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Collisional zonal flow damping for ITG modes

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Abstract

Zonal flow helps reduce and control the level of ion temperature gradient (ITG) turbulence in a tokamak. The collisional damping of zonal flow has been estimated by Hinton and Rosenbluth (H-R). Their calculation shows that the damping of zonal flow is closely related to the frequency response of neoclassical polarization of the plasma. Based on a variational principle, H-R calculated the neoclassical polarization in the low and high collisionality limits. A new approach, based on an eigenfunction expansion of the collision operator, is employed to evaluate the neoclassical polarization and the zonal flow residual for arbitrary collisionality. An analytical expression for the temporal behavior of the zonal flow is also given showing that the damping rate tends to be somewhat slower than previously thought. These results are expected to be useful extensions of the original H-R collisional work that can provide an effective benchmark for numerical codes for all regimes of collisionality.

I. INTRODUCTION

It is known that zonal flow is an important mechanism to suppress ion temperature gradient (ITG) turbulence [1][2][3]. Consequently it is important to understand the damping mechanisms that act on zonal flow. The original Rosenbluth-Hinton (R-H) study showed that zonal flow is modified by the collisionless neoclassical polarization, but significant residual flow survives due to the smallness of this polarization [4]. Later, Hinton-Rosenbluth (H-R) found that collisional effects could significantly reduce the residual zonal flow to a level much smaller [5] than the collisionless kinetic theory predicts [4]. This collisional zonal flow damping has been tested by gyrokinetic simulation [6][7]. The original H-R analytical calculation [5] is based on a variational principle and mathematically rather complicated. In addition, it is only valid in two asymptotic limits: high and low collision frequency [5]. Here we provide a semi-analytical method to calculate the collisional zonal flow damping for arbitrary collisionality based on an eigenfunction expansion of the collision operator. In addition, this new approach is used to obtain a simple analytical expression for the neoclassical polarization, which is not only accurate in the two asymptotic limits, but also captures the main features of the intermediate collision frequencies between these two limits. The associated zonal flow damping is then readily identified by this new approach.

Polarization is a shielding phenomenon associated with the plasma. In perpendicular wavenumber and frequency space, the polarization $\varepsilon_k^{pol}(p)$ is defined by

$$\varepsilon_k^{pol}(p)\left\langle k_\perp^2\right\rangle \phi_k(p) = -4\pi e\left\langle \int d^3v f_k^i(p)\right\rangle, \tag{1}$$

where $p = i\omega$ is the frequency variable, and the density distribution function $f_k^i(p)$ is calculated from the linearized gyrokinetic equation in the frequency domain. Only the ion charge density is considered since the ion polarization is larger than the electron polarization by the mass ratio m_i/m_e for the ITG limit of interest here. The Laplace transforms of ϕ_k and f_k^i are defined by $\phi_k(p) = \int_0^\infty dte^{-pt}\phi_k(t)$ and $f_k^i(p) = \int_0^\infty dte^{-pt} f_k^i(t)$.

The linear polarization charge density is compensated by the nonlinear turbulent density due to quasineutrality. In the H-R model, it is assumed that the turbulence produces a perturbed charge density within a time longer than the ion gyroperiod, but much shorter than the ion bounce time. Therefore, this initial turbulent charge density drives an initial zonal flow potential according to classical polarization $\varepsilon_{k,cl}^{pol}$, which is due to particle departure from the guiding center and can be easily identified as $\varepsilon^{pol}_{k,cl} = \omega^2_{pi}/\omega^2_{ci}$, where ω_{pi} is the ion plasma frequency and ω_{ci} is the ion gyrofrequency.

This initial zonal flow potential will then evolve according to the following equation [5] :

$$\phi_k(t) = \phi_k(t=0)\, K_k(t), \tag{2}$$

where the response kernel for an initial charge perturbation, $K_k(t)$, is defined as

$$K_k(t) = \frac{1}{2\pi i}\int \frac{dpe^{pt}}{p}\frac{\varepsilon^{pol}_{k,cl}}{\varepsilon^{pol}_{k}(p)}, \tag{3}$$

and $\varepsilon^{pol}_k(p) = \varepsilon^{pol}_{k,cl} + \varepsilon^{pol}_{k,nc}(p)$ with $\varepsilon^{pol}_{k,nc}(p)$ the neoclassical polarization that is due to the guiding center departure from flux surface, and the path of the p integration is from $-\infty i$ to $+\infty i$, and to the right of all the singularities of the integrand. From the preceding equation, the long time asymptotic behavior of the zonal flow, the so caller residual, depends on the zero frequency polarization response, i.e.,

$$\frac{\phi_k(t=\infty)}{\phi_k(t=0)} = \frac{\varepsilon^{pol}_{k,cl}}{\varepsilon^{pol}_{k,cl} + \varepsilon^{pol}_{k,nc}(0)}. \tag{4}$$

The remainder of this paper is organized as following. In Sec. II, we review H-R collisional kinetics to calculate the neoclassical polarization. Section III provides a detailed discussion of the eigenfunction expansion method and some properties of the eigenfunctions and eigenvalues of the pitch angle scattering operator. In Sec. IV, we apply this new approach to the calculation of neoclassical polarization. We then calculate the temporal dependence of the zonal flow potential in Sec. V. Section VI considers the energy dependence of the pitch angle scattering operator more carefully, which modifies the Sec.V result by a numerical factor. In Sec. VII the model ion-ion collision operator is improved further by requiring that it conserve momentum. Based on the results of the these sections, we calculate the zonal flow rotation in Sec. VIII. Concluding remarks are given in Sec. IX.

II. TRANSIT AVERAGE KINETIC EQUATION AND NEOCLASSICAL POLARIZATION

When the radial wavelength of zonal flow is much larger than the ion poloidal gyroradius, the neoclassical polarization can be separated from the total plasma polarization. To calculate the neoclassical polarization, we focus on the gyrophase independent part of the ion distribution function f_k^i, which we will denote as f_k for convenience. This gyrophase independent ion distribution function f_k is driven by the axisymmetric zonal flow potential ϕ_k [5]:

$$\frac{\partial f_k}{\partial t} + \left(v_{\shortparallel}\mathbf{b}\cdot\nabla + i\omega_D\right) f_k - C_{ii}\left\{f_k\right\} = -\frac{e}{T_i}F_0\left(v_{\shortparallel}\mathbf{b}\cdot\nabla + i\omega_D\right)\phi_k, \tag{5}$$

where F_0 is a local *Maxwellian*, C_{ii} is the linearized ion-ion collision operator, and the magnetic drift frequency is $\omega_D = \mathbf{k}_\perp \cdot \mathbf{v}_d$. Here we assume all perturbed quantities take a eikonal form, $\phi(\mathbf{r},t) = \sum\limits_k \phi_k e^{iS}$ with the eikonal $S = S(\psi)$ and the radial wave vector $\mathbf{k}_\perp = \nabla S$. The magnetic drift $\mathbf{v}_d = \frac{\mathbf{b}}{\Omega} \times \left(\mu\nabla B + v_\parallel^2 \mathbf{b}\cdot\nabla\mathbf{b}\right)$ has the usual radial component form $\mathbf{v}_d \cdot \nabla\psi = v_{\shortparallel}\mathbf{b}\cdot\nabla\left(\frac{Iv_{\shortparallel}}{\Omega}\right)$. Following *H-R* it is convenient to define $\omega_D = v_{\shortparallel}\mathbf{b}\cdot\nabla Q$ with $Q = IS' v_{\shortparallel}/\Omega$. Notice $Q \sim k_\perp \rho_p$, where $\rho_p = \rho_i q/\varepsilon$ is the poloidal gyroradius. The independent velocity variables used in the preceding equation are kinetic energy $E = v^2/2$ and magnetic moment $\mu = v_\perp^2/2B$. Here we will only consider the collisionless the large aspect ratio circular flux surface limit as in R-H.

To solve this equation, we separate the adiabatic response from the total distribution by letting

$$f_k \equiv -\frac{e}{T_i}\phi_k F_0 + H_k e^{-iQ}. \tag{6}$$

Employing the fact that the zonal flow potential is independent of the position along a field line, the new distribution to be determined, H_k, then satisfies the following equation,

$$\frac{\partial H_k}{\partial t} + v_{\shortparallel}\mathbf{b}\cdot\nabla H_k - e^{iQ}C_{ii}\left\{H_k e^{-iQ}\right\} = e^{iQ}\frac{e}{T_i}F_0\frac{\partial \phi_k}{\partial t}. \tag{7}$$

The presence of collisions substantially complicates solving this equation. Fortunately, there are two small parameters hidden in this equation that we can employ. The first small parameter is ω/ω_t, provided that the driving frequency of zonal flow potential ω is much smaller than the ion thermal transit frequency $\omega_t = v_i/qR_0$, where $v_i = \sqrt{2T_i/m_i}$ is the ion thermal speed. The second small parameter is Q since only large wavelength zonal flows are considered. In the original H-R calculation [5], the equation is solved perturbatively by expanding first in ω/ω_t and then in Q. However, here we employ a different approach by expanding first in Q and then in ω/ω_t.

