\cdot\nabla)\mathbf{u}+q_{T\|}\boldsymbol{\kappa}] - 4q_{T\|}\Big(\frac{\mathbf{b}\times\boldsymbol{\kappa}}{\Omega_c}\Big)\cdot[(\mathbf{b}\cdot\nabla)\mathbf{u}] = 0\,. \quad (61)$$

Next we trace the steps of the previous parallel pressure calculation, substituting for $\mathbf{u}_F$ (44) and using vector identities and the relations (54,55), to arrive at

$$\frac{\partial p_\perp}{\partial t} + \nabla\cdot(p_\perp\mathbf{u}) + p_\perp\Big\{\nabla\cdot\mathbf{u} - \mathbf{b}\cdot[(\mathbf{b}\cdot\nabla)\mathbf{u}]\Big\} + \nabla\cdot(q_{T\|}\mathbf{b}) - q_{T\|}\mathbf{b}\cdot\nabla\ln B +$$

$$+\nabla\cdot\Big\{\frac{\mathbf{b}}{m\Omega_c}\times\Big[2p_\perp\nabla\Big(\frac{p_\perp}{n}\Big) + 4mq_{T\|}(\mathbf{b}\cdot\nabla)\mathbf{u} + \nabla\tilde{r}_{T\perp} + (4\tilde{r}_{B\perp}-\tilde{r}_{T\perp})\boldsymbol{\kappa}\Big]\Big\} -$$

$$-2\Big(\frac{\mathbf{b}\times\boldsymbol{\kappa}}{m\Omega_c}\Big)\cdot\Big\{\frac{p_\perp}{2}\nabla\Big(\frac{p_\|}{n}\Big) + mq_{T\|}\Big[2(\mathbf{b}\cdot\nabla)\mathbf{u}+\mathbf{b}\times\boldsymbol{\omega}\Big] + \nabla\tilde{r}_{B\perp}\Big\} - 2\Big[\frac{\mathbf{b}}{\Omega_c}\times(\mathbf{b}\cdot\nabla)\mathbf{u}\Big]\cdot(p_\perp\mathbf{b}\times\boldsymbol{\omega}+\nabla q_{T\|}) = 0. \quad (62)$$

The perpendicular flux of perpendicular heat, as derived from "perpendicular" (non-gyrotropic) fluid theory[11], is

$$\mathbf{q}_{T\perp} = \frac{\mathbf{b}}{m\Omega_c}\times\Big[2p_\perp\nabla\Big(\frac{p_\perp}{n}\Big) + 4mq_{T\|}(\mathbf{b}\cdot\nabla)\mathbf{u} + \nabla\tilde{r}_{T\perp} + (4\tilde{r}_{B\perp}-\tilde{r}_{T\perp})\boldsymbol{\kappa}\Big] \quad (63)$$

hence, using this result along with (58,59), we obtain the compact form of the perpendicular pressure evolution equation:

$$\frac{\partial p_\perp}{\partial t} + \nabla\cdot(p_\perp\mathbf{u}) + p_\perp\Big\{\nabla\cdot\mathbf{u} - \mathbf{b}\cdot[(\mathbf{b}\cdot\nabla)\mathbf{u}]\Big\} + \nabla\cdot(q_{T\|}\mathbf{b}+\mathbf{q}_{T\perp}) - q_{T\|}\mathbf{b}\cdot\nabla\ln B +$$

$$+ 2\mathbf{b}\cdot\mathsf{P}^{GV}\cdot[(\mathbf{b}\cdot\nabla)\mathbf{u}] + 2\mathbf{q}_{B\perp}\cdot\boldsymbol{\kappa} = 0\,. \quad (64)$$

