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Microstability of TARA Ion Anchor with Belly Band: Preliminary Results

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A code has been developed to find the axial eigenmodes for ion loss cone instabilities (DCLC and ALC), and has been used to determine stability requirements for the TARA ion anchor with a belly band coil. The equilibrium has a Fokker-Planck-like hot ion distribution, with maximum density at the outer magnetic field minimum and a 29% density dip at the midplane due to the belly band, while warm ions are confined between the transition region and the potential peak at the outer magnetic field minimum. The axial eigenmodes are found by solving the dispersion relation (a second-order differential equation) along a given field-line, using WKB across the field, and including the effects of fanning, ion cyclotron harmonics, and finite ion cyclotron resonance width. It is found that modes at lower frequencies ($\omega \lesssim 3\omega_{ci}$) are stabilized by the warm plasma, but that higher frequency modes ($\omega \gtrsim 4\omega_{ci}$, $k_\perp \rho_i \gtrsim 10$) remain unstable with growth rates of a few tenths of ω_{ci}. A preliminary study has been made of the radial and azimuthal mode structure, using a modified ray-tracing technique which is valid even for complex group velocities. Results of this study suggest that the radial mode structure will have an important effect on stability.

I. Introduction

An important requirement for the feasibility of tandem mirror reactors, and for good performance of tandem mirror experiments, is the absence of ion microinstabilities in the outermost cells, where the ion distribution has a loss cone. Such instabilities may be driven by the free energy associated with the loss cone (as in the case of the DCLC and ALC instabilities), or by the ion anisotropy (as in the case of the AIC instability).

In this paper we will be concerned only with instabilities driven unstable by the loss cone. For these modes $k_\perp v_i > \omega$, so $k_\perp v_A \gg \omega$, and (in contrast to anisotropy-driven modes such as AIC) the ion response is electrostatic. If the electrons are cold ($k_\| v_e \ll \omega$ and $k_\perp v_e \ll \omega_{ce}$), then the infinite medium dispersion relation is

$$1 + \frac{\omega_{pe}^2}{\omega_{ce}^2}\left(1 + \frac{\omega_{pe}^2}{k_\perp^2 c^2}\right) - \frac{\omega_{pe}^2 k_\|^2}{\omega^2 k^2}\left(1 + \frac{\omega_{pe}^2}{k_\perp^2 c^2}\right)^{-1} - \frac{\omega_{pe}^2 \hat{B} \cdot (\underline{k} \times \nabla n_e)}{k_\perp^2 \omega \omega_{ce} n_e} + \hat{k}_\perp \cdot \underline{\epsilon}_{ion} \cdot \hat{k}_\perp = 0 \qquad (1)$$

where $\underline{\epsilon}_{ion}$ is the ion dielectric tensor. This dispersion relation gives the drift cyclotron loss cone (DCLC) mode when the term with ∇n_e dominates the term with $k_\|$, and it gives the axial loss cone (ALC) mode when the $k_\|$ term dominates the ∇n_e term. In finite geometry, the $k_\|$ term, ∇n_e term and ion term are all comparable in magnitude for DCLC; if $k_\| L \gtrsim 1$ for DCLC, then there is no qualitative difference between DCLC and ALC.

To estimate the conditions needed for stability of DCLC and ALC, we may use the straight-line orbit approximation for the ion term, which is valid when $Im\, \omega \gtrsim \omega_{ci}$:

$$\hat{k}_\perp \cdot \underline{\epsilon}_{ion} \cdot \hat{k}_\perp \simeq \frac{\omega_{pi}^2}{k_\perp^2} \int d\underline{v} \frac{\underline{k}_\perp \cdot (\partial f_{ion}/\partial \underline{v})}{\omega - \underline{k}_\perp \cdot \underline{v}} \equiv \frac{\omega_{pi}^2}{k_\perp^2} \int_{-\infty}^{\infty} \frac{du \quad \partial F_{ion}/\partial u}{\omega/k_\perp - u}$$

where $F_{ion}(u)$ is the one-dimensional ion velocity distribution in the $k_\perp$ direction. If $\partial F_{ion}/\partial u|_{u=\omega/k_\perp} > 0$, i.e. if $\omega/k_\perp$ is in the loss-cone, then $\hat{k}_\perp \cdot \underline{\epsilon}_{ion} \cdot \hat{k}_\perp$ has a positive imaginary part, and the mode is unstable. A mode will be stable if it has $\omega/k_\perp$ such that $\partial F_{ion}/\partial u|_{u=\omega/k_{\perp,v}} < 0$, either because $\omega/k_\perp$ is outside the loss cone, or because there is warm plasma of sufficient density partially filling the loss cone, with $v_{warm} \gtrsim \omega/k_\perp$.

Taking an ion distribution of the form

$$\int_{-\infty}^{\infty} dv_{\|} \quad f_{ion}(v_{\perp}, v_{\|}) = \frac{v_{\perp}^2}{2\pi v_i^2} \quad \exp(-v_{\perp}^2/2v_i^2)$$

and assuming $k_{\perp}\lambda_D \ll 1$ (certainly true for the most dangerous modes), so that we can neglect the vacuum and electron polarization drift terms, and taking $\underline{k}_{\perp}$ in the diamagnetic direction (the most unstable direction), Eq. (1) becomes

$$\frac{-k_{\|}^2 c^2}{\omega^2}(1 + \frac{k_{\perp}^2 c^2}{\omega_{pe}^2})^{-1} - \frac{\omega_{pe}^2}{\omega_{ce}\omega k_{\perp} R_p} + \frac{\omega_{pe}^4}{\omega_{ce}^2 k_{\perp}^2 c^2} + i\sqrt{\frac{\pi}{2}}\frac{\omega_{pi}^2 \omega}{k_{\perp}^3 v_i^3} = 0 \qquad (2)$$

where R_p is the radial density scale length. For DCLC, the $k_{\|}^2$ term may be neglected, and it follows from Eq. (2) that $\omega/k_{\perp} \simeq v_i \mathrm{min}[(\rho_i/R_p)^{1/2}, \beta_i^{-1}(\rho_i/R_p)]$ (where ρ_i is the ion Lamor radius and $\beta_i = \omega_{pi}^2 v_i^2/\omega_{ci}^2 c^2$), which is much less than v_i if $\rho_i \ll R_p$. Hence it is only necessary to have a relatively small amount of warm plasma, at a relatively low temperature, to stabilize DCLC. Furthermore, when DCLC is flute-like (i.e. when $k_{\|}L \ll 1$, with $k_{\|}$ determined by requiring the $k_{\|}$ term in Eq. (2) to be comparable to both the ∇n_e term and the greatest of the other two terms; this means $L \lesssim \max[5R_p, 12\beta_i^{-1}R_p^{3/4}\rho_i^{1/4}]$), all that matters is the warm plasma distribution integrated along the field line, so the warm plasma can be axially localized (in the inner half of the end cell, for example) and still be effective at stabilizing DCLC.

For ALC, and also for DCLC when it is not flute-like, $k_{\|}L \gtrsim 1$. This means that these modes can localize axially away from the region where there is warm plasma. One way to stabilize these modes would be to have an unconfined stream of warm plasma through the entire length of the machine, as was done in the 2XIIB experiment, but this would not be tolerable in a reactor. Thus it is necessary in a tandem mirror reactor, and desirable in an experiment, to have L, the length over which there is no warm plasma, be sufficiently short that $\omega/k_{\perp} > v_i$ when $k_{\|}L \gtrsim 1$. From Eq. (2), neglecting the ∇n_e term,

$$k_{\|}^2 \simeq \frac{\omega_{pi}^2}{c^2}(1 + \frac{k_{\perp}^2 c^2}{\omega_{pe}^2})(\frac{\omega}{k_{\perp} v_i})^3 \qquad (3)$$

If loss-cone modes are stable for $\omega/k_\perp v_i > 0.3$, and if $k_\parallel L > 1$ is the condition for a mode to be axially localized, then from Eq. (3) unstable loss-cone modes will exist only if the region where there is no warm plasma is longer than the critical length

$$L_c \simeq \frac{6c}{\omega_{pi}}(1 + \frac{k_\perp^2 c^2}{\omega_{pe}^2})^{-\frac{1}{2}}$$

$$= 6\beta_i^{-\frac{1}{2}}\rho_i(1 + \frac{k_\perp^2 \rho_i^2}{\beta_i}\frac{m_e}{m_i})^{-\frac{1}{2}} \quad (4)$$

For typical design parameters of the TARA ion anchor, e.g. $\beta_i = 0.1$, $T_i = 12$ keV, $B = 3.6$ kG, we find $L_c = 60$ cm for small $k_\perp \rho_i$, and less than 60 cm for larger $k_\perp \rho_i$. This is close to the distance from the midplane to the outer mirror throat, a region which would have no warm plasma if the TARA ion anchor were run in a configuration like the TMX end cell. Therefore, it would be desirable to modify the ion anchor of TARA (and of any experiment with similar parameters) to put the peak in potential closer to the outer mirror throat, and to make the potential fall off as sharply as possible inside the peak, to allow as much warm plasma as possible to be confined over as much of the length of the cell as possible. One way to do this would be to put a circular coil with a modest amount of current around the midplane. This "belly band" coil would increase the magnetic field at the midplane by a few percent, and there would be two minima in the magnetic field some distance on either side of the midplane. If neutral beams were injected perpendicular to the field at the outer field minimum, then the potential peak would occur at the outer field minimum, and a substantial amount of warm plasma could be confined at the midplane. An alternative to this belly band configuration would be to inject ions at the midplane, at a sufficiently small angle to the magnetic field (but not too close to the loss cone) to produce a sloshing ion distribution, with two peaks in density (and in potential), one on each side of the midplane. Although sloshing ions are not a possibility in the present design of the TARA ion anchor, because the mirror ratio is too small, they may be possible in other experiments or in reactors.

