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Arbitrary poloidal gyroradius effects in tokamak pedestals and transport barriers

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Abstract. A technique is developed and applied for analyzing pedestal and internal transport barrier (ITB) regions in a tokamak by formulating a special version of gyrokinetics. In contrast to typical gyrokinetic treatments, canonical angular momentum is taken as the gyrokinetic radial variable rather than the radial guiding center location. Such an approach allows strong radial plasma gradients to be treated, while retaining zonal flow and neoclassical (including orbit squeezing) behavior and the effects of turbulence. The new, nonlinear gyrokinetic variables are constructed to higher order than is typically the case. The nonlinear gyrokinetic equation obtained is capable of handling such problems as collisional zonal flow damping with radial wavelengths comparable to the ion poloidal gyroradius, as well as zonal flow and neoclassical transport in the pedestal or ITB. This choice of gyrokinetic variables allows the toroidally rotating Maxwellian solution of the isothermal tokamak limit to be recovered. More importantly, we prove that a physically acceptable solution for the lowest order ion distribution function in the banana regime anywhere in a tokamak and, in particular, in the pedestal must be nearly this same isothermal Maxwellian solution. That is, the ion temperature variation scale must be much greater than the poloidal ion gyroradius. Consequently, in the banana regime the background radial ion temperature profile cannot have a pedestal similar to that of plasma density.

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1. Introduction

Understanding tokamak pedestal physics[1,2] is one of the more crucial challenges currently facing magnetic fusion science. A self-consistent, predictive description of this region is necessary to understand the reason for improved confinement or H mode operation [3] and to gain insight into the Greenwald density limit [4]. As the barrier between the core and scrape-of-layer, the pedestal also helps control particle and heat fluxes [5] to the first wall and divertor [6]. One of the many reasons that the pedestal appears complicated is that the well known kinetic approaches [7-11] fail in the presence of the strong plasma gradients associated with the pedestal [11,12] as well as internal transport barriers (ITB) [13,14]. In these regions, as well as near the magnetic axis [15,16], finite ion orbit [10,11], orbit squeezing [17], and even neutral [18-20] effects on the pedestal may need to be addressed. To deal with the geometrical complications associated with large drift departures from flux surfaces [21], a variation of standard gyrokinetics [9,22,23] using the canonical angular momentum as the radial variable is developed and applied. This alternate description is constructed to exactly preserve conservation of canonical angular momentum and energy and is thereby able to provide key insights into the behavior of the ions in regions with step gradients. Canonical angular momentum has been employed as a variable in drift kinetic quasilinear descriptions [24], but we are not aware of it being used in gyrokinetic descriptions.

Gyrokinetics is a well established formalism capable of handling phenomena with high perpendicular wavenumbers that is being successfully used for studies of turbulence in tokamak core plasmas [25-31]. However, its application to steep gradient regions becomes more transparent if an alternative analytical treatment involving canonical angular momentum is employed. We focus on the development and insights provided by such an electrostatic gyrokinetic formulation that explicitly makes use of the axisymmetric magnetic field of a tokamak while allowing strong radial variation of the background ion profiles so that barrier widths comparable to the poloidal ion gyroradius may be treated in fully turbulent plasmas.

The technique we employ is a generalization of a standard linear gyrokinetic procedure [32,33] and its nonlinear counterpart that is used to consider the shortcomings of gyrokinetic quasineutrality at long wavelengths [34]. By modifying these procedures we construct nonlinear gyrokinetic variables to higher order than is typically done while retaining finite poloidal ion gyroradius effects. The resulting fully nonlinear gyrokinetic equation is not only valid for $k_\perp \rho \sim 1$, as any gyrokinetic approach would be, but also due to our choice of canonical angular momentum as one of the variables, it is naturally separable into departures from flux surfaces caused by neoclassical drifts and classical finite Larmor radius (ρ) effects. This feature is what makes the analysis of the leading order solution for the ion distribution function in a tokamak pedestal and an ITB (and near the magnetic axis) intuitively easy to understand since it precisely retains the isothermal limit [35]. In particular, it allows us to conclude that in the pedestal and an ITB (and near the magnetic axis) the lowest order ion distribution function must be nearly isothermal in the banana regime. As a result, an ion temperature pedestal or internal ion heat transport barrier is not allowed in a tokamak operating in the banana regime.

Having this result, we go further to formulate the gyrokinetic equation for the next order corrections to the ion distribution function. The relevant gyrokinetic equation obtained consistently contains neoclassical effects [7,8,10,11] and zonal flow phenomena [36-39] in the pedestal or an ITB along with the terms responsible for orbit squeezing [40] and potato orbits [15,16]. This gyrokinetic equation is also valid for zonal flow and neoclassical studies in core tokamak plasmas since our full nonlinear gyrokinetic equation with turbulence retained is constructed to smoothly connect to the core where it remains valid.

The remainder of the paper is organized as follows. In sections 2 - 3 we outline the gyrokinetic procedure we use to derive the full nonlinear gyrokinetic equation and discuss how it differs from standard nonlinear gyrokinetics [34,41-44] including a version developed especially for the edge [44]. The expressions for the gyrokinetic variables we employ and the orderings under which they are obtained are given in brief in sections 4 - 5 and in detail in appendices A - C. In section 6 the full nonlinear gyrokinetic equation is derived and its main properties are discussed. An entropy production analysis is employed in section 7 (with some details relegated to appendix D) to obtain the most general form of the leading order solution for the ion distribution function. Section 8 provides further insight into the physics of a pedestal or an ITB with the help of pressure balance equations. The gyrokinetic equation for zonal flow and neoclassical phenomena is presented in section 9. We close with a brief discussion of the results in section 10.

2. Gyrokinetic procedure

An assumption that is a basis of the gyrokinetic procedure to be described is the slow spatial variation of the equilibrium magnetic field. In particular, the background magnetic field of interest is assumed to obey the ordering

$$\delta \equiv \frac{\rho_i}{L} \ll 1\,, \tag{1}$$

where $L \equiv |\nabla \ln(B)|^{-1}$ and $\rho_i \equiv v_i/\Omega_i$ with $v_i \equiv \sqrt{2T_i/M}$ the ion thermal speed and $\Omega_i \equiv ZeB/Mc$ the ion cyclotron frequency. For simplicity, the magnetic field will be also assumed constant in time so that electric field can be treated as electrostatic; however, the slowly evolving induced electric field in a tokamak can easily be retained.

Consider the Vlasov operator written in terms of $\{\vec{r},\vec{v},t\}$ variables:

$$\frac{d}{dt} \equiv \frac{\partial}{\partial t} + \vec{v}\cdot\nabla_r + \left(\Omega\vec{v}\times\hat{n} - \frac{Ze}{M}\nabla\phi\right)\cdot\nabla_v\,. \tag{2}$$

Then, the evolution of the distribution function is given by

$$\frac{df}{dt} = C\{f\}\,, \tag{3}$$

where C is the collision operator. Equation (3) includes the fast time scale associated with the gyromotion of particles in the external magnetic field. Generally, in order to remove this time scale an averaging over gyrophase (φ) is performed. This, in turn, requires switching to a new set of magnetic field aligned variables that includes the gyrophase and then gyrophase averaging (3) written in terms of these variables. If the new variables are denoted by $\{q_1,...q_5,\varphi\}$, then (3) transforms into

$$\frac{\partial f}{\partial t} + \frac{\partial f}{\partial q_1}\frac{dq_1}{dt} + ... + \frac{\partial f}{\partial q_5}\frac{dq_5}{dt} + \frac{\partial f}{\partial \varphi}\frac{d\varphi}{dt} = C\{f\}\,. \tag{4}$$

The gyroaverage to be employed is defined as

$$\langle\cdot\rangle \equiv \frac{1}{2\pi}\oint d\varphi\,(\cdot)\,, \tag{5}$$

where the integration is performed holding the q_j's fixed.

If the new variables are chosen so that $\left\{\frac{dq_1}{dt},...\frac{dq_5}{dt},\frac{d\varphi}{dt}\right\}$ do not depend on φ the averaging of the left side of (4) becomes particularly convenient. However, it is difficult to find variables that possess this property exactly. Fortunately, the existence of the small parameter (1) allows us to construct variables whose total time derivatives are gyroindependent to the desired order in δ. The procedure follows.

We first choose a suitable set of initial variables $\{q_1^{(0)},...q_5^{(0)}\}$ and apply the $\frac{d}{dt}$ operator to them as well as to φ. Then, we extract the gyrodependent part of these total time derivatives and define the corrections $\{q_1^{(1)},...q_5^{(1)},\varphi^{(1)}\}$ such that $\frac{d}{dt}\left(q_j^{(0)}+q_j^{(1)}\right)$ is gyroindependent to next order, where $q_j^{(0)}+q_j^{(1)}$ is the improved variable. This procedure employs the lowest order result

$$\frac{d}{dt}q_j^{(1)} \approx -\Omega\frac{\partial}{\partial\varphi}q_j^{(1)}\,. \tag{6}$$

Thus, we can recover $q_j^{(1)}$ by performing an integration over φ as follows:

$$\Omega\frac{\partial}{\partial\varphi}q_j^{(1)} = \frac{d}{dt}q_j^{(0)} - \left\langle\frac{d}{dt}q_j^{(0)}\right\rangle. \tag{7}$$

This results in $q_j^{(1)} \sim \delta q_j^{(0)}$, thereby allowing us to determine the variables up to any given order by repeating the steps above. What this procedure yields is a particularly convenient set of *gyrokinetic variables*.

Note, that by this procedure we only find the gyrodependent part of $q_j^{(1)}$ that results in the gyroindependency of $\frac{d}{dt}\left(q_j^{(0)}+q_j^{(1)}\right)$. Thus, we can arbitrarily choose $\left\langle q_j^{(1)}\right\rangle \sim \delta q_j^{(0)}$ if it is convenient. Generally, we will set $\left\langle q_j^{(1)}\right\rangle = 0$, but sometimes a clever choice of $\left\langle q_j^{(1)}\right\rangle$ can further simplify (4). This freedom is what allows us to define a magnetic moment variable that will be an adiabatic invariant order by order, as will be demonstrated. Moreover, it is just the freedom needed to replace the regular radial gyrokinetic variable with the canonical angular momentum.

3. An alternative to regular gyrokinetics

Often, the initial set of variables is chosen as [9,32,33,41-44]

$$\vec{r}\,;\ \frac{v^2}{2}+\frac{Ze}{M}\phi(\vec{r}\,)\ \text{or}\ v_{\|};\ \mu_0\equiv\frac{v_\perp^2}{2B};\ \varphi\,.$$

However, in the case of tokamaks it is convenient to make use of conservation of the toroidal component of the canonical angular momentum. To do so we employ

$$\psi_*\equiv\psi-\frac{Mc}{Ze}R\vec{v}\cdot\hat{\zeta} \tag{8}$$

as the radial variable. The other initial variables are chosen to be the poloidal angle θ, the toroidal angle ζ, the magnetic moment μ_0, and the kinetic energy $\frac{v^2}{2}$. The gyrophase is defined such that

$$\vec{v}=v_{\|}\hat{n}+v_\perp\left(\hat{e}_1\cos\varphi+\hat{e}_2\sin\varphi\right) \tag{9}$$

where $v_{\|}\equiv\vec{v}\cdot\hat{n}=\sqrt{v^2-2\mu_0 B}$, $\hat{n}\equiv\vec{B}\big/B$, and $B\equiv\left|\vec{B}\right|$. Also, $\hat{e}_1(\vec{r}\,)$ and $\hat{e}_2(\vec{r}\,)$ are orthogonal unit vectors in the plane perpendicular to $\vec{B}$ such that $\hat{e}_1\times\hat{e}_2=\hat{n}$.

4. Orderings

We desire to develop a formalism to handle both neoclassical (large spatial scale) and turbulent (small spatial scale) phenomena. For this purpose we adopt the ordering used in [34]. Basically, this ordering allows only weak variations along the magnetic field while rapid perpendicular gradients are allowed for small amplitude fluctuations of the potential. Mathematically, our orderings are expressed as

$$\hat{n}\cdot\nabla\sim\frac{1}{L} \tag{10}$$

and

$$\frac{e\left|\varphi_k\right|}{T}\sim\frac{1}{k_\perp L}, \tag{11}$$

where the subscript k denotes a Fourier component. Physically, (11) implies that the $E\times B$ drift can be only of order δv_{th} or smaller.

The distribution function f is ordered analogously to the potential by taking

$$\frac{f_k}{f_0}\sim\frac{1}{k_\perp L}, \tag{12}$$

where the equilibrium solution f_0 is assumed to have spatial scales of order L. These orderings allow perturbations of the potential, density, and temperature with sharp gradients, and are relevant to turbulence, zonal flow, and the pedestal, ITBs, and near the magnetic axis in tokamaks.

In addition to the preceding orderings, we assume the characteristic frequency of the turbulent behavior to be that of drift waves,

$$\omega_*\sim\frac{v_{th}}{L}k_\perp\rho\,, \tag{13}$$

and allow the species collision frequency ν to be of order of its transit frequency,

$$\nu \sim \frac{v_{th}}{L}, \tag{14}$$

where v_{th} is the species thermal speed and ρ is its Larmor radius.