Finally, adding (60) and (64), defining the total perpendicular heat flux $\mathbf{q}_\perp = \mathbf{q}_{B\perp} + \mathbf{q}_{T\perp}$ and recalling the "perpendicular" (non-gyrotropic) fluid theory result[11]

$$\mathsf{P}^{GV} : (\nabla\mathbf{u}) = \mathbf{b}\cdot\mathsf{P}^{GV}\cdot[2(\mathbf{b}\cdot\nabla)\mathbf{u} + \mathbf{b}\times\boldsymbol{\omega}] + q_{T\|}\sigma\,, \quad (65)$$

we recover exactly the well known evolution equation for the mean scalar pressure, which combined with the component on the momentum conservation equation along $\mathbf{u}$ is also the expression of energy conservation:

$$\frac{3}{2}\Big[\frac{\partial p}{\partial t} + \nabla\cdot(p\mathbf{u})\Big] \ + \ \Big(\mathsf{P}^{CGL} + \mathsf{P}^{GV}\Big) : (\nabla\mathbf{u}) \ + \ \nabla\cdot(q_{\|}\mathbf{b} + \mathbf{q}_{\perp}) \ = \ 0 \ . \qquad (66)$$

v) Parallel flux of parallel heat.

The dynamic evolution equation for the parallel flux of parallel heat, $q_{B\|}$, is obtained from the $\alpha = 3$, $\beta = 0$ moment of the drift-kinetic equation. Following the by now familiar procedure, but omitting the details, we get

$$\frac{\partial q_{B\|}}{\partial t} \ + \ \nabla\cdot(q_{B\|}\mathbf{u}) \ + \ 3q_{B\|}\mathbf{b}\cdot[(\mathbf{b}\cdot\nabla)\mathbf{u}] \ + \ \frac{3p_{\|}}{2m}\mathbf{b}\cdot\nabla\Big(\frac{p_{\|}}{n}\Big) \ + \ \frac{1}{m}\nabla\cdot\Big(2\tilde{r}_{B\|}\mathbf{b} + \mathbf{t}_{B\perp}\Big) \ +$$
$$+ \ \frac{3\tilde{r}_{B\perp}}{m}(\mathbf{b}\cdot\nabla\ln B + \sigma) \ + \ \frac{3}{m}\mathbf{b}\cdot\mathsf{P}^{GV}\cdot\Big[\nabla\Big(\frac{p_{\|}}{2n}\Big) - \frac{p_{\|}}{n}\boldsymbol{\kappa}\Big] \ - \ \frac{3}{m}\mathbf{t}_{B\perp}\cdot\boldsymbol{\kappa} \ + \ 3\mathbf{q}_{B\perp}\cdot(\mathbf{b}\times\boldsymbol{\omega}) \ = \ 0 \ , \quad (67)$$

where the perpendicular vector $\mathbf{t}_{B\perp}$ is:

$$\mathbf{t}_{B\perp} \ = \ \frac{\mathbf{b}}{\Omega_c}\times\Big\{\nabla s_{B\perp} \ - \ \frac{q_{B\|}}{n}\nabla p_{\perp} \ - \ \frac{3p_{\|}}{2n}\nabla q_{T\|} \ +$$
$$+ \ \Big[2s_{B\|} - 4s_{B\perp} - \frac{q_{B\|}}{n}(p_{\|} - p_{\perp}) - \frac{3p_{\|}}{n}(q_{B\|} - q_{T\|})\Big]\boldsymbol{\kappa} \ + \ 4\tilde{r}_{B\|}(\mathbf{b}\cdot\nabla)\mathbf{u} \ + \ 3\tilde{r}_{B\perp}\mathbf{b}\times\boldsymbol{\omega}\Big\} \ . \qquad (68)$$

This equation coincides with the one derived from moments of the Vlasov equation[11] after identifying the scalar $\tilde{s}_B$, defined but not evaluated explicitly in Ref.11, as

$$\tilde{s}_B/2 \ = \ \nabla\cdot\mathbf{t}_{B\perp} \ - \ 3\mathbf{t}_{B\perp}\cdot\boldsymbol{\kappa} \ + \ 3\tilde{r}_{B\perp}\sigma \ . \qquad (69)$$

vi) Parallel flux of perpendicular heat.

