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Abstract

Short mean free path descriptions of magnetized plasmas have existed for almost 50 years so it is surprising to find that further modifications are necessary. The earliest work adopted an ordering in which the flow velocity was assumed to be comparable to the ion thermal speed. Later, less well known studies extended the short mean free path treatment to the normally more interesting drift ordering in which the pressure times the mean flow velocity is comparable to the diamagnetic heat flow. Such an ordering is required to properly retain the temperature gradient terms in the viscosity that arise from the gyrophase dependent and independent portions of the distribution function. Our treatment corrects the expressions for the parallel and perpendicular collisional ion viscosities found in these later treatments which used an approximate truncated polynomial expression for the distribution function and neglected the non-linear piece of the collision operator due to its bi-linear form. The modified parallel and perpendicular ion viscosities contain additional terms quadratic in the heat flux. In addition, we solve for the electron parallel and gyro-viscosities which were not considered by previous drift ordered treatments. As in all drift orderings we assume the collision frequency is small compared to the cyclotron frequency. However, we permit the perpendicular scale lengths to be much less than the parallel ones as is the case in many magnetic confinement applications. As a result, our description is valid for turbulent and collisional transport, and also allows stronger poloidal density and temperature variation in a tokamak than the standard Pfirsch-Schlüter ordering.

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I. INTRODUCTION

The short mean free path description of magnetized plasma as originally formulated by Braginskii [1, 2] and Robinson and Bernstein [3] assumes an ordering in which the ion mean flow is on the order of the ion thermal speed. Mikhailovskii and Tsypin [4-6] realized that this ordering is not the one of most interest in many practical situations in which the flow is weaker and on the order of the ion diamagnetic heat flux divided by the pressure. In their drift ordering the ion flow velocity is assumed to be on the order of the diamagnetic drift velocity - the case of interest for most magnetic confinement and fusion devices in general, and the edge of many tokamaks in particular. Indeed, most short mean free path treatments of turbulence in magnetized plasmas must use some version of the Mikhailovskii and Tsypin results to properly treat the temperature gradient terms in the viscous stress tensor. However, the truncated polynomial expansion solution technique of Mikhailovskii and Tsypin makes two assumptions which we remove to obtain completely general results. First, they neglect contributions to the viscosity that arise from the full non-linear form of the collision operator. This modification gives rise to heat flux squared terms in the parallel and perpendicular viscosities that are the same size as terms found by Mikhailovskii and Tsypin. Second, because of their truncation only an approximation to the gyrophase dependent portion of the ion distribution function is retained. This approximate form is not accurate enough to completely and properly evaluate some of the terms in the perpendicular collisional viscosity. The modifcations to the parallel and perpendicular viscosities that we find may alter collisional and turbulent transport in some situations.

In many magnetized devices, including tokamaks, the perpendicular scale lengths can be much shorter than the parallel ones so that the ion gyro-radius over the perpendicular scale length can be comparable to the mean free path over the parallel scale length. By considering this general ordering we obtain a formulation that can safely be used to study turbulent transport in collisional plasmas, and we allow stronger poloidal density and temperature variation in tokamaks than the normal Pfirsch-Schlüter ordering [7-9]. More specifically, we generalize the short mean free path closure procedure for the collision frequency small compared to the gyro-frequency by allowing the parallel scale length $L_\parallel$ to be larger than the perpendicular scale length $L_\perp$. In Sec. II we perform a joint expansion of the kinetic equation in the two small parameters

$$\delta = \rho/L_\perp \quad \text{and} \quad \Delta = \lambda/L_\parallel \tag{1}$$

which we treat as comparable ($\delta \sim \Delta$), where $\rho = v_i/\Omega$ is the ion gyro-radius and $\lambda = v_i/\nu$ is the Coulomb mean free path, with $v_i = (2T/M)^{1/2}$ the ion thermal speed, ν the ion-ion collision frequency, Ω the ion gyro-frequency, and T and M the ion temperature and mass.

We adopt the Mikhailovskii and Tsypin drift ordering for the mean ion flow velocity $\vec{V}$ by assuming it is on the order of the diamagnetic drift velocity which is on the order of the sum of the ion daimagnetic and collisional parallel heat fluxes $\vec{q}$ divided by the ion pressure p = nT with n the ion density. As a result, we order

$$|\vec{V}|/v_i \sim |\vec{q}|/pv_i \sim \delta, \tag{2}$$

with $V_\| \sim |\vec{V}_\perp|$. We then solve for the ion distribution function to high enough order that we can form all components of the ion viscosity as well as the heat flux. An alternate ordering $v_i \sim V_\| >> |\vec{V}_\perp|$ for $L_\perp \sim L_\|$ was considered by Nemov [10].

Our ordering allows turbulent fluctuations to be as large as the unperturbed background plasma quantities. For the background variations, our ordering is consistent with, but more general than, the usual Pfirsch-Schlüter tokamak ordering [7-9]. Recall the standard ion expressions for the parallel heat flux $q_\|$ and diamagnetic heat flux $\vec{q}_\perp$,

$$q_\| = -(125p/32M\nu)\vec{n}\cdot\nabla T \quad \text{and} \quad \vec{q}_\perp = (5p/2M\Omega)\vec{n}\times\nabla T, \tag{3}$$

where we define the unit vector $\vec{n} = \vec{B}/B$ with $\vec{B}$ an arbitrary magnetic field, $B = |\vec{B}|$, $\Omega = eB/Mc$ for singly charged ions of charge e with c the speed of light, and $\nu = 4\pi^{1/2}ne^4\ell n\Lambda/3M^{1/2}T^{3/2}$, with $\ell n\Lambda$ the Coulomb logarithm. Pfirsch-Schlüter transport finds $q_\| \sim |\vec{q}_\perp|$ by assuming $\vec{n}\cdot\nabla\ell nT \sim \tilde{T}/\overline{T}L_\|$, with $\tilde{T}$ a small correction to the lowest order flux function temperature $\overline{T}$. Consequently, $\tilde{T}/\overline{T} \sim \delta/\Delta << 1$ is required. Our ordering does not require density, ion temperature, or electrostatic potential to be lowest order flux functions, so $\vec{n}\cdot\nabla\ell nT \sim 1/L_\|$ is consistent with $q_\| \sim |\vec{q}_\perp|$ for $\delta \sim \Delta$. Indeed, we employ $\vec{n}\cdot\nabla\ell nT \sim 1/L_\|$ and $\vec{n}\times\nabla\ell nT \sim 1/L_\perp$ for turbulent fluctuations as well.

In the next section, we perform a systematic expansion of the ion kinetic equation in the small parameters δ and Δ to determine the ion distribution function to order $\delta^2 \sim \delta\Delta \sim \Delta^2$ in terms of the ion flow velocity and the parallel and diamagnetic heat fluxes of Eq. (3). Section III completes the ion description by evaluating the collisional perpendicular heat flux, and the gyro-viscosity and the collisional parallel and perpendicular viscosities. Our parallel viscosity is shown to contain terms in addition to those found by Mikhailovskii and Tsypin due to the need to retain the full non-linear ion-ion collision operator. Our perpendicular collisional viscosity also corrects their expression. Some of these corrections occur because they used a truncated polynomial approximation rather than the exact gyrophase dependent portion of the ion distribution function, while the others come from the need to retain the non-linear collision terms they neglected. Section IV considers the electron problem which is somewhat simpler because the perpendicular collisional viscosity is negligible and need not be evaluated. Both the electron collisional parallel and gyro-viscosities are explicitly evaluated. We close with a discussion of our results in Sec. V.

II. ION FORMULATION

In this section we systematically solve the Fokker-Planck equation for the ion distribution function f,

$$\frac{\partial f}{\partial t} + \nabla\cdot(\vec{v}f) + \nabla_v \cdot \left[\frac{e}{M}(\vec{E} + \frac{1}{c}\vec{v}\times\vec{B})f\right] = C + C_{ie}, \tag{4}$$

with $\vec{E}$ the electric field, C the ion-ion collision operator, C_{ie} the ion-electron collision operator, and $\nabla_v = \partial/\partial\vec{v}$. To do so it is convenient to make a change of velocity variables to $\vec{w} = \vec{v} - \vec{V}$, where $n\vec{V} = \int d^3v\vec{v}f$, $n = \int d^3vf$, and continuity requires $\partial n/\partial t + \nabla\cdot(n\vec{V}) = 0$. In the new velocity varible the ion kinetic equation becomes

$$\frac{\partial f}{\partial t} + (\vec{w}+\vec{V})\cdot\nabla f + \left[\Omega\vec{w}\times\vec{n} + \frac{e}{M}(\vec{E} + \frac{1}{c}\vec{V}\times\vec{B}) - \frac{\partial\vec{V}}{\partial t} - (\vec{w}+\vec{V})\cdot\nabla\vec{V}\right]\cdot\nabla_w f = C + C_{ie}. \tag{5}$$

where $\nabla_w = \partial/\partial\vec{w}$. To rewrite Eq. (5) we use ion momentum conservation in the form

$$Mn(\frac{\partial\vec{V}}{\partial t} + \vec{V}\cdot\nabla\vec{V}) - en(\vec{E} + \frac{1}{c}\vec{V}\times\vec{B}) = -\nabla p - \nabla\cdot\vec{\vec{\pi}} - \vec{F}, \tag{6}$$

with the ion-electron momentum exchange defined as $\vec{F} = -M\int d^3v\vec{v}C_{ie}$, the ion pressure given by $p = nT = (M/3)\int d^3w w^2 f$, and ion viscosity tensor $\vec{\vec{\pi}}$ defined by

$$\vec{\vec{\pi}} = M\int d^3w(\vec{w}\vec{w} - \frac{1}{3}w^2\vec{\vec{I}})f. \tag{7}$$

In addition, we use the mass ratio expanded form of the ion-electron collision operator to write $C_{ie} = (Mn)^{-1}\vec{F}\cdot\nabla_w f$ since ion-electron equilibration is smaller by $(m/M)^{1/2}$ with m the electron mass. As a result, Eq. (5) becomes

$$\Omega\vec{w}\times\vec{n}\cdot\nabla_w f + [\vec{w}\cdot\nabla f + (Mn)^{-1}\nabla p\cdot\nabla_w f] + [\frac{\partial f}{\partial t} + \vec{V}\cdot\nabla f - \vec{w}\cdot\nabla\vec{V}\cdot\nabla_w f]$$
$$+ (Mn)^{-1}(\nabla\cdot\vec{\vec{\pi}})\cdot\nabla_w f = C, \tag{8}$$

where compared to the explicit Ω term, the terms in the first set of square parenthesis are of order δ or smaller, and those in the second set of square parenthesis are of order δ^2 or smaller. In addition, since $\vec{\vec{\pi}}$ includes parallel and gyro-viscosities with $\nabla\cdot\vec{\vec{\pi}}$ ~ $Mn(\vec{V}\cdot\nabla\vec{V} + \partial\vec{V}/\partial t)$, the explicit $\vec{\vec{\pi}}$ term in Eq. (8) is small by order $\delta^2\Delta \sim \delta\Delta^2 \sim \delta^3$.

