Theory of Electron Cyclotron Heating in the Constance II Experiment

by
Michael E. Mauel

*February 25, 1981**
Plasma Fusion Center
Research Lab of Electronics
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Abstract

The bounce-averaged quasi-linear equation for a non-relativistic mirror-confined plasma interacting with electromagnetic waves is derived for use in the study of ECRH of the Constance II mirror experiment. The derivations follows the more formal examples given by Berk[1] for electostatic waves and Bernstein and Baxter[2] for relativistic plasmas. The validity of the theory is discussed by examining individual particle orbits in an EM field. The local dispersion relation is found while deriving a self-consistent WKB theory which can be used to estimate the power transferred from the launching horn to the plasma.

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1. Introduction .

Current plans to test tandem-mirror reactors require electron cyclotron heating to maintain the temperature difference between the central-cell and plug electrons[3,4,5]. Bulk heating will be applied at the fundamental cyclotron resonance to raise the confining potential of each plug. Second harmonic heating will create a hot-electron thermal barrier which should insulate the plug from the central-cell electrons. In both cases, ECRH is used to control the development of the electron distribution function. To be efficient, the bulk heating must guard against tail heating, and the barrier heating must not permit "hot tail runaway"[3]. ECRH has never been used in mirrors for these applications, and, for this reason, Constance II is conducting experiments to study the development of the electron energy distribution with ECRH[6].

This report derives the ECRH theory that will be used to analyze the data. The theory consists of two parts: (1) the derivation of the correct expression for the resonant energy exchange between the waves and particles, and (2) the WKB theory for the propagation of the wave energy from the launching horns to the absorption layers. On the average, an electron gains energy from the waves only at a few local resonances along its orbit. For collisionless particles, low electric fields, and narrow bandwidths, the particle's gyrophase with respect to the wave frequency is not random, and the electrons are purely reactive. As the electric field increases, the bounce resonances overlap resulting in stochastic energy diffusion[7]. Lieberman and Lichtenberg[8] were the first to derive the diffusion equation for a uniform, stationary electric field. Berk[1] was the first to derive a self-consistent bounce-averaged quasi-linear theory which also included the correct WKB theory for the propagation of electrostatic waves. Bernstein and Baxter[2] were the first to extend the theory to relativistic plasmas and to electromagnetic waves. Finally, Porkolab, *et al.*[9] first performed ray tracing calculations for ECRH in mirrors.

The contents of the report are organized into eight sections and an introduction. First, the geometry of the particle orbits are discussed. Then, an expression for the diffusion equation is intuitively derived for an imposed electromagnetic wave. The fourth section discusses the conditions for the validity of the linear, stochastic theory. And, in the fifth section, the bounced-averaged quasi-linear theory is derived for non-relativistic particles. The sixth section is devoted to the bounce-averaged resonance function used in the quasi-linear theory. Next, the WKB theory for wave propagation is derived from the requirement of energy conservation. This gives the geometric and physical optics solutions to the problem of estimating the field intensity and power absorption at the resonance zones. Finally, the last two sections describe the local resonance function used in the WKB theory and the bounce-averaged energy conservation equation.

2. Geometry .

For simplicity, the geometry of the plasma used in the kinetic theory is assumed to be locally cylindrical. Non-axisymmetric effects are ignored, and only trapped, bouncing particles are treated. The unperturbed orbits used to describe the trapped electrons, bouncing in the mirror, are

$$s = s_m \cos(\omega_B t + \psi) \tag{1}$$

$$y = Y(0) + \rho(s)\sin\left(\int_0^t \omega_c dt' + \phi\right) + \int_0^t \mathrm{V}_D \cdot \hat{y}\, dt' \tag{2}$$

$$x = X(0) + \rho(s)\cos\left(\int_0^t \omega_c dt' + \phi\right) + \int_0^t \mathrm{V}_D \cdot \hat{x}\, dt' \tag{3}$$

