

PSFC/JA-08-14

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December 2007

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Submitted for publication in the Plasma Phys. Control. Fusion (December 2007)

# Electrostatic turbulence in tokamaks on transport time scales

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## Abstract

Simulating electrostatic turbulence in tokamaks on transport time scales requires retaining and evolving a complete turbulence modified neoclassical transport description, including all the axisymmetric neoclassical and zonal flow radial electric field effects, as well as the turbulent transport normally associated with drift instabilities. Neoclassical electric field effects are particularly difficult to retain since they require evaluating the ion distribution function to higher order in gyroradius over background scale length than standard gyrokinetic treatments. To avoid extending gyrokinetics an alternate hybrid gyrokinetic-fluid treatment is formulated that employs moments of the full Fokker-Planck equation to remove the need for a higher order gyrokinetic distribution function. The resulting hybrid description is able to model all electrostatic turbulence effects with wavelengths much longer than an electron Larmor radius such as the ion temperature gradient (ITG) and trapped electron modes (TEM).

PACS numbers: 52.25.Dg, 52.25.Fi, 52.30.Ex, 52.30.Gz, 52.35.Ra

## 1. Introduction

Moment equations are often employed in strongly magnetized plasmas to obtain expressions for the heat fluxes and viscosity requiring less accurate, lower order evaluations of the distribution functions [see, for example, references 1-9]. The heat fluxes and viscosities found in this manner can then be employed in the conservation of charge, number, momentum, and energy equations along with a lower order kinetic equation to obtain a hybrid fluid-kinetic closure. The earliest example of a hybrid closure in a strongly magnetized plasma is due to Kulsrud [10] who employed a large flow ordering with a simple drift kinetic equation for the parallel dynamics.

A hybrid fluid-kinetic description of strongly magnetized plasma, if correctly formulated and properly applied, has advantages over a purely kinetic description since it requires solving a much less accurate kinetic equation for the distribution function and the results are often easier to interpret. Closed hybrid descriptions can consist of and evolve charge, density, momentum, and energy conservation equations, and a kinetic equation for each species, along with Maxwell's equations. To obtain such a closed hybrid description the expressions for the species heat fluxes and viscous stress tensors, as well as the interspecies energy and momentum exchanges, should be written in terms of velocity moments of and require the least possible information about the distribution function, which in turn is found by solving the simplest possible gyrokinetic and/or drift kinetic equations.

Here we formulate a hybrid system that requires solving both an ion gyrokinetic equation and an electron drift kinetic equation. To treat the ions we must also formulate the gyrokinetic extension of some aspects of the recent drift kinetic derivation by Simakov and Catto [9] of the ion viscosity and heat flux for arbitrary collisionality plasmas. Their results are obtained by expanding in ion Larmor radius  $\rho_i$  over background perpendicular scale length  $L_\perp$  and assuming the lowest order distribution function is Maxwellian. We generalize the moment procedure they employed to insure that turbulent gyrokinetic effects with  $k_\perp \rho_i \sim 1$  are retained as well as the long wavelength features that require higher order terms in  $\rho_i/L_\perp$ , where  $k_\perp$  is the characteristic perpendicular wave number of the turbulence with  $k_\perp L_\perp \gg 1$  allowed. Gyrokinetics has not yet been formulated to high enough order in the gyroradius of

background magnetic field and plasma scale lengths to recover these results directly. Indeed, here we show that there is no need to do so because the gyrokinetic extension of the moment procedure used in Ref. [9] retains the desired long wavelength effects in general, and the axisymmetric zonal flow and radial electric field behavior in particular. Moreover, in our formulation a lowest order full  $f$  gyrokinetic equation provides the  $k_{\perp}\rho_i \sim 1$  effects needed to describe turbulent phenomena. The moment equations are required to extend simulations to transport time scales on which long wavelength phenomena with  $k_{\perp}L_{\perp} \sim 1$  must be retained to describe the interaction between the turbulence generated zonal flow and neoclassical modifications to the axisymmetric radial electric field. To retain transport phenomena including low mode number effects as generally as possible we allow perpendicular ( $L_{\perp}$ ) and parallel ( $L_{\parallel} \sim k_{\parallel}^{-1}$ ) scale lengths to be comparable, while for the turbulent fluctuations including the zonal flow our orderings allow  $L_{\parallel} \gtrsim L_{\perp} \gg k_{\perp}^{-1} \sim \rho_i$ . These orderings permit the lowest order distribution  $f_0$  to be Maxwellian as in local core gyrokinetic codes, but by use of a moment approach allow us to keep both transport and turbulent modifications to much higher order than standard gyrokinetics with only the  $\delta f$  of local gyrokinetic codes. In summary, we retain long wavelength phenomena to higher order than standard gyrokinetics, by extending the moment procedure of drift kinetics [1-9] to gyrokinetics to retain and evolve these phenomena as well as turbulence in a fully self-consistent way.

To keep the formulation for describing turbulence on transport time scales as simple as possible we only consider electrostatic turbulence in a tokamak and assume  $k_{\perp}\rho_e \ll 1$  for the electrons. As a result, we employ  $\vec{E} = -\nabla\Phi$  and the axisymmetric, steady state magnetic field form  $\vec{B} = I(\psi)\nabla\zeta + \nabla\zeta \times \nabla\psi = B\vec{b}$ , with  $\psi$  the poloidal flux function ( $\nabla\psi = R\vec{b}_p$ ),  $\zeta$  the toroidal angle ( $\nabla\zeta = 1/R$ ),  $I = RB_t$ ,  $B = |\vec{B}|$ ,  $R$  the major radius, and  $B_t$  and  $B_p$  the toroidal and poloidal components of the magnetic field. These assumptions seem straightforward, but tedious, to remove. Making them simplifies the presentation and avoids obscuring key points.

Sections 2 and 3 present the fluid and kinetic descriptions we employ, while section 4 evaluates the heat flows and viscosities in detail. We close with a brief summary of the equations that must be solved in a hybrid gyrokinetic - fluid description.

## 2. Fluid conservation equations

We consider a quasi-neutral plasma with only a single singly charged ion species of plasma density  $n$ , with  $e$  the magnitude of the charge on an electron. We denote the ion mean velocity by  $\vec{V}$ , the current density by  $\vec{J} = en(\vec{V} - \vec{V}_e)$ , so that the electron mean flow is  $\vec{V}_e = \vec{V} - \vec{J}/en$ , and the ion and electron temperatures and pressures by  $T_i$  and  $T_e$ , and  $p_i = nT_i$  and  $p_e = nT_e$ . In electron momentum conservation we ignore inertial terms and gyroviscosity, as well as perpendicular viscosity, but retain parallel viscosity. We employ the sum of the ion and electron momentum equations rather than the ion momentum equation. As a result, the electron momentum conservation equation (or Ohm's law) and the conservation forms of the number, charge, total momentum, and species energy equations are as follows:

$$\frac{\partial n}{\partial t} + \nabla \cdot (n\vec{V}) = S_n, \quad (1)$$

$$\nabla \cdot \vec{J} = 0, \quad (2)$$

$$\frac{\partial(Mn\vec{V})}{\partial t} + \nabla \cdot [(p_i + p_e)\vec{I} + \vec{\pi}_i + \vec{\pi}_e] = \frac{1}{c}\vec{J} \times \vec{B} + \vec{S}_{mi} + \vec{S}_{me}, \quad (3)$$

$$en(-\nabla\Phi + \frac{1}{c}\vec{V}_e \times \vec{B}) + \nabla \cdot (p_e\vec{I} + \vec{\pi}_e) = \vec{F} + \vec{S}_{me}, \quad (4)$$

$$\frac{3}{2}\frac{\partial p_i}{\partial t} + \nabla \cdot (\bar{q}_i + \frac{5}{2}p_i\vec{V}) = -en\vec{V} \cdot \nabla\Phi + W + S_{pi}, \quad (5)$$

and

$$\frac{3}{2}\frac{\partial p_e}{\partial t} + \nabla \cdot (\bar{q}_e + \frac{5}{2}p_e\vec{V}_e) = en\vec{V} \cdot \nabla\Phi - W + S_{pe}, \quad (6)$$

where the ion viscosity  $\vec{\pi}_i = M \int d^3v f_i (\vec{v}\vec{v} - \vec{I}v^2/3) = M \int d^3v f_i \vec{v}\vec{v} - p_i\vec{I} = \vec{\pi}_{li} + \vec{\pi}_{gi} + \vec{\pi}_{\perp i}$  implicitly retains the Reynolds stress terms, the electron viscosity is simply  $\vec{\pi}_e = \vec{\pi}_{le} = m \int d^3v f_e (\vec{v}\vec{v} - \vec{I}v^2/3) = m \int d^3v f_e \vec{v}\vec{v} - p_e\vec{I} = (p_{\perp e} - p_{\parallel e})(\vec{I}/3 - \vec{b}\vec{b})$  with the overbar on  $f_e$  denoting a gyroaverage holding  $\vec{r}$  fixed, the ion and electron heat fluxes are defined as  $\bar{q}_i = \int d^3v f_i \vec{v}(Mv^2 - 5T_i)/2$  and  $\bar{q}_e = \int d^3v f_e \vec{v}(mv^2 - 5T_e)/2$ , and  $\vec{I}$  is the unit dyad. Arbitrary collisionality expressions for the parallel, gyro, and perpendicular viscosities, along with the ion and electron heat fluxes,  $\bar{q}_i$  and  $\bar{q}_e$ , will be given in

subsequent sections. Momentum exchange of the electrons with the ions is given by the friction term [3,4,6]

$$\vec{F} = mnv_e \vec{V} - 2\gamma_e mn \int d^3v f_e \vec{v} / v^3 = mnv_e (\vec{V} - \vec{V}_{\perp e} + 3\vec{q}_{de}/5p_e) - 2\gamma_e mn \vec{b} \int d^3v \vec{f}_e v_{\parallel} / v^3, \quad (7)$$

that is obtained from  $\vec{F} = \int d^3v m \vec{v} C_{ei}$ , with  $C_{ei}$  the electron-ion collision operator,  $m$  the electron mass,  $\nu_e = 4(2\pi)^{1/2} n e^4 \ell_n \Lambda / 3m^{1/2} T_e^{3/2}$  the electron collision frequency,  $\gamma_e = [3(2\pi)^{1/2} v_e / 4n] (T_e/m)^{3/2}$ ,  $\vec{q}_{de} = -(5p_e / 2m\Omega_e) \vec{b} \times \nabla T_e$ , and  $\Omega_e = eB/mc$ . The final form of  $\vec{F}$  is obtained by assuming drift kinetic electrons, with the last term of equation (7) including implicit parallel electron and ion flow terms in  $\vec{f}_e$  (the ion flow enters because of the electron-ion collision operator). Energy exchange of the ions with the electrons is given by [3,4,6]

$$W = 3mnv_e (T_e - T_i) / M + (mnv_e \vec{V} - \vec{F}) \cdot \vec{V}, \quad (8)$$

where  $W = \int d^3v (Mv^2/2) C_{ie}$  with  $C_{ie}$  the ion-electron collision operator and  $M$  the ion mass. The particle, ion and electron momentum, and ion and electron energy sources,  $S_n$ ,  $\vec{S}_{mi}$ ,  $\vec{S}_{me}$ ,  $S_{pi}$ , and  $S_{pe}$ , are allowed to be the same order as time derivatives in the corresponding equations.