For $Q \ll 1$, we may expand H_k

$$H_k = H_k^{(0)} + H_k^{(1)} + H_k^{(2)} + ...,$$

where $H_k^{(j+1)} = H_k^{(j)}\mathcal{O}(Q)$. The leading order kinetic equation in this expansion becomes

$$\frac{\partial H_k^{(0)}}{\partial t} + v_{\shortparallel}\mathbf{b}\cdot\nabla H_k^{(0)} - C_{ii}\left\{H_k^{(0)}\right\} = \frac{e}{T_i}F_0\frac{\partial \phi_k}{\partial t}. \tag{8}$$

By inspection, we find that the leading order solution is simply,

$$H_k^{(0)} = \frac{e}{T_i}F_0\phi_k, \tag{9}$$

since $\mathbf{b}\cdot\nabla H_k^{(0)} = C_{ii}\left\{H_k^{(0)}\right\} = 0$. This piece of the lowest order distribution simply cancels the adiabatic response in Eq. (6). The next order kinetic equation in the Q expansion gives

$$\frac{\partial H_k^{(1)}}{\partial t} + v_{\shortparallel}\mathbf{b}\cdot\nabla H_k^{(1)} - C_{ii}\left\{H_k^{(1)}\right\} = iQ\frac{e}{T_i}F_0\frac{\partial \phi_k}{\partial t}, \tag{10}$$

since $C_{ii}\left\{H_k^{(0)}Q\right\} = 0$ because the momentum must be conserved in like collisions. In addition, the second order kinetic equation in this Q expansion series becomes

$$\frac{\partial H_k^{(2)}}{\partial t} + v_{\shortparallel}\mathbf{b}\cdot\nabla H_k^{(2)} - C_{ii}\left\{H_k^{(2)}\right\} - iQC_{ii}\left\{H_k^{(1)}\right\} + C_{ii}\left\{iQH_k^{(1)}\right\} = -\frac{1}{2}Q^2\frac{e}{T_i}F_0\frac{\partial \phi_k}{\partial t}. \tag{11}$$

In terms of the $H_k^{(j)}$, the perturbed density function f_k to Q^2 accuracy can be written as,

$$f_k \cong \left(-\frac{e}{T_i}F_0\phi_k iQ + H_k^{(1)}\right)(1-iQ) + \frac{e}{T_i}F_0\phi_k\frac{1}{2}Q^2 + H_k^{(2)} \tag{12}$$

To calculate the polarization constant, it is convenient to make a detour and first calculate the time change of flux-surface averaged polarization density, $\left\langle n_k^{pol}\right\rangle = \left\langle \int d^3v f_k\right\rangle$, to obtain

$$\begin{aligned}\frac{\partial}{\partial t}\left\langle n_k^{pol}\right\rangle &= \frac{\partial}{\partial t}\left\langle \int d^3v f_k\right\rangle \\ &= \left\langle \int d^3v\left[\left(-\frac{ieQF_0}{T_i}\frac{\partial\phi_k}{\partial t} + \frac{\partial H_k^{(1)}}{\partial t}\right)(1-iQ) + \frac{eQ^2F_0}{2T_i}\frac{\partial\phi_k}{\partial t} + \frac{\partial H_k^{(2)}}{\partial t}\right]\right\rangle\end{aligned} \tag{13}$$

Inserting Eqs. (10) and (11) for $\partial H_k^{(1)}/\partial t$ and $\partial H_k^{(2)}/\partial t$ in the preceding equation and utilizing the properties of linear ion-ion collision operator C_{ii}, we find

$$\frac{\partial}{\partial t}\left\langle n_k^{pol}\right\rangle = \left\langle \int d^3v\left(-\frac{e}{T_i}F_0 iQ\frac{\partial\phi_k}{\partial t} + \frac{\partial H_k^{(1)}}{\partial t}\right)(-iQ)\right\rangle.$$

Thus, we obtain

$$\left\langle n_k^{pol}\right\rangle = -\left\langle \int d^3v\left(iQH_k^{(1)} + \frac{e}{T_i}\phi_k F_0 Q^2\right)\right\rangle. \tag{14}$$

This expression for the polarization density is accurate to second order in Q^2, yet we need only solve the first order equation, Eq. (10).

At this stage, a further simplification can be made to the polarization density in Eq. (14) by assuming a large aspect ratio circular flux surface tokamak model. In addition, we define the pitch angle variable $\lambda = v_\perp^2 B_0 / v^2 B$, with B_0 the on axis value of magnetic field and $B_0/B = R/R_0 = 1 + \varepsilon \cos\theta$, then the velocity volume element d^3v can be written as $d^3v = 4\pi B E dE d\lambda / B_0 \left| v_\parallel \right|$. Using $Q = IS' v_\parallel / \Omega$, the polarization density can be written as

$$\left\langle n_k^{pol} \right\rangle = n_0 \frac{e\phi_k}{T_i} k_\perp^2 \rho_i^2 \frac{B_0^2}{B_p^2} \frac{3}{2} \int d\lambda \left(\left\langle \frac{\Omega_0 T_i}{i\sigma I S' v e \phi_k F_0} \oint \frac{d\theta h}{2\pi} H_k^{(1)} \right\rangle_E - \oint \frac{d\theta h^2}{2\pi} \xi \right), \tag{15}$$

where $\xi = \left| v_\parallel \right| / v$, is the dimensionless parallel speed with $\sigma = v_\parallel / \left| v_\parallel \right|$, $h \equiv B_0/B = 1 + \varepsilon \cos\theta$ for a large aspect ratio circular tokamak, and the energy average is defined as

$$\langle A \rangle_E = \frac{\int_0^\infty dE E^{3/2} e^{-mE/T} A}{\int_0^\infty dE E^{3/2} e^{-mE/T}}. \tag{16}$$

The preceding equation can be used in Eq. (1) to obtain the neoclassical polarization in the form,

$$\varepsilon_{k,nc}^{pol}(p) = \frac{\omega_{pi}^2}{\omega_{ci}^2} \frac{q^2}{\varepsilon^2} \frac{3}{2} \int d\lambda \left(\oint \frac{d\theta h^2}{2\pi} \xi - \left\langle \frac{\Omega_0 T_i}{i\sigma I S' v e \phi_k F_0} \oint \frac{d\theta h}{2\pi} H_k^{(1)} \right\rangle_E \right) \tag{17}$$

where $\varepsilon = r/R_0$ is the inverse aspect ratio for a tokamak, $\omega_{pi} = \sqrt{4\pi e^2 n_0 / m_i}$ is the ion plasma frequency, $\omega_{ci} = eB_0/m_i c$ is the ion gyrofrequency at the magnetic axis, $\rho_i = v_i/\omega_{ci}$ is the ion gyroradius. This form of the expression for $\varepsilon_{k,nc}^{pol}(p)$ is convenient to display the λ space structure of total perturbed distribution and determine the contributions from trapped particles and passing particles separately, even though the second term in the preceding equation can easily be integrated by using $\int d^3v v_\parallel^2 F_0 = n_0 T_i / m_i$, to obtain an alternative form for the neoclassical polarization,

$$\varepsilon_{k,nc}^{pol}(p) = \frac{\omega_{pi}^2}{\omega_{ci}^2} \frac{q^2}{\varepsilon^2} \left(1 - \int_0^{1-\varepsilon} d\lambda \left\langle \frac{3\Omega_0 T_i}{2i\sigma I S' v e \phi_k F_0} \oint \frac{d\theta h}{2\pi} H_k^{(1)} \right\rangle_E \right). \tag{18}$$

Next we solve Eq. (10) perturbatively for $H_k^{(1)}$ by expanding in the second small parameter, $\omega/\omega_t \ll 1$,

$$H_k^{(1)} = h_k^{(1)} + h_k^{(2)} + ...$$

Then, the leading order equation $v_\parallel \mathbf{b} \cdot \nabla h_k^{(1)} = 0$ gives that $h_k^{(1)}$ is independent of poloidal angle θ. A transit average of the next order equation gives,

$$\frac{\partial h_k^{(1)}}{\partial t} - \overline{C_{ii}\left\{ h_k^{(1)} \right\}} = i\overline{Q} \frac{e}{T_i} F_0 \frac{\partial \phi_k}{\partial t}, \tag{19}$$

where the transit average is defined as, $\overline{A} = \oint d\tau A / \oint d\tau$, with $d\tau = d\theta / \left(v_{\shortparallel} \mathbf{b} \cdot \nabla \theta \right)$. For trapped particles, this average is over a full bounce; while for passing particles, it is over one complete poloidal circuit. Specifically, for a large aspect ratio circular cross section tokamak, $d\tau \cong qR_0 d\theta / v_{\shortparallel}$, where q is the safety factor. In this case the transit average becomes

$$\overline{A} = \frac{\oint \frac{d\theta}{v_{\shortparallel}} A}{\oint \frac{d\theta}{v_{\shortparallel}}}. \tag{20}$$

The preceding transit averaged equation (19) is what Hinton and Rosenbluth(H-R) solve in two limits to obtain their collisional results [5].

Since the distribution $h_k^{(1)}$ is the leading order approximation to h_k, it is not necessary to evaluate higher order terms, and we need not make a distinction between $h_k^{(1)}$ and $H_k^{(1)}$. The trapped contribution satisfying Eq. (19) is simply $h_k^{(1)} = 0$ because the drive $\overline{Q}$ vanishes. The remaining task is to solve the transit averaged kinetic equation Eq. (19) for the passing particle distribution $h_k^{(1)}$, which is required to calculate neoclassical polarization in Eq. (18). Laplace transforming Eq. (19), the transit averaged kinetic equation can be written as

$$h_k(p) - \frac{1}{p}\overline{C_{ii}\left\{h_k(p)\right\}} = i\overline{Q}\frac{e\phi_k(p)}{T_i}F_0, \tag{21}$$

where the distribution $h_k(p)$ is the Laplace transform of $h_k^{(1)}$ and independent of the poloidal angle θ. Since all the analysis that follows is done in the frequency domain, $h_k(p)$ is abbreviated to h_k without causing any confusion.

