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LINEAR MODE CONVERSION**

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# Symmetries in Dissipation-Free Linear Mode Conversion

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## Abstract

From properties of wave energy flow conservation and wave energy flow under time reversal, we derive the symmetries in dissipation-free mode conversion in an inhomogeneous plasma. These are applied to high-temperature plasma mode conversions near the ion-ion hybrid and upper hybrid resonances.

## 1. Introduction

In a stable inhomogeneous plasma, linear wave propagation in the vicinity of local resonances and nearby cutoffs brings about a complex coupling of the natural wave modes of the homogeneous plasma. This coupling, generally termed *mode conversion* (MC), is particularly important for analyzing the excitation of, or emission from, kinetic waves that exist only inside a hot plasma. In high-temperature plasmas, analysis of MC requires the solution of

the full-wave kinetic Vlasov-Maxwell (integro-partial differential) equations — a very demanding task (even computationally). For a loss-free (LF) mode conversion region (MCR) in a plasma, we derive the coupling symmetries that must be satisfied among the various wave energy flows into and out of such regions, without having to solve the full-wave kinetic equations. We illustrate the general results for cases of interest to heating and current drive in toroidally confined plasmas.

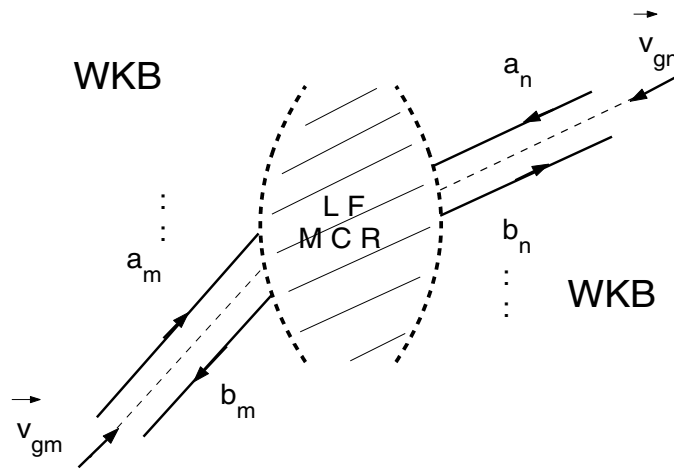


Figure 1:

As shown in Figure 1, outside the MCR we assume the fields can be described generally in a WKB-eikonal form based on a local dispersion rela-

tion, and group velocity ray tracing determines the energy flow of the various modes. For a collisionless (Vlasov) description of a hot plasma, we then take that Landau and/or Doppler-shifted cyclotron resonance dissipations occur outside the mode conversion region, thus making it a LF-MCR.

## 2. Proof of symmetries

In the WKB regions to the right and left of the LF-MCR, let the complex field amplitudes of waves with energy flow into and out of the LF-MCR be, respectively,

$$a_p \sim \exp(i\vec{k}_p \cdot \vec{r} - i\omega t) \quad \text{and} \quad b_p \sim \exp(-i\vec{k}_p \cdot \vec{r} - i\omega t), \quad (1)$$

normalized so that:  $|a_p|^2$  = wave energy flow density into the LF-MCR;  $|b_p|^2$  = wave energy flow density out of the LF-MCR. In Figure 1, such modes on the left of the LF-MCR have  $p = m$  (there can be any number of such modes:  $m_1, m_2, \dots$ ), and on the right of the LF-MCR, similarly,  $p = n$  designating any number of modes ( $n_1, n_2, \dots$ ). For a weakly dissipative mode, the total wave energy flow density (electromagnetic plus kinetic) is given, in general, by [1]

$$\langle \vec{s}_p \rangle = \left[ \frac{1}{2} \text{Re} \left( \vec{E} \times \vec{H}^* \right) - \frac{\epsilon_0}{4} \omega \frac{\partial \chi_{\alpha\beta}^H}{\partial \vec{k}} \right]_p = \vec{v}_{gp} \langle w_p \rangle \quad (2)$$