In the electrostatic limit considered here Eqs. (1) - (6) are ten equations evolving the ten unknowns  $n$ ,  $p_i$  or  $T_i$ ,  $p_e$  or  $T_e$ ,  $\vec{V}$ ,  $\vec{V}_e = \vec{V} - \vec{J}/en$ , and  $\Phi$ . Equations (1), (5) and (6) advance  $n$ ,  $p_i$  or  $T_i$ , and  $p_e$  or  $T_e$ . The momentum and charge conservation equations give the electrostatic potential and the ion and electron flows (or equivalently the ion flow and the current density), with the parallel component of (4) the parallel Ohm's law.

Flux surface averaging  $\nabla \cdot \vec{J} = 0$  and integrating once in  $\psi$  gives the global ambipolarity constraint  $\langle \vec{J} \cdot \nabla \psi \rangle_{\psi} = 0$ , where the flux surface average is defined by

$$\langle \dots \rangle_{\psi} = [\oint d\zeta \oint d\vartheta (\dots) / \vec{B} \cdot \nabla \vartheta] / [2\pi \oint d\vartheta / \vec{B} \cdot \nabla \vartheta] \quad (9)$$

with  $\vartheta$  the poloidal angle. This result is consistent with Ampere's law of course, but our electrostatic assumption means that magnetic fluctuations are ignored so Ampere's law is not employed and is assumed to be satisfied by the steady state fields to whatever order is required.

To see that the radial particle transport reduces to the correct intrinsically ambipolar form [11,12] for an axisymmetric  $\vec{B}$  in the source and sink free limit, we form the  $R^2\nabla\zeta$  component of equation (4) to obtain

$$n\vec{V}_e \cdot \nabla\psi = (c/e)R^2\nabla\zeta \cdot (en\nabla\Phi - \nabla p_e + \vec{F} + \vec{S}_{me}) - (c/e)\nabla \cdot (R^2\vec{\pi}_e \cdot \nabla\zeta) . \quad (10)$$

The lowest order result of flux surface averaging equation (10) in the absence of a significant toroidal momentum source or sink is

$$\langle n\vec{V}_e \cdot \nabla\psi \rangle_\psi = (c/e)\langle en\partial\Phi/\partial\zeta + R^2\nabla\zeta \cdot \vec{F} \rangle_\psi, \quad (11)$$

where  $\langle \partial p_e / \partial \zeta \rangle_\psi = 0$  is employed and any gyroviscous contribution from  $\langle \nabla \cdot (R^2\vec{\pi}_e \cdot \nabla\zeta) \rangle_\psi$  is an order  $k_\perp \rho_e^2 / L_\perp \ll 1$  correction (since any departure from a Maxwellian  $f_{0e}$  will depend on  $\partial f_{0e} / \partial \psi \propto 1/L_\perp$ ) and assumed negligible. Then,  $\langle \vec{J} \cdot \nabla\psi \rangle_\psi = 0$  gives  $\langle n\vec{V} \cdot \nabla\psi \rangle_\psi = \langle n\vec{V}_e \cdot \nabla\psi \rangle_\psi$  and intrinsic ambipolarity [11,12] is maintained even in the presence of fluctuations (as long as any magnetic perturbations remain sufficiently small) provided the electron kinetic equation is solved in a way that insures  $\langle n\vec{V}_e \cdot \nabla\psi \rangle_\psi$  is independent of  $\partial\Phi/\partial\psi$  (in the neoclassical limit  $\partial\Phi/\partial\psi$  terms from  $V_\parallel$  in the electron-ion collision operator and  $\partial f_{0e}/\partial\psi|_E$  exactly cancel). Notice that we do not improperly determine the axisymmetric radial electric field by adjusting it until  $\langle n\vec{V} \cdot \nabla\psi \rangle_\psi \approx 0$  or  $\langle \vec{J} \cdot \nabla\psi \rangle_\psi = 0$ , as is sometimes mistakenly done in tokamaks.

Then the axisymmetric portion of the radial electric field is determined by conservation of toroidal angular momentum as required [3,5,6,13-15]. It is obtained from the  $R^2\nabla\zeta$  component of equation (3), using  $\vec{B} \times \nabla\psi = \vec{I}\bar{B} - B^2R^2\nabla\zeta$ , to find

$$\frac{\partial(MR^2n\vec{V} \cdot \nabla\zeta)}{\partial t} + \nabla \cdot \{R^2[(p_i + p_e)\vec{I} + \vec{\pi}_i + \vec{\pi}_e] \cdot \nabla\zeta\} = c^{-1}\vec{J} \cdot \nabla\psi + R^2(\vec{S}_{mi} + \vec{S}_{me}) \cdot \nabla\zeta . \quad (12)$$

In the steady state in the absence of sources or sinks, the flux surface average of equation (12) followed by a  $\psi$  integration yields the lowest order constraint  $\langle R^2\nabla\zeta \cdot \vec{\pi}_i \cdot \nabla\psi \rangle_\psi \approx 0$ , that determines the neoclassical  $\partial\Phi/\partial\psi$  (or equivalently  $\vec{V} \cdot \nabla\zeta$ ).

In an axisymmetric steady state (ss) lowest order flow is in a flux surface and is given by

$$n\vec{V}|_{ss} = K(\psi)\vec{B} - cn\left[\frac{\partial\Phi}{\partial\psi} + \frac{1}{en}\frac{\partial p_i}{\partial\psi}\right]R^2\nabla\zeta , \quad (13)$$

with  $n\vec{V}|_{ss} \cdot \nabla\psi = 0$ . Consequently, only the flux function  $K$  and the potential need be determined in the ion flow. The  $K$  is found from the lowest order version of parallel momentum conservation, namely  $\langle \vec{B} \cdot (\nabla \cdot \vec{\pi}_{||i}) \rangle_{\psi} \approx 0$ , in the Pfirsch-Schlüter regime, and directly from the ion distribution function in the banana regime [3,4,6]. As already noted, the potential is found from equation (12) [3,5,6,13-17], but it is important to realize that the poloidally varying corrections to equation (13) must be retained in the gyroviscosity when evaluating its contribution to  $\langle R^2 \nabla \zeta \cdot \vec{\pi}_i \cdot \nabla \psi \rangle_{\psi} \approx 0$  in complete generality [16].

Our description contains a steady state current consistent with our axisymmetric  $\vec{B}$ . From Ampere's law and  $\vec{J} \times \vec{B} \approx c \nabla p$ , with  $p = p_i + p_e$  a lowest order flux function, this lowest order current is in a flux surface and given by

$$\vec{J}|_{ss} = -(c/4\pi)(dI/d\psi)\vec{B} - cR^2(dp/d\psi)\nabla\zeta . \quad (14)$$

### 3. Drift and gyrokinetic equations

The electron drift and ion gyrokinetic equations need not be solved in a conservative form since only moments of the distribution functions are needed to provide closure in the conservation of number, momentum, energy and charge forms of our hybrid formulation. The plasma density, mean ion and electron velocities, species pressures, and electrostatic potential are evaluated from fluid equations in conservative form so no extraneous number, charge, momentum and energy sources or sinks will inadvertently be included.

To keep the  $\vec{E} \times \vec{B}$  drift velocity  $\vec{v}_E$  of order  $\delta$  times the ion thermal speed  $v_i$  we must assume  $e\Phi/T \ll 1$  for the fluctuating gyrokinetic part of the electrostatic potential, while allowing  $e\Phi/T \sim 1$  in the long wavelength drift kinetic portion. The restriction on the  $\vec{E} \times \vec{B}$  drift leads us to order

$$e\Phi/T \sim 1/k_{\perp}L_{\perp} , \quad (15)$$

with  $k_{\perp} \sim 1/\rho_i$  for gyrokinetic fluctuations and  $k_{\perp} \sim 1/L_{\perp}$  in the long wavelength limit.

Before considering gyrokinetic effects ( $k_{\perp}\rho_i \sim 1$  and  $k_{\perp}L_{\perp} \gg 1$ ) in more detail it is convenient to consider the drift kinetic limit ( $k_{\perp}\rho \ll 1$  and  $k_{\perp}L_{\perp} \sim 1$ ) to see what



effects are needed to retain turbulent and neoclassical effects on transport time scales of many ion-ion collision times.

### 3.1. Drift kinetic equation

Drift kinetic descriptions in strongly magnetized plasmas assume the species gyrofrequency is much larger than any frequency of interest ( $\Omega \gg \partial/\partial t$ ) and that the species gyroradius  $\rho$  is smaller than any other length scale of interest ( $\delta \equiv \rho/L_\perp \ll 1$  and  $k_\perp \rho \ll 1$ , where  $L_\perp$  and  $k_\perp^{-1}$  are the shortest perpendicular scale length and wavenumbers of interest). The lowest order form of the drift kinetic equation [5,18,19] in kinetic energy  $\varepsilon = v^2/2$  and magnetic moment  $\mu_0 = v_\perp^2/2B$  variables is

$$\frac{\partial \bar{f}}{\partial t} + [(v_\parallel + v_p)\bar{b} + \bar{v}_d] \cdot \nabla_{\varepsilon, \mu_0} \bar{f} - \frac{e}{M} \nabla \Phi \cdot [(v_\parallel + v_p)\bar{b} + \bar{v}_d] \frac{\partial \bar{f}}{\partial \varepsilon} = \bar{C}\{\bar{f}\}, \quad (16)$$

where the spatial gradient is performed holding  $\varepsilon$  and  $\mu_0$  fixed, the gyroaverage is performed holding  $\varepsilon$ ,  $\mu_0$ , and  $\bar{r}$  fixed,  $\bar{C}$  is the gyroaveraged collision operator,  $v_\parallel = (2\varepsilon - 2\mu_0 B)^{1/2}$  is the parallel velocity and  $v_p = (\mu_0 B / \Omega) \bar{b} \cdot \nabla \times \bar{b}$  its correction, and  $\bar{v}_d = \bar{v}_E + \bar{v}_M$  is the total drift velocity with  $\bar{v}_E = c \bar{B} \times \nabla \Phi / B^2$  the electric or  $\bar{E} \times \bar{B}$  drift and  $\bar{v}_M = \Omega^{-1} \bar{b} \times (\mu_0 \nabla B + v_\parallel^2 \bar{\kappa})$  the magnetic drift, with  $\bar{\kappa} = \bar{b} \cdot \nabla \bar{b}$  the curvature. This form of the drift kinetic equation retains only order  $\delta$  corrections and assumes that  $\mu_0$  variation of  $\bar{f}$  is weak compared to the  $\varepsilon$  variation, that is, it assumes  $\partial \bar{f} / \partial \varepsilon \gg B^{-1} \partial \bar{f} / \partial \mu_0$  as is required to allow  $\bar{f} = f_0$  be Maxwellian (or isotropic) to lowest order. The alternate form of the drift kinetic equation obtained by changing variables from  $\varepsilon$  to  $v_\parallel$  may also be employed.