It is known that pitch angle scattering is the dominant collisional process in a large aspect ratio tokamak plasma [8][9]. We therefore approximate the ion-ion collision operator by the model *Lorentz* operator used by H-R by using the simplifying replacement

$$C_{ii}\left\{h_k\right\} = 2\left(\frac{T_i}{m_i E}\right)^{3/2} \nu_{ii}\frac{B_0}{B}\xi\frac{\partial}{\partial\lambda}\lambda\xi\frac{\partial h_k}{\partial\lambda}, \tag{22}$$

with

$$\nu_{ii} = \frac{4\pi e^4 n_i \ln\Lambda}{m_i^{1/2}\left(2T_i\right)^{3/2}} \tag{23}$$

For a large aspect ratio circular tokamak, Eq. (21) can be written as

$$h_k - 2\frac{\nu_{ii}}{p}\left(\frac{T_i}{m_i E}\right)^{3/2}\frac{1}{L(\lambda)}\frac{\partial}{\partial\lambda}D(\lambda)\frac{\partial}{\partial\lambda}h_k = i\frac{2\pi\sigma e\phi_k}{T_i L(\lambda)}\frac{IS'v}{\Omega}F_0, \tag{24}$$

where the functions $L(\lambda)$ and $D(\lambda)$ are defined by

$$L(\lambda) = \oint \frac{d\theta}{\sqrt{1+\varepsilon\cos\theta-\lambda}} \tag{25}$$

$$= 4\sqrt{\frac{k}{2\varepsilon}}K(k), \tag{26}$$

$$D(\lambda) = \oint d\theta\lambda\sqrt{1+\varepsilon\cos\theta-\lambda} \tag{27}$$

$$= 4\lambda\sqrt{\frac{2\varepsilon}{k}}E(k), \tag{28}$$

with k defined as

$$k \equiv \frac{2\varepsilon}{1-\lambda+\varepsilon}. \tag{29}$$

Note that $L(\lambda)$ is proportional to the bounce time. If we define a dimensionless distribution G_k through

$$G_k = \frac{\Omega_0 T_i}{i\sigma e\phi_k I S' v F_0} h_k, \tag{30}$$

then G_k satisfies

$$G_k - 2\frac{\nu_{ii}}{p}\left(\frac{T_i}{m_i E}\right)^{3/2}\frac{1}{L(\lambda)}\frac{\partial}{\partial\lambda}D(\lambda)\frac{\partial}{\partial\lambda}G_k = \frac{2\pi}{L(\lambda)}. \tag{31}$$

Therefore, the neoclassical polarization in Eq. (18) can be written as

$$\varepsilon_{k,nc}^{pol}(p) = \frac{\omega_{pi}^2}{\omega_{ci}^2}\frac{q^2}{\varepsilon^2}\left(1 - \frac{3}{2}\int_0^{1-\varepsilon} d\lambda \left\langle G_k\right\rangle_E\right), \tag{32}$$

when the energy average defined by Eq. (16) is employed.

In the collisionless limit, $G_k = 2\pi/L(\lambda)$. According to Eq. (17), the collisionless neoclassical polarization can be written as

$$\varepsilon_{k,nc}^{pol}(p) = \frac{\omega_{pi}^2}{\omega_{ci}^2}\frac{q^2}{\varepsilon^2}\frac{3}{2}\left(\int_0^{1+\varepsilon} d\lambda \oint \frac{d\theta h^2}{2\pi}\xi - \int_0^{1-\varepsilon} d\lambda\frac{2\pi}{L(\lambda)}\right) \tag{33}$$

The integrand of this equation can be plotted to demonstrate the pitch space structure of the distribution, as shown in Fig. 1. We see the second term in Eq. (33) tends to cancel the first term for the passing particles, leaving an order $\varepsilon^{3/2}$ contribution to the polarization that mainly comes from the trapped contribution of the first term. As a result, the total polarization becomes small and of order $\varepsilon^{3/2}$,

$$\varepsilon_{k,nc}^{pol}(p) = \frac{\omega_{pi}^2}{\omega_{ci}^2}\frac{q^2}{\varepsilon^2}1.6\varepsilon^{3/2}, \tag{34}$$

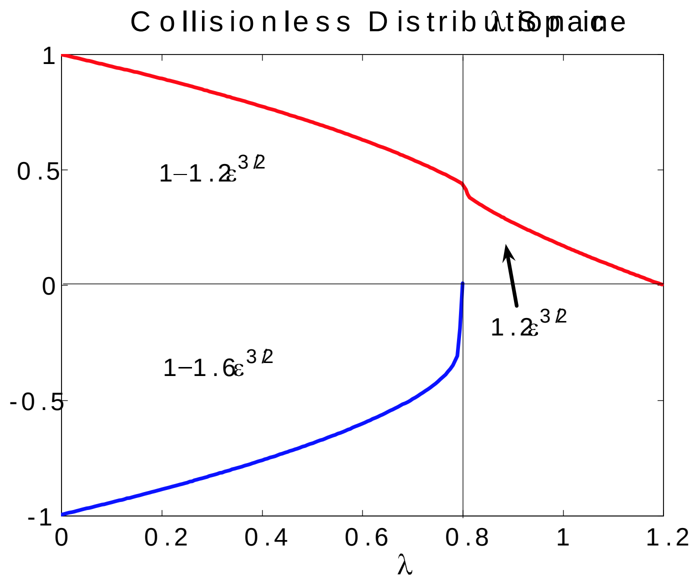

FIG. 1: The collisionless distribution in pitch angle space.

which is the well-know R-H collisionless neoclassical polarization.

For the collisional case, H-R solved Eq. (31) analytically for two asymptotic limits: the high frequency, early time (low collisionality) limit where $p\tau_{ii} \gg 1$, and the low frequency, long time (collisional) limit where $p\tau_{ii} \ll 1$. In the sections that follow we will employ a semi-analytical method to calculate the neoclassical polarization and associated zonal flow damping for arbitrary $p\tau_{ii}$.

III. EIGENFUNCTION EXPANSION

The idea behind this general approach is straight forward. If the eigenfunctions and eigenvalues of the transit averaged collision operator $\frac{\partial}{\partial\lambda} D(\lambda) \frac{\partial}{\partial\lambda}$ can be found, then the transit average kinetic equation of Eq. (21) essentially becomes an algebraic equation. A completely analytical method is desirable, but impractical even for this simple *Lorentz* operator. However, the eigenfunctions and eigenvalues for the *Lorentz* operator can be computed numerically [10][11].

To facilitate numerical implementation, we normalize the passing pitch angle space to $[0, 1]$ by setting $\lambda = (1 - \varepsilon) x$. Then the simplified transit average kinetic equation in Eq.

(31) can be written in terms of the new x variable as

$$\frac{\partial}{\partial x}\tilde{D}\left(x\right)\frac{\partial}{\partial x}G_k - p\tilde{\tau}_{ii}\tilde{L}\left(x\right)G_k = -2\pi p\tilde{\tau}_{ii}\,, \tag{35}$$

where the collision time $\tilde{\tau}_{ii}$ includes the energy dependence implicitly,

$$\tilde{\tau}_{ii} \equiv \frac{1}{\nu_{ii}}\left(\frac{m_i E}{T_i}\right)^{3/2} = \tau_{ii}\left(\frac{m_i E}{T_i}\right)^{3/2}, \tag{36}$$

and the functions $\tilde{D}\left(x\right)$ and $\tilde{L}\left(x\right)$ are defined as

$$\tilde{D}\left(x\right) = \frac{2D\left(\lambda\right)}{\left(1-\varepsilon\right)^2} \tag{37}$$

$$= \frac{8x}{1-\varepsilon}\sqrt{1-x+\varepsilon+x\varepsilon}E\left(\frac{2\varepsilon}{1-x+\varepsilon+x\varepsilon}\right), \tag{38}$$

$$\tilde{L}\left(x\right) = L\left(\lambda\right) = \frac{4}{\sqrt{1-x+\varepsilon+x\varepsilon}}K\left(\frac{2\varepsilon}{1-x+\varepsilon+x\varepsilon}\right). \tag{39}$$

At the trapping boundary $x=1$, the distribution function G_k must vanish to be continuous with the trapped particle distribution. At the other end $x=0$, the velocity particle flux must vanish, $\tilde{D}\left(x\right)\frac{\partial}{\partial x}G_k|_{x=0} = 0$, to avoid a flux into a forbidden region. The boundary condition at this end is automatically satisfied since $\tilde{D}\left(x=0\right) = 0$. In addition, the distribution function G_k is not allowed to diverge to infinity at either boundary.