where  $\chi_{\alpha\beta}^H$  is the Hermitian part of the susceptibility tensor  $\chi_{\alpha\beta}(\vec{k}, \omega)$  with  $\vec{k}$  and  $\omega$  real and the star superscript denotes the complex conjugate;  $\vec{v}_{gp}$  and  $\langle w_p \rangle$  are, respectively, the group velocity and wave energy density of mode  $p$ . Since the full-wave equations describing the LF-MCR are linear with appropriate boundary conditions, the complex field amplitudes  $a_p$  and  $b_p$  are related by a unique scattering matrix  $\overline{\overline{S}}$

$$\vec{b} = \overline{\overline{S}} \cdot \vec{a} \quad (3)$$

where  $\vec{b}$  and  $\vec{a}$  are column vectors containing complex amplitudes of all  $b_p$  and all  $a_p$ , respectively.

From *energy flow conservation* applied to the LF-MCR, we have  $\sum_p (|a_p|^2 - |b_p|^2) = 0$ , where the sum is over all  $m$ 's and  $n$ 's. Using (3), we can express this as  $\vec{a}^\dagger \cdot (\overline{\overline{I}} - \overline{\overline{S}}^\dagger \cdot \overline{\overline{S}}) \cdot \vec{a} = 0$ , where the dagger superscript on  $\overline{\overline{S}}$  denotes the complex-conjugate-transpose of  $\overline{\overline{S}}$ . Since this must hold true for arbitrary  $\vec{a}$ , it follows that

$$\overline{\overline{S}}^\dagger = \overline{\overline{S}}^{-1} . \quad (4)$$

Next, consider *wave energy flow under time reversibility*. For the time reversed system, the direction of time-averaged energy flow density changes sign. In other words, the reversal of time changes time-averaged energy flow

into the mode conversion region to time-averaged energy flow out of the mode conversion region, and vice versa. From (2), energy flow reversal is obtained by setting  $\vec{E} \rightarrow \vec{E}^*$ ,  $\vec{H} \rightarrow -\vec{H}^*$ ,  $\vec{k} \rightarrow -\vec{k}$  [2] and, by (1), time reversal gives  $a_p \rightarrow b_p^*$  and  $b_p \rightarrow a_p^*$ , where the star superscript denotes the complex conjugate. Referring to Figure 1, the effect of time reversal is to change  $a$  to  $b^*$  and  $b$  to  $a^*$ , with arrows pointing in the same direction as indicated in the figure. Thus  $\vec{a}^* = \overline{\overline{S}} \cdot \vec{b}^*$  or, taking the complex conjugate,  $\vec{a} = \overline{\overline{S}^*} \cdot \vec{b}$ . But from (3)  $\vec{a} = \overline{\overline{S}}^{-1} \cdot \vec{b}$ ; hence

$$\overline{\overline{S}^*} = \overline{\overline{S}}^{-1} . \quad (5)$$

Combining (4) and (5), we finally obtain:

$$\overline{\overline{S}}^\dagger = \overline{\overline{S}}^* \quad \text{or equivalently} \quad \overline{\overline{S}}^T = \overline{\overline{S}} \quad (6)$$

where the  $T$  superscript on  $\overline{\overline{S}}$  denotes the transpose of  $\overline{\overline{S}}$ . Hence, *the LF-MCR scattering matrix is symmetric*. The symmetry of the LF-MC scattering matrix,  $S_{ij} = S_{ji}$ , entails important relationships for various power coefficients of the mode conversion process:

$$|S_{ij}|^2 = \left| \frac{b_i}{a_j} \right|^2 = \left| \frac{b_j}{a_i} \right|^2 = |S_{ji}|^2 . \quad (7)$$

We illustrate the symmetries for MCs of waves propagating in  $x$ , across a magnetic field  $\vec{B}_0$ , for the vicinity of the ion-ion hybrid resonance (IIHR) and

associated cutoffs, where fast Alfvén waves (FAWs) couple to ion-Bernstein waves (IBWs), and for the vicinity of the upper-hybrid resonance (UHR) and associated cutoffs, where O and X modes couple to electron-Bernstein waves (EBWs).