For arbitrary collisionality the closure requirements are simplified by choosing the lowest order solution of the drift kinetic equation  $\bar{f}_0$  as an axisymmetric Maxwellian. To see that this is consistent we linearize the drift kinetic equation about  $\bar{f}_0$  and then allow the perpendicular and parallel scale lengths to be comparable ( $k_\perp^{-1} \sim L_\perp \sim L_\parallel$ ). Then the parallel streaming and parallel electric field terms must dominate in the Vlasov operator because the transit time is much faster than any temporal evolution scale for the lowest order distribution  $\bar{f}_0$ . Moreover, if we adopt the arbitrary collisionality ordering on the mean free path  $\lambda$  by taking  $\Delta = \lambda / L_\parallel \sim 1$ , then the collision operator must be retained

to the same order (we always assume  $L_{\perp} \sim L_{\parallel}$  so our results are valid for arbitrary aspect ratio and safety factor). In this case, we see that  $\bar{f}_0$  must be Maxwellian since to lowest order it must satisfy

$$v_{\parallel}[\bar{\mathbf{b}} \cdot \nabla|_{\varepsilon, \mu_0} \bar{f}_0 - \frac{e}{M} \bar{\mathbf{b}} \cdot \nabla \Phi \frac{\partial \bar{f}_0}{\partial \varepsilon}] = \bar{C}\{\bar{f}_0\}. \quad (17)$$

The lowest order solution  $\bar{f}_0$  to the left side of equation (17) can only be a function of total energy  $E_* = v^2/2 + e\Phi/M$  and magnetic flux  $\psi$  [this is rigorously true on irrational flux surfaces, and true by continuity on rational ones], since  $\nabla|_{E_*, \mu_0} = \nabla|_{\varepsilon, \mu_0} - (e/M)\nabla\Phi\partial/\partial\varepsilon$  and  $\bar{\mathbf{b}} \cdot \nabla\psi = 0$ . Moreover, to make the collision operator vanish the right side can only be satisfied by a Maxwellian (the right side does not permit  $\bar{f}_0$  to depend on  $\mu_0$  to lowest order). Consequently, to make the collision and lowest order Vlasov operators vanish,  $\bar{f}_0 = f_0(\psi, E_*, t)$  must be the Maxwellian:

$$f_0 = \eta(M/2\pi T)^{3/2} \exp(-ME_*/T) = n(M/2\pi T)^{3/2} \exp(-Mv^2/2T), \quad (18)$$

with  $\eta = \eta(\psi, t) = n \exp(e\Phi/T)$ ,  $T = T(\psi, t)$ ,  $n = n(\vec{r}, t)$ , and  $\Phi = \Phi(\vec{r}, t)$ . Notice that according to the preceding drift kinetic argument the density and potential need not be flux functions to lowest order since  $\bar{\mathbf{b}} \cdot \nabla\eta = 0$ . For the hybrid gyrokinetic-fluid treatment herein we allow our lowest order Maxwellian to have  $T = T(\vec{r}, t)$ , as well as  $n = n(\vec{r}, t)$  and  $\Phi = \Phi(\vec{r}, t)$  since this form is allowed in the short mean free path limit. By taking  $f$  as Maxwellian to lowest order we can simplify the expressions for the heat fluxes and viscosities to see that only order  $\delta$  and  $\delta^2$  corrections, respectively, to  $f_0$  contribute.

If we write  $\bar{f} = f_0 + \delta\bar{f}(\vec{r}, \varepsilon, \mu_0, t) + \delta\tilde{f}(\vec{r}, \varepsilon, \mu_0, \varphi, t)$  and retain all order  $\delta$  terms as in the Appendix, we would obtain a drift kinetic equation [18,19] for  $\delta\bar{f}$  containing turbulent behavior and neoclassical particle and heat flow as well as the zonal flow generated by the turbulence [20,21], and a gyrophase,  $\varphi$ , dependent contribution  $\delta\tilde{f}$  odd in  $\vec{v}$  giving  $\int d^3v \delta\tilde{f}_i(\vec{v}\vec{v} - \vec{I}v^2/3) = 0$ . As a result, the leading order corrections  $\delta\bar{f}$  and  $\delta\tilde{f}$  to the Maxwellian must give  $\nabla\zeta \cdot \vec{\pi}_i \cdot \nabla\psi = 0$  since  $\int d^3v \delta\tilde{f}_i(\vec{v}\vec{v} - \vec{I}v^2/3)$  is diagonal. Consequently, the radial electric field can only be evaluated by retaining order  $\delta^2$  or higher Larmor radius effects. A higher order version of equation (16) is available [19],

but even it's solution is not good enough to directly evaluate the perpendicular collisional viscosity to the order required to determine the neoclassical electric field.

For the hybrid model outlined herein the drift kinetic equation (16) is only used for the electrons, for which  $C \rightarrow C_{ee} + C_{ei} \equiv C_e$  and  $C_{ei}\{f_e\} = L_{ei}\{f_e - m\vec{V} \cdot \vec{v} f_{0e}/T_e\}$  with  $L_{ei}$  the Lorentz collision operator for electron-ion collisions and  $C_{ee}\{\bar{f}_e\} = C_{ee}\{\bar{f}_e - mV_{||}v_{||}f_{0e}/T_e\}$  for the linearized gyroaveraged electron-electron collision operator. Rewriting the electron kinetic equation as an equation for  $\bar{H}_e \equiv \bar{f}_e - mV_{||}v_{||}f_{0e}/T_e$  or  $\bar{h}_e \equiv \bar{f}_e + (IV_{||}/\Omega_e)\partial f_{0e}/\partial\psi$  insures intrinsic ambipolarity by cancelling the  $\partial\Phi/\partial\psi$  terms in the neoclassical limit since  $\vec{v}_{de} \cdot \nabla\psi \partial f_{0e}/\partial\psi = v_{||}\bar{n} \cdot \nabla[(IV_{||}/\Omega_e)\partial f_{0e}/\partial\psi]$ . The plateau approximation can only be used for  $C_e\{\bar{H}_e\} \rightarrow -v\bar{H}_e$ .

### 3.2. Gyrokinetic equation

Various choices for the gyrokinetic variables forms for the gyrokinetic equation are possible [22-25]. For our purposes the gyrokinetic equation that is the natural extension of the drift kinetic equation (16) is employed [26]

$$\frac{\partial\langle f \rangle}{\partial t} + [\bar{v}_{||}(\vec{R})\bar{b}(\vec{R}) + \bar{v}_d(\vec{R})] \cdot \nabla_{\vec{R}} \langle f \rangle - \frac{e}{M} \nabla_{\vec{R}} \langle \Phi(\vec{r}, t) \rangle \cdot [\bar{v}_{||}(\vec{R})\bar{b}(\vec{R}) + \bar{v}_d(\vec{R})] \frac{\partial\langle f \rangle}{\partial E} = \langle C\{f\} \rangle, \quad (19)$$

where, unlike the drift kinetic gyroaverage, the gyrokinetic gyroaverage denoted by  $\langle \dots \rangle$  is performed holding fixed the gyrokinetic variables

$$E = v^2/2 + (e/M)[\Phi(\vec{r}, t) - \langle \Phi(\vec{r}, t) \rangle] + (c/B)\partial\tilde{\Phi}/\partial t,$$

$$\vec{R} = \vec{r} + \Omega^{-1}\vec{v} \times \vec{b} + \Omega^{-2}\bar{b}[(v_{||}\vec{b} + \frac{1}{8}\vec{v}_{\perp})\vec{v} \times \vec{b} + \vec{v} \times \vec{b}(v_{||}\vec{b} + \frac{1}{8}\vec{v}_{\perp})]: (\nabla\vec{b} \times \vec{b}) \quad (20)$$

$$+ \Omega^{-1}[(v_{||}\vec{b} + \frac{1}{4}\vec{v}_{\perp})\vec{v} \times \vec{b} + \vec{v} \times \vec{b}(v_{||}\vec{b} + \frac{1}{4}\vec{v}_{\perp})] \times \nabla(\vec{b}/\Omega) + (v_{||}/\Omega^2)\vec{v}_{\perp} \cdot \nabla\vec{b} - (c/B\Omega)\nabla_{\vec{R}}\tilde{\Phi} \times \vec{b},$$

and

$$\mu = \frac{v_{\perp}^2}{2B} + \frac{e}{MB}[\Phi(\vec{r}, t) - \langle \Phi(\vec{r}, t) \rangle] - \frac{1}{B}\vec{v} \cdot \vec{v}_M - \frac{v_{||}}{4\Omega B}[\vec{v}_{\perp}\vec{v} \times \vec{b} + \vec{v} \times \vec{b}\vec{v}_{\perp}]: \nabla\vec{b} - \frac{v_{||}v_{\perp}^2}{2\Omega B}\vec{b} \cdot \nabla \times \vec{b}.$$