To solve Eq. (8) we expand f and C in powers of $\delta \sim \Delta$ by writing $f = f_0+f_1+f_2+...$ and $C = C_0+C_1+C_2+....$ For the moment we permit ν and Ω to be comparable and thereby obtain the following hierarchy of equations:

$$\Omega\vec{w}\times\vec{n}\cdot\nabla_w f_0 = C_0, \tag{9}$$

$$\Omega\vec{w}\times\vec{n}\cdot\nabla_w f_1 = C_1 + [\vec{w}\cdot\nabla f_0 + (Mn)^{-1}\nabla p\cdot\nabla_w f_0], \text{ and} \tag{10}$$

$$\Omega\vec{w}\times\vec{n}\cdot\nabla_w f_2 = C_2 + [\vec{w}\cdot\nabla f_1 + (Mn)^{-1}\nabla p\cdot\nabla_w f_1] + [\frac{\partial f_0}{\partial t} + \vec{V}\cdot\nabla f_0 - \vec{w}\cdot\nabla\vec{V}\cdot\nabla_w f_0]. \tag{11}$$

In the Braginskii ordering $\nabla\vec{V}$ and $\vec{V}\cdot\nabla$ terms are one order larger in δ so in his treatment they appear on the right side of Eq. (10). Notice that $C_0 = C_0\{f_0\}$ is the full ion-ion collision operator operating on f_0, $C_1 = C_1\{f_1\}$ is the linearized ion-ion collision operator operating on f_1, and the ion-ion collision operator C_2 must include a term non-linear in f_1 as well as a linearized term operating on f_2 so we can write it as $C_2 = C_1\{f_2\} + C_2\{f_1,f_1\}$.

The non-linear terms $C_2\{f_1,f_1\}$ are neglected by Mikhailovskii and Tsypin, but we will find contributions to the parallel and perpendicular viscosity from C_2 for $\nu << \Omega$. Equations (9) - (11) could also be solved more generally by continuing to permit $\nu \sim \Omega$, but the algebra would become more tedious. We have implicitly assumed that $\delta \sim \Delta \lesssim (m/M)^{1/2}$ so an isotropic temperature equilibration term should enter Eq. (11). However, such a term only leads to an isotropic modification of f, so does not alter $\vec{\pi}$ and is ignored.

The solution to Eq. (9) is the drifting Maxwellian

$$f_0 = n\left(\frac{M}{2\pi T}\right)^{3/2} \exp\left(-\frac{Mw^2}{2T}\right) = n\left(\frac{M}{2\pi T}\right)^{3/2} \exp\left[-\frac{M(\vec{v}-\vec{V})^2}{2T}\right], \tag{12}$$

and we will construct our full solution for f such that f_0 gives the correct density, temperature, and mean velocity; that is, $n = \int d^3vf = \int d^3vf_0$, $nT = p = (M/3)\int d^3vfw^2 = (M/3)\int d^3vf_0w^2$, and $n\vec{V} = \int d^3vf\vec{v} = \int d^3vf_0\vec{v}$. We solve Eqs. (10) and (11) by writing each f_j as a sum of a gyro-averaged $\bar{f}_j$ and gyrophase dependent $\tilde{f}_j$ pieces by letting $f_j = \bar{f}_j + \tilde{f}_j$, where $\bar{f}_j = \langle f_j \rangle$ and $\langle \tilde{f}_j \rangle = 0$ with $\langle ... \rangle$ denoting a gyrophase average. The gyrophase φ is defined by writing $\vec{w} = \vec{w}_\perp + w_\| \vec{n}$ with $\vec{w}_\perp = w_\perp(\vec{e}_1 \cos\varphi + \vec{e}_2 \sin\varphi)$ where the unit vectors $\vec{e}_1$ and $\vec{e}_2$ are orthogonal and normal to $\vec{B}$ such that $\vec{e}_1 \times \vec{e}_2 = \vec{n}$.

Inserting f_0 in Eq. (10) results in

$$C_1 - \Omega\vec{w}\times\vec{n}\cdot\nabla_w f_1 = f_0\left(\frac{Mw^2}{2T} - \frac{5}{2}\right)\vec{w}\cdot\nabla\ell nT, \tag{13}$$

which upon gyro-averaging gives the equation for $\bar{f}_1$ to be

$$\overline{C}_1 = f_0\left(\frac{Mw^2}{2T} - \frac{5}{2}\right)w_\|\,\vec{n}\cdot\nabla\ell nT \tag{14}$$

with $\overline{C}_1 = \langle C_1 \rangle$. The Spitzer problem represented by Eq. (14) can be solved by using an expansion in orthogonal polynomials that depend on $x^2 = Mw^2/2T$ as a trial function solution with its coefficients determined variationally [3, 7, 8]. The solution is of order Δ and may be written as

$$\bar{f}_1 = -\frac{2Mq_\|}{5pT}\left[L_1^{(3/2)}(x^2) - \frac{4}{15}L_2^{(3/2)}(x^2)\right]w_\| f_0 \tag{15}$$

where $\vec{q} = \vec{q}_\perp + q_\|\vec{n}$ with $\vec{q}_\perp$ and $q_\|$ defined by Eq. (3), and where $L_1^{(\alpha)}(x^2) = \alpha + 1 - x^2$ and $L_2^{(\alpha)}(x^2) = [(\alpha+1)(\alpha+2) - 2(\alpha+2)x^2 + x^4]/2$ are generalized Laguerre polynomials. Then, subtracting Eqs. (13) and (14) and assuming $\nu << \Omega$ gives $\Omega\vec{w}\times\vec{n}\cdot\nabla_w f_1 = -f_0[x^2 - (5/2)]\vec{w}_\perp\cdot\nabla\ell nT$, which has the exact order δ solution

$$\tilde{f}_1 = -\frac{f_0}{\Omega}L_1^{(3/2)}(x^2)\vec{w}\times\vec{n}\cdot\nabla\ell nT. \tag{16}$$

Rather than solve for $\tilde{f}_1$ to next order, the order ν/Ω perpendicular heat flux corrections will be evaluated by a moment approach in the next section. Using Eq. (3), the full expression for $f_1 = \bar{f}_1 + \tilde{f}_1$ may be written as

$$f_1 = -\frac{2Mf_0}{5pT}\left[L_1^{(3/2)}(x^2)\vec{q}\cdot\vec{w} - \frac{4}{15}L_2^{(3/2)}(x^2)q_\parallel w_\parallel\right] \tag{17}$$

Notice that Eq. (17) gives the heat flow $\vec{q} = \int d^3v f\vec{w}(Mw^2-5T)/2$ correct to order $\delta \sim \Delta$:

$$\vec{q} = (5p/2M\Omega)\vec{n}\times\nabla T - (125p/32M\nu)\vec{n}\vec{n}\cdot\nabla T\ , \tag{18}$$

where the first and second terms are the usual [1-3] diamagnetic and parallel collisional heat fluxes, respectively, and we define $\vec{q}_\parallel = q_\parallel\vec{n} = \vec{q}\cdot\vec{n}\vec{n}$.

The preceding results are well known [1-6]; however, the solution of Eq. (11) for f_2 that follows is new so we present a few more details. To simplify the right side we first note that we may neglect viscous heating and temperature equilibration in energy conservation,

$$\frac{3n}{2}\left(\frac{\partial T}{\partial t} + \vec{V}\cdot\nabla T\right) + p\nabla\cdot\vec{V} + \nabla\cdot\vec{q} + \vec{\pi}:\nabla\vec{V} = \frac{3mn_e\nu_{ei}}{M}(T_e - T), \tag{19}$$

to obtain

$$\frac{\partial f_0}{\partial t} + \vec{V}\cdot\nabla f_0 = -\frac{2x^2}{3}f_0\nabla\cdot\vec{V} + \frac{f_0}{p}\left(\frac{2x^2}{3} - 1\right)\nabla\cdot\vec{q}\,,$$

where $\nu_{ei} = 4(2\pi)^{1/2}e^4 n_e \ell n\Lambda/3m^{1/2}T_e^{3/2}$ is the electron-ion collision frequency with n_e and T_e the electron density and temperature. In addition, using f_1 gives

$$\vec{w}\cdot\nabla f_1 = -\vec{w}\vec{w}:\nabla\left\{\frac{2Mf_0}{5pT}\left[L_1^{(3/2)}(x^2)\vec{q} - \frac{4}{15}L_2^{(3/2)}(x^2)\vec{q}_\parallel\right]\right\}$$

and

$$\nabla_w f_1 = -\frac{2Mf_0}{5pT}\left[L_1^{(3/2)}(x^2)\vec{q} - \frac{4}{15}L_2^{(3/2)}(x^2)\vec{q}_\parallel\right]$$
$$+\frac{2M^2f_0}{5pT^2}\left[L_2^{(5/2)}(x^2)(\vec{q} - \frac{4}{15}q_\parallel\vec{n}) - \frac{4}{15}L_2^{(3/2)}(x^2)\vec{q}_\parallel\right]\cdot\vec{w}\vec{w}$$

where we use the double dot convention $\vec{a}\vec{b}:\vec{c}\vec{d} = \vec{b}\cdot\vec{c}\ \vec{a}\cdot\vec{d}$. As a result, $\bar{f}_2$ is found by solving the gyro-average of Eq. (11); namely,

$$C_1\{\bar{f}_2\} + \langle C_2\{f_1,f_1\}\rangle = \langle \vec{w}\cdot\nabla f_1 + (Mn)^{-1}\nabla p\cdot\nabla_w f_1\rangle + \langle\frac{\partial f_0}{\partial t} + \vec{V}\cdot\nabla f_0 + \frac{M}{T}f_0\vec{w}\cdot\nabla\vec{V}\cdot\vec{w}\rangle. \tag{20}$$

Notice that $\bar{f}_2$ will contain terms of order Δ^2, $\delta\,\Delta$, and δ^2. Subtracting Eq. (20) from Eq. (11) and assuming $\nu << \Omega$ gives the equation for $\tilde{f}_2$ to be

$$\Omega\vec{w}\times\vec{n}\cdot\nabla_w\tilde{f}_2 + (\vec{w}\vec{w} - \langle\vec{w}\vec{w}\rangle):\vec{S} = 0, \tag{21}$$

where $\vec{S}$ is defined as

$$\vec{S} = \frac{M}{T}f_0\nabla\vec{V} - \nabla\left\{\frac{2Mf_0}{5pT}\left[L_1^{(3/2)}(x^2)\vec{q} - \frac{4}{15}L_2^{(3/2)}(x^2)\vec{q}_\parallel\right]\right\}$$
$$+\frac{2Mf_0}{5p^2T}(\nabla p)\left[L_1^{(5/2)}(x^2)(\vec{q} - \frac{4}{15}q_\parallel\vec{n}) - \frac{4}{15}L_2^{(3/2)}(x^2)\vec{q}_\parallel\right]. \tag{22}$$

We integrate by using

$$\vec{w}\vec{w} - \langle\vec{w}\vec{w}\rangle = -\vec{w}\times\vec{n}\cdot\nabla_w\{[w_\parallel\vec{n} + (1/4)\vec{w}_\perp]\vec{w}\times\vec{n} + \vec{w}\times\vec{n}[w_\parallel\vec{n} + (1/4)\vec{w}_\perp]\}$$

to find

$$\tilde{f}_2 = \Omega^{-1}\{[w_\parallel\vec{n} + (1/4)\vec{w}_\perp]\vec{w}\times\vec{n} + \vec{w}\times\vec{n}[w_\parallel\vec{n} + (1/4)\vec{w}_\perp]\}:\vec{\vec{S}}$$

$$= \frac{1}{8\Omega}(\vec{w}\vec{w} - \frac{1}{3}w^2\vec{\vec{I}}):[\vec{n}\times(\vec{\vec{S}} + \vec{\vec{S}}^T)\cdot(\vec{\vec{I}} + 3\vec{n}\vec{n}) - (\vec{\vec{I}} + 3\vec{n}\vec{n})\cdot(\vec{\vec{S}} + \vec{\vec{S}}^T)\times\vec{n}] \qquad (23)$$

where $\vec{\vec{S}}^T$ is the transpose of $\vec{\vec{S}}$ and $\vec{\vec{I}} = \vec{e}_1\vec{e}_1 + \vec{e}_2\vec{e}_2 + \vec{n}\vec{n}$ the unit dyad. The second form for $\tilde{f}_2$ is convenient since $\vec{n}\times(\vec{\vec{S}} + \vec{\vec{S}}^T)\cdot(\vec{\vec{I}} + 3\vec{n}\vec{n}) - (\vec{\vec{I}} + 3\vec{n}\vec{n})\cdot(\vec{\vec{S}} + \vec{\vec{S}}^T)\times\vec{n}$ is symmetric and traceless with a vanishing $\vec{n}\vec{n}$ component. The solution for $\tilde{f}_2$ contains terms of order δ^2 and $\delta\,\Delta$; the order ν/Ω perpendicular viscosity corrections will be evaluated by a moment approach in the next section. Our solution differs from that of Mikhailovskii and Tsypin [4-6] who use a polynomial approximation for $\tilde{f}_2$ that neglects terms involving $L_2^{(5/2)}(x^2) = L_1^{(5/2)}(x^2) + L_2^{(3/2)}(x^2)$ so their $\tilde{f}_2$ only contains $L_0(x^2) = 1$ and $L_1^{(5/2)}(x^2) = 1 + L_1^{(3/2)}(x^2)$. This shortcoming only appears when they evaluate the perpendicular collisional viscosity since they evaluate the gyro-viscosity by a moment approach.