In order to accurately evaluate the merits of using a belly band coil, or (in other experiments) sloshing ions, it is necessary to calculate the axial eigenmodes numerically, rather than use the rough estimate $k_{\|}L > 1$ and the infinite medium experession for $k_{\|}$ evaluated at some typical axial position, as was done in deriving Eq. (4). An axial eigenmode code has been written by Pearlstein, and has been used to find stability criteria for the DCLC and ALC modes in various designs of MFTF-B. We have written a similar code, which gives results comparable to those of Pearlstein's code for similar equilibria. In addition to increasing our confidence in the correctness of both codes, our code includes some new features, e.g. it evaluates, in an approximate way, the finite width of ion cyclotron harmonic resonances; this is especially important in a belly-band equilibrium, where the magnetic field profile is fairly flat over a large part of the length of the cell, and ions over a broad axial range can be resonant with the same wave. (Our code also lacks some features found in Pearlstein's code, for example ∇B-drift damping from hot electrons.)

In Section II, the belly band equilibrium is described, including the expressions used for the hot and warm ion distribution functions. An iterative procedure is used to numerically calculate a self-consistent warm ion distribution, confined axially to the inner side of the potential peak. In Section III, the dispersion relation (a second order differential equation along a field line) is derived, and the treatment of the finite ion cyclotron harmonic resonance width is discussed. In Section IV, we discuss the boundary conditions which must be used at the mirror throats in order to find the axial normal modes. When the electrons are cold, outgoing wave boundary conditions are appropriate, but these may not be correct when the electrons are warm. In Section V, results are presented for the stability of loss-cone modes along certain field lines. It is found that, for reasonable amounts of warm plasma, the lowest $k_\perp$ modes are stabilized, but that modes with higher $k_\perp$ and higher frequency ($k_\perp \rho_i \gtrsim 10$, and $\omega \gtrsim 4\omega_{ci}$) remain unstable, although at rather low growth rates (no more than a few tenths of ω_{ci}). This behavior is expected from Eq. (4), which shows that modes with $k_\perp \rho_i \gtrsim \beta_i^{1/2}(m_i/m_e)^{1/2}$ require substantially shorter lengths for stability. It is likely, however, that these high $k_\perp$ modes will turn out to be less harmful than low $k_\perp$ modes, when the nonlinear or quasilinear effect on confinement is determined.

The stability of a given three-dimensional equilibrium does not depend on the local stability of each field line and direction of $\underline{k}_\perp$, but on the stability of the three-dimensional normal modes. If the WKB approximation is valid across the magnetic field, then the three-dimensional normal modes can be constructed by ray-tracing. The criteria for stability may be substantially modified, due to the radial convection of wave energy, and the additional constraint on the normal modes that they must satisfy the boundary conditions radially and azimuthally. If ray-tracing across field lines is attempted for the DCLC and ALC modes, it is found that the group velocities have large imaginary parts, so the usual ray-tracing methods do not work. In Section V, we discuss a generalization of the usual ray-tracing equations, which is valid even when the group velocities are complex. A code incorporating this method has been partly completed; preliminary results for loss-cone instabilities in the TARA ion anchor suggest that the radial mode structure has an important effect on stability. A summary and conclusions are presented in Section VI, together with a discussion of effects (e.g., electron Landau damping, ion ∇B drift) which may be important, but which have not yet been included in the codes.

II. Equilibrium

The TMX yin-yang coils used in the TARA anchor have a vacuum field in the vicinity of the midplane given by

$$B(z) = B_0[1 + (z/45)^2] \tag{5}$$

where z is the axial distance in cm. The mirror throats occur at a distance of 57 cm from the midplane, with a mirror ratio of 2.43. A good approximation to the magnetic field in the entire region between the mirror throats may be obtained by using Eq. (5) for $|z| = 45$, and using

$$B(z) = 2.43B_0 - a(z \mp 57)^2 \mp b(z \mp 57)^3$$

for $|z| > 45$, with the ($\mp$) sign used in the vicinity of ($\pm$) 57 cm, and with the coefficients a and b chosen to make $B(z)$ and its first derivative continuous.

The magnetic field due to the belly band is

$$B(z) = B_1[1 + (z/25)^2]^{-\frac{3}{2}}$$

We must choose B_1/B_0 large enough to make a local maximum in the field at the midplane, but small enough to avoid introducing too much bad curvature. We have chosen $B_0 = 2.72$ kG and $B_1 = 1.18$ kG, which gives a midplane "bump" 3.4% greater than the minimum magnetic field, and mirror ratio (between the maximum and minimum field) of 1.75. The total magnetic field (the sum of the fields due to the yin-yang and belly band coils) is then

$$B(z) = 2.72[1 + (z/45)^2] + 1.18[1 + (z/25)^2]^{-\frac{3}{2}} \quad for \quad |z| \leq 45 \tag{6a}$$

$$B(z) = 6.6 - 1.185 \times 10^{-2}(z \mp 57)^2 \mp 3.944 \times 10^{-4}(z \mp 57)^3 \quad for \quad 45 < |z| < 57 \tag{6b}$$

with B in kG and z in cm. Since the plasma radius is about 7.5 cm at the midplane, the long thin approximation is fairly good, and we can ignore radial variations in the vacuum field. Since $\beta \lesssim 0.15$, the diamagnetic corrections to the vacuum field are not too large, and we have used Eq. (6) for $B(z)$ on all field lines.

The field lines are labelled by (α, β) coordinates, which are Cartesian at the midplane. Because of the long, thin approximation and the neglect of diamagnetic corrections to the vacuum field, we will have $\nabla\alpha \cdot \nabla\beta = 0$ everywhere if we choose $\nabla\alpha$ and $\nabla\beta$ to be parallel to the planes of symmetry of the quadrupole field. The fanning factor $\nabla\alpha/\nabla\beta$ is given by

$$\frac{\nabla\alpha}{\nabla\beta} = \exp(z/15) \tag{7}$$

which is equal to 45 at the mirror throat. Equation (7), together with $\nabla\alpha\nabla\beta = B(z)$, defines $\nabla\alpha(z)$ and $\nabla\beta(z)$.

To determine the hot ion distribution, we have made use of a bounce-averaged Fokker-Planck code which calculated the distribution generated by a 15 keV neutral beam injected perpendicular to the field at the field minimum of a 1.75 mirror ratio cell. In the Fokker-

Planck code, and in the equilibrium we have used, the hot ions are assumed to be energetic enough so that they are not affected by the ambipolar potential; this assumption was later checked, and found to be fairly well satisfied. In this case, the hot ion density depends only on B. A good fit to the Fokker-Planck results for the hot ion density is provided by

$$n_{hot}(B) = \frac{2B^{\frac{1}{2}} \quad \arctan[(B_{min}/B - B_{min}/B_{max})^{\frac{1}{2}}(1 + T_{\parallel}/T_{\perp} - B_{min}/B)^{-\frac{1}{2}}]}{B^{\frac{1}{2}}_{min}(1 + T_{\parallel}/T_{\perp} - B_{min}/B)^{\frac{1}{2}}}$$

$$- \frac{2(1 - B/B_{max})^{\frac{1}{2}}}{(1 + T_{\parallel}/T_{\perp} - B_{min}/B_{max})} \tag{8}$$

if we take $T_{\parallel}/T_{\perp} = 0.08$. The dip in density at the midplane (due to the 3.4% bump in magnetic field) is 29%. This rather complicated expression is just the $n_{hot}(B)$ that is associated with the pitch angle distribution

$$g(\nu) = (1 + T_{\parallel}/T_{\perp} - \nu B_{min})^{-1} - (1 + T_{\parallel}/T_{\perp} - B_{min} B_{max})^{-1}$$

which is one of the simplest expressions exhibiting the desired behavior in the vicinity of $\nu = B^{-1}_{min}$ and $\nu = B^{-1}_{max}$. The pitch angle distribution is defined as

$$g(\nu) \equiv \nu^{-\frac{3}{2}} \int_0^\infty d\mu \quad \mu^{\frac{1}{2}} f(H = \mu/\nu, \mu)$$

and is related to $n_{hot}(B)$ by

$$n_{hot}(B) = B^{\frac{1}{2}} \int_{B^{-1}_{max}}^{B^{-1}} d\nu (B^{-1} - \nu)^{-\frac{1}{2}} g(\nu).$$

The Fokker-Planck calculation was done for a mirror cell with a single field minimum. With a belly band, there are two field minima, one on each side of the midplane. If the neutral beam is injected at the outer field minimum (at $H = \mu B_{min}$), then we would expect $n_{hot}(B)$ to be lower than the value given by Eq. (8) for $B < B_{Bump}$ (where B_{bump} is the field at the midplane) at $z < 0$ (around the inner minimum), since ions can become trapped on the inner side of the midplane bump only by diffusing first into the region of velocity space where $H > \mu B_{bump}$, and then back to $H < \mu B_{bump}$ on the other side of the midplane. During the time it takes for this diffusion to occur, some of the ions will be lost by charge exchange, so we expect to find fewer ions trapped on the inner side of the midplane bump than on the outer side. As a guess, we have simply used constant density, i.e.,

$$n_{hot}(B) = n_{hot}(B_{bump}) \tag{9}$$

(with the right hand side given by Eq. (8)) for $B < B_{bump}$ on the inner side ($z < 0$). Recently, Matsuda has developed a multi-region bounce-averaged Fokker-Planck code, and we plan to use this code to see whether Eq. (9) is a good guess. $B(z)$ and $n_{hot}(z)$ are plotted in Fig. 1.

The Fokker-Planck code predicts a perpendicular hot ion velocity distribution at a given axial position which has a sharp peak at the injected neutral beam energy. Such a distribution might be unstable to velocity-space instabilities (such as Dory-Guest-Harris) which would smooth out the distribution, without causing any losses of ions. Since we are not concerned with such instabilities, but only with instabilities that would drive ions into the loss cone, we have used a smoother, broader perpendicular velocity distribution for the hot ions, one of the form

$$f_{\perp}(\mu, B) = \frac{n_{hot}(B)[\exp(-\mu/2\mu_{hot}(B)) - exp(-\mu/2\mu_{hole})]}{\mu_{hot}(B) - \mu_{hole}} \tag{10}$$

This "subtracted Maxwellians" form for $f_{\perp}$ has the advantage that it is easy to incorporate into the ion term of the dispersion relation for loss cone instabilities, and is often used in loss cone stability studies. The justification for using the Fokker-Planck results for $n_{hot}(B)$, but this simplified form for $f_{\perp}$, is the idea that velocity space instabilities will cause changes in $f_{\perp}$ but not in $n_{hot}(B)$; although not strictly true, this should be a reasonable approximation.