5. Gyrokinetic variables for an axisymmetric magnetic field

We next briefly consider the explicit expressions for the gyrokinetic variables that result from the procedure of Sec. 2 along with the orderings of section 4. Gyrokinetic variables resulting from an initial variable $q^{(0)}$ will be denoted as q_* at each order. We perform the calculation up to the second order in δ starting from the initial variables given in section 3. Here we summarize the results correct up to the first order, with the details of the derivation given in appendix A. Second order corrections and details of their derivation are given in appendix B.

5.1. Spatial variables

Applying the gyrokinetic procedure to $\theta_0 \equiv \theta$ and $\zeta_0 \equiv \zeta$ we find

$$\theta_* = \theta + \frac{\vec{v}\times\hat{n}}{\Omega}\cdot\nabla\theta \tag{15}$$

and

$$\zeta_* = \zeta + \frac{\vec{v}\times\hat{n}}{\Omega}\cdot\nabla\zeta\,. \tag{16}$$

No first order correction to the ψ_* of equation (8) is needed. Equations (15) and (16) give the usual θ and ζ coordinates of the gyrocenter, while ψ_* labels the so-called "drift surface" [7,11]. The total time derivatives of the spatial variables to the requisite order are given by

$$\dot{\psi}_* \approx \left\langle \dot{\psi}_* \right\rangle = c\frac{\partial\overline{\phi}}{\partial\zeta_*}, \tag{17}$$

$$\dot{\theta}_* \approx \left\langle \dot{\theta}_* \right\rangle = (v_\parallel^*\hat{n}_* + \vec{v}_d)\cdot(\nabla\theta)_* + \frac{Iv_\parallel}{\Omega}\frac{\partial\left(v_\parallel\hat{n}\cdot\nabla\theta\right)}{\partial\psi}, \tag{18}$$

$$\begin{aligned}\dot{\zeta}_* \approx \left\langle \dot{\zeta}_* \right\rangle &= (v_\parallel^*\hat{n}_* + \vec{v}_d)\cdot(\nabla\zeta)_* + \frac{Iv_\parallel}{\Omega}\frac{\partial\left(v_\parallel\hat{n}\cdot\nabla\zeta\right)}{\partial\psi} = \\ &= \left(\frac{Iv_\parallel}{BR^2}\right)_* + \vec{v}_d\cdot\nabla\zeta + \frac{Iv_\parallel}{\Omega}\frac{\partial}{\partial\psi}\left(\frac{Iv_\parallel}{BR^2}\right),\end{aligned} \tag{19}$$

where

$$\vec{v}_d \equiv -\frac{c}{B}\nabla\overline{\phi}\times\hat{n} + \frac{v_\parallel^2}{\Omega}\hat{n}\times(\hat{n}\cdot\nabla\hat{n}) + \frac{\mu}{\Omega}\hat{n}\times\nabla B\,, \tag{20}$$

$$\overline{\phi} \equiv \langle\phi\rangle \equiv \frac{1}{2\pi}\oint\phi\left(\psi_*,\theta_*,\zeta_*,E_*,\mu_*,\varphi_*\right)d\varphi_*\,, \tag{21}$$

and $I = RB_t$, with B_t the toroidal magnetic field and R the tokamak major radius. The axisymmetric tokamak magnetic field is taken to be

$$\vec{B} = I(\psi)\nabla\zeta + \nabla\zeta\times\nabla\psi\,, \tag{22}$$

so that ψ_* can be rewritten as

$$\psi_* = \psi + \frac{\vec{v}\times\hat{n}}{\Omega}\cdot\nabla\psi - \frac{Iv_\parallel}{\Omega}. \tag{23}$$

Also, in the preceding formulas and throughout the paper we use the following notation. If a certain quantity is given in terms of initial variables by $Q = Q(\psi,\theta,\zeta,E,\mu,\varphi)$, then we define

$$Q_* \equiv Q\big(\psi_*, \theta_*, \zeta_*, E_*, \mu_*, \varphi_*\big). \tag{24}$$

For example,

$$v_{\|}^{*} = \sqrt{E_* - \mu_* B_*}\,. \tag{25}$$

The difference between Q and Q_* is of order δQ and sometimes is unimportant. For instance, in the last term in (18) we can replace $v_{\|}$ by $v_{\|}^{*}$ and still stay within the required precision. However, in the first term of the same equation we must distinguish between these two.

5.2. Energy

Applying the gyrokinetic procedure to $E_0 \equiv v^2/2$ we find

$$E_* = \frac{v^2}{2} + \frac{Ze}{M}\tilde{\phi}\,, \tag{26}$$

and to requisite order

$$\dot{E}_* \approx \big\langle \dot{E}_* \big\rangle = -\frac{Ze}{M}(\dot{\psi}_* \frac{\partial \overline{\phi}}{\partial \psi_*} + \dot{\theta}_* \frac{\partial \overline{\phi}}{\partial \theta_*} + \dot{\zeta}_* \frac{\partial \overline{\phi}}{\partial \zeta_*}) \Big/ \left(1 + \frac{Ze}{M}\frac{\partial \overline{\phi}}{\partial E_*}\right), \tag{27}$$

where

$$\tilde{\phi} \equiv \phi - \overline{\phi}\,. \tag{28}$$

In (27) the expressions for $\dot{\psi}_*$, $\dot{\theta}_*$, and $\dot{\zeta}_*$ are given by (17) – (19), and the small $\partial\overline{\phi}/\partial E_*$ term is given by (B.21) and must be retained to ensure that total energy remains an exact constant of the motion in the steady state.

5.3. Magnetic moment

The gyrokinetic procedure applied to $\mu_0 \equiv v_\perp^2/2B$ gives

$$\mu_1 = -\frac{\vec{v}\cdot\vec{v}_{M}}{B} - \frac{v_{\|}}{4B\Omega}[\,\vec{v}_\perp(\vec{v}\times\hat{n}) + (\vec{v}\times\hat{n})\vec{v}_\perp\,] : \nabla\hat{n} + \frac{Ze}{MB}\tilde{\phi} + \big\langle \mu_1 \big\rangle, \tag{29}$$

where

$$\vec{v}_{M} \equiv \frac{v_{\|}^2}{\Omega}\hat{n}\times(\hat{n}\cdot\nabla\hat{n}) + \frac{\mu_0}{\Omega}\hat{n}\times\nabla B\,. \tag{30}$$

As mentioned at the end of section 2, $\big\langle \mu_1 \big\rangle$ can be chosen arbitrarily as long as $\big\langle \mu_1 \big\rangle \sim \delta\mu_0$. For all the other variables we set the gyroindependent part of the correction equal to zero (notice that ψ_* automatically retains a gyroindependent term). However, as the magnetic moment is an adiabatic invariant [45], we show we can define $\big\langle \mu_1 \big\rangle$ such that $\big\langle \dot{\mu}_* \big\rangle = 0$ order by order. This feature is checked in the appendix C by choosing

$$\big\langle \mu_1 \big\rangle = -\frac{v_\perp^2 v_{\|}}{2B\Omega}\hat{n}\cdot\nabla\times\hat{n} \tag{31}$$

to find

$$\big\langle \dot{\mu} + \dot{\mu}_1 \big\rangle / \mu_0 \sim \delta^3\Omega\,. \tag{32}$$

This choice allows us to neglect the $\partial f/\partial\mu$ term in the gyrokinetic equation even with $k_\perp\rho \sim 1$ potential fluctuations retained.

5.4. Gyrophase

For the ordering we employ, $\partial f/\partial\varphi = 0$ to lowest order. As a result, for our purposes it is adequate to use $\varphi_* = \varphi$ as defined by (9). Then, we find

$$\dot{\varphi}_* \approx \left\langle \dot{\varphi}_* \right\rangle = -\Omega_* - \frac{v_\parallel}{2}\hat{n}\cdot\nabla\times\hat{n} + v_\parallel \hat{n}\cdot\nabla\hat{e}_2\cdot\hat{e}_1 - \frac{Z^2e^2}{M^2c}\frac{\partial\overline{\phi}}{\partial\mu} - \frac{ZeI}{Mv_\parallel}\frac{\partial\overline{\phi}}{\partial\psi_*} - Iv_\parallel\frac{\partial\ln B}{\partial\psi_*} \equiv -\overline{\Omega}\,. \quad (33)$$

The first order correction to the gyrophase is given in appendix A for completeness.

6. Electrostatic gyrokinetic equation

Having defined the gyrokinetic variables we can now insert them into (4) and gyroaverage to find our full nonlinear gyrokinetic equation

$$\frac{\partial\overline{f}}{\partial t} + \dot{\psi}_*\frac{\partial\overline{f}}{\partial\psi_*} + \dot{\theta}_*\frac{\partial\overline{f}}{\partial\theta_*} + \dot{\zeta}_*\frac{\partial\overline{f}}{\partial\zeta_*} + \dot{E}_*\frac{\partial\overline{f}}{\partial E_*} = \langle C\{f\}\rangle, \quad (34)$$

where $\overline{f} \equiv \langle f \rangle$ and expressions (17) - (19) and (27) give $\dot{\psi}_*$, $\dot{\theta}_*$, $\dot{\zeta}_*$, and $\dot{E}_*$. Note that for $\dot{E}_*$ defined by (27) the total energy $\varepsilon \equiv E_* + \frac{Ze}{M}\overline{\phi}$ is exactly conserved by the gyrokinetic Vlasov operator. Consequently, we can construct an exact solution to (34) in the isothermal case in the same way as Catto and Hazeltine in [35].

To do so we observe that for a stationary and axisymmetric plasma any function of ε and ψ_* makes the left side of the equation exactly vanish. On the other hand, to make the right side vanish $\overline{f}$ has to be Maxwellian as ion-ion collisions dominate over those between ions and electrons. Combining these two statements we find an exact solution for arbitrary collisionality to be the rigidly toroidally rotating Maxwellian

$$f_* = n\left(\frac{M}{2\pi T}\right)^{3/2}\exp\left(-\frac{M\left(\vec{v}-\omega R\hat{\zeta}\right)^2}{2T}\right), \quad (35)$$

with the density given by

$$n = \eta\exp\left(-\frac{Ze\phi}{T} + \frac{M\omega^2R^2}{2T} - \frac{Ze}{cT}\omega\psi\right), \quad (36)$$

where T, ω, and η are constants. In terms of the gyrokinetic variables this solution is only a function of the constants of motion ε and ψ_* since

$$f_* = \eta\left(\frac{M}{2\pi T}\right)^{3/2} e^{-\frac{M\varepsilon}{T}-\frac{Ze}{cT}\omega\psi_*}. \quad (37)$$

7. Entropy production

Now we analyze the case with spatially varying T still assuming $\partial/\partial\zeta = 0$. It is convenient to switch to θ_* and ε variables so that our gyrokinetic equation becomes

$$\left.\frac{\partial\overline{f}}{\partial t}\right|_\varepsilon + \dot{\theta}_*\left.\frac{\partial\overline{f}}{\partial\theta_*}\right|_\varepsilon = \langle C\{f\}\rangle - \frac{Ze}{M}\frac{\partial\overline{\phi}}{\partial t}\frac{\partial\overline{f}}{\partial\varepsilon}. \quad (38)$$

In the steady state we let $f = f_0$ and transit average

$$\dot{\theta}_*\left.\frac{\partial\overline{f}_0}{\partial\theta_*}\right|_\varepsilon = \left\langle C\{f_0\}\right\rangle \quad (39)$$

to obtain the solubility constraint

$$\overline{\left\langle C\{f_0\}\right\rangle} = 0\,, \quad (40)$$

where the transit average is defined by

$$\bar{Q} \equiv \frac{\oint Q\, d\theta_* / \dot{\theta}_*}{\oint d\theta_* / \dot{\theta}_*}. \tag{41}$$

The full nonlinear constraint (40) must be satisfied for any physically acceptable stationary solution $f_0 = f_0\left(\psi_*, \theta_*, \varepsilon, \mu_*\right)$, and the transit average is performed holding ψ_*, ε and μ_* fixed by integrating over a complete bounce for trapped particles and a full poloidal circuit for the passing. Next, we use the preceding to determine the lowest order ion distribution function f_0 in a tokamak pedestal and internal transport barrier (ITB).

We define the radial scale w of the distribution function as

$$\left| \nabla\psi \frac{\partial \ln f}{\partial \psi} \right| \equiv \frac{1}{w}. \tag{42}$$

In a pedestal or in an internal barrier region we assume strong spatial gradients by allowing

$$w \sim \rho_{pol} \ll L, \tag{43}$$

where ρ_{pol} is the poloidal ion gyroradius. Gradients along the flux surface will be allowed to be strong as well

$$\left| \nabla\theta \frac{\partial \ln f}{\partial \theta} \right| \lesssim \frac{1}{\rho_{pol}}, \tag{44}$$

although we will demonstrate that only weak derivatives over θ are physically possible in the banana regime. The electrostatic potential ϕ is assumed to scale analogously to f. With these assumptions, we demonstrate that in the pedestal or an ITB the leading order solution to (40) remains Maxwellian (from now on we refer to the pedestal case only as proof for an ITB is exactly the same). Before doing so we remark that the original orderings (11) – (12) we used to derive the axisymmetric gyrokinetic equation imply that the characteristic scale of the leading order axisymmetric distribution function and potential is the size of tokamak *L*. However, all our results remain valid provided $\rho \ll \rho_{pol} \lesssim w$. Indeed, in all the estimates required for the derivation of the gyrokinetic variables we can then replace *L* by ρ_{pol} so that the outcome of the gyrokinetic procedure stays unchanged. However, the comparison among different terms in the gyrokinetic formulas can be affected. In particular, in (18) for $\dot{\theta}_*$ the contribution of the $\vec{E} \times \vec{B}$ term in $\vec{v}_d$ becomes comparable to that due to the $v_\parallel$ if the potential gradient is of order $1/\rho_{pol}$ so that orbit squeezing effects enter [17].