Before solving Eq. (35), we digress briefly to consider the eigenvalue problem that must be solved. We let the eigenfunctions $g_n\left(x\right)$ and the associated eigenvalues μ_n satisfy the following eigenequation:

$$\frac{\partial}{\partial x}\tilde{D}\left(x\right)\frac{\partial}{\partial x}g_n\left(x\right) = -\mu_n\tilde{L}\left(x\right)g_n\left(x\right), \tag{40}$$

where $g_n\left(x\right)$ and μ_n depend on ε implicitly. Each eigenfunction $g_n\left(x\right)$ and eigenvalue μ_n can be computed numerically by a shooting method [10][11]. Because the transit averaged collision operator $\frac{\partial}{\partial x}\tilde{D}\left(x\right)\frac{\partial}{\partial x}$ is self-ajoint, this eigenvalue problem is a *Sturm-Liouville* problem. Therefore, the eigenfunctions form a complete set and are orthogonal to each other,

$$\int_0^1 dx g_m\left(x\right)g_n\left(x\right)\tilde{L}\left(x\right) = M_n\delta_{nm}, \tag{41}$$

where δ_{nm} is the *Kronecker delta*, and the constant M_n is defined as

$$M_n \equiv \int_0^1 dx g_n^2\left(x\right)\tilde{L}\left(x\right). \tag{42}$$

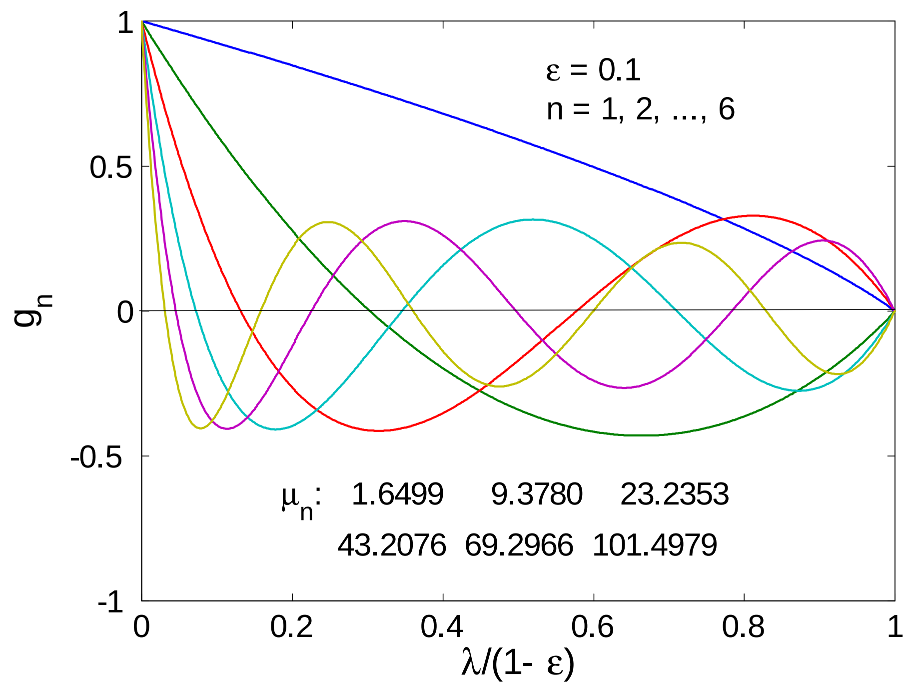

FIG. 2: Eigenfunction g_n and eigenvalue μ_n in Eq.(40) for $n = 1$ through 6, and $\varepsilon = 0.1$.

We can then expand the solution of Eq. (35) in terms of this complete set $\{g_n(x)\}$,

$$G_k = \sum_{l=1}^{\infty} a_l g_l(x). \tag{43}$$

Inserting this series into Eq. (35) and multiplying the equation by $g_n(x)$ and integrating it from 0 to 1, we obtain an algebraic equation

$$a_n(\mu_n + \tilde{\gamma}_0) M_n = 2\pi\tilde{\gamma}_0\beta_n, \tag{44}$$

where the constants $\tilde{\gamma}_0$ and β_n are defined as

$$\tilde{\gamma}_0 \equiv p\tilde{\tau}_{ii}, \tag{45}$$

$$\beta_n \equiv \int_0^1 dx g_n(x). \tag{46}$$

Note that $\tilde{\gamma}_0$ depends on energy implicitly and β_n depends on ε implicitly. From Eq. (44) we know that the eigenfunction expansion coefficient a_n is

$$a_n = \frac{2\pi\tilde{\gamma}_0\beta_n}{(\mu_n + \tilde{\gamma}_0) M_n}, \tag{47}$$

and the distribution function G_k becomes

$$G_k = \sum_{n=1}^{\infty} \frac{2\pi\tilde{\gamma}_0\beta_n}{(\mu_n + \tilde{\gamma}_0) M_n} g_n(x). \tag{48}$$

Knowing the eigenfunctions g_n, we can then calculate β_n and M_n. These quantities, together with the eigenvalues μ_n and driving frequency $\tilde{\gamma}_0$, can be used to determine the distribution G_k and then the neoclassical polarization.

A. Properties of Eigenfunctions and Eigenvalues

Before going further, it is useful to discuss some properties of the eigenfunctions $\{g_n\}$ and eigenvalues $\{\mu_n\}$. We are particularly interested in the properties of the quantities β_n, M_n and μ_n, because these quantities are directly related to the passing particle distribution G_k, as we have shown in the preceding section. It is found that these quantities can be expanded in power series of inverse aspect ratio ε, [10][11]

$$\mu_n(\varepsilon) = \sum_{k=0}^{\infty} u(n,k)\,\varepsilon^{k/2}, \tag{49}$$

$$\beta_n(\varepsilon) = \sum_{k=0}^{\infty} B(n,k)\,\varepsilon^{k/2}, \tag{50}$$

$$M_n(\varepsilon) = \sum_{k=0}^{\infty} M(n,k)\,\varepsilon^{k/2}. \tag{51}$$

In order to obtain the leading order coefficients in the preceding expansions, we can let $\varepsilon \to 0$. Then the eigenequation, Eq. (40), becomes a *Legendre* equation with the variable $\xi = \sqrt{1-x}$,

$$\frac{\partial}{\partial\xi}\left(1-\xi^2\right)\frac{\partial}{\partial\xi}G = -2\mu G. \tag{52}$$

Therefore, we can easily determine the eigenvalues $\mu_n(0) = 2n^2 - n$, and eigenfunctions $g_n^{(0)}(x) = P_{2n-1}\left(\sqrt{1-x}\right)$. With these eigenfunctions, we can evaluate $\beta_n(0)$ and $M_n(0)$ to find $\beta_n(0) = \int_0^1 dx P_{2n-1}\left(\sqrt{1-x}\right) = \frac{2}{3}\delta_{n1}$ and $M_n(0) = \int_0^1 dx P_{2n-1}^2\left(\sqrt{1-x}\right)\frac{2\pi}{\sqrt{1-x}} = \frac{4\pi}{4n-1}$.

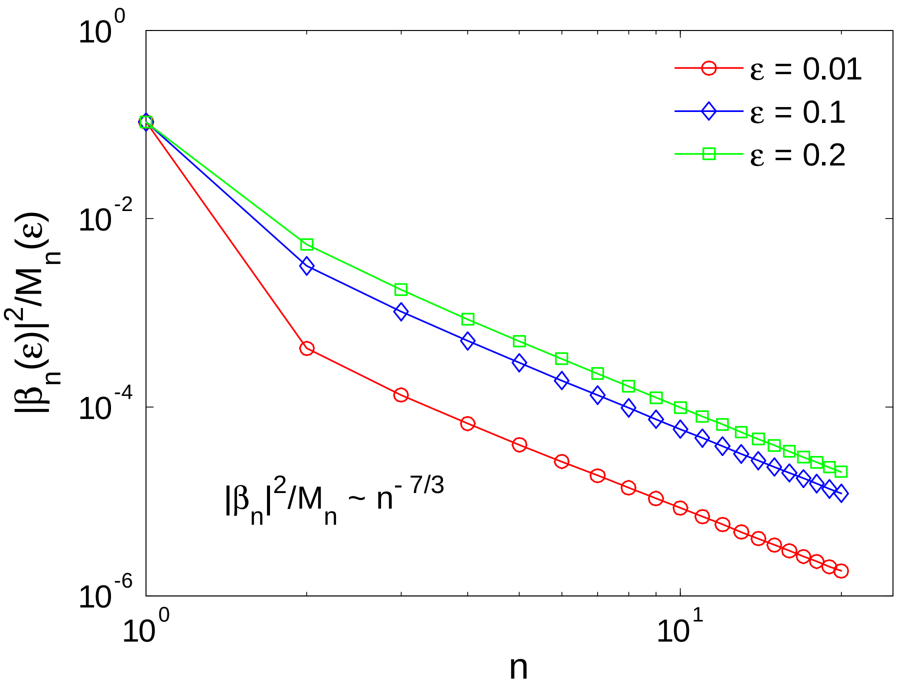

FIG. 3: The quantity $\beta_n(\varepsilon)^2/M_n(\varepsilon)$ versus mode number n on a log-log scale, for $\varepsilon = 0.01$, 0.1, and 0.2.

Therefore, we may write the leading term explicitly to obtain

$$\mu_n(\varepsilon) = 2n^2 - n + \sum_{k=1}^{\infty} u(n,k)\,\varepsilon^{k/2}, \tag{53}$$

$$\beta_n(\varepsilon) = \frac{2}{3}\delta_{n1} + \sum_{k=1}^{\infty} B(n,k)\,\varepsilon^{k/2}, \tag{54}$$

$$M_n(\varepsilon) = \frac{4\pi}{4n-1} + \sum_{k=1}^{\infty} M(n,k)\,\varepsilon^{k/2}. \tag{55}$$

We can compute the quantities $\mu_n(\varepsilon)$, $\beta_n(\varepsilon)$ and $M_n(\varepsilon)$ numerically for various inverse aspect ratios ε and mode numbers n. As we will show, the most interesting quantity for the polarization calculation is $\beta_n(\varepsilon)^2/M_n(\varepsilon)$. Therefore, we plot it as a function of mode number n for various ε, as shown in Fig. 3.