We have assumed μ_{hole} is independent of B, and taken $\mu_{hot}(B)$ to have the form

$$\mu_{hot}(B) = \frac{\mu_{hole}(1 + aT_{\parallel}/T_{\perp} - B_{min}/B_{max})}{(1 + aT_{\parallel}/T_{\perp} - B_{min}/B)}$$

with the free parameter a. In order to choose a reasonable value for a, we wrote a code, FOFHMU, which takes $\mu_{hot}(B)$ and $n_{hot}(B)$ as input, and generates $f_{hot}(H, \mu)$as output. This code is described in Appendix A. We would like to choose a to be as small as possible (in order to make μ_{hot}/μ_{hole} as large as possible, and minimize loss cone instabilities), subject to the constraints 1) that $f_{hot}(H,\mu) \geq 0$ everywhere, and 2) that $f_{hot}(H,\mu)$ be fairly small for H less than the ambipolar potential, justifying our neglect of the ambipolar potential in the hot ion equilibrium. We found that $a = 1.2$ is a reasonable choice, for $T_{\parallel}/T_{\perp} = 0.08$ and $B_{max}/B_{min} = 1.75$. The hot ion distribution functions $f(v_{\perp}, v_{\parallel})$, at both the outer and inner field minima, are shown in Fig. 2.

Although the ion term in the dispersion relation depends mostly on the perpendicular velocity distribution, the parallel velocity distribution does come into the expression for the ion cyclotron harmonic resonance width. The hot ion distribution $f(H,\mu)$ given in Appendix A is not separable into a function of $v_{\perp}$ times a function of $V_{\parallel}$, but as a rough approximation, which greatly simplifies the expression for the ion term in the dispersion relation, we will treat the ion distribution as if it were $f_{\perp}(\mu, B)$ (given by Eq. (10)) times $(2\pi\langle v_{\parallel}^2\rangle)^{-\frac{1}{2}} exp(-v_{\parallel}^2/2\langle v_{\parallel}^2\rangle)$, where $\langle v_{\parallel}^2\rangle$ is the mean $v_{\parallel}^2$ of the actual ion distribution $f(H,\mu)$,

$$\langle v_{\parallel}^2\rangle = [n_{hot}(B)]^{-1}B \int_0^{\infty} d\mu \int_{\mu B}^{\mu B_{max}} dH (H - \mu B)^{\frac{1}{2}} f(H,\mu) \tag{12}$$

From Appendix A, $f(H,\mu)$ may be expressed as a sum of Laguerre polynomials $L_i(\mu/\mu_{hole})$ times functions $N_i(\mu/H)$ which are moments of the perpendicular distribution function $f_{\perp}(\mu, B)$, defined by Eq. (A6). Changing the variable of integration in Eq. (12) from H to $\nu \equiv \mu/H$, and using the identity

$$\int_0^{\infty} dx \quad x \quad \exp(-x/2) L_i(x) = 4(-1)^i (2i+1)$$

to do the μ integration, Eq. (12) becomes

$$\langle v_{\parallel}^{2}\rangle = \frac{4B^{\frac{3}{2}}\mu_{hole}}{\pi n_{hot}(B)} \int_{B_{max}^{-1}}^{B^{-1}} d\nu \quad \nu^{-1}(B^{-1}-\nu)^{\frac{1}{2}} \Sigma \max_{i}(-1)^{i}(2i+1)N_{i}(\nu) \tag{13}$$

Using Eq. (A8) for $N_i(\nu)$, the sum over i in Eq. (13) may be done, yielding

$$\langle v_{\parallel}^{2}\rangle = \frac{4B^{\frac{3}{2}}\mu_{hole}}{\pi n_{hot}(B)} \int_{B_{max}^{-1}}^{B^{-1}} d\nu \quad \nu^{-1}(B^{-1}-\nu)^{\frac{1}{2}} \frac{d}{d\nu} y(\nu) = \frac{4\pi B^{\frac{3}{2}}\mu_{hole}}{\pi n_{hot}(B)} \int_{0}^{y(B^{-1})} dy \quad \nu^{-1}(B^{-1}-\nu)^{\frac{1}{2}} \tag{14}$$

where

$$y(\nu) \equiv \int_{\nu^{-1}}^{B_{max}} dB'(\nu - B'^{-1})^{-\frac{1}{2}} G(B')$$

and

$$G(B) \equiv n_{hot}(B)B^{-\frac{5}{2}}[4 + 4aT_{\parallel}/T_{\perp} - \frac{B_{min}}{B} - \frac{3B_{min}}{B_{max}} + B_{min}(B^{-1} - B_{max}^{-1})(1 + aT_{\parallel}/T_{\perp} - B_{max}^{-1})$$

$$(1 + aT_{\parallel}/T_{\perp} - B_{min}/B)^{-1}][2(1 + aT_{\parallel}/T_{\perp}) - B_{min}(B^{-1} - B_{max}^{-1})]^{-1} \tag{15}$$

Equation (14) may be integrated by parts to yield

$$\langle v_{\parallel}^{2}\rangle = \frac{2B^{\frac{3}{2}}\mu_{hole}}{\pi n_{hot}(B)} \int_{B}^{B_{max}} dB' G(B') \int_{B'^{-1}}^{B^{-1}} \frac{d\nu(2B^{-1}-\nu)\nu^{-2}}{(B^{-1}-\nu)^{\frac{1}{2}}(\nu - B'^{-1})^{\frac{1}{2}}} \tag{16}$$

For $aT_{\parallel}/T_{\perp} \ll 1$, $G(B')$ is a steeply decreasing function of B', so the B' integral is dominated by $B' \simeq B$. Then in the ν integral, $(2B^{-1}-\nu)\nu^{-2} \simeq B$, and the ν integration may be done, yielding

$$\langle v_{\parallel}^{2}\rangle = \frac{2B^{\frac{5}{2}}\mu_{hole}}{n_{hot}(B)} \int_{B}^{B_{max}} dB' G(B') \tag{17}$$

Although the B' integral cannot be expressed in closed form, we can find simple expressions for $\langle v_\parallel^2 \rangle$ in the limiting cases $B \simeq B_{min}$ and $B \simeq B_{max}$, using Eq. (16) for $G(B)$ and Eq. (8) for $n_{hot}(B)$. We find

$$n_{hot}(B) \quad \alpha \quad (1 + T_\parallel/T_\perp - B_{min}/B)^{-\frac{1}{2}}$$

$$\langle v_\parallel^2 \rangle \simeq 8\mu_{hole}B_{min}(1 - B_{min}/B_{max})(1 + T_\parallel/T_\perp - B_{min}/B)^{\frac{1}{2}}(1 + aT_\parallel/T_\perp - B_{min}/B)^{-\frac{1}{2}}$$

$$\simeq 7\mu_{hole}B_{min}(1 - B_{min}/B_{max})$$

for $B \simeq B_{min}$ and

$$n_{hot}(B) \quad \alpha \quad (B_{max}/B - 1)^{\frac{3}{2}}$$

$$\langle v_\parallel^2 \rangle \simeq \frac{16}{5}\mu_{hole}(B_{max} - B)$$

for

$$B \simeq B_{max}$$

.

For the code, we have simply used an arbitrary expression for $\langle v_\parallel^2 \rangle$ which has the correct behavior in these two limits,

$$\langle v_\parallel^2 \rangle = \mu_{hole}(B^{-1} - B_{max}^{-1})(B_{max} - B_{min})^{-1}[\frac{16}{5}B_{max}^2(B - B_{min}) + 7(B_{max} - B)]. \qquad (18)$$

The warm plasma is assumed to be confined to the inner side of the potential peak, which is located at the outer field minimum (where the hot ion density is greatest). On the inner side of the potential peak, we assume a warm ion distribution of the form

$$f_{warm}(H, \mu)\alpha(\mu B_{min} - H) \quad \exp(-H/T_w) \quad \text{for} \quad H < \mu B_{min}$$

$$f_{warm}(H,\mu) = 0 \quad \text{for} H \geq \quad \mu B_{min} \tag{19}$$

where

$$H \equiv \frac{1}{2} m_i v_\perp^2 + \frac{1}{2} m_i v_\parallel^2 + e\phi(z) - e\phi_{max}$$

and

$$\mu \equiv \frac{1}{2} m_i v_\perp^2 / B(z).$$

The warm plasma density is given by

$$n_{warm}(z) = \int_{-\infty}^{\infty} dv_\parallel \int_{0}^{\infty} 2\pi v_\perp dv_\perp f_{warm}(H,\mu)$$

$$a \quad \int_{-[\phi_{max}-\phi(z)]^{\frac{1}{2}}}^{[\phi_{max}-\phi(z)]^{\frac{1}{2}}} dv_\parallel \int_0^{[1-B_{min}/B(z)]^{-\frac{1}{2}}[\phi_{max}-\phi(z)-v_\parallel^2]^{\frac{1}{2}}} dv_\perp v_\perp [\phi_{max} - \phi(z) - v_\parallel^2 - v_\perp^2(1 - B_{min}/B)]$$

$$\exp[-(v_\perp^2 + v_\parallel^2 + \phi(z))/T_w] \tag{20}$$

Doing the $v_\parallel$ and $v_\perp$ integrals, we obtain

$$n_{warm}(z) \quad a \quad [\phi_{warm} - \phi(z)]^{\frac{1}{2}} T_w^{-\frac{1}{2}} + \sqrt{\pi}\Phi([\phi_{max} - \phi(z)]^{\frac{1}{2}} T_w^{-\frac{1}{2}})([\phi_{max} - \phi(z)] T_w^{-1}$$