In the preceding,  $\bar{v}_{||}(\vec{R}) = [2\varepsilon - 2\mu B(\vec{R})]^{1/2} = \langle v_{||} \rangle + v_p(\vec{R})$ ,  $\bar{v}_d(\vec{R}) = \bar{v}_E(\vec{R}) + \bar{v}_M(\vec{R})$ , and  $\nabla_{\vec{R}} = \partial/\partial\vec{R}$ , where  $\langle v_{||} \rangle = \langle [2\varepsilon - 2\mu B(\vec{r})]^{1/2} \rangle$ ,  $\bar{v}_E(\vec{R}) = c\vec{B}(\vec{R}) \times \nabla_{\vec{R}} \langle \Phi(\vec{r}, t) \rangle / B^2(\vec{R})$ ,  $\bar{v}_M(\vec{R}) = \Omega^{-1}(\vec{R})\bar{b}(\vec{R}) \times [\mu \nabla_{\vec{R}} B(\vec{R}) + v_p^2(\vec{R})\vec{k}(\vec{R})]$ ,  $v_p(\vec{R}) = (\mu B/\Omega)\bar{b}(\vec{R}) \cdot \nabla_{\vec{R}} \times \vec{b}(\vec{R})$ , and  $\tilde{\Phi} \equiv \tilde{\Phi}(\vec{R}, E, \mu, t) \equiv \int^{\Phi} d\phi [\Phi(\vec{r}, t) - \langle \Phi(\vec{r}, t) \rangle]$  with this indefinite integral performed holding  $\vec{R}$ ,  $E$ , and  $\mu$  fixed such that  $\langle \tilde{\Phi} \rangle = 0$ . Our vector conventions are  $\vec{a}\vec{c} : \vec{M} = \vec{c} \cdot \vec{M} \cdot \vec{a}$  and  $\vec{a}\vec{c} \times \vec{M} =$

$\vec{a} \times (\vec{c} \cdot \vec{M}) = -\vec{c} \cdot \vec{M} \times \vec{a}$ . Use of the higher order gyrokinetic variables [23,26] given in equations (20) is essential when we evaluate the ion viscosity. The alternate form of the gyrokinetic equation obtained by changing variables from  $E$  to  $\bar{v}_{\parallel}$  may also be employed.

The hybrid model outlined herein employs the gyrokinetic equation for the ions. Like the drift kinetic equation (16), our gyrokinetic equation (19) is derived by neglecting some order  $(\rho/L_{\perp})^2$  corrections even though it allows  $k_{\perp}\rho \sim 1$ , where  $L_{\perp}$  is the local unperturbed density, temperature, potential, or magnetic field scale length. However, our gyrokinetic variables (20) allow us to retain all order  $(\rho/L_{\perp})^2$  gyrophase dependent corrections in the long wavelength limit as shown in Appendix D of [26] by Taylor expanding  $\langle f \rangle$  about the Maxwellian. This feature is essential to allow us to retain neoclassical electric field effects when we evaluate the ion viscosity. In addition, we employ quasi-neutrality to equate the electron and ion densities (so only the plasma density enters) rather than attempting to determine the electrostatic potential by using the gyrokinetic version of quasi-neutrality [24,25] that requires the distribution functions to vary high order in the  $\rho/L_{\perp}$  expansion. In the hybrid description described herein the electrostatic potential is determined by employing conservation equations (1)-(6).

Although we employ a full  $f$  gyrokinetic equation, it is convenient to consider the lowest order distribution Maxwellian to make estimates and to order the electrostatic potential according to (15). As a result, to estimate the characteristic departure of the full gyrokinetic  $\langle f \rangle$  from Maxwellian we use a Maxwell-Boltzmann or adiabatic response [ $n \propto \exp(-e\Phi/T)$ ] for gyrokinetic fluctuations by taking

$$(\langle f \rangle - f_0)/f_0 \sim e\Phi/T \sim \rho_i/L_{\perp} = \delta, \quad (21)$$

since  $k_{\perp} \sim 1/\rho_i$ . In the long wavelength limit  $\langle f \rangle \sim f_0$ . More precisely, when  $k_{\perp}\rho_i \ll 1$ , Eqs. (16) and (19) are identical since  $\langle f \rangle \rightarrow \bar{f} + \Omega^{-1} \vec{v} \times \vec{b} \cdot [\nabla \bar{f} - (e/M) \nabla \Phi \partial \bar{f} / \partial \epsilon] + \dots$ , but of course for  $k_{\perp}\rho_i \sim 1$  they differ and  $(\langle f \rangle - \bar{f})/\bar{f} \sim \delta$ . We stress here that  $\langle f \rangle$  contains all order  $k_{\perp}\rho_i \sim 1$  modifications, but only order  $\rho_i/L_{\perp}$  corrections as in standard drift kinetics - it is missing some order  $(\rho_i/L_{\perp})^2$  gyrophase independent corrections.

A direct evaluation of  $\bar{q}_i = \int d^3v f_i \vec{v} (Mv^2 - 5T_i)/2$  using the local gyrokinetic result of equation (A8) from the Appendix, with  $f_{0i}$  the local Maxwellian and

$$\partial f_{0i}/\partial \psi|_E = f_{0i}[p_i^{-1} \partial p_i / \partial \psi + (e/T_i) \partial \Phi / \partial \psi + (Mv^2/2T_i - 5/2) T_i^{-1} \partial T_i / \partial \psi] ,$$

gives

$$\bar{q}_i \rightarrow \int d^3v \langle h \rangle \bar{v} (Mv^2 - 5T_i)/2 + (5p_i/2M\Omega_i) (\partial T_i / \partial \psi) (\bar{b} \times \nabla \psi - \bar{I}b) ,$$

where  $\langle h \rangle / f_{0i} \sim \rho_i / L_\perp$  and  $\bar{b} \times \nabla \psi - \bar{I}b = -R^2 B \nabla \zeta$ . In the drift kinetic limit  $\langle h \rangle \rightarrow \bar{h} + \Omega^{-1} \bar{v} \times \bar{b} \cdot [\nabla \bar{h} - (e/M) \nabla \Phi \partial \bar{h} / \partial \epsilon] + \dots$  this gives the incomplete result

$$\begin{aligned} \bar{q}_i \rightarrow & \bar{b} \frac{1}{2} \int d^3v \bar{h} v_{\parallel} (Mv^2 - 5T_i) + \frac{5p_i}{2M\Omega_i} \frac{\partial T_i}{\partial \psi} (\bar{b} \times \nabla \psi - \bar{I}b) \\ & + \frac{1}{4\Omega_i} \int d^3v v_{\perp}^2 (Mv^2 - 5T_i) \bar{b} \times [\nabla \bar{h} - (e/M) \nabla \Phi \partial \bar{h} / \partial \epsilon] \end{aligned}$$

that is seen to be missing the perpendicular collisional heat flux (note that the third term on the right is  $\bar{h}/f_{0i} \sim \delta$  smaller than the second). Therefore, to retain neoclassical (and classical) heat flow, as well as the  $k_\perp \rho_i \sim 1$  turbulent heat flow, without the need for higher order gyrokinetics, we evaluate  $\bar{q}_i$  by an alternate, moment approach that generalizes the one used in drift kinetics. Similar, but more complicated difficulties arise for the ion viscosity. These issues are addressed in the next section, while the Appendix presents a simple derivation of the intrinsically ambipolar form of the ion kinetic equation (A4) with zonal flow retained.

#### 4. Heat flows and viscosities

All gyrokinetic, drift kinetic, and moment descriptions in strongly magnetized plasmas take advantage of the species gyrofrequency being much larger than any frequency of interest. This assumption allows various moments of the full Fokker-Planck equation for the ions,

$$\frac{df_i}{dt} \equiv \frac{\partial f_i}{\partial t} + \nabla \cdot (\bar{v} f_i) - \nabla_v \cdot \left[ \frac{e}{M} (\nabla \Phi - \frac{1}{c} \bar{v} \times \bar{B}) f_i \right] = C_i \{f_i\}, \quad (22)$$

to be used to obtain expressions for fluxes in which a less accurate or lower order expression for the ion distribution function  $f_i$  can be employed, where  $C_i \{f_i\}$  is the ion-ion plus ion-electron collision operator [2-9].

##### 4.1. Ion heat flow

For ion heat flow it is convenient to form the  $M\bar{v}v^2/2$  moment of equation (22) and then subtract from it  $T_i$  times the  $\bar{v}$  moment to obtain [9]

$$\begin{aligned} \Omega_i \vec{b} \times \vec{q}_i + \nabla \cdot \left[ \int d^3v f_i (Mv^2 - 5T_i) \vec{v} \vec{v} / 2 \right] + (5/2M)(p_i \vec{I} + \vec{\pi}_i) \cdot \nabla T_i + (e/M) \vec{\pi}_i \cdot \nabla \Phi = \\ (1/2) \int d^3v (Mv^2 - 5T_i) \vec{v} C_i \{f_i\}, \end{aligned} \quad (23)$$

where time derivatives are neglected as small. To lowest order the ion-electron collision term in (23) may be neglected and  $f_i$  thought of as the Maxwellian  $f_{0i}$ . Also, we need only retain the linearized ion-ion collision operator by making the replacement  $C_i \{f_i\} \rightarrow C_{ii}^{\ell} \{f_i - f_{0i}\} = C_{ii}^{\ell} \{f_i\} - C_{ii}^{\ell} \{f_{0i}\} = C_{ii}^{\ell} \{f_i\}$  since  $C_{ii}^{\ell} \{f_{0i}\} = 0$ .

The lowest order terms in (23) are  $\Omega_i \vec{b} \times \vec{q}_i$ , the diamagnetic term  $p_i \nabla T_i$ , and the second term since  $\nabla \cdot [M \int d^3v (Mv^2 - 5T_i) \vec{v} \vec{v} f_i] \sim p_i \nabla T_i$ . Moreover, even though  $T_i$  is a lowest order flux function according to drift kinetics or gyrokinetics, if we imagine  $T_i = \bar{T}_i(\psi) + \tilde{T}_i(\psi, \vartheta, \zeta)$ , then both terms in  $\nabla T_i = \nabla \psi \partial \bar{T}_i / \partial \psi + \nabla \tilde{T}_i$  are comparable since we order  $\tilde{T}_i / \bar{T}_i \sim \rho_i / L_{\perp}$  for gyrokinetic ( $k_{\perp} \rho_i \sim 1$ ) temperature fluctuations, with a tilde on a spatial quantity indicating an order  $\rho_i / L_{\perp}$  correction to the lowest order flux function denoted by an overbar. Consequently,  $\vec{p}_i \nabla \tilde{T}_i \sim (\rho_i / L_{\perp}) \bar{p}_i \nabla \bar{T}_i$ .

To consistently retain neoclassical and classical heat transport all terms in equation (23) must be retained to next order, with  $v_i / \Omega_i \sim \rho_i / L_{\perp}$  to retain collisional heat transport. In the term containing viscosity only the parallel and gyroviscosity (including Reynolds stress) need be retained. Collisional perpendicular viscosity is smaller by  $v_i / \Omega_i$  and need not be kept here, however, it will be needed in momentum conservation. Ion viscosity will be discussed in detail in subsection D of this section.