IV. COLLISIONAL NEOCLASSICAL POLARIZATION

We next proceed to construct the neoclassical polarization from these numerical eigenfunctions. If we define

$$P = \frac{3}{2}\int_0^{1-\varepsilon} d\lambda \left\langle G_k \right\rangle_E , \tag{56}$$

then the neoclassical polarization in Eq. (32) can be written as

$$\varepsilon_{k,nc}^{pol}\left(p\right) = \frac{\omega_{pi}^2}{\omega_{ci}^2}\frac{q^2}{\varepsilon^2}\left(1-P\right). \tag{57}$$

Once we know the term P, we effectively know the neoclassical polarization. Inserting the expression for G_k in Eq. (48) into the preceding equation we obtain

$$P = 3\pi\left(1-\varepsilon\right)\sum_{n=1}^{\infty}\frac{\beta_n^2}{M_n}\left\langle \frac{\tilde{\gamma}_0}{\mu_n+\tilde{\gamma}_0}\right\rangle_E , \tag{58}$$

where the energy average is defined in Eq. (16). Letting $y = m_i E/T_i$, the preceding equation can be explicitly written as

$$P = 4\sqrt{\pi}\left(1-\varepsilon\right)\sum_{n=1}^{\infty}\frac{\beta_n^2}{M_n}\int_0^{\infty}\frac{dy y^3 e^{-y}}{\mu_n/\gamma_0 + y^{3/2}}, \tag{59}$$

where the frequency $\gamma_0 = p\tau_{ii}$ is independent of energy.

Employing the preceding equation and Eq. (57), we can calculate the neoclassical polarization for various driving frequencies or collisionalities $p\tau_{ii}$, and inverse aspect ratios ε, as shown in Fig. 4. From this figure we can see that, for high driving frequencies or low collisionalities, the neoclassical polarization is small, so it is sensitive to inverse aspect ratio ε. On this fast time scale, the ions can only diffuse within a narrow boundary layer so that the diffusion can only modify the collisionless result slightly [5]. However, for low driving frequencies or high collisionalities, the neoclassical polarization becomes large and ignoring the ε dependence is not as critical to the final answer, as shown in Fig. 4. On this slow time scale, the ions have enough time to diffuse over the whole pitch angle space. In this small $p\tau_{ii}$ limit, the distribution function G_k extends over the whole passing space with a magnitude proportional to $p\tau_{ii}$. Therefore, the first term in Eq. (57) dominates, the neoclassical polarization becomes order unity, and the ε dependence becomes a higher order effect.

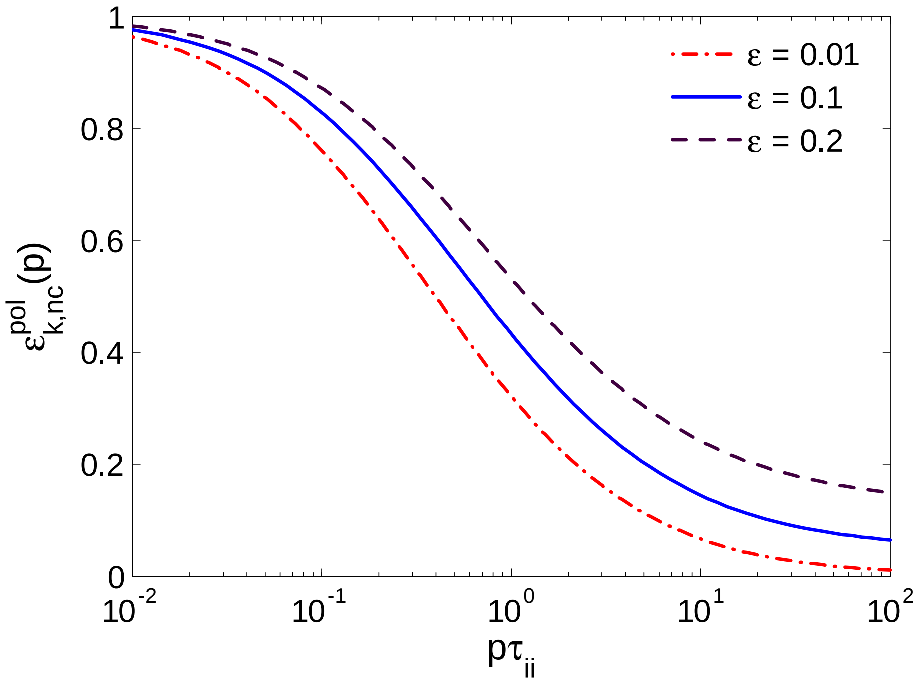

FIG. 4: The neoclassical polarization in units of $\frac{\omega_{pi}^2}{\omega_{ci}^2}\frac{q^2}{\varepsilon^2}$, calculated by Eq.(57) by computing the coefficients μ_n, β_n, and M_n numerically and employing a total of 20 terms to approach the exact value.

A. Approximate Methods for Neoclassical Polarization

The preceding exact eigenfunction expansion method is very accurate, but sometimes a simpler result may be more convenient. We next present a more concise, but approximate method for evaluating the neoclassical polarization.

From Fig. 3, we see that β_1^2/M_1 dominates the other eigenfunction expansion coefficients. Therefore, it is sometimes convenient to retain only the leading order term in the preceding equation, namely

$$P \to 4\sqrt{\pi}\,(1-\varepsilon)\,\frac{\beta_1^2}{M_1}\int_0^\infty \frac{dy y^3 e^{-y}}{\mu_1/\gamma_0 + y^{3/2}}, \tag{60}$$

where $\mu_1 = 1 + 1.461\sqrt{\varepsilon}$ is the leading order eigenvalue of Eq. (40) [11][12]. However, from Fig. 1 we know that in the collisionless limit $P = 1 - 1.635\varepsilon^{3/2}$. This limit suggests that a reasonable approximation satisfying the high frequency asymptotic limit that also has the form of Eq. (60) is

$$P \approx \frac{4}{3\sqrt{\pi}} \left(1 - 1.635\varepsilon^{3/2}\right) \int_0^\infty \frac{dyy^3 e^{-y}}{\alpha/\gamma_0 + y^{3/2}}, \tag{61}$$

where the quantity α is to be determined by the low frequency asymptotic limit. In the low frequency limit $p\tau_{ii} \ll 1$, we know the distribution function $G_k = \pi p\tau_{ii} \left(\frac{m_i E}{T_i}\right)^{3/2} \int_\lambda^{1-\varepsilon} d\lambda' / \left[4\sqrt{1+\varepsilon-\lambda'} E\left(\frac{2\varepsilon}{1+\varepsilon-\lambda'}\right)\right]$ if only the bulk response is considered [5]. According to Eq. (56), we obtain

$$P \approx \frac{8}{\sqrt{\pi}} \gamma_0 \left(1 - 1.461\sqrt{\varepsilon}\right). \tag{62}$$

Comparing this equation to the low frequency limit of Eq. (61), we find α to be

$$\alpha = \mu_1 = 1 + 1.461\sqrt{\varepsilon}, \tag{63}$$

to the accuracy of $\mathcal{O}\left(\sqrt{\varepsilon}\right)$. This is the same as the leading order eigenvalue μ_1 [11][12].Therefore the neoclassical polarization can be calculated from Eq. (57), (61) and (63).

We then can compare the polarization P from this approximation method to that from the exact numerical calculation by summing Eq. (59) to 20 terms. For inverse aspect ratio $\varepsilon = 0.1$, this approximation is very good, as shown in Fig. 5. In fact even for $\varepsilon = 0.2$ the relative error is only 15%. We also plot the H-R collisional polarization P in the same figure as a comparison. It is seen that the H-R analytical results are only valid in either limit, but fail quickly as $p\tau_{ii}$ approaches the other limit.

V. COLLISIONAL ZONAL FLOW DAMPING

Using our rather complete understanding of the role of collisions on neoclassical polarization, we can proceed to investigate the zonal flow damping associated with collisions. Using Eqs.(2) and (3), the evolution of the zonal flow potential can be written as

$$\phi_k(t) = \phi_k(t=0) \frac{1}{2\pi i} \int \frac{dpe^{pt}}{p} \frac{\varepsilon_{k,cl}^{pol}}{\varepsilon_{k,cl}^{pol} + \varepsilon_{k,nc}^{pol}(p)}, \tag{64}$$

where the classical polarization is $\varepsilon_{k,cl}^{pol} = \frac{\omega_{pi}^2}{\omega_{ci}^2} \frac{q^2}{\varepsilon^2}$ in the long wavelength limit, and the neoclassical polarization can be evaluated from Eq. (57) using the approximate P of Eq. (61).