We begin our demonstration by multiplying (39) by $\ln f_0$ and integrating it over ε and μ_* to obtain the steady state result

$$0 = \iint d\varepsilon d\mu_* \oint \frac{d\theta_*}{\dot{\theta}_*} \oint d\varphi_* \ln f_0 C_{ii}\{f_0\}, \tag{45}$$

where we employ

$$\ln f_0 \frac{\partial f_0}{\partial \theta_*} = \frac{\partial}{\partial \theta_*}\left(f_0 \ln f_0 - f_0 \right) \tag{46}$$

to annihilate the left side. Notice that all the integrals in equation (46) are performed holding ψ_* fixed. Next, we recall (18) and ordering (44) to find the leading order result

$$\dot{\theta}_* \approx v_\parallel \hat{n} \cdot \nabla\theta + \vec{v}_E \cdot \nabla\theta \approx \left(v_\parallel + \frac{cI}{B} \frac{\partial \phi}{\partial \psi} \right) \hat{n} \cdot \nabla\theta, \tag{47}$$

where we must retain the $\vec{E} \times \vec{B}$ term as noted at the end of the previous paragraph. Contributions of the other terms from (18) are always one order smaller in ρ/w. Rewriting we obtain

$$-\oint \frac{d\theta}{\vec{B} \cdot \nabla\theta} \iiint_{\psi_*} \frac{d\varepsilon d\mu_* d\varphi_*}{\left[v_\parallel / B + \left(cI / B^2 \right) \partial\phi / \partial\psi \right]} \ln f_0 C_{ii}\{f_0\} = 0, \tag{48}$$

where the inner integrations are performed holding ψ_* fixed.

To clarify the novel features of a pedestal plasma, we first review the analysis of (45) in the weak gradient limit ($w \sim L$) relevant to the core (see [11] for example). In this simpler case we can hold ψ fixed instead of ψ_* without an error to leading order. Then, neglecting the $\partial\phi/\partial\psi$ term in the denominator, equation (45) becomes

$$-\oint \frac{d\theta}{\vec{B}\cdot\nabla\theta}\iiint d\varepsilon d\mu_* d\varphi_* \frac{B}{v_\parallel}\ln f_0 C_{ii}\{f_0\} = 0\,. \tag{49}$$

Finally, employing

$$\frac{d\varepsilon d\mu_* d\varphi_*}{d^3v} \approx \frac{dE_0 d\mu_0 d\varphi_0}{d^3v} = \frac{v_\parallel}{B} \tag{50}$$

we see that the left side of (49) is the flux-surface averaged entropy production on a given flux surface. Thus, we can employ the Boltzmann H-theorem to determine that f_0 is Maxwellian.

In the pedestal $\left(\psi_* - \psi\right)\left(\partial f/\partial\psi\right) \sim \left(\rho_{pol}/w\right) f \sim f$ and integrating holding ψ_* fixed rather than ψ becomes important. To adjust the logic to the pedestal we need to integrate (48) with respect to ψ_* over the entire pedestal region. Then, we can use

$$\frac{d\psi_* d\theta_* d\zeta_* d\varepsilon d\mu_* d\varphi_*}{d^3r d^3v} \approx (\hat{n}\cdot\nabla\theta)\left(v_\parallel + \frac{cI}{B}\frac{\partial\phi}{\partial\psi}\right) \tag{51}$$

(see appendix D for the derivation) to transform (48) into

$$\int\limits_{V_{ped}} d^3r \ln f_0 C_{ii}\{f_0\} = 0\,, \tag{52}$$

where V_{ped} denotes the pedestal volume. As a result, we conclude from the H-theorem that $f_0 = f_0\left(\psi_*, \theta_*, \varepsilon, \mu_*\right)$ must be Maxwellian in the pedestal as well.

It is interesting to notice, that the proof for the core plasma only requires integration over a given flux surface, while for the pedestal plasma we have to integrate over the entire pedestal region (the presence of a separatrix complicates the pedestal case as discussed at the end of this section and in section 10; however for the ITB case this proof is robust). This feature suggests that in the absence of sharp gradients each flux surface equilibrates by itself, while within the pedestal all flux surfaces are coupled. Physically, this coupling is due to the order ρ_{pol} departures of ions from a flux surface. This effect is not important in the core plasma, where spatial variation is weak on the ρ_{pol} scale and therefore we can consider any given flux surface a closed system. However, when the radial gradient scale is as large as $1/\rho_{pol}$ these flux surface departures affect the equilibrating of the neighboring flux surfaces and therefore it is the entire pedestal region that is a closed system rather than its individual flux surfaces.

As a result of the preceding observations, the leading order ion distribution function must be Maxwellian, thereby satisfying constraint (40) and making $\left\langle C\{f_0\}\right\rangle = 0$ as well. Therefore, in the banana regime (39) results in $\partial f_0/\partial\theta_* = 0$ so that f_0 can only depend on ε, ψ_*, and μ_*, and allowing strong poloidal gradients [recall (44)] was unnecessary. The only Maxwellian that satisfies these conditions must be independent of μ_* and given by the relations (35) - (37), in which T, ω, and η are now allowed to be slowly varying compared to ρ_{pol}:

$$\rho_{pol}\nabla\ln T_i \ll 1 \tag{53}$$

and

$$\rho_{pol}\nabla\ln\omega \ll 1\,,\ \rho_{pol}\nabla\ln\eta \ll 1\,. \tag{54}$$

Thus, for the ions we have proven that the solution to (39) in a pedestal or an ITB is an isothermal Maxwellian to lowest order – no other solution is possible. Non-isothermal modifications enter in next order as indicated by (53) and (54). As a result, in the banana regime a pedestal in the background ion temperature

is unlikely to exist in a tokamak. In the Pfirsh – Schlüter regime ion departures from a flux surface are much smaller and an ion temperature pedestal cannot be ruled out. The plateau regime is a transitional case.

In addition, an ion temperature pedestal in the near scrape-of-layer (SOL) (or at the separatrix) is unlikely since our kinetic equation (39) remains valid there and is satisfied by the very same nearly isothermal Maxwellian ion distribution function we find inside the separatrix. As a result, no entropy production or entropy flow occurs to lowest order in the near SOL and no ion temperature pedestal is anticipated there as long as the near SOL remains in the banana regime.

8. Pressure balance in pedestal or ITB

In the previous section we studied pedestal and internal transport barrier plasmas given that the ion distribution function radial gradient is of order $1/\rho_{pol}$. This gradient can only be associated with the density (and potential) as the ion temperature is proven to be slowly varying. In this section we comment on how such large density gradients can be sustained.

We start by noting that from the ion pressure balance equation and (53) we find to lowest order that

$$\omega_i = -c\frac{d\phi}{d\psi} - \frac{cT_i}{Zen}\frac{dn}{d\psi}\,, \tag{55}$$

where $dn/d\psi$ obeys ordering (43). Then we estimate that

$$\omega_i \Big/ \frac{cT_i}{en}\frac{dn}{d\psi} \sim \frac{\omega_i R}{v_i} \tag{56}$$

with $\omega_i R$ the net ion flow. Thus, unless ions are sonic the left side of (55) must be smaller to lowest order than each of the terms on the right. Also, $dn/d\psi < 0$ and therefore $d\phi/d\psi < 0$, so the *electric field in the pedestal is inward*, as observed for pedestals in the presence of subsonic ion flow [46,47]. Also, we notice that, as the left side of (55) is negligible for a plasma having subsonic ion flow, density and potential must be connected through the lowest order radial Boltzmann relation giving

$$\frac{d\phi}{d\psi} \approx -\frac{T_i}{Zen}\frac{dn}{d\psi}\,. \tag{57}$$

Next, we consider electron flows in the pedestal by writing the net electron velocity as

$$\vec{V}_e = \omega_e R\hat{\zeta} + \vec{B}\,K_e(\psi)/n\,, \tag{58}$$

with K_e a flux function so that $\nabla\cdot\left(n\vec{V}_e\right) = 0$ to lowest order. Then, total pressure balance, $\vec{J}\times\vec{B} = c\nabla\left(p_e + p_i\right)$, reduces to the lowest order electron pressure balance result

$$\omega_e = -c\frac{d\phi}{d\psi} + \frac{c}{en_e}\frac{dp_e}{d\psi}\,, \tag{59}$$

when (57) is employed. But here the terms on the right side have the same sign and therefore cannot cancel as in the ion equation. Estimating, $\omega_e \sim \left|c\,\partial\phi/\partial\psi\right| \sim \left(c/en\right)\left|\partial p_e/\partial\psi\right|$ we find a large electron flow,

$$\omega_e R \sim v_i\,. \tag{60}$$

Thus, the electrostatic potential associated with the density gradient in the pedestal or an ITB can only be sustained by a large electron flow. As a result, *it is the electron dynamics that underlies pedestal or ITB physics*, and we can say that ions are electrostatically confined by the electrons. Although it is not clear what establishes the pedestal, it is clear that subsonic ion flow implies the pedestal is maintained by a large electron current with the ions electrostatically confined. Any small small departure of the ions from a radial Maxwell - Boltzmann relation must be due to weak ion temperature variation.

9. Zonal flows and neoclassical transport

Now that we have the leading order solution to (38) we can seek higher order corrections to it. We proceed by writing

$$\overline{f} = f_*\big(\psi_*,\varepsilon\big) + g\big(\psi_*,\theta_*,\mu_*,\varepsilon,t\big), \tag{61}$$

with $g \ll f_*$ and f_* given by (35) - (37) but with T, η, and ω allowed to be slowly varying functions of ψ_*. Then, equation (38) becomes

$$\left.\frac{\partial g}{\partial t}\right|_{\varepsilon} + \big\langle \dot{\theta}_* \big\rangle \left.\frac{\partial g}{\partial \theta_*}\right|_{\varepsilon} - \big\langle C_{ii}\big\{ f_* + g\big\}\big\rangle = \frac{Ze}{M}\frac{\partial \overline{\phi}}{\partial t}\frac{\partial f_*}{\partial \varepsilon}. \tag{62}$$

To evaluate the collision operator term in (62) we expand the slowly varying terms of f_* around ψ to obtain

$$f_* = \eta\big(\psi_*\big)\left(\frac{M}{2\pi T\big(\psi_*\big)}\right)^{3/2} e^{-\frac{M\varepsilon}{T(\psi_*)}-\frac{Ze\omega(\psi_*)}{cT(\psi_*)}\psi_*} \approx \eta(\psi)\left(\frac{M}{2\pi T(\psi)}\right)^{3/2} e^{-\frac{M\varepsilon}{T(\psi)}-\frac{Ze\omega(\psi)}{cT(\psi)}\psi_*}[1 +$$
$$\big(\psi_* - \psi\big)\left(\frac{M\varepsilon}{T^2}\frac{\partial T}{\partial \psi} + \frac{Ze\omega\psi_*}{cT^2}\frac{\partial T}{\partial \psi} - \frac{Ze\psi_*}{cT}\frac{\partial \omega}{\partial \psi} + \frac{1}{\eta}\frac{\partial \eta}{\partial \psi} - \frac{3}{2T}\frac{\partial T}{\partial \psi}\right) + \ldots\ ]. \tag{63}$$

The expression preceding the square parentheses is a toroidally rotating Maxwellian at any given point in space

$$\eta(\psi)\left(\frac{M}{2\pi T(\psi)}\right)^{3/2} e^{-\frac{M\varepsilon}{T(\psi)}-\frac{Ze\omega(\psi)}{cT(\psi)}\psi_*} = n\left(\frac{M}{2\pi T(\psi)}\right)^{3/2}\exp\Big(-\frac{M\big(\vec{v}-\omega(\psi)R\hat{\zeta}\big)^2}{2T(\psi)}\Big) \equiv f_M, \tag{64}$$

where $n = n(\vec{r}\,)$ is given by (36). We use

$$C_{ii}\big\{ f_M\big\} = 0 \tag{65}$$

and employ the linearized ion-ion collision operator C_{ii}^l along with momentum conservation to note that

$$C_{ii}^l\big\{\vec{v} f_M\big\} = 0\,. \tag{66}$$

Recalling that $\psi_* - \psi = -\big(Mc/Ze\big)R\vec{v}\cdot\hat{\zeta}$ and using properties (65) - (66) we find

$$C_{ii}\big\{ f_*\big\} \approx C_{ii}^l\left\{-f_M\left(\frac{M^2cR}{Ze}\frac{v^2\vec{v}\cdot\hat{\zeta}}{2T^2}\frac{\partial T}{\partial \psi} + \frac{Ze\omega}{cT^2}\frac{\partial T}{\partial \psi}\Big(\frac{McR}{Ze}\vec{v}\cdot\hat{\zeta}\Big)^2 - \frac{1}{cT}\frac{\partial \omega}{\partial \psi}\Big(\frac{McR}{Ze}\vec{v}\cdot\hat{\zeta}\Big)^2\right)\right\}. \tag{67}$$

Finally, we can neglect the last two terms in the collision operator for subsonic flows because of (53) - (54) to obtain the simple result

$$\big\langle C_{ii}\big\{ f_*\big\}\big\rangle \approx C_{ii}^l\left\{-\frac{Iv_{\|}}{\Omega} f_M \frac{Mv^2}{2T^2}\frac{\partial T}{\partial \psi}\right\}. \tag{68}$$

Next, we evaluate the $\overline{\phi}$ term on the right side of (62) assuming $\phi = \phi(\psi,t)$ to the requisite order [36--39], and using an eikonal form

$$\phi = \hat{\phi} e^{iS(\psi)}, \tag{69}$$

with $\vec{k}_{\perp} \equiv \nabla S(\psi)$. Then, expanding $S(\psi)$ around ψ_* and gyroaveraging ϕ holding ψ_* fixed yields

$$\overline{\phi} \approx \left\langle \hat{\phi} e^{i\left[S(\psi_*)+(\psi-\psi_*)\frac{\partial S}{\partial \psi_*}+\ldots\right]}\right\rangle \approx \left\langle \hat{\phi} e^{i\left[S(\psi_*)-\frac{\vec{v}\times\hat{n}}{\Omega}\cdot\nabla S+\frac{Iv_{\|}}{\Omega}S'+\ldots\right]}\right\rangle = \phi_* J_0\Big(\frac{k_{\perp}v_{\perp}}{\Omega}\Big)e^{iQ}, \tag{70}$$

where $\phi_* \equiv \hat{\phi} e^{iS(\psi_*)}$ and $Q \equiv \big(Iv_{\|}/\Omega\big)S'$, with $S' \equiv \partial S/\partial\psi$ and S assumed slowly varying.