$$-\frac{3}{2} + B_{min}/B(z)) \quad \exp([\phi_{max} - \phi(z)] T_w^{-1})$$

$$-[1 - B_{min}/B(z)]^{\frac{3}{2}} [B(z)/B_{min}]^{\frac{1}{2}} ReZ([\phi_{max} - \phi(z)]^{\frac{1}{2}} T_w^{-\frac{1}{2}} [B(z)/B_{min} - 1]^{-\frac{1}{2}}) \tag{21}$$

where Φ is the probability integral and ReZ is the real part of the Z function. The potential $\phi(z)$ is also related to the warm plasma density $n_{warm}(z)$ by the Boltzmann relation

$$\phi(z) - \phi_{max} = T_e ln([n_{warm}(z) + n_{hot}(z)]/n_{max}) \tag{22}$$

where n_{max} is the hot plasma density at the outer field minimum. Since $B(z)$ and $n_{hot}(z)$ are known functions, given by Eqs. (6) and (8), we can use Eqs. (21) and (22) to solve for the two unknown functions $n_{warm}(z)$ and $\phi(z)$. The constant of proportionality in Eq. (21) is fixed by specifying n_{warm} at the inner mirror throat, where the anchor joins the transition region. The most obvious numerical scheme for finding $n_{warm}(z)$ and $\phi(z)$ would be to first use Eq. (22) to find $\phi(z)$ at the inner mirror throat, $z = -z_{thr}$. Then $\phi(-z_{thr})$ would be used as an initial guess for $\phi(-z_{thr} + \Delta z)$, for some $\Delta z \ll L$, and Eq. (21) would be used to make an initial guess for $n_{warm}(-z_{thr} + \Delta z)$; this $n_{warm}(-z_{thr} + \Delta z)$ would be put into Eq. (22) to make a revised estimate of $\phi(-z_{thr} + \Delta z)$, and the procedure would be continued until the estimates of $\phi(-z_{thr} + \Delta z)$ and $n_{warm}(-z_{thr} + \Delta z)$ converged. Then $\phi(-z_{thr} + \Delta z)$ would be used as an initial guess for $\phi(-z_{thr} + 2\Delta z)$, and so on. In practice, this scheme often fails to converge since the estimates overshoot the correct solution, and the errors grow larger with each iteration. Instead, the new estimate used for n_{warm} is $(1 - w)$ times the old estimate, plus w times the right hand side of Eq. (21), where w is a weight factor, typically about 0.1, which is adjusted dynamically to maximize the rate of convergence.

Figure 1 shows $n_{warm}(z)$ calculated by this method, using $n_{warm}/n_{max} = 0.8$ at the inner mirror throat.

The warm ion distribution function given by Eq. (19) is difficult to put into the ion term of the dispersion relation; the ion term would be much easier to calculate if the warm ion distribution were bi-Maxwellian. We have therefore treated the warm ion velocity distribution as if it were bi-Maxwellian, with the same $\langle v_\perp^2 \rangle$ and $v\langle_\parallel^2\rangle$ as the distribution given by Eq. (19), for purposes of calculating the ion term in the dispersion relation.

In the limit that $e[1 - B_{min}/B(z)]^{-1}[\phi_{max} - \phi(z)] > T_w$, we find $\langle v_\perp^2 \rangle \simeq T_w/m_i$ for the warm ions, while in the opposite limit, we find

$$\langle v_\perp^2 \rangle \simeq \frac{2}{7}\frac{e}{m_i}[\phi_{max} - \phi(z)][1 - B_{min}/B(z)]^{-1}$$

so in the code we have used

$$\langle v_\perp^2 \rangle = \min(T_w/m_i, \frac{2}{7}\frac{e}{m_i}[\phi_{max} - \phi(z)][1 - B_{min}/B(z)]^{-1}) \tag{23}$$

Similarly, we have used

$$\langle v_\parallel^2 \rangle = \min(T_w/m_i, \frac{2}{7}\frac{e}{m_i}[\phi_{max} - \phi(z)]) \tag{24}$$

In practice, we always have $\frac{2}{7}e[\phi_{max} - \phi(z)] \ll T_w$, and $\frac{2}{7}e[\phi_{max} - \phi(z)][1 - B_{min}/B(z)]^{-1} \ll T_w$ except very close to the field minimum, so T_w has very little effect on the dispersion relation.

III. Local Dispersion Relation

From Maxwell's equations and the definition $4\pi i\omega^{-1}\underline{J} = \underline{\epsilon} \cdot \underline{E}$, the dispersion relation for waves in a plasma is

$$-\frac{c^2}{\omega^2}\nabla \times \nabla \times \underline{E} + \underline{E} + \underline{\underline{\epsilon}} \cdot \underline{E} = 0 \tag{25}$$

To evaluate $\nabla \times \nabla \times \underline{E}$ in (α, β, z) coordinates, we note that these coordinates are orthogonal to the extent that the long, thin approximation is valid and the diamagnetic corrections to the magnetic field can be neglected, i.e.,

$$(ds)^2 = (\nabla\alpha)^{-2}(d\alpha)^2 + (\nabla\beta)^{-2}(d\beta)^2 + (dz)^2$$

In this case, the curl of a vector $\underline{A}$ is given by

$$\nabla \times \underline{A} = (\nabla\beta)\hat{\alpha}[\frac{\partial}{\partial\beta}A_z - \frac{\partial}{\partial z}\frac{1}{\nabla\beta}A_\beta] + (\nabla\alpha)\hat{\beta}[\frac{\partial}{\partial z}\frac{1}{\nabla\alpha}A_\alpha - \frac{\partial}{\partial\alpha}A_z] + (\nabla\alpha\nabla\beta)\hat{z}[\frac{\partial}{\partial\alpha}\frac{1}{\nabla\beta}A_\beta - \frac{\partial}{\partial\beta}\frac{1}{\nabla\alpha}A_\alpha] \tag{26}$$

For loss-cone instabilities, $k_\perp \gg k_\parallel$, so we assume $\underline{E}$ locally is of the form

$$\underline{E}(\alpha, \beta, z) = \exp(ik_\alpha\alpha + ik_\beta\beta)\underline{\overline{E}}(\alpha, \beta, z) \tag{27}$$

where $\underline{\overline{E}}(\alpha, \beta, z)$, varies slowly with α, β, and z, on the scale of the plasma density, while $\exp(ik_\alpha\alpha + ik_\beta\beta)$ is a rapid variation. The components of $\nabla \times \nabla \times \underline{E}$ are then

$$\hat{\alpha}\cdot(\nabla\times\nabla\times\underline{E})=\nabla\alpha\nabla\beta\frac{\partial^2}{\partial\alpha\partial\beta}E_\beta-(\nabla\beta)^2\frac{\partial^2}{\partial\beta^2}E_\alpha-\nabla\beta\frac{\partial}{\partial z}\frac{\nabla\alpha}{\nabla\beta}\frac{\partial}{\partial z}\frac{1}{\nabla\alpha}E_\alpha+\nabla\beta\frac{\partial}{\partial z}\frac{\nabla\alpha}{\nabla\beta}\frac{\partial}{\partial\alpha}E_z \quad (28a)$$

$$\hat{\beta}\cdot(\nabla\times\nabla\times\underline{E})=\nabla\alpha\nabla\beta\frac{\partial^2}{\partial\alpha\partial\beta}E_\alpha-(\nabla\alpha)^2\frac{\partial^2}{\partial\alpha^2}E_\beta-\nabla\alpha\frac{\partial}{\partial z}\frac{\nabla\beta}{\nabla\alpha}\frac{\partial}{\partial z}\frac{1}{\nabla\beta}E_\beta+\nabla\alpha\frac{\partial}{\partial z}\frac{\nabla\beta}{\nabla\alpha}\frac{\partial}{\partial\beta}E_z \quad (28b)$$

$$\hat{z}\cdot(\nabla\times\nabla\times\underline{E})=(\nabla\alpha)^2\frac{\partial}{\partial z}\frac{1}{\nabla\alpha}\frac{\partial}{\partial\alpha}E_\alpha+(\nabla\beta)^2\frac{\partial}{\partial z}\frac{1}{\nabla\beta}\frac{\partial}{\partial\beta}E_\beta-(\nabla\alpha)^2\frac{\partial^2}{\partial\alpha^2}E_z-(\nabla\beta)^2\frac{\partial^2}{\partial\beta^2}E_z \quad (28c)$$

We define $\underline{k}_\perp(z)\equiv\nabla\alpha k_\alpha\hat{\alpha}+\nabla\beta k_\beta\hat{\beta}$, and change from $(\hat{\alpha},\hat{\beta},\hat{z})$ coordinates to $(\hat{k}_\perp,\hat{k}_\perp\times\hat{z},\ \hat{z})$ coordinates, where $\hat{k}_\perp\equiv\underline{k}_\perp/k_\perp$. We note that

$$\frac{\partial\underline{E}}{\partial\alpha}=(ik_\alpha\nabla\alpha\overline{\underline{E}}+\frac{\partial}{\partial\alpha}\overline{\underline{E}})\quad\exp(ik_\alpha\alpha+ik_\beta\beta)$$

and similarly for $\partial\underline{E}/\partial\beta$, where the term $ik_\alpha\nabla\alpha\overline{\underline{E}}$, is greater than the term $\partial\overline{\underline{E}}/\partial\alpha$ by a factor of order $k_\perp R_p$, where R_p is the plasma radius. For each term of each component of $\nabla\times\nabla\times\underline{E}$ in $(\hat{k}_\perp,\hat{k}_\perp\times\hat{z},\hat{z})$ coordinates, we keep only the lowest-order non-vanishing terms in $(k_\perp R_p)^{-1}$. Then, using $B(z)=\nabla\alpha\nabla\beta$,

$$\hat{k}_\perp\cdot(\nabla\times\nabla\times\underline{E})=-\hat{k}_\perp\cdot(\hat{z}\times\nabla)\cdot(\hat{z}\times\nabla)(\hat{k}_\perp E_{k_\perp})+\hat{k}_\perp\cdot(\hat{z}\times\nabla)(ik_\perp E_{k_\perp\times z})$$

$$-\frac{B}{k_\perp}\frac{\partial}{\partial z}\frac{k_\perp^2}{B}\frac{\partial}{\partial z}\frac{1}{k_\perp}E_{k_\perp}+\frac{iB}{k_\perp}\frac{\partial}{\partial z}\frac{k_\perp^2}{B}E_z \quad (29)$$

$$(\hat{k}_\perp\times\hat{z})\cdot(\nabla\times\nabla\times\underline{E})=-ik_\perp(\hat{z}\times\nabla)\cdot(\hat{k}_\perp E_{k_\perp})+k_\perp^2E_{k_\perp\times z} \quad (30)$$

$$\hat{z}\cdot(\nabla\times\nabla\times\underline{E})=ik_\perp^2\frac{\partial}{\partial z}\frac{1}{k_\perp}E_{k_\perp}+k_\perp^2E_z \quad (31)$$