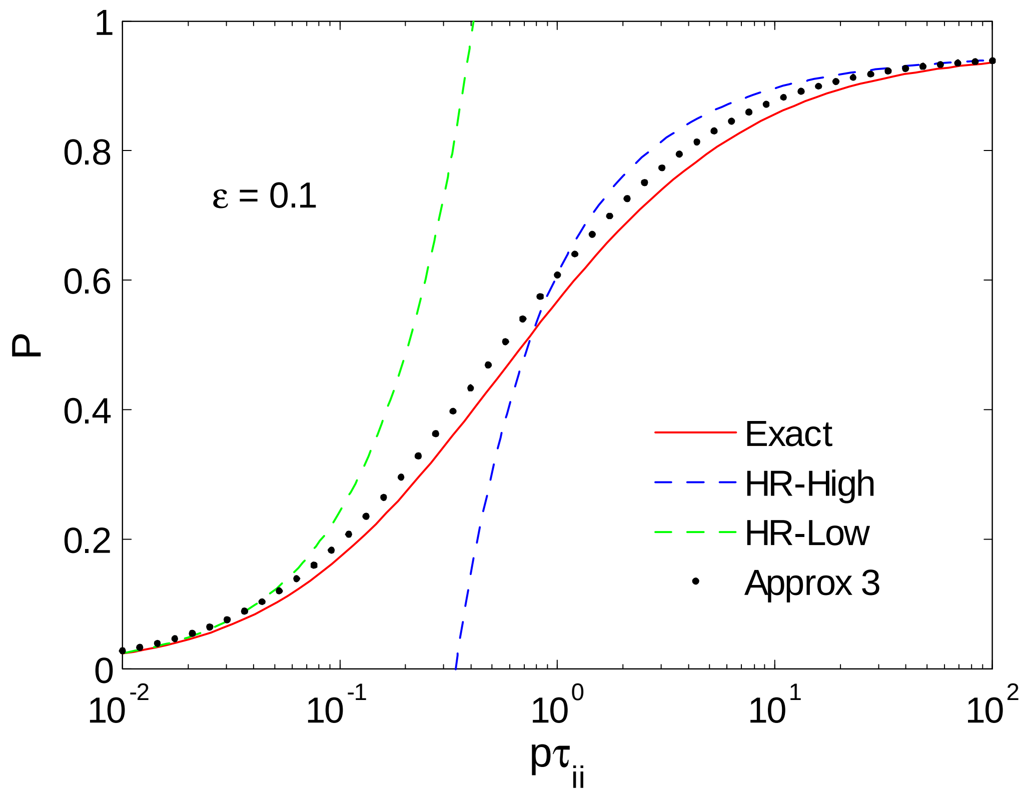

FIG. 5: A comparison of the frequency dependence of the polarization P, which is only part of the neoclassical polarization according to Eq.(57), from different methods for $\varepsilon = 0.1$. The exact line is calculated by Eq.(59). "*HR-High*" is the *H-R* high frequency polarization, while "*HR-Low*" is the *H-R* low frequency polarization [5][12]. The dotted curve "*Approx* " is calculated by Eqs.(61) and (63).

To simplify the results even further, we can use a Pade approximation to the energy integral in Eq. (61). Then, Eq. (61) becomes

$$P \approx \left(1 - 1.6\varepsilon^{3/2}\right) \frac{\gamma_0}{\gamma_0 + \frac{\sqrt{\pi}}{8}\mu_1}. \tag{65}$$

Inserting this equation into Eq. (57), we find the neoclassical polarization to be

$$\varepsilon_{k,nc}^{pol}(p) \approx \frac{\omega_{pi}^2}{\omega_{ci}^2} \frac{q^2}{\varepsilon^2} \frac{1.6\varepsilon^{3/2}\gamma_0 + \frac{\sqrt{\pi}}{8}\mu_1}{\gamma_0 + \frac{\sqrt{\pi}}{8}\mu_1}. \tag{66}$$

In the large aspect ratio limit, the neoclassical polarization is much larger than the classical polarization. Therefore, we may ignore the classical polarization in the denominator of Eq. (3). Inserting the neoclassical polarization in Eq. (66) into Eq. (2), we obtain

$$\phi_k(t) = \phi_k(t=0)\frac{\varepsilon^2}{q^2}\frac{1}{2\pi i}\int\frac{dpe^{pt}}{p}\frac{p\tau_{ii}+\frac{\sqrt{\pi}}{8}\mu_1}{1.6\varepsilon^{3/2}p\tau_{ii}+\frac{\sqrt{\pi}}{8}\mu_1}. \quad (67)$$

Inverting the *Laplace* transform we find

$$\phi_k(t) = \phi_k(t=0)\frac{\varepsilon^2}{q^2}\left[1+\left(\frac{1}{1.6\varepsilon^{3/2}}-1\right)e^{-\frac{\mu_1 t}{7.4\varepsilon^{3/2}\tau_{ii}}}\right]. \quad (68)$$

From this equation we can see that with collisions the zonal flow damps to a level much smaller than the *R-H* collisionless residual in a decay time of order $\varepsilon^{3/2}\tau_{ii}$. Although the decay time is of order $\varepsilon^{3/2}\tau_{ii}$, it is enhanced by the numerical factor 7.4 to roughly the magnitude of ion-ion collision time. Recall that H-R found the average decay time to be $1.5\varepsilon\tau_{ii}$. Therefore, for $\varepsilon = 0.04$ these two decay time are equal. But for most realistic situations, $\varepsilon > 0.04$, and our decay time is slightly larger than the H-R estimate.

VI. ENERGY DEPENDENCE OF COLLISION OPERATOR

In the preceding calculation, we take the pitch angle scattering operator to be the form of Eq. (22) as in H-R. However, the full energy dependence of the pitch angle scattering operator has the following form [13][14]

$$C_{ii}\{h_k\} = 2\frac{H(\sqrt{y})}{y^{3/2}}\nu_{ii}\frac{B_0}{B}\xi\frac{\partial}{\partial\lambda}\lambda\xi\frac{\partial h_k}{\partial\lambda}, \quad (69)$$

where $y = m_i E/T_i$ and $H(z) = \mathrm{erf}(z) - [\mathrm{erf}(z) - z\,\mathrm{erf}'(z)]/(2z^2)$, where $\mathrm{erf}(z) = \frac{2}{\sqrt{\pi}}\int_0^z e^{-x^2}dx$ is the error function. With this improved collision operator, the polarization P in Eq. (59) is modified to become

$$P = 4\sqrt{\pi}(1-\varepsilon)\sum_{n=1}^{\infty}\frac{\beta_n^2}{M_n}\int_0^{\infty}\frac{dyy^3e^{-y}}{H(\sqrt{y})\mu_n/\gamma_0+y^{3/2}}. \quad (70)$$

Thus, the one term approximation for P in Eq. (61) becomes

$$P = \frac{4}{3\sqrt{\pi}}\left(1-1.635\varepsilon^{3/2}\right)\int_0^{\infty}\frac{dyy^3e^{-y}}{H(\sqrt{y})\mu_1/\gamma_0+y^{3/2}}, \quad (71)$$

which still satisfies the two asymptotic limits. Using a Pade approximation, this result simplifies to

$$P = \left(1-1.635\varepsilon^{3/2}\right)\frac{\gamma_0}{\gamma_0+\mu_1/\eta_0}, \quad (72)$$

where the numerical factor

$$\eta_0 = \frac{4}{3\sqrt{\pi}} \int_0^\infty dy y^{3/2} e^{-y} / H\left(\sqrt{y}\right) = 5.38. \tag{73}$$

The neoclassical polarization in Eq. (57) then becomes

$$\varepsilon_{k,nc}^{pol}\left(p\right) = \frac{\omega_{pi}^2}{\omega_{ci}^2} \frac{q^2}{\varepsilon^2} \frac{1.6\varepsilon^{3/2}\gamma_0 + \mu_1/\eta_0}{\gamma_0 + \mu_1/\eta_0} \tag{74}$$

Therefore, the temporal dependence of zonal flow potential becomes

$$\phi_k\left(t\right) = \phi_k\left(t=0\right) \frac{\varepsilon^2}{q^2} \left[1 + \left(\frac{1}{1.6\varepsilon^{3/2}} - 1\right) e^{-\frac{\mu_1 t}{1.6\varepsilon^{3/2}\eta_0\tau_{ii}}}\right], \tag{75}$$

and the decay time becomes $8.8\varepsilon^{3/2}\tau_{ii}$, only slightly longer than the previous estimate.

VII. MOMENTUM CONSERVING MODEL COLLISION OPERATOR

The pitch angle scatter operator employed in the preceding sections doesn't conserve momentum for like particle collisions. For completeness, we consider an improved model collision operator to remove this shortcoming. This Kovrizhnikh model operator takes the following form [15][13][16]

$$C_{ii}\left\{h_k\right\} = \nu_D^{ii} \left[L\left\{h_k\right\} + \frac{m_i}{T_i} v_\parallel u_\parallel F_0\right], \tag{76}$$

where $\nu_D^{ii} L\left\{h_k\right\}$ is the Lorentz operator defined in Eq. (22), but with $\nu_D^{ii} = \nu_{ii} H\left(z\right)/z^3$ and $z \equiv \sqrt{m_i E/T_i} = \sqrt{y}$. To conserve momentum, the parallel flow speed $u_\parallel$ must satisfy

$$u_\parallel = \frac{3}{2} \frac{\int d^3v v_\parallel \nu_D^{ii} h_k}{\int d^3v z^2 \nu_D^{ii} F_0}, \tag{77}$$

which doesn't depend on the sign of the parallel velocity σ. Evaluating $\int d^3v z^2 \nu_D^{ii} F_0 = \nu_{ii} n_0 \alpha_0$ gives the constant $\alpha_0 = \frac{4}{\sqrt{\pi}} \int_0^\infty dz H\left(z\right) z e^{-z^2} = 0.60$. Therefore,

$$u_\parallel = \frac{3}{2\nu_{ii} n_0 \alpha_0} \int d^3v v_\parallel \nu_D^{ii} h_k. \tag{78}$$

