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SYMMETRIES IN DISSIPATION-FREE LINEAR MODE CONVERSION

A. Bers and A. K. Ram

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Plasma Science & Fusion Center
Massachusetts Institute of Technology
Cambridge, Massachusetts 02139, U.S.A.

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Symmetries in Dissipation-Free Linear Mode Conversion

A. Bers and A. K. Ram

Plasma Science & Fusion Center, M.I.T, Cambridge, MA 02139, U.S.A.

ABSTRACT

Linear mode conversions (MC) in loss-free (LF) regions of an inhomogeneous, Vlasov plasma in a magnetic field are shown to obey certain symmetries [1]. These are illustrated and interpreted for situations relevant to plasma heating and/or current drive.

INTRODUCTION

Consider a one-dimensional (in x) generic propagation and MC situation in an inhomogeneous plasma, with unperturbed (equilibrium) parameters (e.g., density, temperature, and magnetic field) that vary in x, as shown schematically in Figure 1. For homogeneity along $\vec{B}_0 = \hat{z}B_0(x)$, Landau and/or Doppler shifted cyclotron resonance absorption for any k_z is assumed to occur outside the LF-MCR.

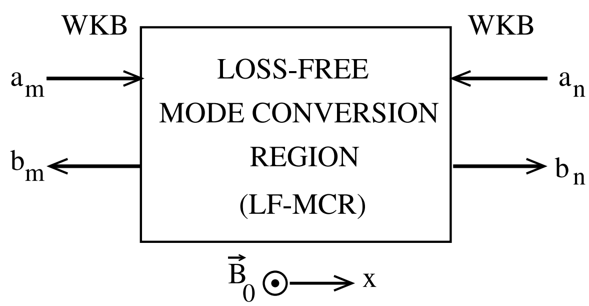

Figure 1:

PROOF OF SYMMETRIES

In the WKB regions to the right and left of the LF-MCR, let the complex field amplitudes of (e.g., forward) waves with energy flow into and out of the LF-MCR be, respectively,

$$a_p \sim \exp(ik_{px}x - i\omega t) \quad \text{and} \quad b_p \sim \exp(-ik_{px}x - i\omega t) \ , \tag{1}$$

normalized so that: $|a_p|^2$ = wave energy flow density into the LF-MCR; $|b_p|^2$ = wave energy flow density out of the LF-MCR. [For backward waves, retaining the energy flow normalizations, the signs of the k_{px}'s in (1) will change.] In Figure 1, such modes on the left of the LF-MCR have $p = m$ (there can be any number of such modes: m_1, $m_2, \ldots$), and on the right of the LF-MCR, similarly, $p = n$ designating any number of modes $(n_1, n_2, \ldots)$. For a weakly dissipative mode, the total wave energy flow density (electromagnetic plus kinetic) is given, in general, by [2]

$$\langle s_x \rangle_p = \left[\frac{1}{2} \mathrm{Re} \, (\vec{E} \times \vec{H}^*)_x - \frac{\varepsilon_0}{4} \omega \frac{\partial \chi^H_{\alpha\beta}}{\partial k_x} E_\alpha E^*_\beta \right]_p \quad , \tag{2}$$

where $\chi^H_{\alpha\beta}$ is the Hermitian part of the susceptibility tensor $\chi_{\alpha\beta}(\vec{k}, \omega)$ with $\vec{k}$ and ω real and the star superscript denotes the complex conjugate. Since the full-wave equations describing the LF-MCR are linear (in general, linear integro-partial-differential equations) with appropriate boundary conditions, the complex field amplitudes a_p and b_p are related by a unique scattering matrix $\overline{\overline{S}}$

$$\vec{b} = \overline{\overline{S}} \cdot \vec{a} \tag{3}$$

where $\vec{b}$ and $\vec{a}$ are column vectors containing complex amplitudes of all b_p and all a_p, respectively.