where $\rho^2(s) = 2B(s)\mu/\omega_c^2(s)$ and V_D is the sum of the ∇ B and curvature drifts. If the particles are deeply trapped, then $\omega_c(s) = \omega_{co}(1 + s^2/L_B^2)$, $\omega_B = V_{\perp,0}/L_B$, and $s_m = V_{\|,0}/\omega_B$. In general, $B(s)$ is not parabolic, so that ω_B is also a function of $v_\|$ and s. For combination electrostatic and magnetic wells, ω_B also depends upon $\Phi(s)$. A particle's phase-space is designated by the variables $(E, \mu, \psi, \phi, X, Y)$ or equivalently $(E, \mu, \phi, \mathrm{R})$, where (X, Y, s) represents the particle's guiding-center position on its drift surface, and $\mathrm{R} = X\hat{x} + Y\hat{y} + s\hat{s}$. ψ is the bounce angle and ϕ is the gyro-phase. The total energy, E, the magnetic moment, μ, and drift surface, $\overline{X}^2 = X^2 + Y^2$, are constants of motion. The velocity gradient and total time derivative are

$$\frac{\partial}{\partial \mathrm{v}} = \frac{\mathrm{v}_\perp - \mathrm{V}_D}{B}\frac{\partial}{\partial \mu} + \mathrm{v}\frac{\partial}{\partial E} + \frac{\hat{\phi}}{v_\perp}\frac{\partial}{\partial \phi} + \frac{1}{\omega_c}\hat{\mathfrak{b}} \times \frac{\partial}{\partial \overline{\mathrm{X}}} \tag{4}$$

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + v_\| \frac{\partial}{\partial s} + \mathrm{V}_D \cdot \nabla + \omega_c \frac{\partial}{\partial \phi} \tag{5}$$

The gradient term in Equation 4 can be written as $(\hat{v}_D/\omega_c)\partial/\partial\overline{X}$ with $\hat{v}_D$ defined as $\hat{b} \times \hat{\bar{x}}$. Furthermore, if $\overline{\Theta}$ is defined as $\tan(Y/X)$, then $\mathrm{V}_D = \hat{v}_D \omega_D \overline{X}$ and $\mathrm{V}_D \cdot \nabla = \omega_D \partial/\partial\overline{\Theta}$ where $\omega_D = d\overline{\Theta}/dt$. The average particle distribution is assumed to be independent of ϕ, ψ, and $\overline{\Theta}$. This simple geometry is adequate for the kinetic theory presented here since the resonant particle effects ultimately depend only upon local gradients. See Horton, *et al.*[10] for a formal derivation of a particle's motion in a mirror.

3. A Monochromatic Wave .

The perturbations to this orbit due to an electromagnetic wave can be analyzed in a manner similar to Jaeger, *et al.*[11]. Consider an electric field, constrained to be

$$\mathrm{E} = \mathrm{E}_k \exp\left(-i\omega t + i\mathrm{k}\cdot\mathrm{x}\right)$$

and

$$\frac{dE}{dt} \approx B\frac{d\mu}{dt} = \frac{q}{m}\mathrm{E}_\perp \cdot \mathrm{v}_\perp \tag{6}$$

where electron Landau damping, for cases when $E_\parallel \neq 0$, is ignored. To solve Equation 6, the right hand side is integrated along a particle's orbit,

$$\Delta\mu_{bounce} = -\frac{q}{m}E_k \sum_n \oint_{bounce} J'_n(k_\perp \rho)\frac{\rho\omega_c}{B}e^{in\phi}\exp\left(-i\int_0^t dt'\nu_n(t')\right)dt \tag{7}$$

where

$$\nu_n(t') = \omega - n\omega_c(t') - k_\parallel v_\parallel(t') - \mathrm{k}_\perp \cdot \mathrm{V}_D(t') \tag{8}$$

and where $\hat{y}$-axis has been aligned with the (assumed to be linearly-polarized) electric field and $\mathrm{k}_\perp = k_\perp \hat{x}$. The primed bessel function means differentiation with respect to its argument, or $J'_n = J_{n-1} - (n/k_\perp\rho)J_n$. For $k_\perp\rho \ll 1$ and $n = 1$, the primed bessel function is approximately $\sim 1/2$. When evaluating Equation 7, it is assumed that $|\Delta\mu| \ll \mu$ since only the first-order change to the unperturbed orbit is evaluated.