The transit average kinetic equation for this model collision operator then becomes

$$G_k - \frac{\nu_D^{ii}}{p} \left[\frac{2}{L\left(\lambda\right)} \frac{\partial}{\partial\lambda} D\left(\lambda\right) \frac{\partial}{\partial\lambda} G_k + \frac{m_i v \hat{u}_\parallel}{T_i} \frac{2\pi}{L\left(\lambda\right)}\right] = \frac{2\pi}{L\left(\lambda\right)}, \tag{79}$$

with the dimensionless distribution function G_k defined in Eq. (30) and the flux function $\hat{u}_\parallel$ defined as

$$\hat{u}_\parallel = \frac{3B_0}{2\nu_{ii}n_0\alpha_0 B}\int d^3v v_\parallel \nu_D^{ii}\sigma G_k F_0. \tag{80}$$

Following the eigenfunction expansion technique in Sec. III by letting $G_k = \sum_n a_n g_n(x)$, we find the coefficient a_n to be

$$a_n = \frac{2\pi\beta_n}{M_n}\frac{p\tau_D^{ii} + \frac{m_i v\hat{u}_\parallel}{T_i}}{p\tau_D^{ii} + \mu_n}, \tag{81}$$

where $\tau_D^{ii} = 1/\nu_D^{ii}$ and $a_n = a_n(z)$ is energy dependent. Noticing that $\hat{u}_\parallel \propto \int d^3v v_\parallel \nu_D^{ii}\sigma \sum_n a_n g_n(x) F_0 \propto \sum_n \beta_n^2/M_n$, we can estimate the effect of $\hat{u}_\parallel$ by recalling Fig. 3 and retaining only the leading term approximation. Therefore,

$$\hat{u}_\parallel \approx \sqrt{\frac{2T_i}{m_i}}\frac{6\sqrt{\pi}\beta_1^2}{\alpha_0 M_1}\int_0^\infty dz\frac{H(z)e^{-z^2}}{p\tau_D^{ii} + \mu_1}\left(p\tau_D^{ii} + \frac{m_i v\hat{u}_\parallel}{T_i}\right). \tag{82}$$

This integral cannot be performed analytically. However, we can again use a Pade approximation to to obtain

$$\hat{u}_\parallel \approx \sqrt{\frac{2T_i}{m_i}}\frac{\beta_1^2}{M_1}\frac{6\sqrt{\pi}\delta_0 p\tau_{ii}\left(p\tau_{ii} + \frac{4\kappa_0\mu_1}{\sqrt{\pi}\alpha_0}\right)}{\left[p\tau_{ii} + \left(\frac{\mu_1}{3\pi} - \frac{\beta_1^2}{M_1}\right)\frac{12\sqrt{\pi}\kappa_0}{\alpha_0}\right](p\tau_{ii} + 2\mu_1\delta_0)}, \tag{83}$$

where the constants $\delta_0 = \int_0^\infty dz H(z) e^{-z^2} = 0.322$ and $\kappa_0 = \int_0^\infty dz H(z)^2 e^{-z^2}/z^2 = 0.43$. When evaluating $\hat{u}_\parallel$ in Eq. (82), the Pade approximation for the second term has a 50% error for $p\tau_{ii} \sim 1$ although it is good for $p\tau_{ii} \ll 1$ and $p\tau_{ii} \gg 1$. However, since the second term is small compared to the $\hat{u}_\parallel$ term on the left side, this 50% error becomes insignificant.

The polarization P in Eq. (56) can then be approximately calculated by using $G_k \approx a_1 g_1(x)$ and Eq. (81)

$$P \approx 3\pi(1-\varepsilon)\frac{\beta_1^2}{M_1}\left\langle\frac{p\tau_D^{ii} + \frac{m_i v\hat{u}_\parallel}{T_i}}{p\tau_D^{ii} + \mu_1}\right\rangle_E. \tag{84}$$

To the satisfy the collisionless limit, the polarization must take the form

$$P \approx \left(1 - 1.635\varepsilon^{3/2}\right)\left\langle\frac{p\tau_D^{ii} + \frac{m_i v\hat{u}_\parallel}{T_i}}{p\tau_D^{ii} + \mu_1}\right\rangle_E, \tag{85}$$

where the energy average can be evaluated as

$$\left\langle\frac{p\tau_D^{ii} + \frac{m_i v\hat{u}_\parallel}{T_i}}{p\tau_D^{ii} + \mu_1}\right\rangle_E = \frac{8}{3\sqrt{\pi}}\int_0^\infty dz H(z) e^{-z^2} z^4 \frac{p\tau_{ii}z^3 + 2zH(z)\sqrt{\frac{m_i}{2T_i}}\hat{u}_\parallel}{p\tau_{ii}z^3 + \mu_1 H(z)}. \tag{86}$$

We can again use a Pade approximation to simplify this expression to obtain

$$P \approx \frac{\left(1 - 1.635\varepsilon^{3/2}\right) p\tau_{ii}}{p\tau_{ii} + \mu_1/\eta_1}, \tag{87}$$

where the constant $\eta_1 = 1.16/\sqrt{\varepsilon} + \eta_0$ with η_0 found in Eq. (73) to be 5.38. The neoclassical polarization in Eq. (57) then becomes

$$\varepsilon_{k,nc}^{pol}(p) \approx \frac{\omega_{pi}^2}{\omega_{ci}^2} \frac{q^2}{\varepsilon^2} \frac{1.6\varepsilon^{3/2}\gamma_0 + \mu_1/\eta_1}{\gamma_0 + \mu_1/\eta_1}. \tag{88}$$

Therefore, the temporal dependence of zonal flow potential becomes

$$\phi_k(t) = \phi_k(t=0) \frac{\varepsilon^2}{q^2} \left[1 + \left(\frac{1}{1.6\varepsilon^{3/2}} - 1\right) e^{-\frac{\mu_1 t}{1.6\varepsilon^{3/2}\eta_1\tau_{ii}}}\right], \tag{89}$$

and the decay time becomes $\left(8.8\varepsilon^{3/2} + 1.9\varepsilon\right)\tau_{ii}$. The parallel flow needed to conserve momentum makes the decay time somewhat longer, but the change is only significant for $\varepsilon < 0.05$ so is normally negligible.

VIII. ZONAL FLOW ROTATION

A. Collisionless Rotation

Zonal flow is more than simply the poloidal rotation of plasma. In truth the zonal flow also includes a toroidal rotation as demonstrated by the following calculation.

The complete solution to the linearized kinetic equation includes both gyrophase independent $\overline{f}$ and gyrophase dependent $\widetilde{f}$ contributions: $f = \overline{f} + \widetilde{f}$. The gyrophase independent part $\overline{f}$ produces a parallel flow, that can be calculated to accuracy of $\mathcal{O}(Q^2)$ in the collisionless limit using $\overline{f} = F_0 + f_1$. Here f_1 is the linearized distribution, $f_1 = \sum_k f_k e^{iS}$. For the demonstration here, only the polarization part of f_1 is of interest. It has been calculated by Ref. [4] or shown in Sec. II in the collisionless limit. To $\mathcal{O}(Q^2)$ accuracy,

$$f_1 = \sum_{\mathbf{k}} \frac{e\phi_k}{T_i} i\left(\overline{Q} - Q\right) F_0 \tag{90}$$

Therefore, the parallel flow $u_\parallel = \frac{1}{n_0}\int d^3v v_{\shortparallel} f_1$ can be calculated as

$$u_\parallel = \frac{eI}{n_0 T_i}\frac{\partial\phi}{\partial\psi}\int d^3v F_0 v_{\shortparallel}\left[\overline{\left(\frac{v_{\shortparallel}}{\Omega}\right)} - \frac{v_{\shortparallel}}{\Omega}\right]. \tag{91}$$

The integration can be carried out to obtain

$$u_{\|} = BF(\psi) - \frac{\partial\phi}{\partial\psi}\frac{cI}{B}, \tag{92}$$

with the flux function $F(\psi)$ defined as

$$F(\psi) = \frac{eI}{n_0 T_i}\frac{\partial\phi}{\partial\psi}\int d^3v F_0 \overline{\left(\frac{v_{\shortparallel}}{\Omega}\right)^2}. \tag{93}$$

The perpendicular flow is given by the gyrophase dependent part $\widetilde{f}$, which is simply the diamagnetic term

$$\widetilde{f} = \frac{1}{\Omega}\mathbf{v}\times\mathbf{b}\cdot\nabla|_{\epsilon}F_0, \tag{94}$$

with gradient taken holding the total energy $\epsilon \equiv v^2/2 + \frac{e}{m_i}\phi$ fixed. Therefore, taking the gradient holding E fixed gives

$$\widetilde{f} = \frac{1}{\Omega}\mathbf{v}\times\mathbf{b}\cdot\left(\nabla|_E F_0 + \frac{e}{T_i}F_0\nabla\phi\right). \tag{95}$$