Now we insert (68) and (70) into (62) and use $\partial f_*/\partial\varepsilon \approx \big(-M/T\big) f_M$ to obtain the equation for g to be

$$\frac{\partial g}{\partial t} + \big\langle \dot{\theta}_* \big\rangle \frac{\partial g}{\partial \theta_*} - C_{ii}^l\left\{ g - \frac{Iv_{\|}}{\Omega} f_M \frac{Mv^2}{2T^2}\frac{\partial T}{\partial \psi}\right\} = -\frac{Ze}{T}\frac{\partial \phi_*}{\partial t} f_M J_0\Big(\frac{k_{\perp}v_{\perp}}{\Omega}\Big)e^{iQ}. \tag{71}$$

Finally, we consider the banana regime in which $\partial g/\partial\theta_* = 0$ to lowest order, so that transit averaging (71) gives

$$\frac{\partial g}{\partial t} - \overline{C_{ii}^l \left\{ g - \frac{I v_\parallel}{\Omega} f_M \frac{M v^2}{2T^2} \frac{\partial T}{\partial \psi} \right\}} = -\frac{Ze}{T} \frac{\partial \phi_*}{\partial t} f_M \overline{J_0 \left(\frac{k_\perp v_\perp}{\Omega} \right) e^{iQ}} \tag{72}$$

with transit average defined as in (41). The distinctions between f_M, Q, $v_\parallel$, J_0 and f_*, Q_*, $v_\parallel^*$, J_0^* respectively is unimportant in (71) and (72). Equation (72) contains both neoclassical and zonal flow drives in an uncoupled manner. The neoclassical drive enters in the collision operator and for it the time derivatives in (72) are negligible. The zonal flow drive is due to the $\partial\phi_*/\partial t$ term that requires keeping $\partial g/\partial t$, but for which the neoclassical drive does not matter. This gyrokinetic equation is capable of retaining finite Larmor radius effects on these phenomena, as well as finite poloidal gyroradius and orbit squeezing effects since it is derived using ψ_* as the radial variable.

10. Discussion

An electrostatic gyrokinetic formalism for tokamaks is developed and its first applications are performed. Based on an entropy production argument that retains orbit squeezing as well as $\vec{E} \times \vec{B}$ shear effects, the most important prediction is that in the banana regime the background ion temperature is not allowed to have a pedestal similar to the ones observed for plasma density, electrostatic potential, and electron temperature since inequality (53) must be satisfied. This prediction seems to be in reasonable agreement with experimental observations [48,49] since currently there are no direct measurements of the background ion temperature in a tokamak pedestal. Even impurity helium near the plateau to banana regime transition tends to exhibit a weaker temperature pedestal for the ions than for the electrons [48]. The majority of existing ion temperature measurements are for impurities which have a smaller ion gyroradius and are more collisional than the background ions. Moreover, it must be kept in mind that in the pedestal temperature equilibration between impurities and background ions is no longer local (flux surface by flux surface) because of finite orbit effects that can allow impurity radial heat transport and equilibration to compete.

Of course, the entropy production proof that the background ions do not have a temperature pedestal has some limitations. First, we can only apply it when the collision operator does not dominate over the streaming term in the kinetic equation. Therefore, our proof is valid in the banana regime, but not in the collisional Pfirsch-Schluter regime (with any plateau regime behavior expected to be transitional). Only in the banana regime does the distribution function being Maxwellian result in it being independent of θ_* and therefore μ_*, which in turn leads to slow radial temperature variation.

Another issue is the implicit assumption of the absence of any significant entropy flow from the pedestal into divertor plates that is needed to obtain (52). This assumption requires the pedestal region to be within the tokamak separatrix in such a way that all the flux surfaces carrying a significant amount of plasma are closed. If the separatrix were to fall part way up the pedestal our proof would no longer be mathematically robust. However, our almost isothermal Maxwellian solution remains valid in the near SOL so entropy flow into the divertor is negligible. Therefore, we expect that in the banana regime it will be difficult to sustain the strong background ion temperature variation comparable to that of plasma density needed to obtain an ion temperature at the top of the pedestal in the 2-4 keV range as assumed in ITER [50].

Other limitations of our proof are associated with the neglect of charge exchange and ionization, and direct orbit loss to physical structures outside the SOL, which may or may not be playing a role in establishing the pedestal [50]. Orbit loss results in non-Maxwellian features that cause the entropy production to be finite so we anticipate that ion orbit loss will have to remain a weak effect in a well defined pedestal in local equilibrium. Moreover, in the short neutral mean free path limit the velocity dependence of the neutral distribution function will become the same as that of the ions causing charge exchange collisions of the ions with the neutrals to produce no entropy. For longer neutral mean free paths we expect little entropy production due to the presence of the neutrals based on a self-similar treatment of the neutrals which finds results roughly in agreement with short mean free path results [51].

Interestingly, we can apply our nonlocal entropy production proof to the case of the so-called “potato regime” near the magnetic axis [15,16] that is the potato analog of the regular banana regime. In this region

of a tokamak ρ_{pol} becomes large so that (53) requires an almost constant ion temperature in the vicinity of the magnetic axis meaning that there is no transport in a conventional sense. This analysis is in agreement with the point made in [16] that near the magnetic axis we should speak about a global solution in the entire region rather than about a local diffusive process. This point is in turn similar to the point about the non-local equilibration of the pedestal that we make in section 7.

Finally, we remark that a favorable consequence of the lack of a background ion temperature pedestal in the banana regime is the probable enhancement of the bootstrap current in the pedestal. To see this effect we employ the usual Z = 1, large aspect ratio expression

$$j_{BS} = -f_t n T_e R\left[1.66\left(1+\frac{T_i}{ZT_e}\right)\frac{d\ln n}{d\psi} + 0.47\,\frac{d\ln T_e}{d\psi} - \frac{0.29}{ZT_e}\frac{d\ln T_i}{d\psi}\right], \tag{73}$$

where f_t is a trapped particles fraction (e.g. see [11]). We use (73) only as an estimate because neoclassical transport in pedestal can be slightly different from this result in the large aspect ratio form due to strong shaping effects in the pedestal. Experiments show that T_e and n profiles are very similar with strong electron temperature variation being allowed by the small poloidal gyroradius of the electrons. We recall that (53) prevents T_i from having a gradient comparable to that of n and T_e, so the ion temperature gradient term is expected to be negligible in the pedestal, but more importantly (53) leads us to expect $T_i/T_e \gg 1$ to hold in the coefficient of the ion density gradient term. Thus, the first term in square parentheses in (73) is expected to be greater in pedestal than in the core resulting in a larger bootstrap current closer to plasma edge.

In summary, the modified gyrokinetic approach we employ promises to be a useful tool for studies of plasma turbulence and transport in tokamaks. The choice of ψ_* as the gyrokinetic radial variable results in a convenient treatment of arbitrary poloidal gyroradius effects in the pedestal, in ITBs, and about the magnetic axis, while still allowing neoclassical collisional effects and zonal flow to enter naturally along with finite Larmor radius phenomena including orbit squeezing. As a result, our formalism is capable of handling such problems as collisional zonal flow damping with $k_\perp \rho_{pol} \sim 1$, zonal flow in a pedestal, and neoclassical transport in a pedestal, as well as turbulent phenomena.

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Appendix A. First order corrections to gyrokinetic variables

We show in this appendix how the gyrokinetic procedure we describe in section 2 is implemented to obtain our gyrokinetic variables correct up to first order in δ.

Spatial variables. Following the steps outlined in section 2 we first apply the Vlasov operator to $\theta_0 \equiv \theta$ to obtain

$$\frac{d\theta_0}{dt} = \vec{v}\cdot\nabla\theta\,. \tag{A.1}$$

Next, we extract gyrodependent part of $d\theta_0/dt$ by writing

$$\frac{d\theta_0}{dt} - \left\langle\frac{d\theta_0}{dt}\right\rangle = \vec{v}_\perp\cdot\nabla\theta\,. \tag{A.2}$$

Then, we have to solve for θ_1 such that to lowest order

$$\frac{d}{dt}(\theta_0+\theta_1)-\left\langle\frac{d}{dt}(\theta_0+\theta_1)\right\rangle=0\,. \tag{A.3}$$

Using (A.2) and (6), (A.3) gives the equation

$$\Omega\frac{\partial\theta_1}{\partial\varphi}=\frac{d\theta_0}{dt}-\left\langle\frac{d\theta_0}{dt}\right\rangle=\vec{v}_\perp\cdot\nabla\theta\,. \tag{A.4}$$

To perform the integration over φ we use $\int\vec{v}_\perp d\varphi=\vec{v}\times\hat{n}$. Thus, setting $\langle\theta_1\rangle=0$ gives $\theta_1=\Omega^{-1}\vec{v}\times\hat{n}\cdot\nabla\theta$, reproducing the relation (15) given in the section V. We get the first order correction to ζ by similar procedure to find $\zeta_1=\Omega^{-1}\vec{v}\times\hat{n}\cdot\nabla\zeta$ and (16).

As has already been mentioned, the ψ_* variable does not require a first order correction. However, if we were to simply take ψ as the initial variable and then proceed analogously to θ and ζ, we find

$$\tilde{\psi}_1=\frac{\vec{v}\times\hat{n}}{\Omega}\cdot\nabla\psi\,. \tag{A.5}$$

If we define

$$\psi_1\equiv\psi_*-\psi=-\frac{Mc}{Ze}R\vec{v}\cdot\hat{\zeta}\,, \tag{A.6}$$

then we see the gyrodependent part of ψ_1 is equal to $\tilde{\psi}_1$. This can be verified by using (22) for the magnetic field in tokamaks to rewrite ψ_1 as $\psi_1=\Omega^{-1}\vec{v}\times\hat{n}\cdot\nabla\psi-Iv_\parallel/\Omega$, as in (23).

Magnetic moment. Here we will only show the derivation of the gyrodependent part of μ_1 denoted as $\tilde{\mu}_1$. The gyroindependent term $\langle\mu_1\rangle$ will be considered in the appendix C. As usual, we first evaluate

$$\begin{aligned}\frac{d\mu_0}{dt}=\frac{d}{dt}\left(\frac{v_\perp^2}{2B}\right)&=-\mu_0v_\parallel\hat{n}\cdot\nabla\ln B-\mu_0v_\parallel\nabla\cdot\hat{n}-\mu_0\vec{v}_\perp\cdot\nabla\ln B-\frac{v_\parallel^2}{B}\hat{n}\cdot\nabla\hat{n}\cdot\vec{v}_\perp\\&\quad-\frac{v_\parallel}{2B}[\vec{v}_\perp\vec{v}_\perp-(\vec{v}\times\hat{n})(\vec{v}\times\hat{n})]:\nabla\hat{n}-\frac{Ze}{MB}\vec{v}_\perp\cdot\nabla\phi\,.\end{aligned} \tag{A.7}$$

We notice that

$$\left\langle\frac{d\mu_0}{dt}\right\rangle=-\mu_0v_\parallel(\hat{n}\cdot\nabla\ln B-\nabla\cdot\hat{n})=-\mu_0v_\parallel\nabla\cdot B=0\,, \tag{A.8}$$

giving $d\mu_0/dt$ as purely gyrodependent. Then, we write

$$\begin{aligned}\Omega\frac{\partial\tilde{\mu}_1}{\partial\varphi}=\frac{d\mu_0}{dt}-\left\langle\frac{d\mu_0}{dt}\right\rangle&=-\frac{v_\perp^2}{2B}\vec{v}_\perp\cdot\nabla\ln B-\frac{v_\parallel^2}{B}\hat{n}\cdot\nabla\hat{n}\cdot\vec{v}\\&\quad-\frac{v_\parallel}{2B}[\vec{v}_\perp\vec{v}_\perp-(\vec{v}\times\hat{n})(\vec{v}\times\hat{n})]:\nabla\hat{n}-\frac{Ze}{MB}\vec{v}_\perp\cdot\nabla\phi\,.\end{aligned} \tag{A.9}$$

Our ordering allows large gradients for the electric potential and therefore the last term in (A.9) must be analyzed carefully. To do so, notice that

$$\left(\frac{\partial\phi}{\partial\varphi}\right)_{\psi_*,\theta_*,\zeta_*}\approx\frac{\partial\phi(\psi_*-\psi_1,\theta_*-\theta_1,\zeta_*-\zeta_1)}{\partial\varphi}=-\frac{\partial\phi}{\partial\psi}\frac{\partial\psi_1}{\partial\varphi}-\frac{\partial\phi}{\partial\theta}\frac{\partial\theta_1}{\partial\varphi}-\frac{\partial\phi}{\partial\zeta}\frac{\partial\zeta_1}{\partial\varphi}\,. \tag{A.10}$$

Using the relations for θ_1, ζ_1, and ψ_1 we obtain

$$\left(\frac{\partial\phi}{\partial\varphi}\right)_{\psi_*,\theta_*,\zeta_*}\approx-\frac{\partial\phi}{\partial\psi}\frac{\vec{v}_\perp\cdot\nabla\psi}{\Omega}-\frac{\partial\phi}{\partial\theta}\frac{\vec{v}_\perp\cdot\nabla\theta}{\Omega}-\frac{\partial\phi}{\partial\zeta}\frac{\vec{v}_\perp\cdot\nabla\zeta}{\Omega}=-\frac{\vec{v}_\perp\cdot\nabla\phi}{\Omega}\,. \tag{A.11}$$

This form is conveniently integrated over φ to find (29).