Since the integrand is highly phase dependent, the largest contributions to the integral arise when $\nu(t^*) \approx 0$. For parabolic, magnetic well and for $\mathrm{V}_D = 0$, this is when

$$\delta\omega_{n0} - n\omega_{c0}\frac{s^{*2}}{L_B^2} - k_\parallel v_\parallel^* \approx 0 \tag{9}$$

where $\delta\omega_{n0} = \omega - n\omega_{c0}$, $s^* = s(t^*)$, and $v_\parallel^* = v_\parallel(t^*)$. Figure 1 illustrates $(v_\parallel, s)$ phase space for $\mathrm{V}_D = 0$, fixed $V_{\perp,0}$ and for $k_\parallel$, $\delta\omega_{n0} \neq 0$. Particles with $V_{\parallel,0} < \delta\omega_{n0}/k_\parallel$ have four stationary points. Particles with larger $V_{\parallel,0}$ have only two resonance points, and those with $V_{\parallel,0} = \delta\omega_{n0}/k_\parallel$ have three. For each resonance crossing, the net change in the magnetic moment will be proportional to the product of an effective time in resonance and a phase dependent term. To calculate the interaction time several cases are considered. When the stationary points are well separated, then $\nu(t') \approx (t' - t^*)\nu'(t^*)$, and the integral can be approximated by the leading term

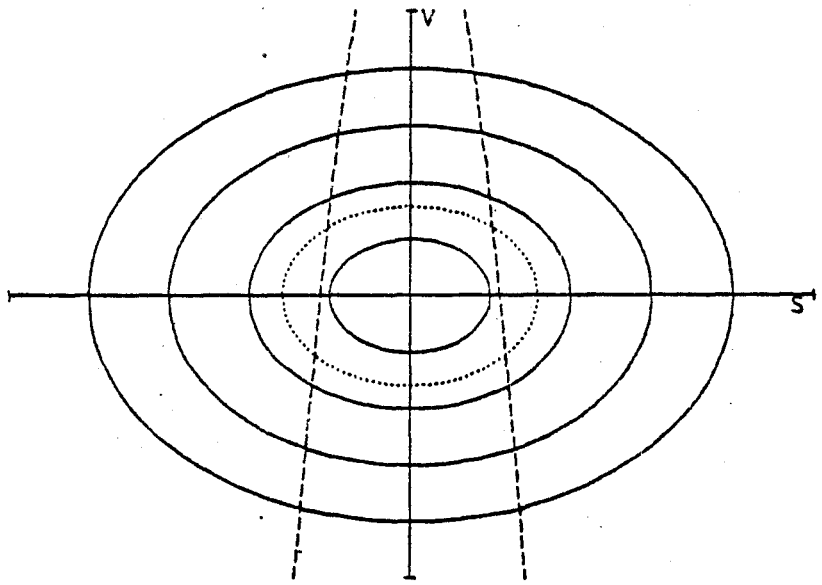

Figure 1. Illustration of resonance points along bounce orbits of the particles in a parabolic well. Verticle axis is $V_{\|,0}$, horizontal is s_m. For $V_D = 0$, fixed $\omega_B = 50 Mrad/s$, $N_\| = 0.4$, $\omega_c 0 = 32 Mrad/s$, and for $\delta\omega_{n0} \neq 0$. This corresponds to $4kev$ electrons in Constance II. The dashed lines are the resonance points, and the dotted line is the boundary between the $p < 0$ and $p > 0$ bounce-resonances.

$$Re\{\Delta\mu^*\} \approx -\frac{q}{m} E_k J'_n(k_\perp \rho) \frac{\rho\omega_c}{B} \pi^{1/2} \tau_{eff} \sin(n\phi + \pi/4) \tag{10}$$

In this case, $\tau_{eff}^{-2} = \nu(t^*)/2$. All of the slowly varying quantities are evaluated at t^*. When $\nu(t^*) < 0$, then the phase of the argument of sine changes by $\pi/2$.