The last moment of the drift-kinetic equation to be considered here is the dynamic evolution equation for the parallel flux of perpendicular heat $q_{T\|}$, which is similarly derived taking $\alpha = 1$ and $\beta = 2$. Omitting again the details, the result is

$$\frac{\partial q_{T\|}}{\partial t} \;+\; \nabla\cdot(q_{T\|}\mathbf{u}) \;+\; q_{T\|}\nabla\cdot\mathbf{u} \;+\; \frac{p_\|}{m}\mathbf{b}\cdot\nabla\Big(\frac{p_\perp}{n}\Big) \;+\; \frac{1}{m}\nabla\cdot\Big(2\tilde{r}_{B\perp}\mathbf{b}+\mathbf{t}_{T\perp}\Big) \;-$$

$$-\;\Big[\frac{p_\perp(p_\|-p_\perp)}{mn}+\frac{2\tilde{r}_{B\perp}-\tilde{r}_{T\perp}}{m}\Big]\mathbf{b}\cdot\nabla\ln B \;+\; \Big(\frac{p_\perp^2}{mn}+\frac{\tilde{r}_{T\perp}}{m}\Big)\sigma \;+\; \frac{1}{m}\mathbf{b}\cdot\mathsf{P}^{GV}\cdot\Big[\nabla\Big(\frac{p_\perp}{n}\Big)+\frac{2p_\|}{n}\boldsymbol{\kappa}\Big] \;+$$

$$+\;\frac{p_\perp}{m}\nabla\cdot\Big(\frac{1}{n}\mathbf{b}\cdot\mathsf{P}^{GV}\Big) \;+\; \frac{1}{m}(2\mathbf{t}_{B\perp}-\mathbf{t}_{T\perp})\cdot\boldsymbol{\kappa} \;+\; 4\mathbf{q}_{B\perp}\cdot[(\mathbf{b}\cdot\nabla)\mathbf{u}] \;+\; \mathbf{q}_{T\perp}\cdot(\mathbf{b}\times\boldsymbol{\omega}) \;=\; 0\;, \qquad (70)$$

where

$$\mathbf{t}_{T\perp} \;=\; \frac{\mathbf{b}}{\Omega_c}\times\Big\{\nabla s_{T\perp} \;-\; \frac{2q_{T\|}}{n}\nabla p_\perp \;-\; \frac{2p_\perp}{n}\nabla q_{T\|} \;+$$

$$+\;\Big[4s_{B\perp}-2s_{T\perp}-\frac{2q_{T\|}}{n}(p_\|-p_\perp)-\frac{4p_\perp}{n}(q_{B\|}-q_{T\|})\Big]\boldsymbol{\kappa} \;+\; 8\tilde{r}_{B\perp}(\mathbf{b}\cdot\nabla)\mathbf{u} \;+\; \tilde{r}_{T\perp}\mathbf{b}\times\boldsymbol{\omega}\Big\}\;. \qquad (71)$$

The sum of (67) and (70) gives the evolution equation for the total parallel heat flux $q_\|$. This equation coincides again with the one derived from moments of the Vlasov equation[11] after identifying the scalar $\tilde{s}$, defined but not evaluated explicitly in Ref.11, as

$$\tilde{s}/2 \;=\; \nabla\cdot(\mathbf{t}_{B\perp}+\mathbf{t}_{T\perp}) \;-\; (\mathbf{t}_{B\perp}+\mathbf{t}_{T\perp})\cdot\boldsymbol{\kappa} \;+\; 3(\tilde{r}_{T\perp}-\tilde{r}_{B\perp})\sigma\;. \qquad (72)$$