The solution of Eq. (20) for $\bar{f}_2$ is more involved since it is a complicated Spitzer problem. We begin by noting that $\langle\vec{w}\vec{w}\rangle - (w^2/3)\vec{\vec{I}} = w^2P_2(\xi)[\vec{n}\vec{n} - (1/3)\vec{\vec{I}}]$, where $\xi = w_\parallel/w$ and $P_2(\xi) = (3\xi^2 - 1)/2$ is a Legendre polynomial. Using the preceding, Eq. (20) becomes

$$C_1\{\bar{f}_2\} = H, \qquad (24)$$

where

$$H = -\langle C_2\{f_1,f_1\}\rangle + 2x^2P_2(\xi)[\vec{n}\vec{n} - (1/3)\vec{\vec{I}}]:\nabla\vec{V} + (2f_0/3p)L_1^{(1/2)}(x^2)\nabla\cdot\vec{q}$$

$$-Mw^2[\vec{\vec{I}} + (3\vec{n}\vec{n} - \vec{\vec{I}})P_2(\xi)]:\nabla\left\{\frac{2f_0}{15pT}\left[L_1^{(3/2)}(x^2)\vec{q} - \frac{4}{15}L_2^{(3/2)}(x^2)\vec{q}_\parallel\right]\right\}$$

$$+\frac{4f_0x^2}{15p^2}\left[L_1^{(5/2)}(x^2)(\vec{q} - \frac{4}{15}\vec{q}_\parallel) - \frac{4}{15}L_2^{(3/2)}(x^2)\vec{q}_\parallel\right]\cdot[\vec{\vec{I}} + (3\vec{n}\vec{n} - \vec{\vec{I}})P_2(\xi)]\cdot\nabla p$$

$$-\frac{2f_0}{5p^2}\left[L_1^{(3/2)}(x^2)\vec{q} - \frac{4}{15}L_2^{(3/2)}(x^2)\vec{q}_\parallel\right]\cdot\nabla p. \qquad (25)$$

To solve Eq. (24) we note the self adjointness of C_1 and define the functional

$$\Lambda = \int d^3w hC_1\{hf_0\} - 2\int d^3w hH \qquad (26)$$

which is variational ($\delta\Lambda = 0$ if $hf_0 = \bar{f}_2$) and maximal ($\delta^2\Lambda \le 0$). We only require the portion of $\bar{f}_2$ that contributes to the parallel viscosity [that is, terms proportional to $P_2(\xi)$]; so we assume a trial function h of the form

$$h = x^2P_2(\xi)[a_0 + a_1L_1^{(5/2)}(x^2)]\ . \qquad (27)$$

The coefficients a_j are determined variationally by minimizing Λ ($\partial\Lambda/\partial a_j = 0$). To perform the integrals we use the orthogonality of Legendre and generalized Laguerre polynomials $\int_0^1 d\xi P_j(\xi)P_k(\xi) = \delta_{jk}/(2k+1)$ and $\int_0^\infty dz z^\alpha L_j^{(\alpha)}(z)L_k^{(\alpha)}(z)\exp(-z) = \delta_{jk}\Gamma(k+\alpha+1)/k!$, where δ_{jk} is the Kronecker delta function and $\Gamma(k+\alpha+1)$ a gamma function. The preceding are used to show [1, 2, 11, 12]

$$\int d^3wx^2P_2(\xi)L_j^{(5/2)}(x^2)C_1\{x^2P_2(\xi)L_k^{(5/2)}(x^2)f_0\} = -\frac{9}{10}n\nu B_{jk} = -\frac{9}{10}n\nu\begin{pmatrix} 1 & \frac{3}{4} & \frac{15}{32} \\ \frac{3}{4} & \frac{205}{48} & \frac{489}{128} \\ \frac{15}{32} & \frac{489}{128} & \frac{11889}{1024} \end{pmatrix}$$

where j and k = 0,1, and 2. As a result, we find

$$\int d^3whC_1\{hf_0\} = -\frac{9}{10}n\nu(a_0^2+\frac{3}{2}a_0a_1+\frac{205}{48}a_1^2) \tag{28}$$

and

$$2\int d^3wh[H + C_2\{f_1,f_1\}] = a_0n(3\vec{n}\cdot\nabla\vec{V}\cdot\vec{n} - \nabla\cdot\vec{V}) + \frac{2a_0}{5T}(3\vec{n}\cdot\nabla\vec{q}\cdot\vec{n} - \nabla\cdot\vec{q}) \tag{29}$$
$$+\frac{7a_1}{5pT}(3\vec{q}_\parallel - \vec{q})\cdot\nabla p - \frac{7a_1}{5T}[\vec{n}\vec{n} - (1/3)\vec{I}]:\{2[3\vec{q} + (2/5)\vec{q}_\parallel]\nabla\ell nT + [3\nabla\vec{q} + (4/5)\nabla\vec{q}_\parallel]\}.$$

Consequently, all that remains to be evaluated are the new q^2 and $q_\parallel^2$ contributions to Eq. (26) from the full non-linear collision term

$$C_2\{f_1,f_1\} = \nabla_w\cdot[\gamma\int d^3w'\, g^{-3}(g^2\vec{I} - \vec{g}\vec{g})\cdot(\nabla_w - \nabla_{w'})(f_1f_1')], \tag{30}$$

where $\gamma = 2\pi e^4\ell n\Lambda/M^2 = 3\pi^{1/2}T^{3/2}\nu/2M^{3/2}n$, $\vec{g} = \vec{w} - \vec{w}'$, and $f_1' = f_1(\vec{r},\vec{w}',t)$.

To evaluate the C_2 contributions to Eq. (26) it is convenient to form the following moments:

$$\int d^3w\vec{w}\vec{w}C_2\{f_1,f_1\} = 2\gamma\int d^3g\int d^3Gf_1f_1'g^{-3}(g^2\vec{I} - 3\vec{g}\vec{g}) \tag{31}$$

and

$$\int d^3ww^2[\vec{w}\vec{w} - (1/3)w^2\vec{I}]C_2\{f_1,f_1\} = 2\gamma\int d^3g\int d^3Gf_1f_1'\{g^{-3}[G^2 + (g^2/4)](g^2\vec{I} - 3\vec{g}\vec{g})$$
$$-(4/3g)(G^2\vec{I} - 3\vec{G}\vec{G}) + (2\vec{G}\cdot\vec{g}/g^3)[2\vec{G}\cdot\vec{g}\vec{I} - 3(\vec{G}\vec{g} + \vec{g}\vec{G})]\}, \tag{32}$$

where to get these forms we have integrated by parts, interchanged primed and unprimed variables (and taken one half the sum), and introduced the new velocity variables $\vec{g} = \vec{w} - \vec{w}'$ and $\vec{G} = (\vec{w} + \vec{w}')/2$. To evaluate the integrals we introduce the dimensionless variables $\vec{u} = (M/T)^{1/2}\vec{g}/2$ and $\vec{c} = (M/T)^{1/2}\vec{G}$ to write

$$\frac{f_1f_1'}{f_0f_0'} = \frac{M}{25p^2T}\Big\{\Big[(c^2+u^2)^2 - 10(c^2+u^2) + 25 - 4(\vec{c}\cdot\vec{u})^2\Big]\Big[(\vec{q}\cdot\vec{c})^2 - (\vec{q}\cdot\vec{u})^2\Big]$$
$$+\frac{1}{225}\Big[(c^2+u^2)^4 - 28(c^2+u^2)^3 + 266(c^2+u^2)^2 - 980(c^2+u^2) + 1225 + 16(\vec{c}\cdot\vec{u})^4$$
$$-8(c^2+u^2)^2(\vec{c}\cdot\vec{u})^2 + 112(c^2+u^2)(\vec{c}\cdot\vec{u})^2 - 504(\vec{c}\cdot\vec{u})^2\Big]\Big[(\vec{q}_\parallel\cdot\vec{c})^2 - (\vec{q}_\parallel\cdot\vec{u})^2\Big] + \frac{2}{15}\Big[(c^2+u^2)^3$$
$$-19(c^2+u^2)^2 + 105(c^2+u^2) - 175 - 4(c^2+u^2)(\vec{c}\cdot\vec{u})^2 + 36(\vec{c}\cdot\vec{u})^2\Big]\Big[\vec{q}\cdot\vec{c}\vec{c}\cdot\vec{q}_\parallel - \vec{q}\cdot\vec{u}\vec{u}\cdot\vec{q}_\parallel\Big]$$
$$+\frac{4\vec{c}\cdot\vec{u}}{15}\Big[(c^2+u^2)^2 - 10(c^2+u^2) + 35 - 4(\vec{c}\cdot\vec{u})^2\Big]\Big[\vec{q}\cdot\vec{c}\vec{u}\cdot\vec{q}_\parallel - \vec{q}\cdot\vec{u}\vec{c}\cdot\vec{q}_\parallel\Big]\Big\}. \tag{33}$$

We have performed the tedious six dimensional integrals (31) and (32) both analytically and with Mathematica to find

$$\int d^3w\vec{w}\vec{w}C_2\{f_1,f_1\} = \frac{-3\nu}{50pT}\{(q^2\vec{\mathrm{I}} - 3\vec{q}\vec{q}) + \frac{7}{100}(q_{\|}^2\vec{\mathrm{I}} - 3\vec{q}_{\|}\vec{q}_{\|}) - \frac{7}{30}[2q_{\|}^2\vec{\mathrm{I}} - 3(\vec{q}\vec{q}_{\|} + \vec{q}_{\|}\vec{q})]\} \quad (34)$$

and

$$\int d^3ww^2[\vec{w}\vec{w} - (1/3)w^2\vec{\mathrm{I}}]C_2\{f_1,f_1\} = \frac{-3\nu}{50pT}\{\frac{121}{30}(q^2\vec{\mathrm{I}} - 3\vec{q}\vec{q})$$

$$+\frac{1463}{5400}(q_{\|}^2\vec{\mathrm{I}} - 3\vec{q}_{\|}\vec{q}_{\|}) - \frac{121}{180}[2q_{\|}^2\vec{\mathrm{I}} - 3(\vec{q}\vec{q}_{\|} + \vec{q}_{\|}\vec{q})]\} \quad (35)$$