To verify that the ion heat flux in the neoclassical and classical limits [11-12] remains independent of the radial electric field, we first consider the collisional term in (23), and employ for  $\tilde{f}_i$  the lowest order gyrophase dependent drift kinetic ( $k_{\perp} \rho_i \ll 1$ ) solution

$$\tilde{f}_i = f_{0i} [(M/T_i) \vec{v} \cdot \vec{V}_{Li} - (1/\Omega_i T_i) (5/2 - x_i^2) \vec{v} \cdot \vec{b} \times \nabla T_i], \quad (24)$$

with

$$\vec{V}_{Li} = (c/B) \vec{b} \times \nabla \Phi + (1/M n_i \Omega_i) \vec{b} \times \nabla p_i \quad (25)$$

to lowest order and  $x_i = (Mv^2/2T_i)^{1/2}$ . To obtain Eqs. (24) and (25), we allow  $f_{0i}$  to depend on  $\vec{r}$  rather than  $\psi$ , and then solve  $\Omega_i \partial \tilde{f}_i / \partial \varphi = \vec{v}_{\perp} \cdot [\nabla f_{0i} + (ef_{0i}/T_i) \nabla \Phi]$ . We notice

that the  $\bar{v}f_{0i}$  dependence of  $\tilde{f}_i$  in  $C_{ii}^\ell\{\tilde{f}_i\}$  does not matter since  $C_{ii}^\ell\{\bar{v}f_{0i}\}=0$  gives  $C_{ii}^\ell\{\tilde{f}_i\}=C_{ii}^\ell\{\tilde{f}_{iT}\}$ , where

$$\tilde{f}_{iT} = (f_{0i}x_i^2/\Omega_i T_i)\bar{v} \cdot \bar{b} \times \nabla T_i \quad (26)$$

is the only gyrophase dependent term that contributes. Next, the self-adjointness of  $C_{ii}^\ell\{f_i\}$ ,  $\int d^3v g C_{ii}^\ell\{h\} = \int d^3v (h/f_{0i}) C_{ii}^\ell\{g f_{0i}\}$ , followed by the use of [9,15]

$$C_{ii}^\ell\{Mv^2 \bar{v} f_{0i}\} = 2v_i T_i Q(x_i) \bar{v} f_{0i} \quad (27)$$

gives

$$\begin{aligned} \int d^3v (Mv^2 - 5T_i) \bar{v} C_{ii}^\ell\{f_i\} &\rightarrow \int d^3v (f_i/f_{0i}) C_{ii}^\ell\{Mv^2 \bar{v} f_{0i}\} \\ &= 2v_i T_i \int d^3v f_i Q(x_i) \bar{v}. \end{aligned} \quad (28)$$

Here  $Q(x) = -[3(2\pi)^{1/2}/x]\{[1 - (5/2x^2)]E(x) + (5/2x)E'(x)\}$ ,  $E(x) = 2\pi^{-1/2} \int_0^x dt \exp(-t^2)$  the error function, and  $E'(x) = dE(x)/dx$ . Notice that in the drift kinetic limit ( $f_i \rightarrow \bar{f}_i + \tilde{f}_i$ ), only the  $\tilde{f}_{iT}$  portion of  $\tilde{f}_i$  and the gyro-independent departure of  $\bar{f}_i$  from the Maxwellian  $f_{0i}$  contribute to (28) as desired since  $\int d^3v f_{0i} Q(x_i) \bar{v} \bar{v} \equiv 0$ .

Returning to equation (23) and solving for  $\bar{q}_{i\perp}$  with the replacement  $f_i \rightarrow \langle f_i \rangle$  to the requisite order in the collisional term gives the result

$$\begin{aligned} \bar{q}_i = \frac{1}{\Omega_i} \bar{b} \times \{ \nabla \cdot [ \int d^3v \langle f_i \rangle (Mv^2 - 5T_i) \bar{v} \bar{v} / 2 ] + \frac{5p_i}{2M} \nabla T_i + \bar{\pi}_i \cdot ( \frac{e}{M} \nabla \Phi + \frac{5}{2M} \nabla T_i ) \} \\ + (v_i/\Omega_i) T_i \int d^3v \langle f_i \rangle Q(x_i) \bar{v} \times \bar{b} + \bar{b} \int d^3v \langle f_i \rangle v_{\parallel} (Mv^2 - 5T_i) / 2, \end{aligned} \quad (29)$$

where  $v_i = (4\pi^{1/2} n_i e^4 \ell n \Lambda) / (3M^{1/2} T_i^{3/2})$  and  $\Omega_i = eB/Mc$ . The last term in (29), the gyrokinetic parallel heat flux, is the same order as the first and second terms on the right (all of order  $p_i v_i \delta$ , since the first and last terms vanish for  $f_{0i}$ ). The third and fourth terms are smaller by  $\delta$  and the same size as the fluctuating contributions to the second term and the fast time average of the first term. The next to the last term ( $\sim p_i v_i \delta v_i / \Omega_i$ ) contains the classical collisional heat flux contribution. In some situations (i.e., short mean free path and long wavelengths) it may be sufficient to only retain the second, fourth, and fifth terms on the right side of (29).

We stress again that a direct evaluation of  $\bar{q}_i \rightarrow \int d^3v \langle f_i \rangle \bar{v} (Mv^2 - 5T_i) / 2$  will not recover the collisional terms in (29) because the gyrokinetic equation (19) is not

sufficiently accurate. However, in the drift kinetic limit we can see the collisional term in (29) does not vanish. In particular  $\int d^3v \langle f_i \rangle Q(x_i) \bar{\mathbf{b}} \times \bar{\mathbf{v}} \rightarrow \int d^3v \tilde{f}_{iT} Q(x_i) \bar{\mathbf{b}} \times \bar{\mathbf{v}}$  since only the  $\tilde{f}_{iT}$  term from  $\langle f_i \rangle \rightarrow \bar{f}_i + \Omega_i^{-1} \bar{\mathbf{v}} \times \bar{\mathbf{b}} \cdot [\nabla f_{0i} + (ef_{0i}/T_i) \nabla \Phi]$  survives to order  $\delta$  because  $\int d^3v f_{0i} Q(x_i) \bar{\mathbf{v}} \equiv 0$ . More generally, the velocity space integrals involving  $\langle f_i \rangle$  must be performed holding  $\bar{\mathbf{r}}$  fixed since  $\langle f_i \rangle$  depends on the gyrokinetic variables (20). Fortunately, in equation (29) it is sufficient to approximate them by the first order forms  $\bar{\mathbf{R}} = \bar{\mathbf{r}} + \Omega^{-1} \bar{\mathbf{v}} \times \bar{\mathbf{b}}$  and  $E = v^2/2 + (e/M)(\Phi - \langle \Phi \rangle)$ , and use  $\mu_0 = v_{\perp}^2/2B$  (since  $\langle f_i \rangle$  is Maxwellian to lowest order).

In the form (29) the neoclassical contributions are implicit, but they can be made explicit by using the  $\nabla \zeta$  component of (23) with  $\bar{\mathbf{b}} \times \nabla \zeta = \nabla \psi / R^2 B$  to obtain the result

$$\begin{aligned} \bar{\mathbf{q}}_i \cdot \nabla \psi = & (Mc/2e) \nabla \cdot [R^2 \int d^3v \langle f_i \rangle (Mv^2 - 5T_i) \bar{\mathbf{v}} \bar{\mathbf{v}} \cdot \nabla \zeta] + (5cR^2 p_i / 2e) \nabla \zeta \cdot \nabla T_i + \\ & cR^2 \nabla \zeta \cdot \bar{\boldsymbol{\pi}}_i \cdot [\nabla \Phi + (5/2e) \nabla T_i] - (v_i / \Omega_i) T_i B R^2 \int d^3v \langle f_i \rangle Q(x_i) \bar{\mathbf{v}} \cdot \nabla \zeta. \end{aligned} \quad (30)$$

The dominant turbulent heat flux in (30) comes from the second term on the right, while the classical (from  $\bar{\mathbf{v}}_{\perp} \cdot \nabla \zeta$ ) and neoclassical (from  $v_{\parallel} \bar{\mathbf{b}} \cdot \nabla \zeta$ ) contributions are given by the collisional term. Flux surface averaging (30) and retaining only these terms gives

$$\langle \bar{\mathbf{q}}_i \cdot \nabla \psi \rangle_{\psi} \approx (5c/2e) \langle p_i \partial T_i / \partial \zeta \rangle_{\psi} - \langle (v_i / \Omega_i) T_i B R^2 \int d^3v \langle f_i \rangle Q(x_i) \bar{\mathbf{v}} \cdot \nabla \zeta \rangle_{\psi}. \quad (31)$$

The two terms on the right compare as:  $\langle \bar{p}_i \partial \tilde{T}_i / \partial \zeta \rangle_{\psi} / p_i T_i \lesssim k_{\perp} L_{\perp} \delta^2 \sim \delta$  versus  $\delta v_i R / v_i \sim \delta$ , with a phase factor possibly reducing the turbulent heat flux.