Energy. Once again, we begin by applying the Vlasov operator to the initial variable to find

$$\frac{dE_0}{dt} \equiv \frac{d}{dt}\left(\frac{v^2}{2}\right) = -\frac{Ze}{M}\vec{v}\cdot\nabla\phi = -\frac{Ze}{M}\vec{v}_\perp\cdot\nabla\phi - \frac{Ze}{M}v_\parallel\hat{n}\cdot\nabla\phi. \tag{A.12}$$

Next, with the help of (A.11) we extract the gyrodependent part of the total time derivative to find

$$\frac{dE_0}{dt} - \left\langle \frac{dE_0}{dt} \right\rangle \approx -\frac{Ze}{M}v_\parallel\hat{n}\cdot\nabla\tilde{\phi} + \frac{Ze}{M}\Omega\frac{\partial\tilde{\phi}}{\partial\varphi}. \tag{A.13}$$

Our orderings allow us to neglect the first term on the right side of (A.13) and therefore the equation for E_1 can be written as

$$\Omega\frac{\partial E_1}{\partial\varphi} = \frac{dE_0}{dt} - \left\langle \frac{dE_0}{dt} \right\rangle = \frac{Ze}{M}\Omega\frac{\partial\tilde{\phi}}{\partial\varphi}. \tag{A.14}$$

Integrating setting $\langle E_1 \rangle = 0$ gives

$$E_1 = \frac{Ze}{M}\tilde{\phi}. \tag{A.15}$$

A useful expression. Before deriving the first order correction to the gyrophase we obtain a useful relation that will also be helpful during the calculation of the second order corrections. Suppose we have a physical quantity given in terms of original spatial variables $Q = Q(\psi,\theta,\zeta)$. Then, according to (24) we define $Q_* \equiv Q\big(\psi_*,\theta_*,\zeta_*\big)$. As it has been already mentioned there is a first order difference between Q and Q_*. For a slowly varying function we have upon Taylor expanding

$$Q \approx Q\big(\psi_* - \psi_1, \theta_* - \theta_1, \zeta_* - \zeta_1,\big) \approx Q_* - \psi_1\frac{\partial Q}{\partial\psi} - \theta_1\frac{\partial Q}{\partial\theta} - \zeta_1\frac{\partial Q}{\partial\zeta}.$$

Note that this expansion is not normally valid for such quantities as electric potential and distribution function because they contain strong spatial gradients. Inserting the relations for θ_1, ζ_1, and ψ_1 we find

$$Q \approx Q_* - \frac{\partial f}{\partial\theta}\frac{\vec{v}\times\hat{n}}{\Omega}\cdot\nabla\theta - \frac{\partial f}{\partial\zeta}\frac{\vec{v}\times\hat{n}}{\Omega}\cdot\nabla\zeta - \frac{\partial f}{\partial\psi}\frac{\vec{v}\times\hat{n}}{\Omega}\cdot\nabla\psi + \frac{\partial f}{\partial\psi}\frac{Mc}{Ze}Rv_\parallel\hat{n}\cdot\hat{\zeta},$$

or defining I as in (22)

$$Q \approx Q_* - \frac{\vec{v}\times\hat{n}}{\Omega}\cdot\nabla Q + \frac{Iv_\parallel}{\Omega}\frac{\partial Q}{\partial\psi}. \tag{A.16}$$

Gyrophase. Evaluating $d\varphi_0/dt$ gives

$$\frac{d\varphi_0}{dt} = \frac{v_\parallel}{v_\perp^2}\vec{v}\cdot\nabla\hat{n}\cdot(\vec{v}\times\hat{n}) + \vec{v}\cdot\nabla\hat{e}_2\cdot\hat{e}_1 + \frac{Ze}{Mv_\perp^2}(\vec{v}\times\hat{n})\cdot\nabla\phi - \Omega. \tag{A.17}$$

To extract the gyrodependent part of $d\varphi_0/dt$ we have to take into account that Ω becomes slightly gyrodependent when expressed in terms of the starred variables. To do so we employ Eq. (A.16) to write

$$\Omega \approx \Omega_* - \frac{\vec{v}\times\hat{n}}{\Omega}\cdot\nabla\Omega + \frac{Iv_\parallel}{\Omega}\frac{\partial\Omega}{\partial\psi}. \tag{A.18}$$

In addition, we use the vector relation

$$\vec{v}\cdot\nabla\hat{n}\cdot(\vec{v}\times\hat{n}) = v_\parallel\vec{v}_\perp\cdot\hat{n}\times\vec{\kappa} - \frac{v_\perp^2}{2}\hat{n}\cdot\nabla\times\hat{n} + \frac{1}{2}[(\vec{v}\times\hat{n})\vec{v}_\perp + \vec{v}_\perp(\vec{v}\times\hat{n})]:\nabla\hat{n}, \tag{A.19}$$

where $\vec{\kappa} \equiv \hat{n}\cdot\nabla\hat{n}$ and the double-dot notation is defined by $\vec{a}\vec{c} : \vec{\vec{T}} \equiv \vec{c}\cdot\vec{\vec{T}}\cdot\vec{a}$.

Finally, we rewrite the $(\vec{v}\times\hat{n})\cdot\nabla\phi$ term so that it can be integrated over φ. For this purpose we notice that

$$\left(\frac{\partial\phi}{\partial\mu}\right)_{\psi_*,\theta_*,\zeta_*} \approx \frac{\partial\phi\big(\psi_* - \psi_1, \theta_* - \theta_1, \zeta_* - \zeta_1\big)}{\partial\mu_*} = -\frac{\partial\phi}{\partial\psi}\frac{\partial\psi_1}{\partial\mu} - \frac{\partial\phi}{\partial\theta}\frac{\partial\theta_1}{\partial\mu} - \frac{\partial\phi}{\partial\zeta}\frac{\partial\zeta_1}{\partial\mu}. \tag{A.20}$$

Using the relations for θ_1, ζ_1, and ψ_1, we find that

$$\left(\frac{\partial\phi}{\partial\mu}\right)_{\psi_*,\theta_*,\zeta_*} \approx -\frac{B}{\Omega v_\perp^2}\vec{v}\times\hat{n}\cdot\nabla\phi - \frac{B}{v_\parallel}\frac{Mc}{Ze}I\frac{\partial\phi}{\partial\psi}. \tag{A.21}$$

or

$$\frac{\vec{v}\times\hat{n}\cdot\nabla\phi}{v_\perp^2} = -\frac{\Omega}{B}\left(\frac{\partial\phi}{\partial\mu} + \frac{B}{v_\parallel}\frac{Mc}{Ze}I\frac{\partial\phi}{\partial\psi}\right). \tag{A.22}$$

On the right side of the last formula the original variables can be replaced by the starred ones without an error to the order of interest. Thus, the only φ dependence in $(\vec{v}\times\hat{n})\cdot\nabla\phi$ enters through the electric potential.

Inserting (A.18), (A.19), and (A.22) into (A.17) and gyroaveraging we obtain $\left\langle d\varphi_0/dt\right\rangle = -\overline{\Omega}$ as given by (33). Extracting the gyrodependent part of $d\varphi_0/dt$ and using $\Omega\,\partial\varphi_1/\partial\varphi_0 = d\varphi_0/dt - \left\langle d\varphi_0/dt\right\rangle$ yields

$$\begin{aligned}\varphi_1 = &-\frac{\vec{v}_\perp}{\Omega}\cdot\left(\frac{v_\parallel^2}{v_\perp^2}\vec{\kappa} - \hat{n}\times\nabla\hat{e}_2\cdot\hat{e}_1 + \nabla\ln B\right) - \frac{v_\parallel^2}{4\Omega v_\perp^2}[\vec{v}_\perp\vec{v}_\perp - (\vec{v}\times\hat{n})(\vec{v}\times\hat{n})]:\nabla\hat{n}\\ &-\frac{Ze}{MB}\left(\frac{\partial\tilde{\Phi}}{\partial\mu} + \frac{B}{v_\parallel}\frac{Mc}{Ze}I\frac{\partial\tilde{\Phi}}{\partial\psi_*}\right),\end{aligned} \tag{A.23}$$

where

$$\tilde{\Phi} \equiv \frac{1}{2\pi}\int^{\varphi}\phi\left(\psi_*,\theta_*,\zeta_*,E_*,\mu_*,\varphi_0'\right)d\varphi_0' \tag{A.24}$$

with $\left\langle\tilde{\Phi}\right\rangle = 0$.

Expression for $\nabla_\perp\phi$. To complete appendix A we give an expression that will be used in appendix C to prove that the magnetic moment correction given by (29) and (31) makes μ a good adiabatic invariant. This expression is obtained by using the relations (A.11) and (A.22) to decompose the perpendicular component of electric field as

$$\nabla_\perp\phi = -\frac{\Omega}{v_\perp^2}\frac{\partial\phi}{\partial\varphi}\vec{v}_\perp - \frac{\Omega}{B}\left(\frac{\partial\phi}{\partial\mu} + \frac{BI}{\Omega v_\parallel}\frac{\partial\phi}{\partial\psi_*}\right)\vec{v}\times\hat{n}. \tag{A.25}$$

Appendix B Second order corrections to gyrokinetic variables

In this appendix we perform a second iteration to evaluate the gyrokinetic variables to second order in δ. To carry out this calculation we apply the gyrokinetic procedure to the variables correct up the first order that were calculated in appendix A.

Spatial variables. We begin by evaluating

$$\frac{d}{dt}(\theta_0+\theta_1) = v_\parallel\hat{n}\cdot\nabla\theta + \frac{\vec{v}\times\hat{n}\vec{v}}{\Omega}:\nabla\nabla\theta - \vec{v}\cdot\nabla\left(\frac{\hat{n}}{\Omega}\right)\times\vec{v}\cdot\nabla\theta - \frac{c}{B}\nabla\phi\times\hat{n}\cdot\nabla\theta. \tag{B.1}$$

Here, the first term is one order larger than the others and therefore it needs to be expressed in terms of the new variables up to order δ. To do so for $\hat{n}\cdot\nabla\theta$, we employ (A.16),

$$\hat{n}\cdot\nabla\theta = (\hat{n}\cdot\nabla\theta)_* - \frac{\vec{v}\times\hat{n}}{\Omega}\cdot\nabla(\hat{n}\cdot\nabla\theta) + \frac{McI}{Ze}v_\parallel\frac{\partial}{\partial\psi_*}(\hat{n}\cdot\nabla\theta). \tag{B.2}$$

In addition, $v_\parallel$ requires some special care. Writing

$$v_\parallel = \sqrt{2(E_0 - \mu_0 B(\vec{r}))} \tag{B.3}$$

and using $v_\parallel^*$ from (25) we expand to obtain

$$v_{\|}^{*}-v_{\|}=\frac{\left(\mu_0-\mu_*\right)B_*+\left(B-B_*\right)\mu_*+\left(E_*-E_0\right)}{v_{\|}}. \tag{B.4}$$

Using (29), (31), and (A.15) and applying (A.16) to B we find

$$\begin{aligned} v_{\|}^{*}-v_{\|}=\frac{\vec{v}\cdot\vec{v}_{M}}{v_{\|}}-\mu_*\frac{\vec{v}\times\hat{n}}{\Omega v_{\|}}\cdot\nabla B+\mu_*\frac{I}{\Omega}\frac{\partial B}{\partial\psi_*}+\frac{\vec{v}_{\perp}\vec{v}_{\perp}}{4\Omega}:(\hat{n}\times\nabla\hat{n}-\nabla\hat{n}\times\hat{n})+ \\ +\frac{v_{\perp}^2}{2\Omega}\hat{n}\cdot\nabla\times\hat{n}. \end{aligned} \tag{B.5}$$