Using the preceding results we obtain

$$\int d^3whC_2\{f_1,f_1\} = -\frac{9Mn\nu}{200p^2T}\left[a_0(q^2 - \frac{331}{150}q_{\|}^2) + a_1(\frac{89}{60}q^2 - \frac{14833}{5400}q_{\|}^2)\right] . \quad (36)$$

From the form of Eq. (28), we see $\partial\Lambda/\partial a_j = 0$ gives two equations coupling a_0 and a_1. To form the parallel viscosity only a_0 is needed:

$$a_0 = -\frac{379Mq^2}{8900Tp^2} + \frac{8837Mq_{\|}^2}{89000Tp^2} - \frac{1025}{534\nu}[\vec{n}\cdot\nabla\vec{V}\cdot\vec{n} - (1/3)\nabla\cdot\vec{V}] - \frac{331}{267p\nu}[\vec{n}\cdot\nabla\vec{q}\cdot\vec{n} - (1/3)\nabla\cdot\vec{q}]$$

$$-\frac{56}{445p\nu}[\vec{n}\cdot\nabla\vec{q}_{\|}\cdot\vec{n} - (1/3)\nabla\cdot\vec{q}_{\|}] - \frac{952}{1335pT\nu}\vec{q}_{\|}\cdot\nabla T - \frac{14}{89p^2\nu}\vec{q}\cdot\nabla p + \frac{42}{89p^2\nu}\vec{q}_{\|}\cdot\nabla p , \quad (37)$$

where the q^2 and $q_{\|}^2$ terms are from $C_2\{f_1,f_1\}$.

The complete solution for f to the accuracy we require is given by adding Eqs. (12), (17), (23) and (27). This solution will be used in the following sections to evaluate the collisional heat flux and the various viscosities.

III. ION VISCOSITY AND COLLISIONAL PERPENDICULAR ION HEAT FLUX

The collisional contribution to the perpendicular ion heat flux is formally smaller by ν/Ω than the order Δ parallel collisional heat flux and the order δ diamagnetic heat flux. It is most conveniently evaluated [8,9] by forming the $(Mw^2/2)\vec{w}$ moment of Eq. (4), which using the definition $\vec{q} = (1/2)\int d^3vf(Mw^2 - 5T)\vec{w}$ gives to the two lowest orders

$$\Omega\vec{n}\times\vec{q} + (5p/2M)\nabla T = \int d^3w(Mw^2/2)\vec{w}C_1\{f_1\} . \quad (38)$$

Crossing by $\vec{n}$, substituting in for f_1, evaluating the integrals using [13]

$$\int d^3wx^2\vec{w}C_1\{x^2\vec{w}f_0\} = -(2\nu p/M)\vec{\mathrm{I}} ,$$

recalling that to lowest order $\vec{q}_{\perp} = (5p/2M\Omega)\vec{n}\times\nabla T$, and adding in $q_{\|}$ yields the familiar expressions for the collisional perpendicular, diamagnetic, and parallel ion heat fluxes:

$$\vec{q} = (5p/2M\Omega)\vec{n}\times\nabla T - (2p\nu/M\Omega^2)\nabla_{\perp}T - (125p/32M\nu)\vec{n}\vec{n}\cdot\nabla T . \quad (39)$$

Equation (39) along with $\vec{\pi}$, evaluated next, and $\vec{F}$, evaluated in the next section, completes the closure of the energy conservation equation, which in conservation form is

$$\frac{\partial}{\partial t}\left(\frac{3}{2}p+\frac{1}{2}MnV^2\right)+\nabla\cdot\left[\left(\frac{5}{2}p+\frac{1}{2}MnV^2\right)\vec{V}+\vec{\vec{\pi}}\cdot\vec{V}+\vec{q}\right]=(en\vec{E}-\vec{F})\cdot\vec{V}+\frac{3mn_e\nu_{ei}}{M}(T_e-T).$$

It is considerably more involved to evaluate the ion viscosity $\vec{\vec{\pi}}$, which is also needed to close the momentum conservation equation (6), which in conservation form is

$$\frac{\partial}{\partial t}(Mn\vec{V})+\nabla p+\nabla\cdot(\vec{\vec{\pi}}+Mn\vec{V}\vec{V})=en(\vec{E}+\frac{1}{c}\vec{V}\times\vec{B})-\vec{F}\ .$$

Closure requires evaluating the collisional parallel viscosity and collisionless gyro-viscosity, which can be performed directly using $\bar{f}_2$ and $\tilde{f}_2$, respectfully. In addition, the collisional perpendicular ion viscosity is most conveniently evaluated by a moment approach.

We begin by evaluating the parallel ion viscosity

$$\vec{\vec{\pi}}_{\|}=M\int d^3w\langle\vec{w}\vec{w}-\frac{1}{3}w^2\vec{\vec{I}}\rangle\bar{f}=(\vec{n}\vec{n}-\frac{1}{3}\vec{\vec{I}})(p_{\|}-p_{\perp})\ . \qquad (40)$$

where $p_{\|}=p+\int d^3w\bar{f}_2Mw_{\|}^2$, $p_{\perp}=p+\int d^3w\bar{f}_2Mw_{\perp}^2/2$, and

$$p_{\|}-p_{\perp}=\int d^3ww^2P_2(\xi)\bar{f}_2\ .$$

Substituting in $\bar{f}_2=hf_0$ with h given by Eq. (27) and using the orthogonality properties of the Legendre and generalized Laguerre polynomials gives $p_{\|}-p_{\perp}=3pa_0/2$ so that

$$p_{\|}-p_{\perp}=-\frac{1137Mq^2}{17800Tp}+\frac{26511\,Mq_{\|}^2}{178000Tp}+\frac{1025p}{1068\nu}[\nabla\cdot\vec{V}-3\vec{n}\cdot\nabla\vec{V}\cdot\vec{n}]-\frac{476}{445T\nu}\vec{q}_{\|}\cdot\nabla T$$
$$+\frac{331}{534\nu}[\nabla\cdot\vec{q}-3\vec{n}\cdot\nabla\vec{q}\cdot\vec{n}]+\frac{28}{445\nu}[\nabla\cdot\vec{q}_{\|}-3\vec{n}\cdot\nabla\vec{q}_{\|}\cdot\vec{n}]-\frac{21}{89p\nu}[\vec{q}\cdot\nabla p-3\vec{q}_{\|}\cdot\nabla p]\,,$$

with $\vec{q}$ and $\vec{q}_{\|}$ given by their lowest order forms (3) or (18). The first two terms are proportional to q^2 and $q_{\|}^2$ and arise because our $\delta\sim\Delta$ ordering requires us to retain $C_2\{f_1,f_1\}$ in Eq. (24). Like the $\nabla\cdot\vec{q}_{\|}-3\vec{n}\cdot\nabla\vec{q}_{\|}\cdot\vec{n}$ term, they have small coefficients, but are formally of the same order as the remaining terms previously obtained by Mikhailovskii and Tyspin [4-6]. Indeed, the $\vec{q}_{\|}\cdot\nabla T$ and $q_{\|}^2$ are exactly of the same form and so can be combined to write

$$\vec{\vec{\pi}}_{\|}=(\vec{n}\vec{n}-\frac{1}{3}\vec{\vec{I}})\left\{\frac{1025}{1068\nu}[p\nabla\cdot\vec{V}+(2/5)\nabla\cdot\vec{q}-3p\vec{n}\cdot\nabla\vec{V}\cdot\vec{n}-(6/5)\vec{n}\cdot\nabla\vec{q}\cdot\vec{n}]+\frac{319417Mq_{\|}^2}{890000Tp}\right.$$
$$\left.-\frac{21}{89\nu}[\vec{q}\cdot\nabla\ell np-\nabla\cdot\vec{q}+3\vec{n}\cdot\nabla\vec{q}\cdot\vec{n}-3\vec{q}_{\|}\cdot\nabla\ell np]+\frac{28}{445\nu}[\nabla\cdot\vec{q}_{\|}-3\vec{n}\cdot\nabla\vec{q}_{\|}\cdot\vec{n}]-\frac{1137Mq_{\perp}^2}{17800Tp}\right\} \qquad (41)$$

or, in a more compact double dot notation form similar to Mikhailovskii and Tsypin's,

$$\vec{\vec{\pi}}_{\|}=\frac{0.960}{\nu}(\vec{\vec{I}}-3\vec{n}\vec{n}):\left\{(p\nabla\vec{V}+\frac{2}{5}\nabla\vec{q})+0.246(\nabla\vec{q}-\vec{q}\nabla\ell np+\frac{4}{15}\nabla\vec{q}_{\|})\right\}(\vec{n}\vec{n}-\frac{1}{3}\vec{\vec{I}})$$
$$+\frac{M}{pT}\left[0.412q_{\|}^2-0.064q^2\right](\vec{n}\vec{n}-\frac{1}{3}\vec{\vec{I}})\ . \qquad (42)$$

Notice that for our ordering the $p\nabla\cdot\vec{V}_\perp$, $\nabla\cdot\vec{q}_\perp$, and $\vec{q}_\perp\cdot\nabla\ell np$ are larger by $L_\parallel/L_\perp$ than the remaining terms. However, these terms only appear in the combinations $5p\nabla\cdot\vec{V}_\perp + 2\nabla\cdot\vec{q}_\perp$ and $p\nabla\cdot\vec{q}_\perp - \vec{q}_\perp\cdot\nabla p$ which are the same order as all the other terms in $\vec{\pi}_\parallel$.

The gyro-viscosity is evaluated by using $\tilde{f}_2$ in Eq. (7):

$$\vec{\pi}_g = M\int d^3w(\vec{w}\vec{w} - \frac{1}{3}w^2\vec{I})\tilde{f}_2 = M\int d^3w\vec{w}\vec{w}\tilde{f}_2\ . \tag{43}$$

Inserting Eq. (23) for $\tilde{f}_2$, using $\int d^3w\vec{w}\vec{w}\vec{w}\vec{w}\nabla Q = \nabla\int d^3w\vec{w}\vec{w}\vec{w}\vec{w}Q$ and

$$\int d^3w\ w_\alpha w_\beta w_\sigma w_\gamma f_0 = n(T/M)^2[\delta_{\alpha\beta}\delta_{\sigma\gamma} + \delta_{\alpha\gamma}\delta_{\sigma\beta} + \delta_{\alpha\sigma}\delta_{\beta\gamma}],$$

noting $L_2^{(3/2)}(x^2) = L_2^{(5/2)}(x^2) - L_1^{(5/2)}(x^2)$, and using the orthogonality relations for the Legendre and generalized Laguerre polynomials to show

$$\int d^3w\ \vec{w}\vec{w}\vec{w}\vec{w}L_j^{(5/2)}(x^2)f_0 = 0\ ,$$

we find the Mikhailovskii and Tyspin [4-6] result for the gyro-viscosity, namely,

$$\vec{\pi}_g = \frac{1}{4\Omega}\left\{\vec{n}\times\left[\left(p\nabla\vec{V} + \frac{2}{5}\nabla\vec{q}\right) + \left(p\nabla\vec{V} + \frac{2}{5}\nabla\vec{q}\right)^T\right]\cdot\left(\vec{I} + 3\vec{n}\vec{n}\right)\right.$$
$$\left. - \left(\vec{I} + 3\vec{n}\vec{n}\right)\cdot\left[\left(p\nabla\vec{V} + \frac{2}{5}\nabla\vec{q}\right) + \left(p\nabla\vec{V} + \frac{2}{5}\nabla\vec{q}\right)^T\right]\times\vec{n}\right\}, \tag{44}$$

with $\vec{q}$ given to lowest order by Eq. (18). Equation (44) is the normal definition of the gyro-viscosity $\vec{\pi}_g$ as found from the gyrophase dependent $\tilde{f}_2$ part of f by assuming $\nu << \Omega$. However, it is not completely diamagnetic since it depends on collisions through $q_\parallel$ due to the $\bar{f}_1$ contributions to Eq. (11). As a result, the $q_\parallel$ terms in this form of $\vec{\pi}_g$ and Eq. (23) for $\tilde{f}_2$ cannot be obtained from the strictly collisionless gyrophase dependent term used to derive the Hazeltine drift kinetic equation [14, 15]. We also remark that the radial flux of toroidal angular momentum in the Pfirsch-Schlüter regime is thought to be due to poloidal variation of $\vec{\pi}_g$ caused by the poloidal variation of B in a tokamak for $\delta << \Delta$ [7].