The first term in the preceding equation gives the diamagnetic flow, which combines with the parallel neoclassical flow to give a divergence free flow. As usual, this piece along with other neoclassical contributions are ignored as small when evaluating the zonal flow. The second term in the preceding equation gives the poloidal zonal flow that is of interest here, namely

$$\begin{aligned}\mathbf{u}_\perp &= \frac{1}{n_0}\int d^3v\mathbf{v}_\perp\widetilde{f} \\ &= \frac{e}{n_0 T_i\Omega}\mathbf{b}\times\nabla\phi\cdot\int d^3v\mathbf{v}_\perp\mathbf{v}_\perp F_0,\end{aligned} \tag{96}$$

which not surprisingly simply turns out to be

$$\mathbf{u}_\perp = \frac{c}{B}\mathbf{b}\times\nabla\phi. \tag{97}$$

Combining the parallel flow in Eq. (92) and perpendicular flow from the preceding equation, we find a divergence free flow

$$\mathbf{u} = -c\frac{\partial\phi}{\partial\psi}R\widehat{\zeta} + \mathbf{B}F(\psi), \tag{98}$$

with $\widehat{\zeta} = R\nabla\zeta$, the unit vector in the toroidal direction and $\mathbf{B} = I\nabla\zeta + \nabla\zeta\times\nabla\psi$. From the preceding equation, we see that the zonal flow not only contains poloidal rotation, but also toroidal rotation. In the collisionless limit, the function $F(\psi)$ can be evaluated for a circular tokamak in the same way we calculate the collisionless polarization in the previous section to find

$$F(\psi) = \frac{\partial \phi}{\partial \psi}\frac{cI}{B_0^2}(1 - 1.6\varepsilon^{3/2}), \tag{99}$$

where $B_0 = I/R_0$. Therefore, the total flow becomes

$$\mathbf{u} = -cR\frac{\partial \phi}{\partial \psi}\widehat{\zeta} + \frac{cR_0}{B_0}\frac{\partial \phi}{\partial \psi}(1 - 1.6\varepsilon^{3/2})\mathbf{B}, \tag{100}$$

giving the toroidal component

$$u_\zeta = -R_0 c\frac{\partial \phi}{\partial \psi}\left(2\varepsilon\cos\theta + 1.6\varepsilon^{3/2}\right), \tag{101}$$

and the poloidal component

$$u_{pol} = R_0 c\frac{\partial \phi}{\partial \psi}\frac{\varepsilon}{q}(1 - 1.6\varepsilon^{3/2}), \tag{102}$$

where the potential ϕ is related to the initial potential $\phi(t=0)$ by $\phi = \phi(t=0)\,\varepsilon^2/\left(q^2 1.6\varepsilon^{3/2}\right)$, according to Eqs.(2), (3) and (34) or Ref. [4], since the neoclassical polarization dominates over the classical. Because the potential ϕ takes a local eikonal form and hence the radial variable ε is fixed in calculation, we can write the total collisionless flow as

$$\mathbf{u} = -cR\frac{\varepsilon^2/q^2}{1.6\varepsilon^{3/2}}\frac{\partial \phi(t=0)}{\partial \psi}\widehat{\zeta} + \frac{cR_0}{B_0}\frac{\varepsilon^2}{q^2}\left(\frac{1}{1.6\varepsilon^{3/2}} - 1\right)\frac{\partial \phi(t=0)}{\partial \psi}\mathbf{B}. \tag{103}$$

As is seen from Eqs.(101) and (102), toroidal rotation and poloidal rotation both exist for zonal flow and are of similar magnitude, that is $O\left(\varepsilon c R_0 \frac{\partial \phi}{\partial \psi}\right)$. The toroidal flow is largest on the outboard side where it is also larger than the poloidal flow. In the collisionless limit, the toroidal zonal flow may be more effective than the poloidal in controlling turbulence on the outboard side.

B. Collisional Damping

For the collisional case, in the frequency domain the perturbed distribution function becomes

$$f_1 = \sum_{\mathbf{k}}\left(h_k - \frac{e\phi_k}{T_i}iQF_0\right), \tag{104}$$

where h_k has been obtained by solving Eq. (21). Therefore, in the frequency domain, the parallel flow $u_\parallel = \frac{1}{n_0}\int d^3v v_\parallel f_1$ can be calculated as

$$u_\parallel(p) = \frac{cI}{B}\frac{\partial \phi(p)}{\partial \psi}(P - 1), \tag{105}$$

where P is defined by Eq. (56) and approximately calculated by Eq. (72). The perpendicular flow still takes the familiar $E \times B$ form, as in the collisionless case. Therefore, the total zonal flow becomes

$$\mathbf{u}(p) = -cR\frac{\partial\phi(p)}{\partial\psi}\hat{\zeta} + \mathbf{B}\frac{cI}{B_0^2}\frac{\partial\phi(p)}{\partial\psi}P, \tag{106}$$

where $\phi(p)$ is related to the initial zonal flow potential $\phi(t=0)$ by [5]

$$\phi(p) = \frac{\phi(t=0)}{p}\frac{\varepsilon_{k,cl}^{pol}}{\varepsilon_{k,cl}^{pol} + \varepsilon_{k,nc}^{pol}}. \tag{107}$$

Again, ignoring the classical polarization in the denominator of this equation, and inserting Eq. (88) we can obtain an expression $\phi(p)$. This expression along with Eq. (87) allows Eq. (106) to be written as

$$\begin{aligned}\mathbf{u}(p) = &-cR\frac{\partial\phi(t=0)}{\partial\psi}\frac{\varepsilon^2}{q^2 p}\frac{p\tau_{ii} + \mu_1/\eta_1}{1.6\varepsilon^{3/2}p\tau_{ii} + \mu_1/\eta_1}\hat{\zeta} \\ &+ \mathbf{B}\frac{cI}{B_0^2}\frac{\partial\phi(t=0)}{\partial\psi}\frac{\varepsilon^2}{q^2}\frac{\left(1 - 1.6\varepsilon^{3/2}\right)\tau_{ii}}{1.6\varepsilon^{3/2}p\tau_{ii} + \mu_1/\eta_1}.\end{aligned} \tag{108}$$

The inverse Laplace transform of this equation gives the time evolution of the zonal flow

$$\begin{aligned}\mathbf{u}(t) = &-cR\frac{\partial\phi(t=0)}{\partial\psi}\frac{\varepsilon^2}{q^2}\left[1 + \left(\frac{1}{1.6\varepsilon^{3/2}} - 1\right)e^{-\frac{\mu_1 t}{1.6\varepsilon^{3/2}\eta_1\tau_{ii}}}\right]\hat{\zeta} \\ &+ \mathbf{B}\frac{cI}{B_0^2}\frac{\partial\phi(t=0)}{\partial\psi}\frac{\varepsilon^2}{q^2}\left(\frac{1}{1.6\varepsilon^{3/2}} - 1\right)e^{-\frac{\mu_1 t}{1.6\varepsilon^{3/2}\eta_1\tau_{ii}}}.\end{aligned} \tag{109}$$

From this equation, we see that the poloidal rotation of zonal flow decays to zero in a time of $\left(8.80\varepsilon^{3/2} + 1.86\varepsilon\right)\tau_{ii}$, but there is a long time residual toroidal rotation, of order $cR\frac{\partial\phi(t=0)}{\partial\psi}\frac{\varepsilon^2}{q^2}$.

IX. CONCLUSION AND DISCUSSION

In the preceding sections we have employed a semi-analytical method to efficiently calculate the arbitrary collision frequency response of the collisional neoclassical polarization based on an eigenfunction expansion of the collision operator. Our analytical formula for the collisional neoclassical polarization is valid for the whole range of $p\tau_{ii}$. The formula is extremely good for the asymptotic limits $p\tau_{ii} \ll 1$ and $p\tau_{ii} \gg 1$. For values of $p\tau_{ii}$ between these two limits, our analytical result captures the leading order frequency dependence of the neoclassical polarization in the ε expansion so is able to keep the relative error much smaller than H-R. In the original H-R work, the validity for the weak collision case $p\tau_{ii} \gg 1$

requires $\sqrt{\varepsilon p\tau_{ii}} \gg 1$ and the validity for the strong collision case requires $p\tau_{ii} \ll 1$. Their results fail quickly for intermediate values of $p\tau_{ii}$ as shown in Fig. 5. Consequently, for zonal flow damping, it is advantageous to employ our formula since it treats intermediate collisionalities more accurately. Earlier work [17] proposed a plateau in the frequency response of neoclassical polarization at intermediate values of $p\tau_{ii}$, leading to an order $\sqrt{\varepsilon}$ modification in Eq. (32) compared to the collisionless value of order $\varepsilon^{3/2}$ given by Eq. (34) and collisional value of unity implied by Eq. (59). Although an increase in the neoclassical polarization with collisions is found here, the calculations herein actually indicate the transitional behavior of the frequency response of neoclassical polarization is smooth with no evidence of the existence of such a plateau.

As an extension of the original H-R work, we consider the full energy dependence of the pitch angle scatter operator, which effectively increases the decay time by a small factor. We then check the effect of retaining momentum conservation in ion-ion collisions by using a momentum conserving model collision operator. This improvement gives an enhanced zonal flow decay time, presumably because some of the initial momentum lost by pitch angle scattering operator is restored by this new term. However, the correction is negligible except at very large aspect ratios.

In addition, we carefully consider the zonal flow poloidal and toroidal rotation components. Our kinetic results show that in the initial phase, the zonal flow contains not only poloidal, but also toroidal rotation, and that they are of roughly the same magnitude. The toroidal flow is largest on the outboard side where it is also numerically larger than the poloidal flow. Eventually the poloidal rotation damps away to zero while the toroidal rotation damps to a constant residual plateau that is of order $\mathcal{O}\left(\sqrt{\varepsilon}\right)$ compared to the initial magnitude.

Our result for the collisional neoclassical polarization and the zonal flow damping is simple and can presumably be readily verified by numerical simulations. Indeed, it should provide a useful benchmark for turbulence codes.

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