Having (B.2) and (B.5), we can now gyroaverage $v_{\|}\hat{n}\cdot\nabla\theta$ by writing

$$v_{\|}\hat{n}\cdot\nabla\theta=v_{\|}^{*}(\hat{n}\cdot\nabla\theta)_*+\left(v_{\|}-v_{\|}^{*}\right)(\hat{n}\cdot\nabla\theta)_*+\left[\hat{n}\cdot\nabla\theta-(\hat{n}\cdot\nabla\theta)_*\right]v_{\|}^{*}, \tag{B.6}$$

and after some algebra find

$$\left\langle v_{\|}\hat{n}\cdot\nabla\theta\right\rangle=v_{\|}^{*}(\hat{n}\cdot\nabla\theta)_*+\frac{Iv_{\|}}{\Omega}\frac{\partial\left(v_{\|}\hat{n}\cdot\nabla\theta\right)}{\partial\psi}-\frac{v_{\perp}^2}{2\Omega}(\hat{n}\cdot\nabla\times\hat{n})(\hat{n}\cdot\nabla\theta). \tag{B.7}$$

Next, we need to gyroaverage the rest of the terms in (B.1). These calculations give

$$\left\langle\frac{\vec{v}\times\hat{n}\vec{v}}{\Omega}:\nabla\nabla\theta\right\rangle=\frac{\vec{\vec{I}}\times\hat{n}}{\Omega}:\nabla\nabla\theta=0, \tag{B.8}$$

$$\left\langle\vec{v}\cdot\nabla\left(\frac{\hat{n}}{\Omega}\right)\times\vec{v}\cdot\nabla\theta\right\rangle=-\left[\hat{n}\times\left(\frac{v_{\|}^2}{\Omega}\vec{\kappa}+\frac{v_{\perp}^2}{2\Omega}\nabla\ln B\right)+\frac{v_{\perp}^2}{2\Omega}\hat{n}(\hat{n}\cdot\nabla\times\hat{n})\right]\cdot\nabla\theta, \tag{B.9}$$

and

$$\left\langle\frac{c}{B}\nabla\phi\times\hat{n}\cdot\nabla\theta\right\rangle=\frac{c}{B}\nabla\overline{\phi}\times\hat{n}\cdot\nabla\theta. \tag{B.10}$$

Collecting the terms we reproduce the relation (18) for $\left\langle d\left(\theta_0+\theta_1\right)/dt\right\rangle=\left\langle\dot{\theta}_*\right\rangle$.

Now, we can extract the gyrodependent part of (B.1) and, using

$$\Omega\frac{\partial\theta_2}{\partial\varphi_0}=\frac{d}{dt}(\theta_0+\theta_1)-\left\langle\frac{d}{dt}(\theta_0+\theta_1)\right\rangle,$$

integrate it over φ setting $\left\langle\theta_2\right\rangle=0$ to obtain θ_2 as

$$\begin{aligned} \theta_2=\frac{\vec{v}\times\hat{n}\vec{v}\times\hat{n}-\vec{v}_{\perp}\vec{v}_{\perp}}{4}:\frac{\nabla\nabla\theta}{\Omega^2}+\frac{1}{\Omega}\left[\left(\frac{\vec{v}_{\perp}}{4}+\vec{v}_{\|}\right)\vec{v}\times\hat{n}+\vec{v}\times\hat{n}\left(\frac{\vec{v}_{\perp}}{4}+\vec{v}_{\|}\right)\right]:\nabla\left(\frac{\hat{n}}{\Omega}\right)\times\nabla\theta \\ +\frac{\hat{n}\cdot\nabla\theta}{\Omega^2}\left[v_{\|}\vec{v}\cdot\vec{\kappa}+\frac{1}{8}(\vec{v}_{\perp}\vec{v}\times\hat{n}+\vec{v}\times\hat{n}\vec{v}_{\perp}):\nabla\hat{n}\right]+\frac{v_{\|}}{\Omega^2}\vec{v}_{\perp}\cdot\nabla\hat{n}\cdot\nabla\theta-\frac{c}{B\Omega}\nabla\tilde{\Phi}\times\hat{n}\cdot\nabla\theta. \end{aligned} \tag{B.11}$$

The calculation of ζ_2 involves exactly the same procedure as used for θ_2 giving (19) as well as

$$\begin{aligned} \zeta_2=\frac{\vec{v}\times\hat{n}\vec{v}\times\hat{n}-\vec{v}_{\perp}\vec{v}_{\perp}}{4}:\frac{\nabla\nabla\zeta}{\Omega^2}+\frac{1}{\Omega}\left[\left(\frac{\vec{v}_{\perp}}{4}+\vec{v}_{\|}\right)\vec{v}\times\hat{n}+\vec{v}\times\hat{n}\left(\frac{\vec{v}_{\perp}}{4}+\vec{v}_{\|}\right)\right]:\nabla\left(\frac{\hat{n}}{\Omega}\right)\times\nabla\zeta \\ +\frac{\hat{n}\cdot\nabla\zeta}{\Omega^2}\left[v_{\|}\vec{v}\cdot\vec{\kappa}+\frac{1}{8}(\vec{v}_{\perp}\vec{v}\times\hat{n}+\vec{v}\times\hat{n}\vec{v}_{\perp}):\nabla\hat{n}\right]+\frac{v_{\|}}{\Omega^2}\vec{v}_{\perp}\cdot\nabla\hat{n}\cdot\nabla\zeta-\frac{c}{B\Omega}\nabla\tilde{\Phi}\times\hat{n}\cdot\nabla\zeta, \end{aligned} \tag{B.12}$$

where $\nabla\zeta=\hat{\zeta}/R$ and $\nabla\nabla\zeta=-\left(\hat{\zeta}\hat{R}+\hat{R}\hat{\zeta}\right)/R^2$.

The total time derivative of ψ_* has been already given in the appendix A. Here we only have to extract the gyrodependent part of $\dot{\psi}_*$ in order to obtain $\left(\psi_*\right)_2$,

$$\dot{\psi}_*-\left\langle\dot{\psi}_*\right\rangle=c\frac{\partial\tilde{\phi}}{\partial\zeta}. \tag{B.13}$$

Integrating $\Omega\,\partial\left(\psi_*\right)_2/\partial\varphi_0=\dot{\psi}_*-\left\langle\dot{\psi}_*\right\rangle$ along with using $\left\langle\left(\psi_*\right)_2\right\rangle=0$ gives

$$\left(\psi_*\right)_2 = \frac{c}{\Omega}\frac{\partial\tilde{\Phi}}{\partial\zeta_*}, \tag{B.14}$$

where to second order $\psi_* \to \psi - \left(Mc/Ze\right)R\vec{v}\cdot\hat{\zeta} + \left(\psi_*\right)_2$.

Energy. To evaluate the Vlasov operator with the required precision it is convenient to write

$$\frac{d\left(E_0+E_1\right)}{dt} = -\frac{Ze}{M}\vec{v}\cdot\nabla\phi + \frac{Ze}{M}\frac{d\tilde{\phi}}{dt} \equiv \frac{Ze}{M}(-\frac{d\phi}{dt} + \left(\frac{\partial\phi}{\partial t}\right)_{\vec{r}}) + \frac{Ze}{M}\frac{d\tilde{\phi}}{dt} = \frac{Ze}{M}(-\frac{d\overline{\phi}}{dt} + \left(\frac{\partial\phi}{\partial t}\right)_{\vec{r}}). \tag{B.15}$$

We can express the total time derivative in terms of the starred variables as

$$\frac{d\overline{\phi}}{dt} = \left(\frac{\partial\overline{\phi}}{\partial t}\right)_* + \frac{\partial\overline{\phi}}{\partial\zeta_*}\dot{\zeta}_* + \frac{\partial\overline{\phi}}{\partial\theta_*}\dot{\theta}_* + \frac{\partial\overline{\phi}}{\partial\psi_*}\dot{\psi}_* + \frac{\partial\overline{\phi}}{\partial E_*}\dot{E}_* + \frac{\partial\overline{\phi}}{\partial\mu_*}\dot{\mu}_*, \tag{B.16}$$

where the $\partial\overline{\phi}/\partial\mu_*$ term can be neglected since $\dot{\mu}_* = 0$ to the requisite order. Also, using

$$\left(\frac{\partial\overline{\phi}}{\partial t}\right)_{\vec{r}} - \left(\frac{\partial\overline{\phi}}{\partial t}\right)_* \sim \delta\left(\frac{\partial\overline{\phi}}{\partial t}\right)_{\vec{r}} \sim \delta^2\Omega k_\perp\rho\overline{\phi}$$

and inserting (B.16) into (B.15) we find

$$\frac{d\left(E_0+E_1\right)}{dt} = \frac{Ze}{M}\frac{\partial\tilde{\phi}}{\partial t} - \frac{Ze}{M}\left(\frac{\partial\overline{\phi}}{\partial\zeta_*}\dot{\zeta}_* + \frac{\partial\overline{\phi}}{\partial\theta_*}\dot{\theta}_* + \frac{\partial\overline{\phi}}{\partial\psi_*}\dot{\psi}_* + \frac{\partial\overline{\phi}}{\partial E_*}\dot{E}_*\right). \tag{B.17}$$

Gyroaveraging and using $\left\langle\left(\dot{E}_0+\dot{E}_1\right)\right\rangle \approx \dot{E}_*$, we solve for $\dot{E}_*$ to find (27).

Next, we extract the gyrodependent part of $d\left(E_0+E_1\right)/dt$ to obtain the equation for E_2 to be

$$\Omega\frac{\partial E_2}{\partial\varphi} = \dot{E}_* - \left\langle\dot{E}_*\right\rangle = \frac{Ze}{M}\frac{\partial\tilde{\phi}}{\partial t}, \tag{B.18}$$

which upon integrating and setting $\left\langle E_2\right\rangle = 0$ yields

$$E_2 = \frac{c}{B}\frac{\partial\tilde{\Phi}}{\partial t}. \tag{B.19}$$

To finish this section we analyze the $\partial\overline{\phi}/\partial E_*$ term

$$\frac{\partial\overline{\phi}}{\partial E_*} \approx \frac{\partial\overline{\phi}\left(\psi_* - \frac{\vec{v}\times\hat{n}}{\Omega}\cdot\nabla\psi + \frac{Mc}{Ze}Rv_{\|}\hat{n}\cdot\hat{\zeta}, \theta_* - \frac{\vec{v}\times\hat{n}}{\Omega}\cdot\nabla\theta, \zeta_* - \frac{\vec{v}\times\hat{n}}{\Omega}\cdot\nabla\zeta\right)}{\partial E_*}. \tag{B.20}$$

Note, that in conventional gyrokinetics the first order corrections to the spatial variables involve only $v_\perp$ and therefore do not depend on E in leading order. Here, the correction to ψ_* involves $v_{\|}$ and therefore this term needs to be retained. From (B.20) we find

$$\frac{\partial\overline{\phi}}{\partial E_*} \approx \frac{Mc}{Ze}R\hat{n}\cdot\hat{\zeta}\frac{\partial\overline{\phi}}{\partial\psi_*}\frac{\partial v_{\|}}{\partial E} \approx \frac{I}{\Omega v_{\|}}\frac{\partial\overline{\phi}}{\partial\psi_*}. \tag{B.21}$$

This expression will be helpful for proving that the magnetic moment is a good invariant. Also, for numerical simulations the right side of the relation (B.21) may be more preferable to use than $\partial\phi/\partial E_*$. Indeed, the E_* dependence of $\overline{\phi}$ is weaker than the ψ_* dependence of $\overline{\phi}$ and therefore numerical evaluation of $\partial\phi/\partial E_*$ potentially contains a greater error than that of $\partial\overline{\phi}/\partial\psi_*$.

Appendix C. Magnetic moment

This appendix verifies that the corrections to the magnetic moment we employ allow us to neglect $\partial f/\partial\mu$ term in the kinetic equation. To do so, we need to prove that $\langle \dot{\mu}_* \rangle/\mu_0 \sim \delta^3\Omega$. This has been already proven for the case without electric potential [23,34,45]. Here, we need only check that the first and second order terms of $\dot{\mu}$ explicitly involving the electric potential gyroaverage away. These terms are given by

$$(\dot{\mu})_\phi \equiv -\frac{Ze}{MB}\vec{v}_\perp\cdot\nabla\phi + \frac{d}{dt}\left(\frac{Ze}{MB}\tilde{\phi}\right) - \frac{Ze}{M}\nabla\phi\cdot\nabla_v\left(\mu_1\big|_{\vec{r},\vec{v}}\right), \tag{C.1}$$

where we define

$$\mu_1\big|_{\vec{r},\vec{v}} \equiv -\frac{\vec{v}\cdot\vec{v}_M}{B} - \frac{v_\parallel}{4B\Omega}[\,\vec{v}_\perp(\vec{v}\times\hat{n}) + (\vec{v}\times\hat{n})\vec{v}_\perp\,]:\nabla\hat{n} - \frac{\mu_0 v_\parallel}{\Omega}\hat{n}\cdot\nabla\times\hat{n}\,. \tag{C.2}$$

It is convenient to consider the first two terms on the right side of (C.1) together

$$-\frac{Ze}{MB}\vec{v}_\perp\cdot\nabla\phi + \frac{d}{dt}\left(\frac{Ze}{MB}\tilde{\phi}\right) = -\frac{Ze}{MB}\left(\frac{d\overline{\phi}}{dt} - \left(\frac{\partial\phi}{\partial t}\right)_r - v_\parallel\hat{n}\cdot\nabla\phi\right) + \tilde{\phi}\frac{d}{dt}\left(\frac{Ze}{MB}\right).$$

Using the preceding allows us to rewrite $(\dot{\mu})_\phi$ as

$$(\dot{\mu})_\phi \equiv -\frac{Ze}{MB}\left(\frac{d\overline{\phi}}{dt} - \left(\frac{\partial\phi}{\partial t}\right)_r - v_\parallel\hat{n}\cdot\nabla\phi\right) - \frac{Ze}{MB}\tilde{\phi}\vec{v}\cdot\nabla\ln B - \frac{Ze}{M}\nabla\phi\cdot\nabla_v\left(\mu_1\big|_{\vec{r},\vec{v}}\right). \tag{C.3}$$

In the following subsections we evaluate each term of (C.3) up to order $\delta^2\Omega\mu_0$ in terms of the starred variables and then gyroaverage.