IV. Summary.

A form of the collisionless drift-kinetic equation in a moving reference frame (32-35), accurate to the first order in the FLR expansion and valid for fully electromagnetic nonlinear dynamics, sonic macroscopic flows and far-from-Maxwellian distribution functions has been put forward. The expression of its coefficient functions (34-35) is rather compact and makes the phase-space conservation (36) clearly manifest. Taking the complete macroscopic flow velocity of the species under consideration as the velocity of the moving reference frame where this drift-kinetic equation applies, it has been shown that its fluid moments reproduce exactly the corresponding FLR macroscopic equations for the gyrotropic fluid variables as previously derived from moments of the Vlasov equation. When working in this reference frame of the macroscopic flow, the electric field is eliminated algebraically and the functions of space-time involved in the coefficient functions of the drift-kinetic equation are the magnetic field $\mathbf{B}$, the flow velocity $\mathbf{u}$ and the five lowest gyrotropic scalars n, $p_{\|}$, $p_{\perp}$, $q_{B\|}$ and $q_{T\|}$, which are also the variables needed to specify the complete stress tensor $\mathsf{P} = \mathsf{P}^{CGL} + \mathsf{P}^{GV}$. Of these, $\mathbf{B}$ and $\mathbf{u}$ must be obtained from Maxwell's equations and from the fluid momentum conservation equations for each species, so that the result is a hybrid fluid-kinetic closed plasma description. For the five gyrotropic scalars, one has the choice of either evaluating them as moments of the drift-kinetic distribution function solution (which requires an implicit solution scheme since these low moments are themselves part of the drift-kinetic coefficient functions), or from their fluid evolution equations (46,60,64,68,70) which involve the additional fourth and fifth rank gyrotropic moments to be determined by the drift-kinetic solution (hence an explicit scheme since these higher moments are not part of the drift-kinetic coefficient functions).

The present form of the drift-kinetic equation can also be used in the traditional way, whereby the working reference frame is taken as either the laboratory (provided the flows are subsonic) or the frame of the electric drift. In these cases, the drift-kinetic coefficient functions involve explicitly the electric field that has to be determined separatetly and whose sufficiently accurate evaluation becomes the trickier part of the problem.

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Appendix A: Elimination of the time derivative of the magnetic unit vector.

The operators $\Lambda_{\pm1}$ (17) and the functions $\lambda_{\pm1}$ (20) involve the components of $\partial\mathbf{b}/\partial t$ along $\mathbf{e}_1$ and $\mathbf{e}_2$ which, according to Faraday's law, are

$$(\mathbf{e}_1 \pm i\mathbf{e}_2)\cdot\frac{\partial\mathbf{b}}{\partial t} \;=\; -\frac{1}{B}(\mathbf{e}_1 \pm i\mathbf{e}_2)\cdot(\nabla\times\mathbf{E})\;. \tag{A.1}$$

Using (12) we can eliminate $\mathbf{E}$ in favor of $\mathbf{F}$:

$$\mathbf{E} \;=\; -\mathbf{u}\times\mathbf{B} \;+\; \frac{\mathbf{F}}{en} \;+\; \frac{m}{e}\Big[\frac{\partial\mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u}\Big]\;, \tag{A.2}$$

whence

$$-\frac{1}{B}(\mathbf{e}_1 \pm i\mathbf{e}_2)\cdot(\nabla\times\mathbf{E}) = (\mathbf{e}_1 \pm i\mathbf{e}_2)\cdot\Big\{(\mathbf{b}\cdot\nabla)\mathbf{u} - (\mathbf{u}\cdot\nabla)\mathbf{b} - \frac{1}{\Omega_c}\;\nabla\times\Big[\frac{\mathbf{F}}{mn} + \frac{\partial\mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u}\Big]\Big\}. \tag{A.3}$$