To complete the ion description we need to evaluate the collisional portion of the perpendicular ion viscosity $\vec{\pi}_\perp$. To this end we use a moment approach to evaluate $\vec{\pi}_\perp$. Forming the $\vec{w}\vec{w}$ moment of Eq. (4) or (8) gives

$$\Omega(\vec{\pi}\times\vec{n} - \vec{n}\times\vec{\pi}) + M\int d^3w\vec{w}\vec{w}C = \vec{I}[\partial p/\partial t + \nabla\cdot(p\vec{V})] + \nabla\cdot(M\int d^3wf\vec{w}\vec{w}\vec{w}) + p\nabla\vec{V} + p(\nabla\vec{V})^T,$$

where the contribution from C_{ie} vanishes and we have neglected higher order terms involving time and space derivatives of $\vec{\pi}$ that are small by $\delta^2\Omega/\nu << 1$ (recall that we assume $\nu >> \partial/\partial t \sim \delta^2\Omega$ to find f). The trace of the preceding equation is

$$\partial p/\partial t + \nabla\cdot(p\vec{V}) + \nabla\cdot[(M/3)\int d^3wfw^2\vec{w}] + (2p/3)\nabla\cdot\vec{V} = 0,$$

since energy must be conserved in like particle collisions. Combining these two equations to eliminate $\partial p/\partial t$ gives the desired moment form

$$\Omega(\vec{\pi}\times\vec{n} - \vec{n}\times\vec{\pi}) = \vec{K} \tag{45}$$

where $\overleftrightarrow{K}$ is the symmetric and traceless tensor

$$\overleftrightarrow{K} = p\nabla\vec{V}+p(\nabla\vec{V})^T-(2p/3)\overleftrightarrow{I}\nabla\cdot\vec{V}+(2/5)[\nabla\vec{q}+(\nabla\vec{q})^T-(2/3)\overleftrightarrow{I}\nabla\cdot\vec{q}]-M\int d^3w\vec{w}\vec{w}C.$$

and we have substituted in for f to find

$$\nabla\cdot\{M\int d^3wf_1\vec{w}[\vec{w}\vec{w}-(w^2/3)\overleftrightarrow{I}]\}=(2/5)[\nabla\vec{q}+(\nabla\vec{q})^T-(2/3)\overleftrightarrow{I}\nabla\cdot\vec{q}] .$$

From Eq. (45) we see that $\overleftrightarrow{K}$ must have the property $\vec{n}\cdot\overleftrightarrow{K}\cdot\vec{n}=0$. To make this true term by term we can make the replacement $\overleftrightarrow{K}\rightarrow\overleftrightarrow{K}+(1/2)(\overleftrightarrow{I}-3\vec{n}\vec{n})\vec{n}\cdot\overleftrightarrow{K}\cdot\vec{n}$. As a result, $\overleftrightarrow{K}$ becomes

$$\overleftrightarrow{K}=\left(p\nabla\vec{V}+\frac{2}{5}\nabla\vec{q}\right)+\left(p\nabla\vec{V}+\frac{2}{5}\nabla\vec{q}\right)^T-\frac{2}{3}\overleftrightarrow{I}\left(p\nabla\cdot\vec{V}+\frac{2}{5}\nabla\cdot\vec{q}\right)-M\int d^3w\vec{w}\vec{w}C$$
$$+(\overleftrightarrow{I}-3\vec{n}\vec{n})[\vec{n}\cdot(p\nabla\vec{V}+\frac{2}{5}\nabla\vec{q})\cdot\vec{n}-\frac{1}{3}(p\nabla\cdot\vec{V}+\frac{2}{5}\nabla\cdot\vec{q})-\frac{1}{2}M\int d^3ww_\parallel^2C], \tag{46}$$

where the terms not involving C lead to the gyro-viscous contribution and the C terms will yield the collisional corrections to the perpendicular viscosity. To see this behavior we solve Eq. (45) to find [15] $\overleftrightarrow{\pi}=(1/4\Omega)[\vec{n}\times\overleftrightarrow{K}\cdot(\overleftrightarrow{I}+3\vec{n}\vec{n})-(\overleftrightarrow{I}+3\vec{n}\vec{n})\cdot\overleftrightarrow{K}\times\vec{n}]+\overleftrightarrow{\pi}_\parallel$ or upon using (44),

$$\overleftrightarrow{\pi}=\overleftrightarrow{\pi}_\parallel+\overleftrightarrow{\pi}_g+\frac{1}{4\Omega}[\vec{n}\times\overleftrightarrow{K}_\nu\cdot(\overleftrightarrow{I}+3\vec{n}\vec{n})-(\overleftrightarrow{I}+3\vec{n}\vec{n})\cdot\overleftrightarrow{K}_\nu\times\vec{n}], \tag{47}$$

where we define $\overleftrightarrow{K}_\nu=-M\int d^3w\vec{w}\vec{w}C-(\overleftrightarrow{I}-3\vec{n}\vec{n})(M/2)\int d^3ww_\parallel^2C$. In writing down the solution to Eq. (45) we added in a homogeneous solution which can only contain terms proportional to $\overleftrightarrow{I}$ and $\vec{n}\vec{n}$ and must equal $\overleftrightarrow{\pi}_\parallel$ since no isotropic term is allowed in $\overleftrightarrow{\pi}$.

To evaluate the collisional terms we first define $C=\overline{C}+\tilde{C}$ with $\overline{C}=C_1\{\overline{f}_2\}+\langle C_2\{f_1,f_1\}\rangle$ and $\tilde{C}=C_1\{\tilde{f}_2\}+C_2\{f_1,f_1\}-\langle C_2\{f_1,f_1\}\rangle$. Using Eqs. (24) and (25) for $\overline{C}$ and recalling $\langle\vec{w}\vec{w}\rangle-(w^2/3)\overleftrightarrow{I}=w^2P_2(\xi)[\vec{n}\vec{n}-(1/3)\overleftrightarrow{I}]$ we see that

$$\overleftrightarrow{K}_0\equiv-M\int d^3w\vec{w}\vec{w}\overline{C}-(\overleftrightarrow{I}-3\vec{n}\vec{n})(M/2)\int d^3ww_\parallel^2C=0 . \tag{48}$$

The gyrophase dependent collisional terms are evaluated by using Eq. (34) to find

$$M\int d^3w\vec{w}\vec{w}\langle C_2\{f_1,f_1\}\rangle=\frac{3}{2}(\vec{n}\vec{n}-\frac{1}{3}\overleftrightarrow{I})\vec{n}\vec{n}:M\int d^3w\vec{w}\vec{w}C_2\{f_1,f_1\}$$
$$=-\frac{9M\nu}{100pT}\left(q^2-\frac{331}{150}q_\parallel^2\right)(\vec{n}\vec{n}-\frac{1}{3}\overleftrightarrow{I}).$$

We require the symmetric and traceless combination

$$\overleftrightarrow{K}_2\equiv-M\int d^3w\vec{w}\vec{w}[C_2\{f_1,f_1\}-\langle C_2\{f_1,f_1\}\rangle]$$
$$=-\frac{9M\nu}{50pT}\left[\vec{q}\vec{q}-\frac{7}{30}(\vec{q}\vec{q}_\parallel+\vec{q}_\parallel\vec{q})-\frac{1}{2}q^2(\overleftrightarrow{I}-\vec{n}\vec{n})+\frac{1}{2}q_\parallel^2\left(\overleftrightarrow{I}-\frac{31}{15}\vec{n}\vec{n}\right)\right], \tag{49}$$

where $\vec{n}\cdot\overleftrightarrow{K}_2\cdot\vec{n}=0$. The tensor $\overleftrightarrow{K}_2$ contains all the new quadratic heat flux terms.

Using the self-adjointness of the linearized collision operator the final integral required is

$$M\int d^3w\vec{w}\vec{w}C_1\{\tilde{f}_2\}=M\int d^3w(\tilde{f}_2/f_0)C_1\{\vec{w}\vec{w}f_0\} \tag{50}$$

Using the procedure in Appendix C of Ref. 13 we write

$$C_1\{\vec{w}\vec{w}f_0\} = J(x)(\vec{w}\vec{w} - \frac{1}{3}w^2\overleftrightarrow{I}) \tag{51}$$

with

$$J(x) = -\frac{9\pi^{1/2}\nu f_0}{2^{1/2}x^3}\left[\left(1 - \frac{3}{2x^2}\right)E(x) + \frac{3E'(x)}{2x}\right] \tag{52}$$

and $E(x) = 2\pi^{-1/2}\int_0^\infty dt\,\exp(-t^2)$ the error function and E'(x) its derivative. Using the preceding for a symmetric and traceless tensor $\overleftrightarrow{T}_k^\alpha$ gives

$$\overleftrightarrow{T}_k^\alpha : \int d^3w L_k^{(\alpha)}(x^2)\vec{w}\vec{w}C_1\{\vec{w}\vec{w}f_0\} = (2/15)\overleftrightarrow{T}_k^\alpha \int d^3w L_k^{(\alpha)}(x^2)w^4 J(x) \tag{53}$$

$$= \frac{1}{5}\overleftrightarrow{T}_k^\alpha \int d^3w L_k^{(\alpha)}(x^2)\left(\vec{w}\vec{w} - \frac{1}{3}w^2\overleftrightarrow{I}\right) : C_1\{\vec{w}\vec{w}f_0\} = -\frac{12nT^2\nu}{5M^2}\overleftrightarrow{T}_k^\alpha \begin{cases} 1 & k = 0, \quad \alpha \\ 3/4 & k = 1, \quad \alpha = 5/2 \\ -9/32 & k = 2, \quad \alpha = 3/2 \end{cases}$$

since

$$\int_0^\infty dx x^6 J(x) L_k^{(\alpha)}(x^2) = -\frac{9n\nu}{8\pi}\left(\frac{M}{2T}\right)^{3/2} \begin{cases} 1 & k = 0 \quad \alpha \\ 3/4 & k = 1 \quad \alpha = 5/2 \\ -9/32 & k = 2 \quad \alpha = 3/2 \end{cases} \tag{54}$$