From *energy flow conservation* applied to the LF-MCR, we have $\sum_p (|a_p|^2 - |b_p|^2) = 0$, where the sum is over all m's and n's. Using (3), we can express this as $\vec{a}^\dagger \cdot (\overline{\overline{I}} - \overline{\overline{S}}^\dagger \cdot \overline{\overline{S}}) \cdot \vec{a} = 0$, where the dagger superscript on $\overline{\overline{S}}$ denotes the complex-conjugate-transpose of $\overline{\overline{S}}$. Since this must hold true for arbitrary $\vec{a}$, it follows that

$$\overline{\overline{S}}^\dagger = \overline{\overline{S}}^{-1} \; . \tag{4}$$

Next, consider *wave energy flow under time reversibility.* For the time reversed system, the direction of time-averaged energy flow density changes sign. In other words, the reversal of time changes time-averaged energy flow into the mode conversion region to time-averaged energy flow out of the mode conversion region, and vice versa. From (2), energy flow reversal is obtained by setting $\vec{E} \to \vec{E}^*$, $\vec{H} \to -\vec{H}^*$, $\vec{k} \to -\vec{k}$ and, by (1), time reversal gives $a_p \to b^*_p$ and $b_p \to a^*_p$, where the star superscript denotes the complex conjugate. Referring to Figure 1, the effect of time reversal is to change a to b^* and b to a^*, with arrows pointing in the same direction as indicated in the figure. Thus $\vec{a}^* = \overline{\overline{S}} \cdot \vec{b}^*$ or, taking the complex conjugate, $\vec{a} = \overline{\overline{S}}^* \cdot \vec{b}$. But from (3) $\vec{a} = \overline{\overline{S}}^{-1} \cdot \vec{b}$; hence

$$\overline{\overline{S}}^* = \overline{\overline{S}}^{-1} \; . \tag{5}$$

Combining (4) and (5), we finally obtain:

$$\overline{\overline{S}}^\dagger = \overline{\overline{S}}^* \quad \text{or equivalently} \quad \overline{\overline{S}}^T = \overline{\overline{S}} \tag{6}$$

where the T superscript on $\overline{\overline{S}}$ denotes the transpose of $\overline{\overline{S}}$. Hence, *the LF-MCR scattering matrix is symmetric*. The symmetry of the LF-MC scattering matrix, $S_{ij} = S_{ji}$, entails important relationships for various power coefficients of the mode conversion process:

$$|S_{ij}|^2 = \left|\frac{b_i}{a_j}\right|^2 = \left|\frac{b_j}{a_i}\right|^2 = |S_{ji}|^2 \ . \tag{7}$$

For MCs near the upper-hybrid resonance involving ordinary, extraordinary and electron Bernstein waves, the symmetries have been described in [3,4]. Here we illustrate the symmetries in two scenarios of MC near the ion-ion hybrid resonance (IIHR).

MODE CONVERSIONS AT THE IIHR

We assume conditions such that the individual ion-cyclotron resonances are outside the MCR containing the IIHR. MC is between fast Alfvén waves (FAW) and ion Bernstein waves (IBW).

1. Cutoff on High-Field Side Following IIHR is Within MCR

The local dispersion relation in the LF-MCR for given (ω, k_z), and the WKB modes outside its boundaries, are illustrated in Figure 2. The associated scattering matrix is given by:

$$\begin{pmatrix} b_B \\ b_F \end{pmatrix} = \begin{pmatrix} S_B & S_{FB} \\ S_{BF} & S_F \end{pmatrix} \begin{pmatrix} a_B \\ a_F \end{pmatrix} \ . \tag{8}$$

From (7): $|S_{FB}|^2 = |S_{BF}|^2$ gives the symmetry in excitations by MCs between FAW and IBW. In addition, (5) gives a reflectivity symmetry $|S_B|^2 = |S_F|^2$.