Using the ordering, appropriate for ion loss-cone instabilities, that

$$\hat{z}\cdot\underline{\underline{\epsilon}}\cdot\hat{z} \simeq \frac{k_\perp^2 c^2}{\omega^2} \gg (\hat{k}_\perp \times \hat{z})\cdot\underline{\underline{\epsilon}}\cdot\hat{k}_\perp \gtrsim \hat{k}_\perp\cdot\underline{\underline{\epsilon}}\cdot\hat{k}_\perp \simeq \frac{k_\parallel^2 c^2}{\omega^2} \tag{32}$$

the components of Eq. (25) are

$$-\overline{E}_{k_\perp} - \epsilon_{xx}\overline{E}_{k_\perp} - \frac{c^2}{\omega^2}\frac{B}{k_\perp}\frac{\partial}{\partial z}\frac{k_\perp^2}{B}\frac{\partial}{\partial z}\frac{1}{k_\perp}\overline{E}_{k_\perp} - \frac{c^2}{\omega^2}(\hat{k_\perp}\times\hat{z})\cdot\nabla[(\hat{z}\times\nabla)\cdot(\hat{k_\perp}E_{k_\perp}]$$

$$+\frac{ic^2}{\omega^2}(\hat{k}_\perp\times\hat{z})\cdot\nabla(k_\perp\overline{E}_{k_\perp\times z} - \epsilon_{xy}\overline{E}_{k_\perp\times z} + \frac{ic^2}{\omega^2}\frac{B}{k_\perp}\frac{\partial}{\partial z}\frac{k_\perp^2}{B}\overline{E}_z = 0 \tag{33}$$

$$-\frac{ik_\perp c^2}{\omega^2}(\hat{z}\times\nabla)\cdot(\hat{k_\perp}E_{k_\perp} + \epsilon_{xy}\overline{E}_{k_\perp} + \frac{k_\perp^2 c^2}{\omega^2}\overline{E}_{k_\perp\times z} = 0 \tag{34}$$

$$\frac{ik_\perp^2 c^2}{\omega^2}\frac{\partial}{\partial z}\frac{1}{k_\perp}\overline{E}_{k_\perp} + \frac{k_\perp^2 c^2}{\omega^2}E_z - \epsilon_{zz}E_z = 0 \tag{35}$$

where $\epsilon_{xx} \equiv \hat{k}_\perp\cdot\underline{\underline{\epsilon}}\cdot k_\perp = (\hat{k}_\perp\times\hat{z})\cdot\underline{\underline{\epsilon}}\cdot(\hat{k}_\perp\times\hat{z})$, and $\epsilon_{xy} \equiv \hat{k}_\perp\cdot\underline{\underline{\epsilon}}\cdot(\hat{k}_\perp\times\hat{z}) = -(\hat{k}_\perp\times\hat{z})\cdot\underline{\underline{\epsilon}}\cdot\hat{k}_\perp$, and $\epsilon_{zz} \equiv \hat{z}\cdot\underline{\underline{\epsilon}}\cdot\hat{z}$. For ion loss-cone instabilities, the electrons dominate ϵ_{xy} and ϵ_{zz}, with

$$\epsilon_{xy} = \frac{i\omega_{pe}^2}{\omega\omega_{ce}}, \quad \epsilon_{zz} = \frac{-\omega_{pe}^2}{\omega^2} \tag{36}$$

for cold electrons, while ϵ_{xx} includes the ion term $\epsilon_{xx,i}$ and the electron polarization drift $\omega_{pe}^2/\omega_{ce}^2$. Eliminating $E_{k_\perp\times z}$ and E_z from Eqs. (33) through (35), we obtain the dispersion relation

$$[1 + \frac{\omega_{pe}^2}{\omega_{ce}^2}(1 + \frac{\omega_{pe}^2}{k_\perp^2 c^2}) + \epsilon_{xx,i}]\overline{E}_{k_\perp} + \frac{B}{k_\perp}\frac{\partial}{\partial z}\frac{1}{B}\frac{\omega_{pe}^2}{\omega^2}(1 + \frac{\omega_{pe}^2}{k_\perp^2 c^2})^{-1}\frac{\partial}{\partial z}\frac{1}{k_\perp}\overline{E}_{k_\perp}$$

$$-(\hat{k}_\perp\times\hat{z})\cdot\nabla(\frac{\omega_{pe}^2}{\omega\omega_{ce}k_\perp}\overline{E}_{k_\perp}) = 0 \tag{37}$$

which is a second-order differential equation (or an integral equation if $\epsilon_{xx,i}$ is an integral operator) for the axial eigenmode $\overline{E}_{k_\perp}(z)$. If we replace $\partial/\partial z$ by $ik_\parallel$, we recover Eq. (1), the infinite medium dispersion relation. Because of the ordering $k_\perp^2 c^2/\omega^2 \gg \epsilon_{xx,i}$ which we assumed [Eq. (32)] in deriving Eq. (37), this dispersion relation does not include modes such as the Alfven ion cyclotron (AIC) instability, for which the ion response is electromagnetic, but only modes (such as DCLC and ALC) for which the ion response is electrostatic. The assumption of cold electrons is valid if $\omega/k_{\parallel}v_e \gg 1$, otherwise ϵ_{zz} is an integral operator and the dispersion relation will be an integral equation. The assumption that ϵ_{xy} is dominated by the electrons is valid if $k_{\perp}\rho_i \gg 1$ (where ρ_i is the ion Larmor radius) or $\omega \gg \omega_{ci}$, otherwise ϵ_{xy} will be modified (and usually reduced in magnitude) by the ions, and the terms $(\omega_{pe}^4/k_{\perp}^2 c^2\omega_{ce}^2)\overline{E}_{k_{\perp}}$ and $(\hat{k}_{\perp} \times \hat{z}) \cdot \nabla(\overline{E}_{k_{\perp}}\omega_{pe}^2/\omega\omega_{ce}k_{\perp})$ will be modified.

The ion term $\epsilon_{xx,i}$ may be evaluated in the usual way by integrating the linearized Vlasov equation along the unperturbed ion orbits to obtain the perturbed ion distribution function and integrating the perturbed distribution function over velocity space to obtain the perturbed ion current in the $\hat{k}_{\perp}$ direction, and hence $\epsilon_{xx,i}$:

$$\epsilon_{xx,i}\overline{E}_{k_{\perp}}(z) = \frac{-i\omega_{pi}^2}{\omega}\int d\underline{v} \quad v_x \int_{-\infty}^{0} d\tau \frac{\partial f_i(v')}{\partial v'_x}\overline{E}_{k_{\perp}}(z') \quad \exp[-i\omega\tau + i\underline{k}'_{\perp}\cdot(\underline{x}'-\underline{x}_{gc}) - i\underline{k}_{\perp}\cdot(\underline{x}-\underline{x}_{gec})] \tag{38}$$

where $v_x \equiv \underline{v}\cdot\hat{k}_{\perp}$, $v'_x \equiv \underline{v}'\cdot\hat{k}'_{\perp}$, $\underline{k}'_{\perp} \equiv \underline{k}(z'(\tau))$, $\underline{v}'(\tau) \equiv (d/dt)\underline{x}'(\tau)$, and $\underline{x}'(\tau,\underline{v})$ is the orbit of an ion which is located at position $\underline{x}$ and has velocity $\underline{v}$ at $\tau = 0$. For adiabatic ion motion, neglecting cross-field drifts, we have

$$\underline{x}' - \underline{x}_{gc} = -\frac{\hat{z}\times\underline{v}'}{\omega_{ci}(\tau)}$$

$$\underline{v}' = \hat{z}v'_{\parallel} + v'_{\perp}[\hat{k}_{\perp}\sin(\int_0^{\tau} d\tau'\omega_{ci}(\tau') - \varphi) - (\hat{k}_{\perp}\times\hat{z})\cos(\int_0^{\tau} d\tau'\omega_{ci}(\tau') - \varphi)]$$

where φ is the azimuthal angle of $\underline{v}$. The integral over φ may be done by using the Bessel function identity

$$\exp(ia\sin\theta) = \sum_{n=-\infty}^{\infty} J_n(a) \quad \exp(in\theta)$$

The result is

$$\epsilon_{xx,i}\overline{E}_{k_\perp}(z) = \frac{4\pi i\omega_{pi}^2\omega_{ci}}{\omega k_\perp} \int_{-\infty}^{\infty} dv_\| \int_0^{\infty} dv_\perp v_\perp \int_{-\infty}^{0} d\tau \frac{\partial f_i(v'_\perp, v'_\|)}{\partial v'^2_\perp}$$

$$\sum_{n=-\infty}^{\infty} \exp[-i\omega\tau + in\int_0^{\tau} d\tau'\omega_{ci}(\tau')][n^2\frac{\omega_{ci}(\tau)}{k'_\perp}(\hat{k}'_\perp\cdot\hat{k}_\perp)J_n(\frac{k'_\perp v'_\perp}{\omega_{ci}(\tau)})$$

$$+inv'_\perp\hat{k}'_\perp\cdot(\hat{k}_\perp\times\hat{z})J'_n(\frac{k'_\perp v'_\perp}{\omega_{ci}(\tau)})]J_n(\frac{k_\perp v_\perp}{\omega_{ci}})[\hat{k}'_\perp\cdot(\hat{k}_\perp + i\hat{k}_\perp\times\hat{z})]^n\overline{E}_{k\perp}(z') \tag{39}$$