### 3. Mode conversions at the IIHR

#### 3.1. Cutoff on high-field side following IIHR is within MCR [3]

The local dispersion relation in the LF-MCR for given  $(\omega, k_z)$ , and the WKB modes outside its boundaries, are illustrated in Figure 2a. The

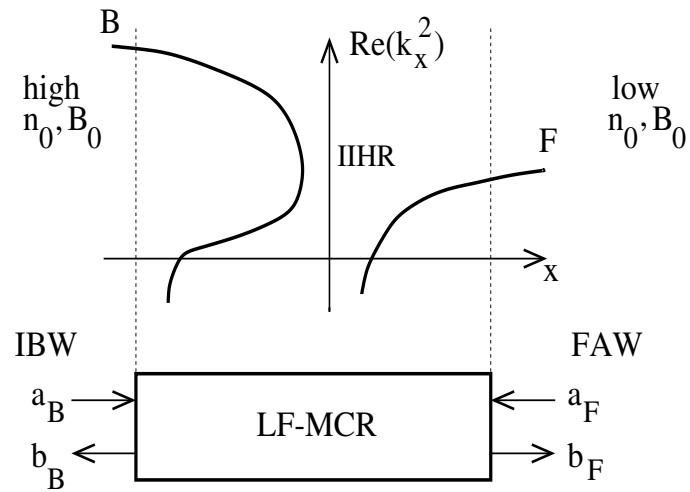


Figure 2a:



associated scattering matrix is given by:

$$\begin{pmatrix} b_B \\ b_F \end{pmatrix} = \begin{pmatrix} S_B & S_{FB} \\ S_{BF} & S_F \end{pmatrix} \begin{pmatrix} a_B \\ a_F \end{pmatrix}. \quad (8)$$

From (7) it follows that  $|S_{FB}|^2 = |S_{BF}|^2$ . This implies that the fraction of FAW power MC to IBW is the same as the power MC from IBW (emission) to the FAW. In addition, from (5) we obtain the reflection symmetry  $|S_B|^2 = |S_F|^2$ ; this indicates that the fraction of (emitted) IBW power that is reflected back into the high- $(n_0, B_0)$  plasma is the same as the fraction of an incident FAW power reflected back toward the low- $(n_0, B_0)$  side of the plasma.

### 3.2. No cutoff on high-field side following IIHR in MCR [4]

The local dispersion relation in the LF-MCR for given  $(\omega, k_z)$ , and the boundaries of the LF-MCR with WKB mode fields outside of its boundaries are shown in Figure 2b. The associated scattering matrix is given by:

$$\begin{pmatrix} b_B \\ b_H \\ b_L \end{pmatrix} = \begin{pmatrix} S_B & S_{BH} & S_{BL} \\ S_{HB} & S_H & S_{HL} \\ S_{LB} & S_{LH} & S_L \end{pmatrix} \begin{pmatrix} a_B \\ a_H \\ a_L \end{pmatrix}. \quad (9)$$

Here, from (7),  $|S_{BH}|^2 = |S_{HB}|^2$  entails that the power MC from a FAW incident on the high- $(n_0, B_0)$  side to the IBW is the same as the power MC