Bringing this expression to (17) and (20), we can write

$$\Lambda_1 \;=\; \Lambda^*_{-1} \;=\; \frac{v'_\perp}{2}(\mathbf{e}_1 - i\mathbf{e}_2)\cdot\frac{\partial}{\partial\mathbf{x}} \;+\; \frac{v'_\perp}{2}(\mathbf{e}_1 - i\mathbf{e}_2)\cdot\mathbf{Z}\;\frac{\partial}{\partial v'_\parallel} \;+\; \frac{1}{2}(\mathbf{e}_1 - i\mathbf{e}_2)\cdot\Big(\frac{\mathbf{F}}{mn} - v'_\parallel\;\mathbf{Y}\Big)\;\frac{\partial}{\partial v'_\perp} \tag{A.4}$$

and

$$\lambda_1 \;=\; \lambda^*_{-1} \;=\; \frac{i}{2v'_\perp}(\mathbf{e}_1 - i\mathbf{e}_2)\cdot\Big(\frac{\mathbf{F}}{mn} - v'_\parallel\mathbf{Y}\Big) \;-\; \frac{v'_\perp}{2}\mathbf{b}\cdot\Big[\nabla\times(\mathbf{e}_1 - i\mathbf{e}_2)\Big]\;, \tag{A.5}$$

where

$$\mathbf{Z}(v'_\parallel,\mathbf{x},t) \;=\; -\mathbf{b}\times\boldsymbol{\omega} \;+\; v'_\parallel\boldsymbol{\kappa} \;-\; \frac{1}{\Omega_c}\;\nabla\times\Big[\frac{\mathbf{F}}{mn} + \frac{\partial\mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u}\Big] \tag{A.6}$$

and

$$\mathbf{Y}(v'_\parallel,\mathbf{x},t) \;=\; 2(\mathbf{b}\cdot\nabla)\mathbf{u} \;+\; v'_\parallel\boldsymbol{\kappa} \;-\; \frac{1}{\Omega_c}\;\nabla\times\Big[\frac{\mathbf{F}}{mn} + \frac{\partial\mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u}\Big]\;. \tag{A.7}$$

For our present purpose of deriving a first-order FLR drift-kinetic equation valid for sonic flows, it is sufficient to keep $\Lambda_{\pm1}$ and $\lambda_{\pm1}$ within the accuracy of $\Lambda_{\pm1}\sim\lambda_{\pm1}\sim v_{th}/L$, therefore it is sufficient to keep

$$\mathbf{Z}(v'_\parallel,\mathbf{x},t) \;=\; -\mathbf{b}\times\boldsymbol{\omega} \;+\; v'_\parallel\boldsymbol{\kappa} \;+\; O(\delta v_{th}/L) \tag{A.8}$$

and

$$\mathbf{Y}(v'_\parallel,\mathbf{x},t) \;=\; 2(\mathbf{b}\cdot\nabla)\mathbf{u} \;+\; v'_\parallel\boldsymbol{\kappa} \;+\; O(\delta v_{th}/L)\;. \tag{A.9}$$

Appendix B: Operator commutations.

Our derivation of the FLR drift-kinetic equation bypasses the explicit calculation of the gyrophase dependent part (i.e. $l \neq 0$ harmonics) of the first-order distribution function and requires only the calculation of the three operator commutators defined by the three terms inside square brackets in Eq.(31). Using the expressions (A.4,A.5) for $\Lambda_{\pm 1}$ and $\lambda_{\pm 1}$, and (18,21) for $\Lambda_{\pm 2}$ and $\lambda_{\pm 2}$, the result is