#### 4.2. Electron heat flow

The heat flow associated with the electrons is simplified because we assume  $k_{\perp} \rho_e \ll 1$  and can therefore adopt a drift kinetic procedure. Otherwise, the same basic procedure is used for the electrons as is used for the ions. We start with [9]

$$\begin{aligned} \Omega_e \bar{\mathbf{q}}_e \times \bar{\mathbf{b}} + \nabla \cdot [ \int d^3v f_e (mv^2 - 5T_e) \bar{\mathbf{v}} \bar{\mathbf{v}} / 2 ] + (5/2m) (p_e \bar{\mathbf{I}} + \bar{\boldsymbol{\pi}}_e) \cdot \nabla T_e - (e/m) \bar{\boldsymbol{\pi}}_e \cdot \nabla \Phi = \\ (1/2) \int d^3v (mv^2 - 5T_e) \bar{\mathbf{v}} C_e \{ f_e \}, \end{aligned} \quad (32)$$

where  $C_e$  includes electron-electron and electron-ion collisions. Only the diagonal part of the stress tensor is required so  $\bar{\boldsymbol{\pi}}_e \rightarrow m \int d^3v \bar{f}_e (\bar{\mathbf{v}} \bar{\mathbf{v}} - \bar{\mathbf{I}} v^2 / 3) = (p_{\parallel e} - p_{\perp e}) (\bar{\mathbf{b}} \bar{\mathbf{b}} - \bar{\mathbf{I}} / 3)$  with  $p_{\parallel e} = m \int d^3v \bar{f}_e v_{\parallel}^2$  and  $p_{\perp e} = mB \int d^3v \bar{f}_e \mu$ . Similarly, we may employ  $f_e \rightarrow \bar{f}_e$  in second term in equation (32), which is then the same order as the collisional term for arbitrary



mean free path. Then, solving for the perpendicular electron heat flux and adding in the parallel component gives

$$\begin{aligned} \bar{q}_e = & -\frac{1}{\Omega_e} \bar{b} \times \{ \nabla \cdot [ \frac{1}{2} \int d^3 v \bar{f}_e (mv^2 - 5T_e) \bar{v} \bar{v} ] + \frac{5p_e}{2m} \nabla T_e - \bar{\pi}_e \cdot ( \frac{e}{m} \nabla \Phi - \frac{5}{2m} \nabla T_e ) \} \\ & + (1/2\Omega_e) \int d^3 v (mv^2 - 5T_e) \bar{b} \times \bar{v} C_e \{ f_e \} + \bar{b} \int d^3 v f_e v_{\parallel} (mv^2 - 5T_e) / 2, \end{aligned} \quad (33)$$

where  $\bar{v} \bar{v} \equiv (v_{\perp}^2/2)(\bar{I} - \bar{b} \bar{b}) + v_{\parallel}^2 \bar{b} \bar{b}$ . To make the neoclassical terms explicit we use the  $\nabla \zeta$  component of (32) to find

$$\begin{aligned} \bar{q}_e \cdot \nabla \psi = & - (mc/2e) \nabla \cdot [ R^2 \int d^3 v \bar{f}_e (mv^2 - 5T_e) \bar{v} \bar{v} \cdot \nabla \zeta ] - (5c/2e) (p_e \partial T_e / \partial \zeta) \\ & + cR^2 \nabla \zeta \cdot \bar{\pi}_e [ \nabla \Phi - (5/2e) \nabla T_e ] + (mcR^2/2e) \int d^3 v (mv^2 - 5T_e) \bar{v} \cdot \nabla \zeta C_e \{ f_e \}. \end{aligned} \quad (34)$$

Flux surface averaging in the drift kinetic limit removes the first term on the right, the second term due to turbulence is expected to dominate, the third or viscous term is an order smaller than the second term, and the last term contains neoclassical ( $v_{\parallel} \bar{b} \cdot \nabla \zeta$ ) as well as classical ( $\bar{v}_{\perp} \cdot \nabla \zeta$ ) contributions which tend to be small.

To make the collision terms more explicit we keep only the linearized collision operators by writing  $C_e \{ f_e \} = C_{ee} \{ f_e \} + C_{ei} \{ f_e \} \rightarrow C_{ee}^{\ell} \{ f_e \} + C_{ei}^{\ell} \{ f_e \}$ , with  $f_e$  equal to the Maxwellian  $f_{0e}$  to lowest order. As before, for like collisions we must be careful to extract terms that give no contribution to  $C_{ee}^{\ell}$  so we employ  $f_e = \bar{f}_e + \tilde{f}_e$  with

$$\tilde{f}_e = f_{0e} [ (m/T_e) \bar{v} \cdot \bar{V}_{\perp e} + (1/\Omega_e T_e) (5/2 - x_e^2) \bar{v} \cdot \bar{b} \times \nabla T_e ], \quad (35)$$

where  $\bar{V}_{\perp e} = (c/B) \bar{b} \times \nabla \Phi - (1/mn_e \Omega_e) \bar{b} \times \nabla p_e$ . Then all that survives is

$$\int d^3 v (mv^2 - 5T_e) \bar{v} C_{ee} \{ f_e \} \rightarrow 2^{1/2} v_e T_e \int d^3 v (\bar{f}_e + \tilde{f}_{eT}) Q(x_e) \bar{v}, \quad (36)$$

with

$$\tilde{f}_{eT} = - (f_{0e} x_e^2 / \Omega_e T_e) \bar{v} \cdot \bar{b} \times \nabla T_e, \quad (37)$$

and  $x_e = (mv^2/2T_e)^{1/2}$ . The extra  $\sqrt{2}$  in (36) arises because of the differing numerical factors in the definitions of  $v_i$  and  $v_e$ . For electron-ion collisions

$$C_{ei}^{\ell} \{ f_e \} = L_{ei} \{ f_e \} + (2\gamma_e mn/T_e v^3) f_{0e} \bar{v} \cdot \bar{V} = L_{ei} \{ f_e - m \bar{V} \cdot \bar{v} f_{0e} / T_e \}, \quad (38)$$

with

$$L_{ei} \{ f_e \} = \gamma_e n \nabla_{v \cdot} \cdot ( \nabla_{v \cdot} \nabla_{v \cdot} v \cdot \nabla_{v \cdot} f_e ). \quad (39)$$

The Lorentz operator has the property

$$L_{ei} \{ (mv^2 - 5T_e) \bar{v} f_{0e} \} = 2\gamma_e p_e (5 - 2x_e^2) \bar{v} f_{0e} / v^3. \quad (40)$$

Using Eqs. (35) and (40), the self-adjointness of  $L_e\{f_e\}$  is used to find

$$\int d^3v(mv^2 - 5T_e)\bar{v}_\perp L_{ei}\{\bar{f}_e\} = (3p_e v_e)\bar{V}_{\perp e} + (13p_e v_e/2m\Omega_e)\bar{b} \times \nabla T_e. \quad (41)$$

Carrying out the complete evaluation of the collisional terms as in [9] gives the final expression for the electron heat flux to be

$$\bar{q}_e = \frac{-1}{\Omega_e} \bar{b} \times \left\{ \nabla \cdot \left[ \frac{1}{2} \int d^3v \bar{f}_e (mv^2 - 5T_e) \bar{v} \bar{v} \right] + \frac{5p_e}{2m} \nabla T_e - \bar{\pi}_e \cdot \left( \frac{e}{m} \nabla \Phi - \frac{5}{2m} \nabla T_e \right) \right\} \quad (42)$$

$$- [(13/4) + \sqrt{2}] (p_e v_e / m \Omega_e^2) \nabla_\perp T_e - (3p_e v_e / 2\Omega_e) \bar{b} \times (\bar{V} - \bar{V}_e) + \bar{b} \int d^3v \bar{f}_e v_\parallel (mv^2 - 5T_e) / 2.$$

### 4.3. Electron viscosity

Next, we consider viscosities. The electrons are drift kinetic with  $k_\perp \rho_e \ll 1$  so we need only retain the parallel viscosity

$$\bar{\pi}_e = m \int d^3v \bar{f}_e (\bar{v} \bar{v} - \bar{I} v^2 / 3) = (p_{\parallel e} - p_{\perp e}) (\bar{b} \bar{b} - \bar{I} / 3) \quad (43)$$

with

$$p_{\parallel e} = m \int d^3v \bar{f}_e v_\parallel^2 \quad \text{and} \quad p_{\perp e} = mB \int d^3v \bar{f}_e \mu. \quad (44)$$

### 4.4. Ion viscosity

The viscosity of the ions is more difficult to evaluate than the electron viscosity since we allow  $k_\perp \rho_i \sim 1$ . As for the ion heat flux, a direct evaluation of the ion viscosity  $\bar{\pi}_i = M \int d^3v f_i [(\bar{v} \bar{v} - \bar{I} v^2 / 3) + (\bar{v} \bar{v} - \bar{v} \bar{v})]$  using  $M \int d^3v \langle f_i \rangle (\bar{v} \bar{v} - \bar{I} v^2 / 3)$  does not obtain the collisional perpendicular viscosity needed to properly evaluate the neoclassical radial flux of toroidal angular momentum, where we define  $\bar{v} \bar{v} = (v_\perp^2 / 2) (\bar{I} - \bar{b} \bar{b}) + v_\parallel^2 \bar{b} \bar{b}$ . As a result, moments of equation (22) are again required to avoid having to solve a more accurate gyrokinetic equation [5-9]. Forming the  $M \bar{v} \bar{v}$  moment of equation (22) and following the procedure in Simakov and Catto [9] requires solving an equation for  $\bar{\pi}_i$  of the usual form [5-9]:  $\Omega_i (\bar{\pi}_i \times \bar{b} - \bar{b} \times \bar{\pi}_i) = \bar{K}_i = \bar{K}_{gi} + \bar{K}_{\perp i}$ . The solution gives the following expressions for the ion gyroviscosity  $\bar{\pi}_{gi}$  and perpendicular viscosity  $\bar{\pi}_{\perp i}$ :

$$\bar{\pi}_{gi} = (4\Omega_i)^{-1} [\bar{b} \times \bar{K}_{gi} (\bar{I} + 3\bar{b} \bar{b}) - (\bar{I} + 3\bar{b} \bar{b}) \cdot \bar{K}_{gi} \times \bar{b}] \quad (45)$$

and

$$\bar{\pi}_{\perp i} = (4\Omega_i)^{-1} [\bar{b} \times \bar{K}_{\perp i} (\bar{I} + 3\bar{b} \bar{b}) - (\bar{I} + 3\bar{b} \bar{b}) \cdot \bar{K}_{\perp i} \times \bar{b}], \quad (46)$$

with

$$\bar{K}_{gi} = \nabla \cdot (M \int d^3v \bar{v} \bar{v} \bar{v} f_i) + (en \nabla \Phi + F_\parallel \bar{b}) \bar{V} + \bar{V} (en \nabla \Phi + F_\parallel \bar{b}), \quad (47)$$

and

$$\vec{K}_{\perp i} = -M \int d^3 v \vec{v} \vec{v} C_{ii} \{f_i\} . \quad (48)$$

Diagonal contributions to  $\vec{K}_{gi}$  and  $\vec{K}_{\perp i}$  are omitted since they do not contribute to  $\vec{\pi}_{gi}$  and  $\vec{\pi}_{\perp i}$  and only  $\vec{b} \cdot \vec{F} = F_{\parallel}$  is retained since  $\vec{F}_{\perp}$  is proportional to the electron Larmor radius and therefore negligible. Diagonal viscous terms are not contained in  $\vec{\pi}_{gi} + \vec{\pi}_{\perp i} \equiv M \int d^3 v f_i (\vec{v} \vec{v} - \overline{\vec{v} \vec{v}})$  since  $\vec{b} \cdot [\int d^3 v f_i (\vec{v} \vec{v} - \overline{\vec{v} \vec{v}})] \cdot \vec{b} = 0 = \vec{I} : [\int d^3 v f_i (\vec{v} \vec{v} - \overline{\vec{v} \vec{v}})]$ . They are all contained in the parallel viscosity defined by

$$\vec{\pi}_{\parallel i} \equiv M \int d^3 v f_i (\overline{\vec{v} \vec{v}} - \vec{I} v^2 / 3) \approx M \int d^3 v \langle f_i \rangle (\overline{\vec{v} \vec{v}} - \vec{I} v^2 / 3) = (p_{\parallel i} - p_{\perp i}) (\overline{\vec{b} \vec{b}} - \vec{I} / 3) , \quad (49)$$

with

$$p_{\parallel i} = M \int d^3 v \langle f_i \rangle v_{\parallel}^2 \quad \text{and} \quad p_{\perp i} = M B \int d^3 v \langle f_i \rangle \mu . \quad (50)$$