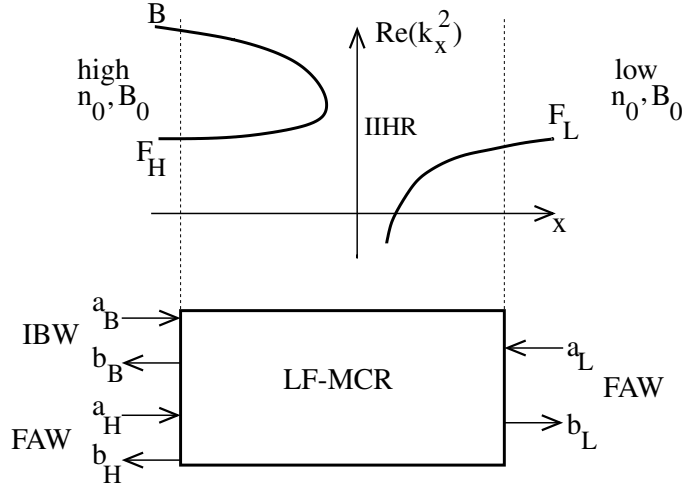


Figure 2b:

from an IBW (emission) to a FAW on the high- $(n_0, B_0)$  side. In an analogous manner,  $|S_{BL}|^2 = |S_{LB}|^2$  gives the equality in MC power from an incident FAW on the low- $(n_0, B_0)$  side to the IBW, and the power MC from an IBW (emission) to a FAW on the low- $(n_0, B_0)$  side.  $|S_{HL}|^2 = |S_{LH}|^2$  expresses the equality in FAW fractional power transmission from the low- $(n_0, B_0)$  side to the high- $(n_0, B_0)$  side and the reverse.

#### 4. Mode conversions at the UHR

Figure 3 shows the local dispersion relation (relevant to the plasma edge of high- $\beta$  spherical tokamaks) for two  $(\omega, k_z)$  situations: solid lines, when

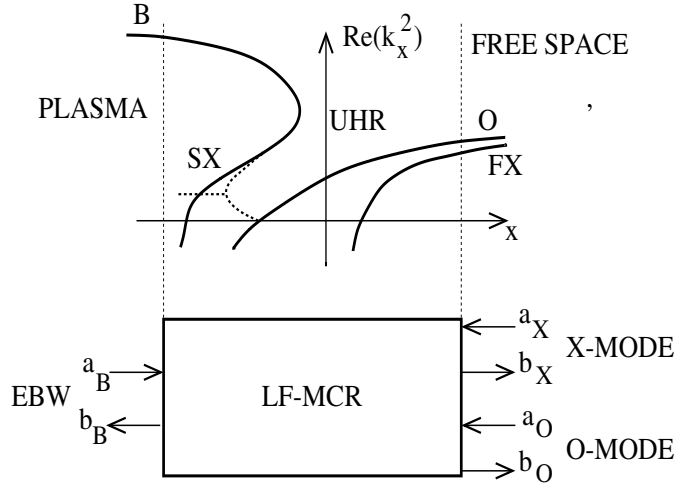


Figure 3:

O mode and X mode cutoffs are well-separated, and solid lines with part of SX branch modified as dashed lines to O mode cutoff; these correspond to optimal MC couplings of X-EBW and O-EBW, respectively [5].

The scattering matrix is given by:

$$\begin{pmatrix} b_B \\ b_X \\ b_O \end{pmatrix} = \begin{pmatrix} S_B & S_{BX} & S_{BO} \\ S_{XB} & S_X & S_{XO} \\ S_{OB} & S_{OX} & S_O \end{pmatrix} \begin{pmatrix} a_B \\ a_X \\ a_O \end{pmatrix}. \quad (10)$$

The symmetries required by (7) give three relations. The first entails  $|S_{BX}|^2 = |S_{XB}|^2$ , i.e., that MC power excitation of EBWs by X-modes is the same as MC power excitation of X-modes by EBWs (emission). The second,

$|S_{BO}|^2 = |S_{OB}|^2$ , is of a similar nature for the O-EBW MC power and EBW (emission) MC power to O-mode. The third relation,  $|S_{XO}|^2 = |S_{OX}|^2$ , states a reflection symmetry: the fractional power reflected out of the X-mode by an excitation of the O-mode is the same as the fractional power reflected out of the O-mode by an excitation of the X-mode.

An approximate kinetic formulation of the full-wave problem of coupling among O, X, and EBWs near the UHR has been solved computationally [5]. The results exhibit all of the above symmetries exactly.

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