If the contribution to the τ integral is substantial only for $|z'-z| \ll L$, where L is the axial scale length of equilibrium quantities and of $\overline{E}_{k_\perp}(z)$, then we can replace z' by z everywhere (and hence $\underline{v}'$ by $\underline{v}$, and $\underline{k}'_\perp$ by $\underline{k}_\perp$), and we recover the infinite medium result for modes with $k_\| = 0$. If $\omega \approx n\omega_{ci}$, however, it is necessary to keep the τ dependence in $\exp[-i\omega\tau + in\int_0^\tau d\tau'\omega_{ci}(\tau')]$, although we can still set $z' = z$ everywhere else. For this term, because only $|z'-z| \ll L$ contributes, we can make the approximation

$$\omega_{ci}(\tau) = \omega_{ci} + v_\|\tau\frac{d\omega_{ci}}{dz} + \frac{1}{2}v_\|^2\tau^2\frac{d^2\omega_{ci}}{dz^2}$$

With this set of assumptions, which is called the impulse approximation, Eq. (39) becomes

$$\epsilon_{xx,i} = \frac{4\pi i\omega_{pi}^2\omega_{ci}^2}{\omega k_\perp^2} \int_{-\infty}^{\infty} dv_\| \int_0^{\infty} dv_\perp v_\perp \frac{\partial f_i(v_\perp, v_\|)}{\partial v_\perp^2} \sum_{n=-\infty}^{\infty} n^2 J_n^2(\frac{k_\perp v_\perp}{\omega_{ci}})$$

$$\int_{-\infty}^{0} d\tau \quad \exp[i(n\omega_{ci}-\omega)\tau + \frac{1}{2}nv_\|\tau^2\frac{d\omega_{ci}}{dz} + \frac{i}{6}nv_\|^2\tau^3\frac{d^2\omega_{ci}}{dz^2}] \tag{40}$$

The $v_\|$ integral may be done analytically if the $v_\|$ dependence is $\exp(-v_\|^2/2\langle v_\|^2\rangle)$, independent of $v_\perp$. Although the actual $v_\|$ dependence is not of this form, we have treated each component s of ions (where $s = hot, hole,$ or $warm$) as if its $v_\parallel$ dependence is $\exp(-v_\parallel^2/2v_{\parallel s}^2)$, where $v_{\parallel,hot}^2$ and $v_{\parallel,hole}^2$ are both given by Eq. (18), and $v_{\parallel,warm}^2$ is given by Eq. (24). Then the contribution to $\epsilon_{xx,i}$ from a given ion component s is

$$\epsilon_{xx,s} = \frac{4\pi i \omega_{ps}^2 \omega_{ci}^2}{\omega k_\perp^2} \int_0^\infty dv_\perp v_\perp \frac{\partial f_{\perp s}(v_\perp)}{\partial v_\perp^2} \sum_{n=-\infty}^{\infty} n^2 J_n^2\left(\frac{k_\perp v_\perp}{\omega_{ci}}\right)$$

$$\int_{-\infty}^{0} d\tau \left(1 - \frac{in}{3} v_{\parallel s}^2 \tau^3 \frac{d^2\omega_{ci}}{dz^2}\right)^{-1} \exp\left[i(n\omega_{ci} - \omega)\tau - \frac{n^2}{8} v_{\parallel s}^2 \tau^4 \left(\frac{d\omega_{ci}}{dz}\right)^2 \left(1 - \frac{in}{3} v_{\parallel s}^2 \tau^3 \frac{d^2\omega_{ci}}{dz^2}\right)^{-1}\right] \quad (41)$$

If the terms involving $d\omega_{ci}/dz$ and $d^2\omega_{ci}/dz^2$ are neglected, then the τ integral just gives the resonant denominator, $-i(n\omega_{ci} - \omega)^{-1}$. In this case, $\epsilon_{xx,s}$ is purely real except at the points $\omega = n\omega_{ci}(z)$ where the wave interracts resonantly with the ions. When we include the $d\omega_{ci}/dz$ and $d^2\omega_{ci}/dz^2$, terms, the resonance is spread out over some characteristic width Δ_s, i.e. $\epsilon_{xx,s}$ has a significant imaginary part whenever $|\omega - n\omega_{ci}(z)| \lesssim \Delta_s$. The τ integral cannot be done analytically, but we have done it numerically for the two cases $d^2\omega_{ci}/dz^2 = 0$ and $d\omega_{ci}/dz = 0$. We find

$$\Delta_s \simeq (0.5 n v_{i\parallel} |d\omega_{ci}/dz|)^{\frac{1}{2}} \quad \text{for} \quad d^2\omega_{ci}/dz^2 = 0$$

$$\Delta_s \simeq (0.022 n v_{i\parallel}^2 |d^2\omega_{ci}/dz^2|)^{\frac{1}{3}} \quad \text{for} \quad d\omega_{ci}/dz = 0$$

In the general case where $d\omega_{ci}/dz \neq 0$ and $d^2\omega_{ci}/dz^2 \neq 0$, a good approximation to the resonance width Δ_s is given by

$$\Delta_s \simeq [0.5 n v_{i\parallel} |d\omega_{ci}/dz| + (0.022 n v_{i\parallel}^2 |d^2\omega_{ci}/dz^2|)^{\frac{2}{3}}]^{\frac{1}{2}} \quad (42)$$

Since it would be very time-consuming to compute the exact shape of the resonance function in Eq. (41), but the finite width has important physical consequences, we have approximated the τ integral in the code by $-i(n\omega_{ci} - \omega - i\Delta_s)^{-1}$, with Δ_s given by Eq. (42).

In going from Eq. (38) to Eq. (39), we have treated f_i as a function only of $v_\perp$ and $v_\parallel$, but in fact the ions have a net current in the diamagnetic direction, due to the pressure gradient, and this causes additional terms to appear in Eq. (40) if $\underline{k}_\perp$ has a non-vanishing component in the diamagnetic direction. In the case where there is only a density gradient (no temperature gradient), the $v_\perp$ integral can be done analytically. The result, which is used in the code, is

$$\epsilon_{xx,i} = \sum\nolimits_{s=hot,hole,warm} \frac{2\omega_{ps}^2}{k_\perp^2 v_{s\perp}^2} \sum\nolimits_{n=-\infty}^{\infty} (n\omega_{ci} - \omega - i\Delta_s)^{-1} I_n(k_\perp^2 v_{\perp s}^2/2\omega_{ci}^2)\exp(-k_\perp^2 v_{\perp s}^2/2\omega_{ci}^2)$$

$$[\omega + \omega_{pi}^{-2}(\underline{k}_\perp \times \hat{z}) \cdot \nabla\omega_{pi}^2(v_{\perp s}^2/2\omega_{ci} + n\omega/k_\perp^2)] \tag{43}$$

where Δ_s is given by Eq. (42), and $\omega_{ps}^2 = 4\pi n_s e^2/m_i$, with $n_{hole} = -n_{hot}\mu_{hole}/\mu_{hot}$, given by Eq. (11), and n_{warm} calculated numerically from Eqs. (21) and (20), as explained in Sec. II.

Equation (43) is only valid for growing modes ($Im\omega > 0$), or for modes with very slight damping rates, $|Im\omega| \ll \Delta$. If $Im\omega \lesssim -\Delta$, then the τ integral in Eq. (39) will not be dominated by $|\tau| \ll L/v_{i\parallel}$, but will be dominated by τ in the distant past, when the wave amplitude was much larger. In this case the impulse approximation is no longer valid, and the τ integral in Eq. (39) must be done exactly, including the bounce motion of the ions in $\omega_{ci}(\tau)$. Then $\epsilon_{xx,i}$ will be exponentially large, and will be a rapidly oscillating function of z. Any normal mode with $Im\omega \lesssim -\Delta$ will have very spiky eigenmodes $\overline{E}_{k\perp}(z)$, with the ions retaining detailed memory of their distant past history. Such eigenmodes would be almost impossible to calculate numerically, but fortunately theyt are not needed. If only the normal mode frequency ω is desired, but not $\overline{E}_{k\perp}(z)$, then the differential equation for a damped mode can be solved along a path in the complex z plane for which $Im(\omega - n\omega_{ci}(z)) > -\Delta$, but the code is not set up to do this at present. We have only looked at modes for which $Im\omega > 0$.

IV. Boundary Conditions for Axial Eigenmodes

To determine whether a given field line is locally stable, we solve the second-order differential equation, Eq. (37), along that field line, subject to appropriate boundary conditions at the inner and outer mirror throats. For a given $\underline{k}_\perp$ and a given field line, there will be only one value of complex ω, or a discrete set of values of ω, satisfying both boundary conditions. A field line is locally stable if $Im\omega \leq 0$ for all magnitudes and directions of $\underline{k}_\perp$.