The order of  $\vec{\pi}_{\perp i}$  is  $p_i \delta v_i / \Omega_i$  (it is order  $\delta$  smaller in the drift kinetic limit). The first term in the  $\vec{K}_{gi}$  contribution to  $\vec{\pi}_{gi}$  can be as large as order  $p_i \delta$  (order  $p_i \delta^2$  in the drift kinetic limit), while the other terms are order  $p_i \delta^2$ . We remark that the off diagonal component  $R^2 \nabla \zeta \cdot \vec{\pi}_{gi} \cdot \nabla \psi$  that depends on the axisymmetric radial electric field can be rewritten as

$$\begin{aligned} (2e/Mc) R^2 \nabla \zeta \cdot \vec{\pi}_{gi} \cdot \nabla \psi = & -(R^2 - I^2 B^{-2}) [\nabla \cdot (M \int d^3 v v^2 \vec{v} f_i / 2) + \vec{V} \cdot (e n \nabla \Phi + \vec{b} F_{\parallel})] \\ & - (3I^2 B^{-2} - R^2) \{ \vec{b} \cdot [\nabla \cdot (M \int d^3 v \vec{v} \vec{v} \vec{v} f_i / 2)] \cdot \vec{b} + V_{\parallel} (e n \vec{b} \cdot \nabla \Phi + F_{\parallel}) \} \\ & + \nabla \cdot [M \int d^3 v (R^2 \vec{v} \cdot \nabla \zeta)^2 \vec{v} f_i] + 2R^2 \vec{V} \cdot \nabla \zeta (e n \partial \Phi / \partial \zeta + I F_{\parallel} / B) . \end{aligned} \quad (51)$$

In this form the Reynold's stress terms  $(n \partial \Phi / \partial \zeta) \vec{V} \cdot \nabla \zeta$  and  $n \vec{V} \cdot \nabla \Phi$  are expected to be the lowest order gyroviscous terms containing the radial electric field. They enter as order  $\delta^2$  corrections to the ion pressure, and are smaller still by the phase factor relating the fluctuating density and potential. They can compete with the  $\partial \Phi / \partial \psi$  term in  $R^2 \nabla \zeta \cdot \vec{\pi}_{\perp i} \cdot \nabla \psi$  as noted in [9]. Great care must be taken when dealing with the ion viscosity to insure that the axisymmetric radial electric field is properly evaluated. The other flows in (51) [namely  $\int d^3 v v^2 \vec{v} f_i$ ,  $\int d^3 v (\vec{v} \cdot \nabla \zeta)^2 \vec{v} f_i$ , and  $\int d^3 v v_{\parallel}^2 \vec{v} f_i$ ] are various contributions to heat flow that do not depend explicitly on  $\partial \Phi / \partial \psi$ .

We next consider  $\vec{\pi}_{\perp i}$ . In equation (48), it is clear that in the contribution to  $C_{ii} \{f_i\}$  from the linearized collision operator  $C_{ii}^{\ell} \{f_i\}$ , only order  $\delta^2$  gyrophase

dependent portions of  $f_i$  will contribute since the order  $\delta$  corrections are odd in  $\vec{v}$  and the gyrophase independent portions of  $f_i$  are irrelevant since they give diagonal contributions. To retain order  $\delta^2$  effects in the nonlinear collision operator contribution  $C_{ii}^{n\ell}\{f_i-f_{0i},f_i-f_{0i}\} \approx C_{ii}^{n\ell}\{\langle f_i \rangle - f_{i0}, \langle f_i \rangle - f_{i0}\}$ , only order  $\delta$  contributions to  $f_i$  are needed (the density associated with  $f_{i0}$  in  $C_{ii}^{n\ell}$  may be taken to be the full density since the error will be negligible). Therefore, we may write

$$\vec{K}_{\perp i} = -M \int d^3 v \vec{v} [C_{ii}^{\ell}\{\langle f_i \rangle\} + C_{ii}^{n\ell}\{\langle f_i \rangle - f_{i0}, \langle f_i \rangle - f_{i0}\}] \quad (52)$$

where the replacement  $f_i \rightarrow \langle f_i \rangle$  is used in  $C_{ii}^{\ell}$  with the understanding that our gyrokinetic variables (20) must be retained in it to order  $\delta^2$  to capture all gyrophase dependent corrections in the drift kinetic limit along with arbitrary  $k_{\perp} \rho_i \sim 1$  (gyrophase independent terms give only inconsequential diagonal contributions).

In the drift kinetic limit, use of equation (52) for  $\vec{K}_{\perp i}$  in equation (46) yields the result of [9], while in the gyrokinetic limit it retains additional physics due to  $k_{\perp} \rho_i \sim 1$ . To see that (52) properly recovers the drift kinetic limit  $\langle f_i \rangle$  must be Taylor expanded in  $C_{ii}^{\ell}$  to order  $\delta^2$  using the variables of (20) as in [26]. The result is the gyrophase dependent term found in [19] and given by

$$\begin{aligned} \tilde{f}_i = & \vec{v} \cdot (\vec{g}_i - \vec{v}_d \frac{1}{B} \frac{\partial \bar{f}_i}{\partial \mu}) - \frac{v_{\parallel}}{4B\Omega_i} \frac{\partial \bar{f}_i}{\partial \mu} (\vec{v}_{\perp} \vec{v} \times \vec{b} + \vec{v} \times \vec{b} \vec{v}_{\perp}) : \nabla \vec{b} \\ & + \frac{1}{8\Omega_i} \vec{v} \vec{v} : [\vec{b} \times (\vec{h} + \vec{h}^T) \cdot (\vec{I} + 3\vec{b}\vec{b}) - (\vec{I} + 3\vec{b}\vec{b}) \cdot (\vec{h} + \vec{h}^T) \times \vec{b}], \end{aligned} \quad (53a)$$

where  $\vec{h} = \nabla \vec{g}_i - (e/M)(\nabla \Phi) \partial \vec{g}_i / \partial \epsilon$  and  $\vec{g}_i = \Omega_i^{-1} \vec{b} \times [\nabla_{\epsilon, \mu} \langle f_i \rangle - (e/M) \nabla \Phi \partial \langle f_i \rangle / \partial \epsilon]$ , superscript T denotes transpose, and we use  $f_{0i}$  in the second order terms containing  $\nabla f_i$  or  $\partial f_i / \partial E$ . The only term missing in equation (53a) is an additive second order ( $\delta^2 \sim \delta v_i / \Omega_i$ ) classical collisional contribution from

$$\tilde{f}_i^c = \Omega_i^{-1} \int^{\Phi} d\phi [\langle C_{ii}^{\ell}\{f_i\} \rangle - C_{ii}^{\ell}\{f_i\}] \approx \Omega_i^{-1} C_{ii}^{\ell}\{\vec{v} \cdot \vec{g}_i \times \vec{b}\} \quad (53b)$$

that must be retained as in Ref. [9] when forming the gyroviscosity to retain all classical collisional effects.

The linear contribution to (52) can be rewritten more conveniently using the self-adjointness of the linearized collision operator and the following result from [9]:

$$C_{ii}^{\ell}\{\bar{v}\bar{v}f_{0i}\} = v_i F(x_i) f_{0i} [\bar{v}\bar{v} - (v^2/3)\bar{I}] \quad (54)$$

with  $F(x) = -[9(2\pi)^{1/2}/2x^3]\{[1 - (3/2x^2)]E(x) + (3/2x)E'(x)\}$ . Then, (52) becomes

$$\bar{K}_{\perp i} = -M \int d^3\bar{v}\bar{v}\bar{v}\{v_i \langle f_i \rangle F(x_i) + C_{ii}^{n\ell}\{\langle f_i \rangle - f_{i0}, \langle f_i \rangle - f_{i0}\}\}, \quad (55)$$

where we have dropped a diagonal term since it cannot contribute to  $\bar{\pi}_{\perp i}$ .

Next, we consider  $\bar{\pi}_{gi}$ . We need only rewrite the second term in (47) by making the replacement  $f_i \rightarrow \langle f_i \rangle$  to obtain the desired form

$$\bar{K}_{gi} = \nabla \cdot (M \int d^3\bar{v}\bar{v}\bar{v}\langle f_i \rangle) + (e n \nabla \Phi + \bar{b} F_{\parallel}) \bar{V} + \bar{V} (e n \nabla \Phi + \bar{b} F_{\parallel}), \quad (56)$$

which will recover the proper drift kinetic result for  $\bar{\pi}_{gi}$  to order  $\delta^3$  for  $k_{\perp} \rho_i \ll 1$ . Consequently, using (56) in (45) gives the desired gyrokinetic expression for  $\bar{\pi}_{gi}$ . Retaining all classical heat flux corrections to the gyroviscosity [16] requires the replacement  $\langle f_i \rangle \rightarrow \langle f_i \rangle + \tilde{f}_i^c$  in equation (53b).

In the linear terms  $\int d^3\bar{v}\bar{v}\bar{v}v_i \langle f_i \rangle F(x_i)$  in  $\bar{K}_{\perp i}$  and  $\int d^3\bar{v}\bar{v}\bar{v}\langle f_i \rangle$  in  $\bar{K}_{gi}$  when the integrals are performed holding  $\bar{r}$  fixed with  $\langle f_i \rangle$  a function of  $\bar{R}$ ,  $E$  and  $\mu$ , the full gyrokinetic change of variables (20) must be employed to carefully relate these gyrokinetic variables to the  $\bar{r}$ ,  $\varepsilon = v^2/2$ ,  $\mu_0 = v_{\perp}^2/2B$ , and  $\varphi$  variables to insure the proper neoclassical electric field is recovered. In the nonlinear  $C_{ii}^{n\ell}$  term in  $\bar{K}_{\perp i}$  the lower order relations  $\bar{R} = \bar{r} + \Omega^{-1}\bar{v} \times \bar{b}$ ,  $E = v^2/2 + (e/M)(\Phi - \langle \Phi \rangle)$  and  $\mu_0 = v_{\perp}^2/2B$  are adequate. However, these lower order relations could be used in  $\int d^3\bar{v}\bar{v}\bar{v}v_i \langle f_i \rangle F(x_i)$  and  $\int d^3\bar{v}\bar{v}\bar{v}\langle f_i \rangle$  if the order  $\delta^2$  gyrophase dependent terms are extracted from  $\langle f_i \rangle$  [recall equation (53)] and evaluated analytically as in [9] by assuming  $k_{\perp} \rho_i \sim 1$  corrections to these terms are unimportant.