$$\frac{1}{i\Omega_c}\Big[(\Lambda_1 - i\lambda_1)\Lambda_{-1} - (\Lambda_{-1} + i\lambda_{-1})\Lambda_1\Big] \;=\; \frac{1}{\Omega_c}\Big\{\mathbf{b}\times\Big(\frac{\mathbf{F}}{mn} - v'_\parallel \mathbf{Y}\Big) - \frac{v'^2_\perp}{2}[\mathbf{b}\cdot(\nabla\times\mathbf{b})]\;\mathbf{b}\Big\}\cdot\frac{\partial}{\partial \mathbf{x}} +$$
$$+\frac{1}{\Omega_c}\Big\{\Big[\mathbf{b}\times\Big(\frac{\mathbf{F}}{mn} - v'_\parallel\mathbf{Y}\Big)\Big]\cdot\mathbf{Z} + \frac{v'^2_\perp}{2}\nabla\cdot(\mathbf{Z}\times\mathbf{b})\Big\}\;\frac{\partial}{\partial v'_\parallel} +$$
$$\frac{v'_\perp}{2\Omega_c}\Big\{\Big(\mathbf{b}\times\boldsymbol{\kappa}\Big)\cdot\Big(\frac{\mathbf{F}}{mn} - v'_\parallel\mathbf{Y}\Big) + \nabla\cdot\Big[\Big(\frac{\mathbf{F}}{mn} - v'_\parallel\mathbf{Y}\Big)\times\mathbf{b}\Big] + \Big[\mathbf{b}\times\Big(\mathbf{Y} + v'_\parallel\boldsymbol{\kappa}\Big)\Big]\cdot\mathbf{Z}\Big\}\;\frac{\partial}{\partial v'_\perp}\;, \qquad \text{(B.1)}$$

$$\frac{v'_\perp}{2i\Omega_c}\Big[(\mathbf{e}_1 - i\mathbf{e}_2)\cdot\nabla\ln B\;\Lambda_{-1} - (\mathbf{e}_1 + i\mathbf{e}_2)\cdot\nabla\ln B\;\Lambda_1\Big] \;=\; \frac{v'^2_\perp}{2\Omega_c}(\mathbf{b}\times\nabla\ln B)\cdot\frac{\partial}{\partial\mathbf{x}} +$$
$$+\frac{v'^2_\perp}{2\Omega_c}\Big[(\mathbf{b}\times\nabla\ln B)\cdot\mathbf{Z}\Big]\frac{\partial}{\partial v'_\parallel} \;+\; \frac{v'_\perp}{2\Omega_c}\Big[(\mathbf{b}\times\nabla\ln B)\cdot\Big(\frac{\mathbf{F}}{mn} - v'_\parallel\mathbf{Y}\Big)\Big]\frac{\partial}{\partial v'_\perp} \qquad \text{(B.2)}$$

and

$$\frac{1}{i\Omega_c}\Big[(\frac{1}{2}\Lambda_2 - i\lambda_2)\Lambda_{-2} - (\frac{1}{2}\Lambda_{-2} + i\lambda_{-2})\Lambda_2\Big] \;=\; \frac{v'^2_\perp}{2}\sigma\;\frac{\partial}{\partial v'_\parallel}\;, \qquad \text{(B.3)}$$

where $\sigma(\mathbf{x},t)$ is the scalar introduced in Ref.11:

$$\sigma \;=\; \frac{1}{4\Omega_c}\epsilon_{jkl}b_j\left(\frac{\partial b_k}{\partial x_m} + \frac{\partial b_m}{\partial x_k}\right)(\delta_{mn} - b_m b_n)\left(\frac{\partial u_l}{\partial x_n} + \frac{\partial u_n}{\partial x_l}\right)\;. \qquad \text{(B.4)}$$

As can be seen, the outcome of these commutations does not depend on $(\mathbf{e}_1, \mathbf{e}_2)$ and involves only the intrinsic geometry of the magnetic field along with the velocity field $\mathbf{u}$ and the force density $\mathbf{F}$. Substituting the expressions (A.8,A.9) for $\mathbf{Z}$ and $\mathbf{Y}$, bringing the result to Eq.(31) and carrying some straightforward algebra, one obtains the final form of the first-order contribution to the drift-kinetic equation.

Appendix C: Drift-kinetic equation in the laboratory frame.