Moreover, again using Eqs. (51) and (52) we find

$$\int d^3w f_0^{-1}(\nabla\vec{Q})[\vec{w}\vec{w} - (1/3)w^2\overleftrightarrow{I}] : C_1\{\vec{w}\vec{w}f_0\} = (2/3)\int d^3w w^4 f_0^{-1} J(x)\nabla\vec{Q}$$
$$= (2n/3)\nabla[\int d^3w w^4 J(x) n^{-1} f_0^{-1}\vec{Q}] + (2\nu/3M)(\nabla T)\int d^3w \vec{Q} w^2 (\partial/\partial x)[x^3 J(x)/\nu f_0]\,, \tag{55}$$

where to evaluate the final integral we also need

$$\nu\int_0^\infty dx x^4 L_k^{(\alpha)}(x^2) f_0 (\partial/\partial x)[x^3 J(x)/\nu f_0] = -\frac{27n\nu}{16\pi}\left(\frac{M}{2T}\right)^{3/2} \begin{cases} 1 & k = 0 \quad \alpha \\ -5/4 & k = 1 \quad \alpha = 5/2 \\ -5/32 & k = 2 \quad \alpha = 3/2 \end{cases}. \tag{56}$$

Inserting Eq. (23) into Eq. (50) and using Eqs. (53)-(56) to perform the integrals yields

$$\overleftrightarrow{K}_1 \equiv -M\int d^3w \vec{w}\vec{w}C_1\{\tilde{f}_2\} = \frac{3p\nu}{10\Omega}\left[\vec{n}\times\overleftrightarrow{W}\cdot(\overleftrightarrow{I} + 3\vec{n}\vec{n}) - (\overleftrightarrow{I} + 3\vec{n}\vec{n})\cdot\overleftrightarrow{W}\times\vec{n}\right] \tag{57}$$

where it is convenient to define $\overleftrightarrow{W}$ as a symmetric, traceless tensor with $\vec{n}\cdot\overleftrightarrow{W}\cdot\vec{n} = 0$:

$$p\overleftrightarrow{W} = \overleftrightarrow{W}_* + \overleftrightarrow{W}_*^T + (\overleftrightarrow{I} - 3\vec{n}\vec{n})\vec{n}\cdot\overleftrightarrow{W}_*\cdot\vec{n} - (\overleftrightarrow{I} - \vec{n}\vec{n})\overleftrightarrow{I} : \overleftrightarrow{W}_*$$

where

$$\overleftrightarrow{W}_* = p\nabla\vec{V} + \frac{2}{5}\nabla\vec{q} - \frac{3(p\nabla\vec{q} - \vec{q}\nabla p)}{10p} - \frac{(3p\nabla\vec{q}_\parallel + 5\vec{q}_\parallel\nabla p)}{100p} - \frac{(90\vec{q} - 13\vec{q}_\parallel)\nabla T}{400T}\;. \tag{58}$$

The preceding results allow us to use Eq. (47) to form the full collisional perpendicular viscosity. To do so, we define the full ion viscosity as

$$\overleftrightarrow{\pi} = \overleftrightarrow{\pi}_\parallel + \overleftrightarrow{\pi}_g + \overleftrightarrow{\pi}_\perp = \overleftrightarrow{\pi}_\parallel + \overleftrightarrow{\pi}_g + \overleftrightarrow{\pi}_{\perp 1} + \overleftrightarrow{\pi}_{\perp 2} \tag{59}$$

where the individual contributions to $\overleftrightarrow{\pi}_\perp$ are given by

$$\overleftrightarrow{\pi}_{\perp k} = \frac{1}{4\Omega}[\vec{n}\times\overleftrightarrow{K}_k\cdot(\overleftrightarrow{I} + 3\vec{n}\vec{n}) - (\overleftrightarrow{I} + 3\vec{n}\vec{n})\cdot\overleftrightarrow{K}_k\times\vec{n}] \tag{60}$$

with the $\overleftrightarrow{K}_k$ for k = 1 and 2 given by Eqs. (49) and (57) (recall $\overleftrightarrow{K}_0$ = 0). The subscript "1" denotes terms in $\overleftrightarrow{\pi}_\perp$ from the linearized collision operator, while the "2" subscript denotes the new terms that are quadratic in the heat fluxes $\vec{q}$ and $\vec{q}_{\|}$ from the non-linear collision operator. Because $\vec{\vec{I}}$ and $\vec{n}\vec{n}$ in $\vec{\vec{K}}_2$ do not contribute, we find the new terms in the collisional perpendicular viscosity to be

$$\vec{\vec{\pi}}_{\perp 2} = -\frac{9M\nu}{200pT\Omega}[\vec{n}\times\vec{q}(\vec{q}+\frac{31}{15}\vec{q}_{\|}) - (\vec{q}+\frac{31}{15}\vec{q}_{\|})\vec{q}\times\vec{n}] \tag{61}$$

These terms quadratic in the heat fluxes $\vec{q}$ and $\vec{q}_{\|}$ were not obtained by previous treatments. The lowest order forms of Eq. (3) or (18) are to be employed for $\vec{q}$ and $\vec{q}_{\|}$ here and elsewhere in $\vec{\vec{\pi}}_{\perp}$.

Inserting Eq. (57) into Eq. (60) and using $\vec{n}\cdot\vec{\vec{W}}\cdot\vec{n} = 0 = \vec{\vec{W}}:\vec{\vec{I}}$ to show that $\vec{n}\times\vec{\vec{W}}\times\vec{n} = (\vec{\vec{I}}-\vec{n}\vec{n})\cdot\vec{\vec{W}}\cdot(\vec{\vec{I}}-\vec{n}\vec{n})$ gives the form for $\vec{\vec{\pi}}_{\perp 1}$ to be

$$\vec{\vec{\pi}}_{\perp 1} = -\frac{3\nu p}{10\Omega^2}[\vec{\vec{W}} + 3(\vec{n}\,\vec{n}\cdot\vec{\vec{W}} + \vec{\vec{W}}\cdot\vec{n}\,\vec{n})] \, .$$

Ignoring homogeneous terms proportional $\vec{\vec{I}}$ and/or $\vec{n}\vec{n}$ that are ν/Ω corrections to the $\vec{\vec{\pi}}_{\|}$ of Eqs. (41) or (42), completes the description for the viscosity giving

$$\vec{\vec{\pi}}_{\perp 1} = -\frac{3\nu}{10\Omega^2}[\vec{\vec{W}}_* + \vec{\vec{W}}_*^T + 3\vec{n}(\vec{n}\cdot\vec{\vec{W}}_* + \vec{\vec{W}}_*\cdot\vec{n}) + 3(\vec{n}\cdot\vec{\vec{W}}_* + \vec{\vec{W}}_*\cdot\vec{n})\vec{n})] \tag{62}$$

or using Eq. (58)

$$\begin{aligned}\vec{\vec{\pi}}_{\perp 1} = -\frac{3\nu}{10\Omega^2}\Big\{ & p\nabla\vec{V} + \frac{2}{5}\nabla\vec{q} + (p\nabla\vec{V} + \frac{2}{5}\nabla\vec{q})^T - \frac{3}{10p}\Big[p\nabla\vec{q} - \vec{q}\nabla p + (p\nabla\vec{q} - \vec{q}\nabla p)^T\Big] \\ & -\frac{1}{100p}\Big[3p\nabla\vec{q}_{\|} + 5\vec{q}_{\|}\nabla p + (3p\nabla\vec{q}_{\|} + 5\vec{q}_{\|}\nabla p)^T\Big] - \frac{1}{40)T}\Big[(90\vec{q} - 13\vec{q}_{\|})\nabla T + (\nabla T)(90\vec{q} - 13\vec{q}_{\|})\Big] \\ & +3\vec{n}\Big[\vec{n}\cdot\Big(p\nabla\vec{V} + \frac{1}{10}\nabla\vec{q} - \frac{3}{100}\nabla\vec{q}_{\|}\Big) + \Big(p\nabla\vec{V} + \frac{1}{10}\nabla\vec{q} - \frac{3}{100}\nabla\vec{q}_{\|}\Big)\cdot\vec{n}\Big] \\ & +3\Big[\vec{n}\cdot\Big(p\nabla\vec{V} + \frac{1}{10}\nabla\vec{q} - \frac{3}{100}\nabla\vec{q}_{\|}\Big) + \Big(p\nabla\vec{V} + \frac{1}{10}\nabla\vec{q} - \frac{3}{100}\nabla\vec{q}_{\|}\Big)\cdot\vec{n}\Big]\vec{n} \\ & +\frac{3q_{\|}}{4p}[\vec{n}\nabla p + (\nabla p)\vec{n}] + \frac{9}{10p}[\vec{n}\vec{q} + \vec{q}\vec{n} - \frac{1}{3}q_{\|}\vec{n}\vec{n}]\vec{n}\cdot\nabla p \\ & -\frac{231q_{\|}}{400T}[\vec{n}\nabla T + (\nabla T)\vec{n}] - \frac{27}{40T}[\vec{n}\vec{q} + \vec{q}\vec{n} - \frac{13}{45}q_{\|}\vec{n}\vec{n}]\vec{n}\cdot\nabla T\Big\} \, .\end{aligned} \tag{63}$$

This portion of $\vec{\vec{\pi}}_{\perp}$ does not agree in detail with the result of Mikhailovskii and Tsypin [4-6] because they used an approximate form for $\tilde{f}_2$, while we use the exact result of Eq. (23). Some of the discrepancies are as follows: $(1/10)\nabla\vec{q}_{\|}$ not -0.27$\nabla\vec{q}_{\|}$, $\vec{q}_{\|}\nabla p/6$ rather than zero, and $[\vec{q}-(13/90)\vec{q}_{\|}]\nabla T$ not $(8/3)\left(\vec{q}-0.27\vec{q}_{\|}\right)\nabla T$. They occur because Mikhailovskii and Tsypin neglect x^4 and x^6 terms in the coefficients of $\nabla\vec{q}_{\|}$, $\vec{q}_{\|}\nabla p$, and $\vec{q}\nabla T$ and $\vec{q}_{\|}\nabla T$, respectively, in Eq. (22) for $\vec{\vec{S}}$ in $\tilde{f}_2$.

The ion description is now complete except for the momentum exchange term $\vec{F}$ that is evaluated in the next section when we consider the closure for the electrons.

IV. ELECTRON FORMULATION

The treatment of the electrons shares many similarities with that of the ions so fewer details will be presented. It is included for completeness since electrons were not considered in Refs. [4-6]. Only, the perpendicular viscosity will be assumed negligible.