## 5. Discussion

The hybrid gyrokinetic - fluid description is now complete. It contains all the drift kinetic features of [9] and insures intrinsic ambipolarity in the neoclassical limit, and yet retains all the gyrokinetic turbulence modifications associated with drift wave turbulence for  $k_{\perp} \rho_e \ll 1$ . The system of equations that must be solved consists of the ion gyrokinetic equation (19), the electron drift kinetic equation (16), and the conservation

equations (1)-(6); along with the momentum and heat exchange terms (7) and (8), the ion and electron heat flows (29) and (42), the electron viscosity (43) and (44), the parallel ion viscosity or pressure anisotropy (49) and (50), the ion gyroviscosity (45) and (56), and the perpendicular ion viscosity (46) and (55). To relate the gyrokinetic variables to the drift kinetic variables  $\bar{\mathbf{r}}$ ,  $\epsilon$ ,  $\mu_0$ ,  $\phi$  in the first term of  $\bar{\mathbf{K}}_{gi}$  in the ion gyroviscosity and  $\bar{\mathbf{K}}_{\perp i}$  in the perpendicular viscosity requires use of equation (20). Elsewhere [including the collisional correction of (53b) in the gyroviscosity], only the lower order relations  $\bar{\mathbf{R}} = \bar{\mathbf{r}} + \Omega^{-1}\bar{\mathbf{v}} \times \bar{\mathbf{b}}$ ,  $E = v^2/2 + (e/M)(\Phi - \langle \Phi \rangle)$  and  $\mu_0 = v_{\perp}^2/2B$  are required. Solutions to the ion gyrokinetic and electron drift kinetic equations are used to evaluate heat flows, viscosities, and collisional exchange, but densities, particle flows, pressures, and the electrostatic potential are evaluated from conservative forms of the moment equations to avoid introducing non-physical sources and sinks.

In some situations certain terms may be neglected. For example, terms associated with classical and neoclassical particle and electron heat transport are expected to be negligible in most cases; and classical ion heat transport will often be small. Also, the perpendicular ion viscosity can be ignored if the axisymmetric radial electric field terms in the Reynold's stress portion of the gyroviscosity dominate (although this seems unlikely to be the case for ion temperature gradient modes - or other electrostatic modes with little radial particle flux - as noted by Simakov and Catto [9]). Even when this is not the case simplification should be possible if only neoclassical (and not classical) effects need be retained and the poloidal magnetic field is small [17,27].

Implementing this hybrid description requires integrating the expertise developed by dealing with both gyrokinetic and extended magnetohydrodynamic codes, but seems the only practical way to evolve turbulence simulations on transport time scales since no extensions of gyrokinetics are needed and only the standard conservation forms of the number, charge, momentum, and energy equations are employed.

### **Acknowledgements**

Research supported by U.S. Department of Energy grants DE-FG02-91ER-54109 at MIT and DE-AC52-06NA-25396 at Los Alamos National Laboratory, and by the Center for

Multiscale Plasma Dynamics at the University of Maryland. We are indebted to Xianzhu Tang (LANL) and Jesus Ramos (MIT) for closure discussions.

### Appendix. Local gyrokinetics

Local gyrokinetic codes such as GS2 [28], PG3EQ [29], and GTC [30] normally assume a stationary lowest order Maxwellian that depends only on  $\varepsilon = v^2/2$  and  $\psi$ . To streamline the derivation of this equation it is convenient to suppress species subscripts and define  $f_* \equiv f_*(\psi_*, E_*)$  with  $E_* = v^2/2 + e\Phi/M$  the total energy and  $\psi_* = \psi - (Mc/e)R^2\nabla\zeta \cdot \vec{v}$  the canonical angular momentum. Using  $df_*/dt = (dE/dt)\partial f_*/\partial E + (d\psi_*/dt)\partial f_*/\partial \psi_* = (e/M)(\partial\Phi/\partial t)(\partial f_*/\partial E) + c(\partial\Phi/\partial\zeta)(\partial f_*/\partial \psi_*)$ , (A1) with  $d/dt$  defined by equation (22), taking  $f_*$  as the Maxwellian  $f_0$  to lowest order, and Taylor expanding  $f_*$  about  $f_0 \equiv f_0(\psi, E)$  as given by equation (18), yields

$$f_* \equiv \eta_*(M/2\pi T_*)^{3/2} \exp(-ME_*/T_*) = f_{0i} - (Mc/e)(R^2\nabla\zeta \cdot \vec{v})\partial f_{0i}/\partial \psi|_{E_*} + \dots, \quad (\text{A2})$$

with  $\eta_* = \eta_*(\psi_*) = n(\vec{r}, t) \exp[e\Phi(\vec{r}, t)/T_*(\psi_*)] = \eta_i(\psi) + \dots$  and  $T_* = T_*(\psi_*) = T_i(\psi) + \dots$

Using  $f_i = f_* + h$  in the ion equation (22) gives  $h$  as satisfying

$$\frac{\partial h}{\partial t} + \vec{v} \cdot \nabla h - \frac{e}{M}(\nabla\Phi - \frac{1}{c}\vec{v} \times \vec{B}) \cdot \nabla_v h \approx C_{ii}^{\ell} \{f_* - f_{0i} + h\} + \frac{ef_{0i}}{T_i} \frac{\partial\Phi}{\partial t} - c \frac{\partial\Phi}{\partial\zeta} \frac{\partial f_{0i}}{\partial \psi} \Big|_{E_*}, \quad (\text{A3})$$

where higher order corrections in  $\rho_i/L_{\perp}$  are neglected in  $f_{0i}$  terms on the right except in the collision operator where the leading order non-vanishing contribution  $f_* - f_{0i} \rightarrow -(M^2cR^2/2eT_i^2)(\partial T_i/\partial \psi)f_{0i}v^2\vec{v} \cdot \nabla\zeta$  must be retained to keep neoclassical and classical heat transport. Changing the left side to the gyrokinetic variables  $\vec{R}$ ,  $E$ ,  $\mu$ , and  $\varphi$  and gyroaveraging holding  $\vec{R}$ ,  $E$ , and  $\mu$  fixed gives the intrinsically ambipolar form

$$\frac{\partial \langle h \rangle}{\partial t} + [\bar{v}_{\parallel}(\vec{R})\bar{b}(\vec{R}) + \bar{v}_d(\vec{R})] \cdot \nabla_R \langle h \rangle - \frac{e}{M} \nabla_R \langle \Phi(\vec{r}, t) \rangle \cdot [\bar{v}_{\parallel}(\vec{R})\bar{b}(\vec{R}) + \bar{v}_d(\vec{R})] \frac{\partial \langle h \rangle}{\partial E} \approx \langle C_{ii}^{\ell} \{H\} \rangle + \frac{ef_{0i}}{T_i} \frac{\partial \langle \Phi \rangle}{\partial t} - c \frac{\partial \langle \Phi \rangle}{\partial \zeta} \frac{\partial f_{0i}}{\partial \psi} \Big|_{E_*}, \quad (\text{A4})$$

where the distinction between  $\vec{R}$  and  $\vec{r}$  is negligible in  $f_{0i}$  terms on the right side and  $H \equiv h - (Iv_{\parallel}f_{0i}/\Omega_i T_i)[(Mv^2/2T_i) - (5/2)](\partial T_i/\partial \psi)$ . In the plateau regime the replacement  $C_{ii}^{\ell} \{H\} \rightarrow -vH$  may be employed.

For the drift wave drive terms, the  $\partial\langle\Phi\rangle/\partial t$  term is actually the same order as the  $\partial\langle\Phi\rangle/\partial\zeta$  term since the axisymmetric part of  $\langle\Phi\rangle \sim T_i/e$  can only evolve on the slower transport time scale, while the non-axisymmetric  $k_\perp\rho_i \sim 1$  contributions evolve at the diamagnetic drift frequency but are smaller by  $1/k_\perp L_\perp \sim \rho_i/L_\perp$ . As a result,  $\langle h \rangle/f_{0i} \sim \rho_i/L_\perp$ . To the order we have derived our gyrokinetic equation

$$f_i = f_{0i} - (Mc/e)(R^2\nabla\zeta \cdot \vec{v})\partial f_{0i}/\partial\psi|_{E_*} + \langle h \rangle, \quad (\text{A5})$$

with  $f_{0i}$  the local Maxwellian and

$$\partial f_{0i}/\partial\psi|_E = f_{0i}[p_i^{-1}\partial p_i/\partial\psi + (e/T_i)\partial\Phi/\partial\psi + (Mv^2/2T_i - 5/2)T_i^{-1}\partial T_i/\partial\psi]. \quad (\text{A6})$$

Using  $BR^2\nabla\zeta \cdot \vec{v} = Iv_\parallel - \vec{v} \times \vec{b} \cdot \nabla\psi$  gives  $\psi_* = \psi + \Omega^{-1}\vec{v} \times \vec{b} \cdot \nabla\psi - Iv_\parallel/\Omega \equiv \Psi - Iv_\parallel/\Omega$  with  $\Psi \equiv \psi + \Omega^{-1}\vec{v} \times \vec{b} \cdot \nabla\psi$  the gyrokinetic radial variable so that

$$f_i = f_{0i} - [Iv_\parallel/\Omega_i - \Omega_i^{-1}\vec{v} \times \vec{b} \cdot \nabla\psi]\partial f_{0i}/\partial\psi|_E + \langle h \rangle = f_{0i}(\Psi, \varepsilon) - (Iv_\parallel/\Omega_i)\partial f_{0i}/\partial\psi|_{E_*} + \langle h \rangle \quad (\text{A7})$$

or upon defining  $\delta f_i$ ,

$$\langle f_i \rangle = f_{0i}(\Psi, \varepsilon) - (Iv_\parallel/\Omega_i)\partial f_{0i}/\partial\psi|_{E_*} + \langle h \rangle \equiv f_{0i}(\Psi, \varepsilon) + \delta f_i. \quad (\text{A8})$$

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