Third term. Here, we express $v_\parallel\hat{n}\cdot\nabla\phi$ in terms of starred variables before considering the first three terms together. We start by writing

$$\nabla\phi = \frac{\partial\phi}{\partial\psi_*}\nabla\psi_* + \frac{\partial\phi}{\partial\theta_*}\nabla\theta_* + \frac{\partial\phi}{\partial\zeta_*}\nabla\zeta_* + \frac{\partial\phi}{\partial E_*}\nabla E_* + \frac{\partial\phi}{\partial\mu_*}\nabla\mu_* + \frac{\partial\phi}{\partial\varphi_*}\nabla\varphi_*. \tag{C.4}$$

To evaluate the right side of (C.3) to the required order, relations (17), (15), (16), and (26) must be inserted for ψ_*, θ_*, ζ_*, and E_*, respectively. To the same order, for μ_* and φ_* we only need insert the zero order expressions in terms of δ. As a result, (C.4) becomes

$$\begin{aligned}\nabla\phi \approx{}& \frac{\partial\phi}{\partial\zeta_*}\nabla\zeta + \frac{\partial\phi}{\partial\theta_*}\nabla\theta + \frac{\partial\phi}{\partial\psi_*}\nabla\psi + \frac{\partial\phi}{\partial\zeta_*}\left(-\nabla\left(\frac{\hat{n}}{\Omega}\right)\times\vec{v}\cdot\nabla\zeta + \frac{\vec{v}\times\hat{n}}{\Omega}\cdot\nabla\nabla\zeta\right)\\ &+ \frac{\partial\phi}{\partial\theta_*}\left(-\nabla\left(\frac{\hat{n}}{\Omega}\right)\times\vec{v}\cdot\nabla\theta + \frac{\vec{v}\times\hat{n}}{\Omega}\cdot\nabla\nabla\theta\right) - \frac{\partial\phi}{\partial\psi_*}\frac{Mc}{Ze}\nabla\left(R\hat{\zeta}\right)\cdot\vec{v} + \frac{\partial\phi}{\partial E_*}\frac{Ze}{M}\nabla\tilde{\phi}\\ &- \frac{\partial\phi}{\partial\mu_*}\left(\mu_0\nabla\ln B + \frac{v_\parallel}{B}\nabla\hat{n}\cdot\vec{v}\right) + \frac{\partial\phi}{\partial\varphi_*}\left(\frac{v_\parallel}{v_\perp^2}\nabla\hat{n}\cdot(\vec{v}\times\hat{n}) + \nabla\hat{e}_2\cdot\hat{e}_1\right).\end{aligned} \tag{C.5}$$

The first three terms in the preceding equation are one order larger than the rest so the difference between $\nabla\zeta$, $\nabla\theta$, $\nabla\psi$ and $(\nabla\zeta)_*$, $(\nabla\theta)_*$, $(\nabla\psi)_*$ has to be taken into account. To do this we employ (23) and (A.16) so that Eq. (C.5) transforms into

$$\begin{aligned}\nabla\phi = {}& \frac{\partial\phi}{\partial\psi_*}(\nabla\psi)_* + \frac{\partial\phi}{\partial\theta_*}(\nabla\theta)_* + \frac{\partial\phi}{\partial\zeta_*}(\nabla\zeta)_* + \frac{Iv_\parallel}{\Omega}\left(\frac{\partial\phi}{\partial\zeta_*}\frac{\partial(\nabla\zeta)}{\partial\psi} + \frac{\partial\phi}{\partial\theta_*}\frac{\partial(\nabla\theta)}{\partial\psi} + \frac{\partial\phi}{\partial\psi_*}\frac{\partial(\nabla\psi)}{\partial\psi}\right)\\ &-\nabla\left(\frac{\hat{n}}{\Omega}\right)\times\vec{v}\cdot\left(\frac{\partial\phi}{\partial\zeta_*}\nabla\zeta + \frac{\partial\phi}{\partial\theta_*}\nabla\theta + \frac{\partial\phi}{\partial\psi_*}\nabla\psi\right) - \frac{\partial\phi}{\partial\psi_*}\frac{Mc}{Ze}\nabla\left(v_\parallel R\hat{\zeta}\cdot\hat{n}\right) + \frac{\partial\phi}{\partial E}\frac{Ze}{M}\nabla\tilde{\phi}\\ &- \frac{\partial\phi}{\partial\mu_*}\left(\mu_*\nabla\ln B + \frac{v_\parallel}{B}\nabla\hat{n}\cdot\vec{v}\right) + \frac{\partial\phi}{\partial\varphi_*}\left(\frac{v_\parallel}{v_\perp^2}\nabla\hat{n}\cdot(\vec{v}\times\hat{n}) + \nabla\hat{e}_2\cdot\hat{e}_1\right).\end{aligned} \tag{C.6}$$

To the required order we can write

$$\nabla\left(\frac{\hat{n}}{\Omega}\right)\times\vec{v}\cdot\left(\frac{\partial\phi}{\partial\psi_*}\nabla\psi+\frac{\partial\phi}{\partial\theta_*}\nabla\theta+\frac{\partial\phi}{\partial\zeta_*}\nabla\zeta\right)\approx\nabla\left(\frac{\hat{n}}{\Omega}\right)\times\vec{v}\cdot\nabla\phi=$$
$$=\frac{\nabla\hat{n}\times\vec{v}\cdot\nabla\phi}{\Omega}+\frac{\nabla\ln B}{\Omega}\vec{v}\times\hat{n}\cdot\nabla\phi\,,$$

and with the help of relation (A.25) we get

$$\nabla\left(\frac{\hat{n}}{\Omega}\right)\times\vec{v}\cdot\nabla\phi=-\left(\frac{\partial\phi}{\partial\mu}+\frac{BI}{\Omega v_{\|}}\frac{\partial\phi}{\partial\psi_*}\right)\left(\frac{v_\perp^2}{B}\nabla\ln B+\frac{v_{\|}}{B}\nabla\hat{n}\cdot\vec{v}\right)+\frac{v_{\|}}{v_\perp^2}\frac{\partial\phi}{\partial\varphi}\nabla\hat{n}\cdot\left(\vec{v}\times\hat{n}\right).\quad\text{(C.7)}$$

Finally, by inserting (C.7) into (C.6) we end up with

$$\nabla\phi\approx\frac{\partial\phi}{\partial\psi_*}(\nabla\psi)_*+\frac{\partial\phi}{\partial\theta_*}(\nabla\theta)_*+\frac{\partial\phi}{\partial\zeta_*}(\nabla\zeta)_*+\frac{Iv_{\|}}{\Omega}\left(\frac{\partial\phi}{\partial\psi_*}\frac{\partial(\nabla\psi)}{\partial\psi}+\frac{\partial\phi}{\partial\theta_*}\frac{\partial(\nabla\theta)}{\partial\psi}+\frac{\partial\phi}{\partial\zeta_*}\frac{\partial(\nabla\zeta)}{\partial\psi}\right)$$
$$+\frac{I}{\Omega v_{\|}}\frac{\partial\phi}{\partial\psi_*}\left(v_\perp^2\nabla(\ln B)+v_{\|}\nabla\hat{n}\cdot\vec{v}\right)+\mu_0\frac{\partial\phi}{\partial\mu}\nabla\ln B+\frac{\partial\phi}{\partial\varphi}\nabla\hat{e}_2\cdot\hat{e}_1-\frac{Mc}{Ze}\frac{\partial\phi}{\partial\psi_*}\nabla\left(v_{\|}R\hat{\zeta}\cdot\hat{n}\right)+\frac{Ze}{M}\frac{\partial\phi}{\partial E}\nabla\tilde{\phi}.$$

In the preceding expression the first three terms are one order larger than the rest.

Then, relating $v_{\|}$ and $v_{\|}^*$ and $\hat{n}$ and $\hat{n}_*$ using (A.16) and (B.5), noting that $\hat{n}_*\cdot(\nabla\psi)_*=(\hat{n}\cdot\nabla\psi)_*=0$, and observing that the $\left(\partial\phi\big/\partial E_*\right)v_{\|}\hat{n}\cdot\nabla\tilde{\phi}$ term is higher order, we evaluate $v_{\|}\hat{n}\cdot\nabla\phi$ to find

$$v_{\|}\hat{n}\cdot\nabla\phi\approx v_{\|}^*\hat{n}_*\cdot\left[\frac{\partial\phi}{\partial\zeta_*}(\nabla\zeta)_*+\frac{\partial\phi}{\partial\theta_*}(\nabla\theta)_*\right]+\frac{Iv_{\|}^2}{\Omega}\hat{n}\cdot\left(\frac{\partial\phi}{\partial\zeta_*}\frac{\partial(\nabla\zeta)}{\partial\psi}+\frac{\partial\phi}{\partial\theta_*}\frac{\partial(\nabla\theta)}{\partial\psi}+\frac{\partial\phi}{\partial\psi_*}\frac{\partial(\nabla\psi)}{\partial\psi}\right)$$
$$+\frac{I}{\Omega}\frac{\partial\phi}{\partial\psi_*}\left(v_\perp^2\hat{n}\cdot\nabla(\ln B)+v_{\|}\vec{\kappa}\cdot\vec{v}\right)+\mu_0 v_{\|}\frac{\partial\phi}{\partial\mu_*}\hat{n}\cdot\nabla\ln B+v_{\|}\frac{\partial\phi}{\partial\varphi_*}\hat{n}\cdot\nabla\hat{e}_2\cdot\hat{e}_1$$
$$-\frac{Mc}{Ze}\frac{\partial\phi}{\partial\psi_*}v_{\|}\hat{n}\cdot\nabla\left(v_{\|}R\hat{\zeta}\cdot\hat{n}\right)-v_{\|}\left(-\frac{\partial\hat{n}}{\partial\psi}\frac{Iv_{\|}}{\Omega}+\frac{\vec{v}\times\hat{n}}{\Omega}\cdot\nabla\hat{n}\right)\cdot\nabla\phi$$
$$-\left(\frac{\vec{v}\cdot\vec{v}_{M}}{v_{\|}}-\mu_*\frac{\vec{v}\times\hat{n}}{\Omega v_{\|}}\cdot\nabla B+\mu_*\frac{I}{\Omega}\frac{\partial B}{\partial\psi_*}+\frac{\vec{v}_\perp\vec{v}_\perp}{4\Omega}:(\hat{n}\times\nabla\hat{n}-\nabla\hat{n}\times\hat{n})+\frac{v_\perp^2}{2\Omega}\hat{n}\cdot\nabla\times\hat{n}\right)\hat{n}\cdot\nabla\phi\quad\text{(C.8)}$$

Then, we use

$$v_{\|}^2\frac{\partial\hat{n}}{\partial\psi}\cdot\nabla\phi+v_{\|}^2\hat{n}\cdot\left[\frac{\partial\phi}{\partial\zeta_*}\frac{\partial(\nabla\zeta)}{\partial\psi}+\frac{\partial\phi}{\partial\theta_*}\frac{\partial(\nabla\theta)}{\partial\psi}+\frac{\partial\phi}{\partial\psi_*}\frac{\partial(\nabla\psi)}{\partial\psi}\right]-\mu_0\frac{\partial B}{\partial\psi}\hat{n}\cdot\nabla\phi=$$
$$v_{\|}\left[\frac{\partial\phi}{\partial\theta_*}\frac{\partial}{\partial\psi}\left(v_{\|}\hat{n}\cdot\nabla\theta\right)+\frac{\partial\phi}{\partial\zeta_*}\frac{\partial}{\partial\psi}\left(v_{\|}\hat{n}\cdot\nabla\zeta\right)\right]$$

and

$$\hat{n}\cdot\nabla\left(v_{\|}R\hat{\zeta}\cdot\hat{n}\right)=R\hat{\zeta}\cdot\left[\hat{n}\cdot\nabla\left(\vec{v}\cdot\hat{n}\hat{n}\right)\right]=\frac{I}{B}\vec{v}\cdot\vec{\kappa}+v_{\|}R\hat{\zeta}\cdot\vec{\kappa}\,.$$