Introducing the shifted electron velocity variable $\vec{w} = \vec{v} - \vec{V}_e$ with $n\vec{V}_e = \int d^3v\vec{v}f_e$, $n = n_e = \int d^3vf_e$, $\partial n/\partial t + \nabla\cdot(n\vec{V}_e) = 0$, and using electron momentum conservation

$$mn(\frac{\partial\vec{V}_e}{\partial t} + \vec{V}_e\cdot\nabla\vec{V}_e) + en(\vec{E} + \frac{1}{c}\vec{V}_e\times\vec{B}) = -\nabla p_e - \nabla\cdot\vec{\vec{\pi}}_e + \vec{F} \tag{64}$$

with $p_e = nT_e = m\int d^3wf_e w^2/3$, $\vec{\vec{\pi}}_e = m\int d^3w[\vec{w}\vec{w} - \vec{\vec{I}}(w^2/3)]f_e$, and $\vec{F} = m\int d^3v\vec{v}C_{ei}$, gives the kinetic equation for the electron distribution function f_e to be

$$\begin{aligned}\Omega_e\vec{w}\times\vec{n}\cdot\nabla_w f_e + C_e = [\vec{w}\cdot\nabla f_e + (mn)^{-1}(\nabla p_e - \vec{F})\cdot\nabla_w f_e] +\\ [\frac{\partial f_e}{\partial t} + \vec{V}_e\cdot\nabla f_e - \vec{w}\cdot\nabla\vec{V}_e\cdot\nabla_w f_e] + (mn)^{-1}(\nabla\cdot\vec{\vec{\pi}}_e)\cdot\nabla_w f_e\ .\end{aligned} \tag{65}$$

Electron quantities are denoted by subscript "e" to distinguish then from the unsubscripted ion quantities, with $\Omega_e = eB/mc$. The collision operator $C_e = C_{ee} + C_{ei}$ is the sum of like and unlike particle contributions, with $C_{ei} = L + D$. The Lorentz operator L is given by

$$L\{f_e\} = [3(2\pi)^{1/2}(T_e/m)^{3/2}\nu_{ei}/4]\nabla_w\cdot(\nabla_w\nabla_w w\cdot\nabla_w f_e). \tag{66}$$

The operator D is a small correction to pitch angle scattering associated with the difference in the mean flows between the ions and electrons. To lowest order it is given by

$$D\{f_e\} \equiv C_{ei} - L = -[3(2\pi)^{1/2}(T_e/m)^{3/2}\nu_{ei}/4]\nabla_w\cdot[(\vec{V} - \vec{V}_e)\cdot\nabla_w\nabla_w\nabla_w w\cdot\nabla_w f_e + \ldots], \tag{67}$$

where the terms not shown are mass ratio corrections which lead to isotropic ion-electron equilibration modifications that do not alter $\vec{\vec{\pi}}_e$, and $\vec{V}$ is the ion mean velocity.

To determine f_e it is convenient to expand using $f_e = f_{0e} + f_{1e} + f_{2e} + \ldots$ We first solve the lowest order equation

$$\Omega_e\vec{w}\times\vec{n}\cdot\nabla_w f_{0e} + C_{0ee} + L_{0ei} = 0$$

to find that f_{0e} is a Maxwellian drifting at the mean velocity of the electrons, namely,

$$f_{0e} = n\left(\frac{m}{2\pi T_e}\right)^{3/2}\exp\left[-\frac{m(\vec{v} - \vec{V}_e)^2}{2T_e}\right]. \tag{68}$$

Notice that $C_{0ee} = C_{0ee}\{f_{0e}\} = 0$ and $L_0 = L\{f_{0e}\} = 0$.

To next order

$$\Omega_e\vec{w}\times\vec{n}\cdot\nabla_w f_{1e} + C_{1ee} + L_1 = [\vec{w}\cdot\nabla f_{0e} + (mn_e)^{-1}(\nabla p_e - \vec{F})\cdot\nabla_w f_{0e}] - D_0, \tag{69}$$

where $L_1 = L\{f_{1e}\}$, $D_0 = D\{f_{0e}\}$, and $C_{1ee} = C_{1ee}\{f_{1e}\}$ is the linearized electron-electron collision operator. To find the lowest order gyrophase dependent portion of f_{1e} we need only solve

$$\Omega_e \vec{w} \times \vec{n} \cdot \nabla_w f_{1e} = f_{0e}\left(\frac{mw^2}{2T_e} - \frac{5}{2}\right)\vec{w}_\perp \cdot \nabla \ell n T$$

to obtain

$$\tilde{f}_{1e} = -\frac{f_{0e}}{\Omega_e}\left(x_e^2 - \frac{5}{2}\right)\vec{w} \times \vec{n} \cdot \nabla \ell n T_e \tag{70}$$

where $x_e^2 = mw^2/2T_e$.

The equation for gyrophase independent portion of f_{1e} is more involved since $F_\| = \vec{n} \cdot \vec{F}$ and D_0 must be retained when solving its lowest order form

$$\overline{C}_{1ee} + \overline{L}_1 = f_{0e}\left(x_e^2 - \frac{5}{2}\right)w_\| \vec{n} \cdot \nabla \ell n T + w_\| F_\| f_{0e}/p_e - \overline{D}_0, \tag{71}$$

where $C_{1ei} = L_1 + D_0$ and $D_0 = [3(2\pi)^{1/2}(T_e/m)^{1/2}\nu_{ei}/2w^3](\vec{V} - \vec{V}_e) \cdot \vec{w} f_{0e}$. We solve for $\bar{f}_{1e}$ variationally since C_{1ee} and L_1 are self-adjoint. Using a trial function that does not alter the mean flow,

$$\bar{f}_{1e} = [b_1 L_1^{(3/2)}(x_e^2) + b_2 L_2^{(3/2)}(x_e^2)]w_\| f_{0e}, \tag{72}$$

we find the variationally determined coefficients to be

$$b_1 = \left[\frac{12(373 + 389\sqrt{2})}{16447 + 15912\sqrt{2}}\right]\frac{m}{T_e}(V_\| - V_{\|e}) + \left[\frac{56995 + 29360\sqrt{2}}{32984 + 31824\sqrt{2}}\right]\nu_{ei}^{-1}\vec{n} \cdot \nabla \ell n T_e$$

and

$$b_2 = \left[\frac{12(4\sqrt{2} - 2)}{505 + 604\sqrt{2}}\right]\frac{m}{T_e}(V_\| - V_{\|e}) - \left[\frac{30(23 + 4\sqrt{2})}{505 + 604\sqrt{2}}\right]\nu_{ei}^{-1}\vec{n} \cdot \nabla \ell n T_e .$$

Knowing f_{1e} we can evaluate the electron heat flux $\vec{q}_e = \int d^3v f_e \vec{w}(mw^2 - 5T_e)/2$ to lowest order to find the diamagnetic and parallel contributions of Braginskii [1, 2]:

$$\vec{q}_e = -(5p_e/2m\Omega_e)\vec{n} \times \nabla T_e - 3.162(p_e/m\nu_{ei})\vec{n}\vec{n} \cdot \nabla T_e - 0.711 p_e(\vec{V}_\| - \vec{V}_{\|e}). \tag{73}$$

A moment approach can be used to evaluate the collisional perpendicular heat flux. Accounting for momentum exchange and unlike collisions the electron version of (38) is

$$\Omega_e \vec{q}_e \times \vec{n} + (5p_e/2m)\nabla T_e = T_e \int d^3w \vec{w}\{x_e^2 C_{1ee} + [x_e^2 - (5/2)](L_1 + D_0)\},$$

where the momentum exchange term $\vec{F}$ gives rise to the $(5/2)(L_1+D_0)$ terms. Carrying out the integrals, noting that only $\tilde{f}_{1e}$ is required, and adding in $\vec{q}_{\|e}$ gives the standard Braginskii result

$$\begin{aligned}\vec{q}_e = &-(5p_e/2m\Omega_e)\vec{n} \times \nabla T_e - 3.162(p_e/m\nu_{ei})\vec{n}\vec{n} \cdot \nabla T_e - 0.711 p_e(\vec{V}_\| - \vec{V}_{\|e}) \\ &- (4.66\nu_{ei}p_e/m\Omega_e^2)\nabla_\perp T_e - (3p_e\nu_{ei}/2\Omega_e)\vec{n} \times (\vec{V} - \vec{V}_e) .\end{aligned} \tag{74}$$

The preceding is to be inserted in the electron energy balance equation

$$\frac{3n}{2}\left(\frac{\partial T_e}{\partial t} + \vec{V}_e \cdot \nabla T_e\right) + p_e \nabla \cdot \vec{V}_e + \nabla \cdot \vec{q}_e + \vec{\pi}_e : \nabla \vec{V}_e = -\frac{3mn\nu_{ei}}{M}(T_e - T) + (\vec{V} - \vec{V}_e) \cdot \vec{F} \tag{75}$$

or in conservation form

$$\frac{\partial}{\partial t}\left(\frac{3}{2}p_e+\frac{1}{2}mnV_e^2\right)+\nabla\cdot\left[\left(\frac{5}{2}p_e+\frac{1}{2}mnV_e^2\right)\vec{V}_e+\vec{\pi}_e\cdot\vec{V}_e+\vec{q}_e\right]$$
$$=-en\vec{E}\cdot\vec{V}_e+\vec{F}\cdot\vec{V}-\frac{3mn\nu_{ei}}{M}(T_e-T)\ .$$

In addition, we must evaluate the lowest order momentum exchange term using

$$\vec{F}=m\int d^3w\vec{w}C_{1ei}=m\int d^3w\vec{w}[L_1+D_0]=mn\nu_{ei}(\vec{V}-\vec{V}_e)-\vec{F}_*$$

with

$$\vec{F}_*=\frac{3\pi^{1/2}T_e^{3/2}\nu_{ei}}{(2m)^{1/2}}\int\frac{d^3w}{w^3}f_{1e}\vec{w}$$
$$=(3n\nu_{ei}/2\Omega_e)\vec{n}\times\nabla T_e+0.71n\vec{n}\vec{n}\cdot\nabla T_e+0.49mn\nu_{ei}(\vec{V}_\parallel-\vec{V}_{\parallel e}) \tag{76}$$

to find the Braginskii expressions for the friction and thermal force

$$\vec{F}=mn\nu_{ei}[(\vec{V}_\perp-\vec{V}_{\perp e})+0.51(\vec{V}_\parallel-\vec{V}_{\parallel e})]-(3n\nu_{ei}/2\Omega_e)\vec{n}\times\nabla T_e-0.71\,n\vec{n}\vec{n}\cdot\nabla T_e\ . \tag{77}$$

This result for the momentum exchange between electrons and ions is to be used in both the electron and ion conservation equations. Higher order corrections are not required since $|\vec{F}\cdot\vec{V}|\sim|\nabla\cdot\vec{q}_e|$ for $\vec{q}_e$ the collisional perpendicular heat flux.

To evaluate the electron viscosity to complete the closure of the electron momentum and energy equations it is convenient to use the lowest order expression for $\vec{q}_e$ and define

$$\vec{q}_*=1.581(p_e/m\nu_{ei})\vec{n}\vec{n}\cdot\nabla T_e-0.0807p_e(V_\parallel-V_{\parallel e})\vec{n} \tag{78}$$

to obtain the form

$$f_{1e}=-(2mf_{0e}/5p_eT_e)[L_1^{(3/2)}(x_e^2)\vec{q}+L_2^{(3/2)}(x_e^2)\vec{q}_*]\cdot\vec{w}\ . \tag{79}$$

Notice that our electron orderings implicitly assume $\vec{q}_e/p_ev_e\sim\vec{V}_e/v_e\sim\rho_e/L_\perp\sim\lambda\vec{n}\cdot\nabla\ell nT_e$ $<<\lambda/L_\parallel$ so that parallel electron temperature variations are weak and an adiabatic electron response is allowed, where λ is the mean free path, $v_e=(2T_e/m)^{1/2}$ and $\rho_e=v_e/\Omega_e$.