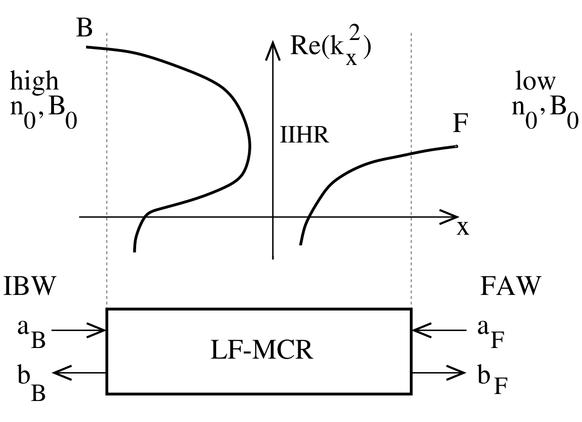

Figure 2:

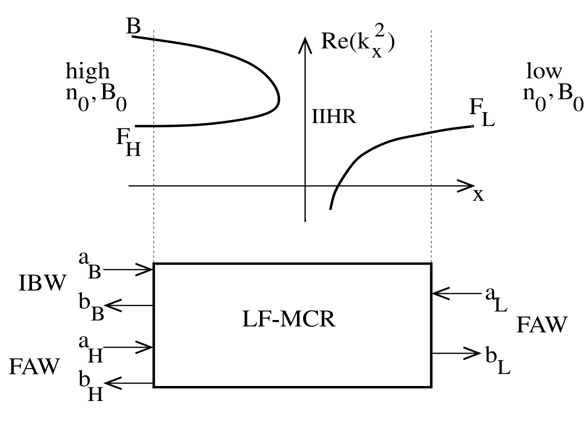

Figure 3:

2. No Cutoff on High-Field Side Following IIHR in MCR

The local dispersion relation in the LF-MCR for given (ω, k_z), and the boundaries of the LF-MCR with WKB mode fields outside of its boundaries are shown in Figure 3. The associated scattering matrix is given by:

$$\begin{pmatrix} b_B \\ b_H \\ b_L \end{pmatrix} = \begin{pmatrix} S_B & S_{BH} & S_{BL} \\ S_{HB} & S_H & S_{HL} \\ S_{LB} & S_{LH} & S_L \end{pmatrix} \begin{pmatrix} a_B \\ a_H \\ a_L \end{pmatrix} . \tag{9}$$

From (7): $|S_{BH}|^2 = |S_{HB}|^2$ and $|S_{BL}|^2 = |S_{LB}|^2$ give symmetries, respectively, in excitations by MCs between high-field side FAW and IBW, and low-field side FAW and IBW; $|S_{HL}|^2 = |S_{LH}|^2$ gives the symmetry in transmissions of FAWs.

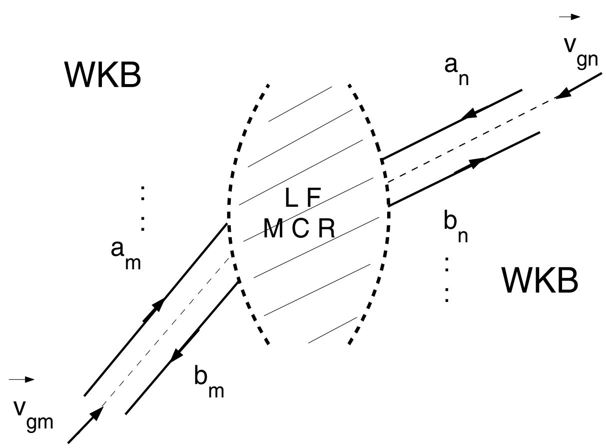

Figure 4:

GENERALIZATION

For 3-D propagation and mode conversion, the LF-MCR is identified by the breakdown of the eikonal description of modes. Outside the LF-MCR, where WKB eikonal descriptions are assumed to apply, and weakly dissipative modes are found to approach the LF-MCR by ray tracing, wave energy flow density is given by [2]

$$\langle \vec{s}_p \rangle = \left[\frac{1}{2} \mathrm{Re} \left(\vec{E} \times \vec{H}^* \right) - \frac{\epsilon_0}{4} \omega \frac{\partial \chi^H_{\alpha\beta}}{\partial \vec{k}} \right]_p = \vec{v}_{gp} \langle \mathrm{w}_p \rangle \tag{10}$$

where $\vec{v}_{gp}$ and $\langle \mathrm{w}_p \rangle$ are, respectively, the group velocity and wave energy density of mode p. Defining the mode amplitudes (a_p, b_p) along $\vec{v}_{gp}$ (see Figure 4), the symmetry of their scattering matrix $\overline{\overline{S}}$ is proven along lines identical to (3)–(6).

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