Gyroaveraging then gives

$$\left\langle v_{\|}\hat{n}\cdot\nabla\phi\right\rangle\approx v_{\|}^*\hat{n}_*\cdot\left[\frac{\partial\overline{\phi}}{\partial\zeta_*}(\nabla\zeta)_*+\frac{\partial\overline{\phi}}{\partial\theta_*}(\nabla\theta)_*\right]+\frac{Iv_{\|}}{\Omega}\left(\frac{\partial\overline{\phi}}{\partial\zeta_*}\frac{\partial\left(v_{\|}\hat{n}\cdot\nabla\zeta\right)}{\partial\psi}+\frac{\partial\overline{\phi}}{\partial\theta_*}\frac{\partial\left(v_{\|}\hat{n}\cdot\nabla\theta\right)}{\partial\psi}\right)$$
$$+\left(\frac{Iv_\perp^2}{\Omega}\frac{\partial\overline{\phi}}{\partial\psi_*}+\mu_0 v_{\|}\frac{\partial\overline{\phi}}{\partial\mu_*}\right)\hat{n}\cdot\nabla\ln B-\frac{Mc}{Ze}\frac{\partial\overline{\phi}}{\partial\psi_*}Rv_{\|}^2\hat{\zeta}\cdot\vec{\kappa}-\frac{v_\perp^2}{2\Omega}(\hat{n}\cdot\nabla\times\hat{n})\left(\hat{n}\cdot\nabla\overline{\phi}\right)\quad\text{(C.9)}$$
$$-\left\langle v_{\|}\frac{\vec{v}\times\hat{n}}{\Omega}\cdot\nabla\hat{n}\cdot\nabla\phi\right\rangle.$$

First three terms. Next, we analyze $d\overline{\phi}\big/dt$. We start by writing

$$\frac{d\overline{\phi}}{dt}=\frac{\partial\overline{\phi}}{\partial t}+\dot{\psi}_*\frac{\partial\overline{\phi}}{\partial\psi_*}+\dot{\theta}_*\frac{\partial\overline{\phi}}{\partial\theta_*}+\dot{\zeta}_*\frac{\partial\overline{\phi}}{\partial\zeta_*}+\dot{E}_*\frac{\partial\overline{\phi}}{\partial E_*},\quad\text{(C.10)}$$

where we insert (17) - (19) for $\dot{\psi}_*$, $\dot{\theta}_*$, and $\dot{\zeta}_*$, respectively, while for $\dot{E}_*$ we need only the leading order result

$$\dot{E}_* \approx -\frac{Ze}{M} v_\parallel \hat{n}\cdot\nabla\overline{\phi}\,. \tag{C.11}$$

To eliminate the terms quadratic in $\overline{\phi}$ we use (B.21) along with the observation that

$$\frac{\partial\overline{\phi}}{\partial\theta_*}\vec{v}_E\cdot\nabla\theta + \frac{\partial\overline{\phi}}{\partial\zeta_*}\vec{v}_E\cdot\nabla\zeta \approx -\frac{\partial\overline{\phi}}{\partial\psi_*}\vec{v}_E\cdot\nabla\psi\,,$$

where $\vec{v}_E \equiv \big(c/B\big)\hat{n}\times\nabla\overline{\phi}$. Then, (C.10) becomes

$$\begin{aligned}\frac{d\overline{\phi}}{dt} = \left(\frac{\partial\overline{\phi}}{\partial t}\right)_* &+ \frac{\partial\overline{\phi}}{\partial\zeta_*}\left[\big(v_\parallel^*\hat{n}_* + \vec{v}_M\big)\cdot\nabla\zeta + \frac{Iv_\parallel}{\Omega}\frac{\partial}{\partial\psi}\left(\frac{Iv_\parallel}{BR^2}\right)\right]\\ &+\frac{\partial\overline{\phi}}{\partial\theta_*}\left[\big(v_\parallel^*\hat{n}_* + \vec{v}_M\big)\cdot\nabla\theta + \frac{Iv_\parallel}{\Omega}\frac{\partial\big(v_\parallel\hat{n}\cdot\nabla\theta\big)}{\partial\psi}\right].\end{aligned} \tag{C.12}$$

Combining (C.12) with (C.9) we obtain

$$\begin{aligned}\frac{d\overline{\phi}}{dt} - \left\langle\left(\frac{\partial\phi}{\partial t}\right)_r\right\rangle - \big\langle v_\parallel\hat{n}\cdot\nabla\phi\big\rangle \approx \vec{v}_M\cdot\nabla\zeta\frac{\partial\overline{\phi}}{\partial\zeta_*} + \vec{v}_M\cdot\nabla\theta\frac{\partial\overline{\phi}}{\partial\theta_*} - \left(\frac{Iv_\perp^2}{\Omega}\frac{\partial\overline{\phi}}{\partial\psi_*} + \mu_0 v_\parallel\frac{\partial\overline{\phi}}{\partial\mu_*}\right)\hat{n}\cdot\nabla\ln B\\ + \frac{Mc}{Ze}Rv_\parallel^2\hat{\zeta}\cdot\vec{\kappa}\frac{\partial\overline{\phi}}{\partial\psi_*} + \frac{v_\perp^2}{2\Omega}(\hat{n}\cdot\nabla\times\hat{n})\big(\hat{n}\cdot\nabla\overline{\phi}\big) - \left\langle\frac{v_\parallel}{\Omega}\vec{v}\times\hat{n}\cdot\nabla\hat{n}\cdot\nabla\phi\right\rangle.\end{aligned} \tag{C.13}$$

Remaining terms. Finally, we analyze the last two terms in (C.3). As in the conventional gyrokinetics we find

$$\begin{aligned}-B\nabla\phi\cdot\nabla_v\Big(\mu_1\big|_{\vec{r},\vec{v}}\Big) = \frac{\vec{v}_\perp\cdot\nabla\phi}{\Omega}(\vec{v}\times\hat{n})\cdot\nabla\ln B + \vec{v}_M\cdot\nabla\phi + \frac{v_\parallel}{2\Omega}(\nabla\phi\times\hat{n})\cdot\nabla\hat{n}\cdot\vec{v}_\perp\\ + \frac{v_\parallel}{2\Omega}(\vec{v}\times\hat{n})\cdot\nabla\hat{n}\cdot\nabla\phi + \frac{v_\parallel}{2\Omega}(\vec{v}_\perp\cdot\nabla\phi)(\hat{n}\cdot\nabla\times\hat{n})\\ + \left[\frac{2v_\parallel}{\Omega}(\vec{v}\times\hat{n})\cdot(\hat{n}\cdot\nabla\hat{n}) + \frac{1}{4\Omega}[\vec{v}_\perp(\vec{v}\times\hat{n}) + (\vec{v}\times\hat{n})\vec{v}_\perp] : \nabla\hat{n} + \frac{v_\perp^2}{2\Omega}\hat{n}\cdot\nabla\times\hat{n}\right]\hat{n}\cdot\nabla\phi.\end{aligned}$$

Then, we notice that

$$\left\langle\frac{\vec{v}_\perp\cdot\nabla\phi}{\Omega}(\vec{v}\times\hat{n})\cdot\nabla\ln B\right\rangle = -\left\langle\frac{\partial\phi}{\partial\varphi}(\vec{v}\times\hat{n})\cdot\nabla\ln B\right\rangle = \big\langle\phi\vec{v}_\perp\big\rangle\cdot\nabla\ln B\,,$$

and therefore

$$\begin{aligned}\left\langle-\frac{\tilde{\phi}\vec{v}\cdot\nabla\ln B}{B} - \nabla\phi\cdot\nabla_v\Big(\mu_1\big|_{\vec{r},\vec{v}}\Big)\right\rangle = \frac{\vec{v}_M\cdot\nabla\phi}{B} + \frac{\mu_0}{\Omega}(\hat{n}\cdot\nabla\times\hat{n})\big(\hat{n}\cdot\nabla\overline{\phi}\big)\\ + \left\langle\frac{v_\parallel}{2\Omega B}(\nabla\phi\times\hat{n})\cdot\nabla\hat{n}\cdot\vec{v}_\perp + \frac{v_\parallel}{2\Omega B}(\vec{v}\times\hat{n})\cdot\nabla\hat{n}\cdot\nabla\phi\right\rangle\end{aligned} \tag{C.14}$$

Next, we combine the terms in the triangle brackets from (C.14) with the $\left\langle v_\parallel\frac{\vec{v}\times\hat{n}}{\Omega}\cdot\nabla\hat{n}\cdot\nabla\phi\right\rangle$ term from (C.13). With the help of relation (A.25) and $\vec{v}_\perp\vec{v}_\perp + (\vec{v}\times\hat{n})(\vec{v}\times\hat{n}) = v_\perp^2\big(\vec{\vec{I}} - \hat{n}\hat{n}\big)$, we obtain

$$\left\langle\frac{v_\parallel}{2\Omega}(\nabla\phi\times\hat{n})\cdot\nabla\hat{n}\cdot\vec{v}_\perp - \frac{v_\parallel}{2\Omega}(\vec{v}\times\hat{n})\cdot\nabla\hat{n}\cdot\nabla\phi\right\rangle = -v_\parallel\mu_0\left(\frac{\partial\overline{\phi}}{\partial\mu} + \frac{BI}{\Omega v_\parallel}\frac{Mc}{Ze}\frac{\partial\overline{\phi}}{\partial\psi_*}\right)\hat{n}\cdot\nabla\ln B\,. \tag{C.15}$$

Combining terms. Finally, we combine the results from the subsections of this appendix to obtain

$$\frac{MB}{Ze}(\dot{\mu})_\phi = \frac{\partial\overline{\phi}}{\partial\psi_*}\vec{v}_M\cdot\nabla\psi + \frac{Iv_\perp^2}{2\Omega}\frac{\partial\overline{\phi}}{\partial\psi_*}\hat{n}\cdot\nabla\ln B - \frac{\partial\overline{\phi}}{\partial\psi_*}\frac{Mc}{Ze}Rv_\parallel^2\hat{\zeta}\cdot\vec{\kappa}\,.$$

Noticing that $\vec{v}_{\nabla B}\cdot\nabla\psi = -\Big(Iv_\perp^2\big/2\Omega\Big)\hat{n}\cdot\nabla\ln B$ and $\vec{v}_\kappa\cdot\nabla\psi = \Big(Mc\big/Ze\Big)Rv_\parallel^2\hat{\zeta}\cdot\vec{\kappa}$ we find to the requisite order the desired result

$$(\dot{\mu})_\phi = 0\,. \tag{C.16}$$

Appendix D. Jacobian in the strong potential gradient case

To follow is the derivation of the leading order Jacobian of the transformation from the original set of variables to the one consisting of ψ_*, θ_*, ζ_*, ε, μ_*, and φ_*. We start by writing

$$\frac{1}{J}\equiv\left|\frac{\partial\big(\psi_*,\theta_*,\zeta_*,\varepsilon,\mu_*,\varphi_*\big)}{\partial\big(\vec{r},\vec{v}\,\big)}\right| = \left|\begin{array}{c|c|c|c|c|c} \nabla\psi_* & \nabla\theta_* & \nabla\zeta_* & \nabla\varepsilon & \nabla\mu_* & \nabla\varphi_* \\ \hline \nabla_v\psi_* & \nabla_v\theta_* & \nabla_v\zeta_* & \nabla_v\varepsilon & \nabla_v\mu_* & \nabla_v\varphi_* \end{array}\right|. \tag{D.1}$$

Keeping only the leading order terms in all the blocks yields

$$\frac{1}{J} = \left|\begin{array}{c|c|c|c|c|c} \nabla\psi & \nabla\theta & \nabla\zeta & \dfrac{Ze}{M}\nabla\phi & 0 & 0 \\ \hline -\dfrac{I}{\Omega}\hat{n} & 0 & 0 & \vec{v} & \dfrac{\vec{v}_\perp}{B} & -\dfrac{\vec{v}\times\hat{n}}{v_\perp^2} \end{array}\right|. \tag{D.2}$$

In the absence of sharp potential gradient we would neglect the $\nabla\phi$ term in the upper-right block to obtain the usual expression for the leading order Jacobian, namely,

$$\frac{1}{J} = -\left[\left(\vec{v}\times\frac{\vec{v}_\perp}{B}\right)\cdot\frac{\vec{v}\times\hat{n}}{v_\perp^2}\right](\nabla\psi\times\nabla\theta\cdot\nabla\zeta) = \frac{v_\parallel}{B}\vec{B}\cdot\nabla\theta\,. \tag{D.3}$$

To calculate the determinant for $w\sim\rho_{pol}$ we multiply the first column of matrix (D.2) by $\Big(Ze\big/M\Big)\Big(\partial\phi\big/\partial\psi\Big)$, the second by $\Big(Ze\big/M\Big)\Big(\partial\phi\big/\partial\theta\Big)$ and the third by $\Big(Ze\big/M\Big)\Big(\partial\phi\big/\partial\zeta\Big)$, add them together and subtract the resulting linear combination from the fourth column of matrix (D.2) to obtain

$$\frac{1}{J} = \left|\begin{array}{c|c|c|c|c|c} \nabla\psi & \nabla\theta & \nabla\zeta & 0 & 0 & 0 \\ \hline -\dfrac{I}{\Omega}\hat{n} & 0 & 0 & \vec{v}+\dfrac{cI}{B}\dfrac{\partial\phi}{\partial\psi}\hat{n} & \dfrac{\vec{v}_\perp}{B} & -\dfrac{\vec{v}\times\hat{n}}{v_\perp^2} \end{array}\right|. \tag{D.4}$$

The preceding determinant is easily evaluated to find

$$\frac{1}{J} = \left(\frac{v_\parallel}{B}+\frac{cI}{B^2}\frac{\partial\phi}{\partial\psi}\right)\vec{B}\cdot\nabla\theta\,. \tag{D.5}$$

Notice, that if $1\big/w\equiv\Big|\Big(e\big/T\Big)\Big(\partial\phi\big/\partial\psi\Big)\nabla\psi\Big|$ is of order $1\big/\rho_{pol}$ the two terms on the right side of (D.5) are comparable.

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