To evaluate the parallel electron viscosity we need to find $\bar{f}_{2e}$ by solving the lowest order gyroaveraged equation

$$\bar{C}_{2ee}+\bar{L}_2=\langle\vec{w}\cdot\nabla f_{1e}+(mn)^{-1}(\nabla p_e-\vec{F})\cdot\nabla_w f_{1e}\rangle+\langle\frac{\partial f_{0e}}{\partial t}+\vec{V}_e\cdot\nabla f_{0e}-\vec{w}\cdot\nabla\vec{V}_e\cdot\nabla_w f_{0e}\rangle, \tag{80}$$

where $\bar{L}_2=L\{\bar{f}_{2e}\}$ and $\bar{C}_{2ee}=C_{1ee}\{\bar{f}_{2e}\}+\langle C_{2ee}\{f_{1e},f_{1e}\}\rangle$ with $\langle C_{2ee}\{f_{1e},f_{1e}\}\rangle$ the full gyroaveraged non-linear electron-electron collision operator acting on f_{1e} and C_{1ee} the linearized collision operator acting on $\bar{f}_{2e}$. Notice that $\bar{D}_1$ can be neglected because the flows are small compared to the electron thermal speed and we are not interested in the isotropic equilibration contributions to $\bar{f}_{2e}$. The self-adjointness of $C_{1ee}\{\bar{f}_{2e}\}$ and $\bar{L}_2=L\{\bar{f}_{2e}\}$ allows us to solve for $\bar{f}_{2e}$ variationally. The technique is the same as for the ions except that $\bar{L}_2$ must be retained. As for the ions, we only need terms proportional to $P_2(\xi)$ to form the parallel viscosity so we need only consider

$$\bar{f}_{2e}/f_{0e} = x_e^2 P_2(\xi)[c_0 + c_1 L_1^{(5/2)}(x_e^2)]. \tag{81}$$

Solving as for the ions we find

$$\vec{\vec{\pi}}_{\|e} = m\int d^3w\langle\vec{w}\vec{w} - \frac{1}{3}w^2\vec{\vec{I}}\rangle\bar{f}_{2e} = (\vec{n}\vec{n} - \frac{1}{3}\vec{\vec{I}})(p_{\|e} - p_{\perp e}) \tag{82}$$

with

$$p_{\|e} - p_{\perp e} = \int d^3w w^2 P_2(\xi)\bar{f}_{2e} = \frac{3}{2}pc_0 = \left[\frac{3m}{4430200 p_e T_e}\right]\left[(6056+24169\sqrt{2})(3q_{\|e}^2 - q_e^2)\right.$$
$$\left.+5(32854\sqrt{2} - 26035)\vec{q}_e\cdot\vec{q}_* + \frac{105}{32}(16433\sqrt{2} - 36)q_*^2\right] + 42\left[\frac{417\sqrt{2} - 439}{22151\nu_{ei}}\right]\left[\nabla\cdot(\vec{q}_e - \vec{q}_*)\right.$$
$$\left.-3\vec{n}\cdot\nabla(\vec{q}_e - \vec{q}_*)\cdot\vec{n} + (2\vec{q}_e - \vec{q}_* - 6\vec{q}_{\|e} + 3\vec{q}_{\|*})\cdot\nabla\ell nT_e - (\vec{q}_e - 3\vec{q}_{\|e})\cdot(\nabla\ell np_e - p_e^{-1}\vec{F})\right]$$
$$+\frac{5}{6}\left[\frac{23479\sqrt{2} - 13775}{22151\nu_{ei}}\right]\left[p_e\nabla\cdot\vec{V}_e + \frac{2}{5}\nabla\cdot\vec{q}_e - 3p_e\vec{n}\cdot\nabla\vec{V}_e\cdot\vec{n} - \frac{6}{5}\vec{n}\cdot\nabla\vec{q}_e\cdot\vec{n}\right]. \tag{83}$$

The preceding can be written more compactly using double dot notation as

$$p_{\|e} - p_{\perp e} = \frac{0.731}{\nu_{ei}}(\vec{\vec{I}} - 3\vec{n}\vec{n}):\left\{(p_e\nabla\vec{V}_e + \frac{2}{5}\nabla\vec{q}_e) - 0.391[\vec{q}_e(\nabla\ell np_e - p_e^{-1}\vec{F}) - \nabla(\vec{q}_e - \vec{q}_*)\right.$$
$$\left.-(2\vec{q}_e - \vec{q}_*)\nabla\ell nT_e\right\} + \frac{m}{p_e T_e}\left[0.027(3q_{\|e}^2 - q_e^2) + 0.069\vec{q}_e\cdot\vec{q}_* + 0.052q_*^2\right]. \tag{84}$$

In $p_{\|e} - p_{\perp e}$, the lowest order expressions for $\vec{q}_e$ and $\vec{q}_{\|e}$ are to be used. The preceding result for $\vec{\vec{\pi}}_{\|e}$ assumes the weak parallel variation of the electron temperature ordering $\vec{q}_e/p_e v_e \sim \vec{V}_e/v_e \sim \rho_e/L_\perp \sim \lambda\vec{n}\cdot\nabla\ell nT_e$. However, if $\vec{n}\cdot\nabla\ell nT_e \sim 1/L_\|$ then we may adopt this strong parallel variation of the electron temperature ordering $\lambda/L_\| \sim \vec{q}_e/p_e v_e >> \rho_e/L_\perp \sim \vec{V}_e/v_e$ to simplify $\vec{\vec{\pi}}_{\|e}$ considerably.

The evaluation of the electron gyro-viscosity is similar to that for the ions since $\tilde{f}_{2e}$ satisfies

$$\Omega_e\vec{w}\times\vec{n}\cdot\nabla_w\tilde{f}_{2e} - (\vec{w}\vec{w} - \langle\vec{w}\vec{w}\rangle):\vec{\vec{S}}_e = 0 \tag{85}$$

with

$$\vec{\vec{S}}_e = \frac{m}{T_e}f_{0e}\nabla\vec{V}_e - \nabla\left\{\frac{2mf_{0e}}{5p_eT_e}\left[L_1^{(3/2)}(x_e^2)\vec{q} + L_2^{(3/2)}(x_e^2)\vec{q}_*\right]\right\}$$
$$+\frac{2mf_{0e}}{5p_e^2T_e}(\nabla p_e - \vec{F})\left[L_1^{(5/2)}(x_e^2)\vec{q} + L_2^{(5/2)}(x_e^2)\vec{q}_*\right]. \tag{86}$$

The preceding forms are similar to Eqs. (21) and (23) with the result that

$$\tilde{f}_{2e} = -\Omega_e^{-1}\{[w_\|\vec{n} + (1/4)\vec{w}_\perp]\vec{w}\times\vec{n} + \vec{w}\times\vec{n}[w_\|\vec{n} + (1/4)\vec{w}_\perp]\}:\vec{\vec{S}}_e\ . \tag{87}$$

.

Forming the electron gyroviscosity, $\vec{\vec{\pi}}_{ge} = m\int d^3w\vec{w}\vec{w}\tilde{f}_{2e}$, by the same procedure used for the ions yields

$$\vec{\vec{\pi}}_{ge} = \frac{-1}{4\Omega_e}\left\{\vec{n}\times\left[\left(p_e\nabla\vec{V}_e + \frac{2}{5}\nabla\vec{q}_e\right) + \left(p_e\nabla\vec{V}_e + \frac{2}{5}\nabla\vec{q}_e\right)^T\right]\cdot\left(\vec{\vec{I}} + 3\vec{n}\vec{n}\right)\right.$$
$$\left. -\left(\vec{\vec{I}} + 3\vec{n}\vec{n}\right)\cdot\left[\left(p_e\nabla\vec{V}_e + \frac{2}{5}\nabla\vec{q}_e\right) + \left(p_e\nabla\vec{V}_e + \frac{2}{5}\nabla\vec{q}_e\right)^T\right]\times\vec{n}\right\} . \tag{88}$$

The electron parallel and gyro-viscosities are needed for the energy and momentum balance equations, where the conservation form of the electron momentum balance equation is

$$\frac{\partial}{\partial t}(mn_e\vec{V}_e) + \nabla p_e + \nabla\cdot(\vec{\vec{\pi}}_e + mn_e\vec{V}_e\vec{V}_e) = -en_e(\vec{E} + \frac{1}{c}\vec{V}_e\times\vec{B}) + \vec{F} .$$

The combined ion-electron short mean free path closure is now complete. Only perpendicular electron viscosity has been neglected.

IV. DISCUSSION

Our short mean free path description of magnetized plasmas considers the normally more interesting situation when the pressure times the mean flow velocity is allowed to be comparable to the diamagnetic and collisional parallel heat flows; and the mean flow is on the order of the diamagnetic and magnetic drift speeds: the drift ordering. It removes shortcomings of and extends the pioneering work by Mikhailovskii and Tsypin [4-6] who first realized the limitations of the Braginskii [1-2] and Robinson and Bernstein [3] formulations which are only valid for flows on the order of the ion thermal speed (often referred to as an MHD ordering). As in all drift ordering treatments we assume the collision frequency to be small compared to the cyclotron frequency. However, we permit the perpendicular scale lengths to be much less than the parallel ones as is the case in many magnetized plasma applications. As a result, our description is valid for both turbulent and collisional transport treatments.

Our treatment of the ions finds additional terms in the parallel viscosity, Eq. (42), that are quadratic in the heat flux and were missed by earlier treatments which neglected the full non-linear collision operator term $C_2\{f_1,f_1\}$ in the kinetic equation, Eqs. (24) and (25), for $\bar{f}_2$. Quadratic heat flux terms also enter the perpendicular viscosity $\vec{\vec{\pi}}_\perp = \vec{\vec{\pi}}_{\perp 1} + \vec{\vec{\pi}}_{\perp 2}$, which we evaluate by a moment approach using Eq. (45) that thereby involves moments of the non-linear collision operator $C_2\{f_1,f_1\}$. As a result, our $\vec{\vec{\pi}}_{\perp 2}$ term is new and not contained in previous treatments. Furthermore, our result for $\vec{\vec{\pi}}_{\perp 1}$ differs from that of Mikhailovskii and Tyspin [4-6] since they truncate their Laguerre polynomial expansion after only two terms while we keep the full gyrophase dependent expression for $\tilde{f}_2$. Their truncated approximation for $\tilde{f}_2$ does not affect their gyro-viscosity $\vec{\vec{\pi}}_g$ since they evaluate it from a moment approach, equivalent to (48), that only requires f_1. The complete description for the ions consists of the conservation of number, ion momentum, and ion energy equations with the ion heat flux $\vec{q}$ given by Eq. (39), with the ion viscosity $\vec{\vec{\pi}} = \vec{\vec{\pi}}_{\|} + \vec{\vec{\pi}}_g + \vec{\vec{\pi}}_{\perp 1} + \vec{\vec{\pi}}_{\perp 2}$ given by Eqs. (41) or (42), (44), (61) and (63), and with the momentum exchange between the ions and electrons, $\vec{F}$, given by Eq. (77). Equations (58) and (61) are a more compact version of Eq. (63).

Our evaluation of the electron parallel and gyro-viscosities is new since electrons were not considered in earlier drift ordering work. Of course, in the large flow limit our electron (and ion) results agree with Braginskii [1-2] and Robinson and Bernstein [3]. We have not evaluated the perpendicular electron viscosity since it is small and unlikely to be of interest. The calculation could be performed by the simliar procedure as used for the ions. Our complete description of the electrons is the conservation of electron momentum and energy equations with the electron heat flux $\vec{q}_e$ given by Eq. (74), the electron viscosity $\vec{\vec{\pi}} = \vec{\vec{\pi}}_{\|e} + \vec{\vec{\pi}}_{ge}$ given by Eqs. (82), (83) or (84) and (88), plus the momentum exchange term of Eq. (77) and the expression for $\vec{q}_*$ as given by Eq. (78). We have assumed singly charged ions and invoked quasi-neutrality so that the electron-electron collision frequency equals the electron-ion collision frequency ν_{ei}.

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