If the flow is sufficiently slow so that $E \ll v_{th}B$, the drift-kinetic analysis can be carried out in the laboratory frame. Then, the slow flow, first-order FLR drift-kinetic equation derived in Ref.6, is recovered as a special case of the present results. Setting $\mathbf{u} = 0$, $\mathbf{v}' = \mathbf{v}$, $\mathbf{F} = en\mathbf{E}$ and $\mathbf{u}_F = \mathbf{u}_E$, our drift-kinetic equation (32-35) becomes

$$\frac{\partial \bar{f}(v_\parallel, v_\perp, \mathbf{x}, t)}{\partial t} \; + \; \dot{\mathbf{x}} \cdot \frac{\partial \bar{f}}{\partial \mathbf{x}} \; + \; \dot{v}_\parallel \, \frac{\partial \bar{f}}{\partial v_\parallel} \; + \; \dot{v}_\perp \, \frac{\partial \bar{f}}{\partial v_\perp} \; = \; 0 \; , \tag{C.1}$$

with

$$\dot{\mathbf{x}} \; = \; \mathbf{u}_E \; + \; v_\parallel \mathbf{b} \; + \; \frac{v_\perp^2}{2} \nabla \times \Big(\frac{\mathbf{b}}{\Omega_c} \Big) \; + \; \Big(v_\parallel^2 - \frac{v_\perp^2}{2} \Big) \frac{\mathbf{b} \times \boldsymbol{\kappa}}{\Omega_c} \; , \tag{C.2}$$

$$\dot{v}_\parallel \; = \; \frac{e}{m} \mathbf{b} \cdot \mathbf{E} \; - \; v_\parallel \mathbf{b} \cdot [(\mathbf{b} \cdot \nabla) \mathbf{u}_E] \; - \; \frac{v_\perp^2}{2} \mathbf{b} \cdot \nabla \ln B \; + \; \frac{v_\parallel v_\perp^2}{2} \nabla \cdot \Big(\frac{\mathbf{b} \times \boldsymbol{\kappa}}{\Omega_c} \Big) \tag{C.3}$$

and

$$\dot{v}_\perp = \frac{v_\perp}{2} \Big\{ \mathbf{b} \cdot [(\mathbf{b} \cdot \nabla) \mathbf{u}_E] \; - \; \nabla \cdot \mathbf{u}_E \; + \; v_\parallel \mathbf{b} \cdot \nabla \ln B \; - \; v_\parallel^2 \nabla \cdot \Big(\frac{\mathbf{b} \times \boldsymbol{\kappa}}{\Omega_c} \Big) \Big\} \; . \tag{C.4}$$

In the traditional literature, it is customary to use as phase-space variables the kinetic energy $\varepsilon(v_\parallel, v_\perp) = m(v_\parallel^2 + v_\perp^2)/2$ and the magnetic moment $\mu(v_\perp, \mathbf{x}, t) = m v_\perp^2/(2B)$. Making the change of variables from $(t, \mathbf{x}, v_\parallel, v_\perp)$ to $(t, \mathbf{x}, \varepsilon, \mu)$, Eqs.(C.1-C.4) become

$$\frac{\partial \bar{f}(\varepsilon, \mu, \mathbf{x}, t)}{\partial t} \; + \; \dot{\mathbf{x}} \cdot \frac{\partial \bar{f}}{\partial \mathbf{x}} \; + \; \dot{\varepsilon} \, \frac{\partial \bar{f}}{\partial \varepsilon} \; + \; \dot{\mu} \, \frac{\partial \bar{f}}{\partial \mu} \; = \; 0 \; , \tag{C.5}$$

with

$$\dot{\mathbf{x}} \; = \; \mathbf{u}_E \; + \; \Big[\frac{2}{m} (\varepsilon - \mu B) \Big]^{1/2} \mathbf{b} \; + \; \mu B \nabla \times \Big(\frac{\mathbf{b}}{m \Omega_c} \Big) \; + \; (2\varepsilon - 3\mu B) \frac{\mathbf{b} \times \boldsymbol{\kappa}}{m \Omega_c} \; , \tag{C.6}$$