The boundary conditions we have used are outgoing wave boundary conditions at both the inner and outer mirror throats. That is, we write Eq. (37) in the form

$$\frac{B}{k_\perp}\frac{\partial}{\partial z}\frac{1}{B}\frac{\omega_{pe}^2}{\omega^2}(1+\frac{\omega_{pe}^2}{k_\perp^2 c^2})^{-1}\frac{\partial}{\partial z}\frac{1}{k_\perp}\overline{E}_{k_\perp}+\frac{\omega_{pe}^2}{k_\perp^2\omega^2}(1+\frac{\omega_{pe}^2}{k_\perp^2 c^2})^{-1}Q(z)\overline{E}_{k_\perp}=0$$

which has the WKB solutions $k_\parallel(z) = \pm Q^{\frac{1}{2}}$. We then use the boundary conditions

$$\frac{d\overline{E}_{k_\perp}}{dz} = -i[Q(-z_{thr})]^{\frac{1}{2}}\overline{E}_{k_\perp} \quad \text{at} \quad z=-z_{thr} \quad \text{(the inner throat)}$$

and

$$\frac{d\overline{E}_{k_\perp}}{dz} = +i[Q(z_{thr})]^{\frac{1}{2}}\overline{E}_{k_\perp} \quad \text{at} \quad z=+z_{thr} \quad \text{(the outer throat).}$$

These boundary conditions, which are the usual ones used in studying ion loss-cone instabilities, correspond to outgoing wave energy at the mirror throats. (This would be true even if the wave energy were negative at the mirror throats, since negative energy waves would have group velocity in the direction opposite to their phase velocity; outgoing phase velocity would then correspond to ingoing group velocity, and to outgoing wave energy.) The usual justification for using outgoing wave boundary conditions at the outer mirror throat is that, for cold electrons, the wave becomes a Langmuir wave in the limit of very low density; that the WKB approximation becomes more and more valid as we get closer to the outer mirror throat (i.e., $k_\parallel(z)|z-z_{thr}| \to \infty$ as $z \to z_{thr}$); and that eventually, very close to the outer mirror throat, finite electron temperature becomes important, and the wave energy is absorbed by electron Landau damping. For the TARA ion anchor, with $T_e \simeq 500$eV, the situation is quite different. Finite electron temperature effects would be important at some distance away from the mirror throat, where WKB is not valid. In these circumstances, the WKB approximation would not become more valid as we approach the outer mirror throat; rather, WKB would never be valid, and it would be necessary to solve an integral equation to find the appropriate boundary conditions. However, an outgoing wave boundary condition can be justified at the outer mirror throat of the TARA ion anchor if there is a steam of warm plasma flowing past the potential peak and out through the outer mirror throat. We would expect such a stream to come from the central cell, with a current of about 200 A. If its streaming velocity at the outer mirror throat is given by the drop in potential from the potential peak to the outer mirror throat (about 5 T_e), then its density at the outer mirror throat would be about 10^{11} cm^{-3}, or about 2% of n_{max}, the peak hot ion density. The perpendicular temperature of the stream would be about 400 eV, the central cell temperature. The contribution of these steaming ions to the perpendicular ion dielectric function $\epsilon_{xx,i}$ would always be small compared to either the hot ion contribution or the vacuum term, for any of the unstable modes found, so the stream ions would not contribute directly to the dispersion relation. But they would keep the electron density from going below 2% of n_{max}, even at the outer mirror throat. Then $k_{\|}(z)$ would not keep increasing as we approach the outer mirror throat, but would remain less than ω/v_e, and finite electron temperature effects will remain unimportant at the outer mirror throat and beyond. The wave will become a Langmuir wave, which will travel out past the outer mirror throat with the stream. Due to various damping processes outside the outer mirror throat (e.g. slight Landau damping by the tail of the electron distribution, cyclotron harmonic damping by the stream ions, collisional damping in a cool sheath near the end wall) the wave will not be able to reflect and come back into the anchor cell, so an outgoing wave boundary condition is justified at the outer mirror throat.

Near the inner mirror throat, the WKB dispersion relation would be

$$k_{\|}^2 \simeq \frac{k_\perp^2 \omega^2}{\omega_{pe}^2}\left(1 + \frac{\omega_{pe}^2}{k_\perp^2 c^2}\right)\epsilon_{xx,i}$$

if $k_{\|} \ll \omega/v_e$. The ion term $\epsilon_{xx,i}$ is dominated by the warm ions; in order of magnitude, $\epsilon_{xx,i} \simeq \omega_{pi}^2/k_\perp^2 v_w^2$. For the unstable modes which we found, $\omega_{pe}^2/k_\perp^2 c^2 \leq 2$ at the midplane, so $\omega_{pe}^2/k_\perp^2 c^2 \ll 1$ near the mirror throats, where $k_\perp$ is greater due to the fanning and the greater magnetic field. Then

$$k_{\|}^2 \simeq \frac{\omega_{pi}^2}{\omega_{pe}^2}\frac{\omega^2}{v_w^2} \simeq \frac{7}{2}\frac{\omega^2}{v_e^2}\frac{T_e}{\phi_{max}} - \phi(z)(1 - \frac{B_{min}}{B(z)})$$

where we have used Eq. (23) for v_w^2. Since, from Eq. (22), $[\phi_{max} - \phi(Z)]/T_e = \ell n[n_{max}/n(z)] \lesssim 1$ for a reasonable amount of warm plasma near the inner mirror throat, we would always have $k_{\|} > \omega/v_e$ near the inner mirror throat, in fact anywhere where the warm ions dominate $\epsilon_{xx,i}$. This means that the electrons cannot be treated as cold in the part of the anchor where the warm ions dominate, and Eq. (37) is not valid in this region. Instead, the term $-k_{\|}^2\omega_{pe}^2/k_{\perp}^2\omega^2$ should be replaced by $(\omega_{pe}^2/k_{\perp}^2 v_e^2)Z'(\omega/k_{\|}v_e)$ if WKB is valid; if WKB is not valid, the second-order differential operator in Eq. (37) must be replaced by the appropriate integral operator. This would make the solution of the dispersion relation much more difficult.

Instead, we note that the main physical effect of finite electron temperature ought to be strong electron Landau damping in the entire region where the warm ions dominate. There is already strong damping in this region due to cyclotron harmonic resonance with the warm ions, since $\omega/k_{\perp}v_w \sim 1$, so modes must be localized away from this region in order to be unstable; this may be seen in the normal modes $\overline{E}_{k_{\perp}}(z)$ plotted in Fig. 3, for example. Therefore, we do not expect the electron Landau damping to have much effect on these modes. To avoid having to solve an integral equation, we have used the cold electron dispersion relation Eq. (37), and we have arbitrarily used an outgoing wave boundary condition at the inner mirror throat (as well as at the outer mirror throat). The results ought to be quite insensitive to the boundary condition at the inner mirror throat; if they are not, then we should not be using Eq. (37) at all, but should be using the integral equation. It is likely that some of the less axially localized modes, such as that shown in Fig. 3a, will be stabilized by electron Landau damping; it is also possible that there are negative energy modes which are destabilized by electron Landau damping, although this seems unlikely because of the large width of the ion cyclotron harmonic resonances. We plan to explore these possibilities in the future; for the results described in this paper, however, we have used the cold electron dispersion relation, Eq. (37), with outgoing wave boundary conditions at both mirror throats.

Local Stability Results

We have examined an equilibrium with a circular plasma cross-section, of radius 7.5 cm, at the midplane, i.e., $n_{hot}(\alpha,\beta) = n_0 \exp[-(\alpha^2+\beta^2)/(7.5cm)^2]$, with n_{warm} having the same radial dependence as n_{hot}. The hot ion Larmor radius at the field minimum, ρ_{i0}, is 4 cm, the peak β_i is 0.15, T_e is 500 eV, and the parameter T_w used in Eq. (19) is 1 keV (but, as pointed out in Sec. II, the warm ion distribution is insensitive to T_w, except very near the field minima).

In our local stability study, we have concentrated on the field line $\alpha = 2$cm, $\beta = 0$. We expect this to be more unstable than the center line ($\alpha = 0, \beta = 0$), because of the additional free energy available from the density gradient. Further from the center line the density gradient is even greater, but the assumption that the ion Larmor radius is small compared to the radial scale length, used in deriving the ion density gradient terms in Eq. (42), starts to break down, and the results are less reliable. For field lines still further from the center line, the density is much lower, and these field lines should be more stable. We have calculated some normal modes for field lines 3 or 4 cm from the central line; we find somewhat greater growth rates than for the field line at 2 cm, but qualitatively similar behavior.

In an axisymmetric equilibrium, the most unstable modes have $\underline{k}_\perp$ in the azimuthal direction, since such modes can make full use of the free energy of the radial gradients. With quadrupole fields, the situation is more complicated, because $k_\perp(z)$ depends strongly on the direction of $\underline{k}_\perp$ at the midplane, due to the dependence of the fanning factor $\nabla\alpha/\nabla\beta$ on z. For the field line at $\alpha = 2$ cm, $\beta = 0$, we have found that the most unstable modes generally have a radial component of $\underline{k}_\perp$ comparable in magnitude to the azimuthal component at the midplane.

In Fig. 4, we show the normal mode frequencies and growth rates as a function of k_β (the azimuthal component of $\underline{k}_\perp$ at the midplane), holding k_α (the radial component) constant at 1 cm^{-1}, i.e., $k_\alpha \rho_{io} = 4$. The warm plasma density at the inner mirror throat was 0.5 n_{max} for Fig. 4a, and 0.8 n_{max} for Fig. 4b (the value of n_{warm} in the equilibrium shown in Fig. 1). Only modes with $Im\omega > 0$ are shown, since, as discussed in Sec. III, the dispersion relation is only valid for $Im\omega > 0$.

As expected, the high density of warm plasma in the inner part of the anchor precludes the existence of unstable flute-like DCLC modes, with $k_\parallel L \gg 1$. Roughly, we would expect unstable ALC modes (or DCLC modes with $k_\parallel L \gtrsim 1$) to occur when $\omega/k_\perp v_i \lesssim 0.3$, at $k_\parallel L_0 = 1, 2, 3 \cdots$, with L_0 the length of the outer part of the anchor, where there is relatively little warm ion density, and $k_\parallel$ given by Eq. (3), evaluating $k_\perp, \omega_{pe}$, and ω_{pi} at some point in the outer part of the anchor. This rough estimate, which ignores the effects of ion cyclotron harmonic structure and the dependence of the shape of the eigenmode on $k_\perp$ and ω, gives a fairly good fit to the modes shown in Fig. 4, if we take $k_\parallel L_0 = 1$ and $L_0 = 60$ cm. (Unstable modes with $k_\parallel L = 2, 3, \cdots$, probably exist at higher frequencies, $\omega/\omega_{ci} > 10$, which we did not examine since we only kept terms up to $n = 10$ in the sum over ion cyclotron harmonics in Eq. (43).) The growth rates of these modes are considerably reduced from those of ALC modes in the absence of warm plasma, which have growth rates γ comparable to the real frequency ω_r. For the modes shown in Fig. 4, γ is only a few percent of ω_r. These reduced growth rates are due to the fact that the modes cannot localize entirely away from the warm ions. For modes with $\gamma \lesssim 0.3\omega_{ci}$, the ion cyclotron harmonic structure becomes important. The damping due to warm ions is much stronger at those axial positions for which $n\omega_{ci}(z) \simeq \omega$. Since the cyclotron resonance is strongest at the midplane, where $d\omega_{ci}/dz = 0$, unstable modes do not occur at $\omega = n\omega_{ci0}$ (where ω_{ci0} is the midplane value of ω_{ci}), but occur only in bands of ω around $(n + \frac{1}{2})\omega_{ci0}$, when $\gamma \lesssim 0.3\omega_{ci}$. The cyclotron harmonic structure also affects the shape of the axial eigenmode $\bar{E}_{k\perp}(z)$, and this influences stability. For example, $\bar{E}_{k\perp}(z)$ for the mode at $\omega_r = 3.7\omega_{ci0}$ and $k_\beta \rho_{i0} = 11.3$ is shown in Fig. 5a. This mode has a large amplitude in the inner part of the anchor, where there is a large fraction of warm ions, and we might expect it to be strongly damped; however, it manages to be marginally unstable because it has a node at $z = -37$ cm, where $\omega = 3\omega_{ci}(z)$, which is the only cyclotron harmonic resonance present. Such unstable modes can only exist in narrow bands of $k_\perp$. At higher frequencies, where $\gamma \gtrsim 0.5\omega_{ci}$ and the effect of ion cyclotron harmonics is smeared out, unstable modes must be localized more in the outer part of the anchor, away from the warm ions. A plot of $\bar{E}_{k\perp}(z)$ for a typical mode of this type is shown in Fig. 5b. As might be expected, such modes cannot be easily stabilized by increasing the density of warm ions in the inner half of the anchor, since the mode is localized almost entirely in the region where there are no ions. A comparison of Figs. 4a and 4b shows that $\omega_r(k_\beta)$ and $\gamma(k_\beta)$ are almost the same for the two cases $n_{warm}(-z_{thr}) = 0.8n_{max}$ and $n_{warm}(-a_{thr}) = 0.5n_{max}$ for $k_\beta \rho_{io} \gtrsim 60$. The modes in Fig. 4a at smaller k_β, which are less well localized away from the warm ions, are easily stabilized by adding more warm plasma; these modes are stable, or have greatly reduced growth rates in Fig. 4b.

These results for local stability are consistent with recent results of Pearlstein, who used a similar axial eigenmode code to find the amount of warm plasma needed to stabilize DCLC and ALC in the anchor of MFTF-B. Pearlstein's equilibrium includes hot electrons localized near the midplane, which cause a large dip in ambipolar potential, allowing more warm ions to be trapped near the midplane, and also has an excess of hot electrons over hot ions near the outer mirror throat, making the ambipolar potential negative there, and allowing warm ions to be trapped near the outer mirror throat. With this equilibrium, all ion loss-cone modes (DCLC and ALC) become stable with a warm plasma density at the midplane which is about 75% of the hot ion density. In our equilibrium, there is no warm plasma near the outer mirror throat, and the warm plasma density at the midplane is only about 30% of the hot ion density, so it is not surprising that some modes remain unstable.

The modes that remain unstable when $n_{warm}(-z_{thr}) = 0.8n_{max}$ all have either very short perpendicular wavelengths or very small growth rates; they all have $\gamma/k_\beta^2 \lesssim 3 \times 10^{-4}\omega_{ci}\rho_i^2$, which means $\gamma/k_\perp^2 \lesssim 10^{-4}\omega_{ci}\rho_i^2$ at the axial position where the mode is localized (since $k_\perp \gtrsim 2k_\beta$ there, due to fanning). If $\gamma/k_\perp^2$ is taken as an estimate of the cross-field diffusion that the mode would cause when it is nonlinearly saturated, then these unstable modes may not cause much degradation in confinement.

Finally, it is possible that the unstable modes found locally on certain field lines may become stable when the mode structure across field lines (the radial and azimuthal mode structure) is taken into account. This may happen if the wave energy is convected away radially at a greater rate than energy is going into the mode from the free energy of the ion loss-cone. (It is also possible that radial convection of wave energy may destabilize a negative energy wave.) This possibility will be examined in Sec. VI.

If the local instabilities which we have found saturate nonlinearly at a low level, or are stabilized by radial convection of wave energy, then the TARA ion anchor with belly band should perform satisfactorily as far as ion loss-cone instabilities are concerned. If the nonlinear behavior is worse than expected from $\gamma/k_{\perp}^2$, and radial convection does not substantially reduce the growth rate, then confinement in the ion anchor may be degraded by ion loss-cone instabilities. In this case, it may be necessary to use ECRH to deepen the ambipolar potential dip, allowing more warm to be trapped, as in the ion anchor design investigated by Pearlstein for MFTF-B. Otherwise, it may be necessary to use a hot electron anchor. Hot electron anchors must also have moderately hot ions (~ 1 keV), which would have a loss-cone and be subject to the DCLC or ALC instabilities, but the density would be lower than in the ion anchor, and the critical length for ALC would be longer, according to Eq. (4). So it is likely that the warm plasma trapped on the inner side of the ambipolar peak will be enough to stabilize all ion loss-cone instabilities in a hot electron anchor.

Appendix A

Given the perpendicular distribution function $f_\perp(\mu, B)$, defined by

$$f_\perp(\mu, B) = B\int_{\mu B}^{\mu B_{max}} dH(H-\mu B)^{-\frac{1}{2}} f(H,\mu) \tag{A1}$$

we wish to find $f(H,\mu)$. The perpendicular distribution is normalized to the density $n_{hot}(B)$, i.e.,

$$n_{hot}(B) = \int_0^\infty d\mu f_\perp(\mu, B) \tag{A2}$$

To do this, we expand $f(H,\mu)$ and $f_\perp(\mu,B)$ in Laguerre polynomials of μ/μ_{hole},

$$f(H,\mu) = \pi^{-1}H^{-\frac{3}{2}}(\mu/\mu_{hole}) \quad \exp(-\mu/2\mu_{hole})\sum_{i=0}^{\infty} L_i(\mu/\mu_{hole})N_i(\mu/H) \tag{A3}$$

$$f_\perp(\mu,B) = \exp(-\mu/2\mu_{hole})\sum_{i=0}^{\infty} L_i(\mu/\mu_{hole})C_i(B) \tag{A4}$$

Since $\int dxe^{-x}L_i(x)L_j(x) = \delta_{ij}$, we can put Eq. (A3) and (A4) into Eq. (A1), multiply each side by $\exp(-\mu/2\mu_{hole})$, integrate over μ and change variables of integration from H to $\nu \equiv \mu/H$, to obtain

$$\pi^{-1}B^{\frac{1}{2}}\int_{B_{max}^{-1}}^{B^{-1}} d\nu(B^{-1}-\nu)^{-\frac{1}{2}}N_i(\nu) = C_i(B) \tag{A5}$$

for all positive integers i. The problem then reduces to that of inverting Eq. (A5) to find $N_i(\nu)$ in terms of $C_i(B)$; Eq. (A3) can then be used to find $f(H,\mu)$ in terms of $C_i(B)$, and Eq. (A4) can be easily inverted to find $C_i(B)$ in terms of $f_\perp(\mu,B)$. The inversion of Eq. (A5) is

$$N_i(\nu) = \left[\frac{d}{dx}\int_{x^{-1}}^{B_{max}} dB \quad B^{-\frac{5}{2}}(x-B^{-1})^{-\frac{1}{2}}C_i(B)\right]_{x=\nu} \tag{A6}$$

where

$$C_i(B) = \int_0^\infty d\mu \quad \mu_{hole}^{-1} \quad \exp(-\mu/2\mu_{hole}) L_i(\mu/\mu_{hole}) f_\perp(\mu, B) \qquad (A6)$$

For the form of $f_\perp(\mu,B)$ given by Eq. (10), we find

$$C_0(B) = n_{hot}[\mu_{hot}/\mu_{hole} + 1]^{-1} \qquad (A7a)$$

and

$$C_i(B) = 2n_{hot}\mu_{hot}\mu_{hole}(-1)^i(\mu_{hot} - \mu_{hole})^{i-1}(\mu_{hot} + \mu_{hole})^{-i-1} \text{for} i \geq 1 \qquad (A7b)$$

Using Eq. (11) for μ_{hot}, Eqs. (A5) through (A7) yield

$$N_0(\nu) = [\frac{d}{dx}\int_{x^{-1}}^{B_{max}} dB \quad n_{hot}(B) \frac{B^{-\frac{5}{2}}(x - B^{-1})^{-\frac{1}{2}}(1 + aT_\parallel/T_\perp - B_{min}/B)}{2(1 + aT_\parallel/T_\perp) - B_{min}(B^{-1} + B_{max}^{-1})}]_{x=\nu} \qquad (A8a)$$

$$N_i(\nu) = (-1)^i 2[\frac{d}{dx}\int_{x^{-1}}^{B_{max}} dB \quad n_{hot}(B) B^{-\frac{5}{2}}(x - B^{-1})^{-\frac{1}{2}}(1 + aT_\parallel/T_\perp - B_{min}/B)$$

$$\frac{(1 + aT_\parallel/T_\perp - B_{min}/B_{max})B_{min}^{i-1}(B^{-1} + B_{max}^{-1})^{i-1}}{[2(1 + aT_\parallel/T_\perp) - B_{min}(B^{-1} + B_{max}^{-1})]^{i+1}}]_{x=\nu} \quad \text{for} \quad i \geq 1 \qquad (A8b)$$

Equations (A3) and (A8) are used by the code FOFHMU to find $f(H,\mu)$ and to make a contour plot of it, using a, $T_\parallel/T_\perp$, and B_{max}/B_{min} as input. For $n_{hot}(B)$, the code uses Eq. (8) or Eq. (9) for $B < B_{bump}$, depending on whether we are looking at $z > 0$ or $z < 0$. Typical results are shown in Fig. 2.