$$\dot{\varepsilon} \; = \; \Big[\frac{2}{m} (\varepsilon - \mu B) \Big]^{1/2} e \; \mathbf{b} \cdot \mathbf{E} \; - \; \mu B \; \nabla \cdot \mathbf{u}_E \; + \; (2\varepsilon - 3\mu B) \; \mathbf{u}_E \cdot \boldsymbol{\kappa} \tag{C.7}$$

and

$$\dot{\mu} \; = \; \frac{\mu}{m \Omega_c} \Big[\tau \; \mathbf{b} \cdot (e\mathbf{E} - \mu \nabla B) \; + \; 2(\varepsilon - \mu B) \; \mathbf{B} \cdot \nabla \Big(\frac{\tau}{B} \Big) \Big] \; , \tag{C.8}$$

where $\tau = \mathbf{b}\cdot(\nabla\times\mathbf{b})$. The advantage of using the $(\varepsilon,\,\mu)$ variables is clear if one does not go beyond the zero-Larmor-radius approximation where $\dot{\mu}$ vanishes. The first-order FLR contribution to $\dot{\mu}$ (C.8) would also vanish if the magnetic twist function $\tau(\mathbf{x},t)$ (or equivalently the parallel current) were equal to zero, but this is seldom of interest. The $(\varepsilon,\,\mu)$ variables become less attractive if we consider the FLR equations with non-zero parallel current. This situation worsens when we include the fast inhomogeneous flows, in which case the form of the FLR drift-kinetic equation in terms of $(\varepsilon,\,\mu)$ becomes much more unwieldly than our form (32-35) in terms of $(v_\parallel, v_\perp)$, besides adding the complication of a B-dependent Jacobian. It is possible to obtain a more accurate adiabatic invariant than μ, that is conserved to any desired order in the FLR expansion, thus making its associated drift-kinetic coefficient function vanish. However, the expression of this adiabatic invariant and the corresponding Jacobian in our required first order already turn out to be quite unappealing. This was the reason for our favoring the $(v_\parallel, v_\perp)$ variables in this work.

Introducing the magnetic gradient drift velocity

$$\mathbf{V}_{\nabla\mathbf{B}}(\varepsilon,\mu,\mathbf{x},t) \;=\; \frac{1}{m\Omega_c}\Big\{\mathbf{b}\times[\mu\nabla B + 2(\varepsilon-\mu B)\boldsymbol{\kappa}] + \mu\tau\mathbf{B}\Big\} \tag{C.9}$$

and using Faraday's law and some vector identities, the coefficient functions (C.6-C.8) can be cast in a form equivalent to the one given in Ref.6 in its first order:

$$\dot{\mathbf{x}} \;=\; \mathbf{u}_E \;+\; \Big[\frac{2}{m}(\varepsilon-\mu B)\Big]^{1/2}\mathbf{b} \;+\; \mathbf{V}_{\nabla\mathbf{B}}\;, \tag{C.10}$$

$$\dot{\varepsilon} \;=\; e\,\mathbf{E}\cdot\dot{\mathbf{x}} \;+\; \mu\,\frac{\partial B}{\partial t} \tag{C.11}$$

and

$$\dot{\mu} \;=\; \mu\Big\{\frac{\tau}{B}\,\mathbf{b}\cdot\mathbf{E} \;+\; 2(\varepsilon-\mu B)^{1/2}\,\mathbf{b}\cdot\nabla\Big[(\varepsilon-\mu B)^{1/2}\frac{\tau}{m\Omega_c}\Big]\Big\}\;, \tag{C.12}$$

where the gradient operator $\nabla = \partial/\partial\mathbf{x}$ now acts at constant ε and μ.

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