When two successive resonances are separated by a time of the order of τ_{eff}, then $\nu(t^*) \approx 0$, and the approximation leading to Equation 10 breaks down. In this case, $\nu(t') \approx \nu(t^*) + (t' - t^*)^2 \nu''(t^*)/2$ and

$$Re\{\Delta\mu_*\} \approx -\frac{q}{m} E_k J'_n(k_\perp \rho) \frac{\rho\omega_c}{B} 2\pi\tau_{eff} \sin(n\phi + \pi/2) Ai(\nu^* \tau_{eff}) \tag{11}$$

where, now, $\tau_{eff}^{-3} = \nu''(t^*)/2$ and Ai is the Airy function. When ν'' is negative, the real branch is used for τ_{eff}. For the parabolic well illustrated in Figure 1, $\nu' = 0$ when $s^* = 0$ and $v_\|^* = k_\| \omega_B^2 L_B^2 / 2n\omega_{c0}$. Notice

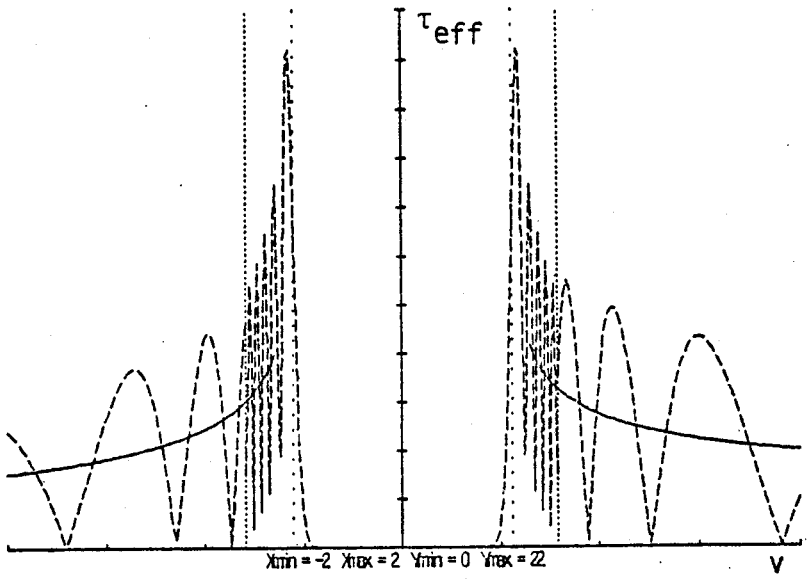

Figure 2. The magnitude of the the effective time in resonance per half-bounce as a function of $V_{\|,0}$ for the case shown in Figure 1. Verticle scale is $10^{-9} sec$. The dashed line is the Airy approximation for the interaction time. The solid line is the uncorrelated sum of the stationary phase approximation. The dotted lines show the $p = 0$ bounce resonance and the turning-point resonance.

that Equations 10 and 11 are identical in form, the only difference being that the effective time spent in resonance is redefined from $(\nu^{*\prime}/2\pi)^{-1/2}$ to $2\pi(\nu^{*\prime\prime}/2)^{-1/3}Ai[\nu^*(\nu^{*\prime\prime}/2)^{-1/3}]$. In fact, Berk[1] used the Airy function approximation at $s^* = 0$ for all particles, since he considered only $\delta\omega_{n0} \approx 0$ and $k_\| = 0$. A more general approximation is given by the rules: (1) when $v_\|^* < \delta\omega_0/k_\|\omega_B$, then expand about $s^* = 0$ for the particles with $|V_{\|,0}| > \min\{|\delta\omega_0/k_\|\omega_B|,\ V_{\perp,0}(2\delta\omega_0/n\dot{\omega}_{c0})^{1/2}\}$ and expand about $v_\|^* = k_\|\omega_B^2 L_B^2/2n\omega_{c0}$ for the remaining particles, and (2) when $v_\|^* > \delta\omega_0/k_\|\omega_B$, then expand about $s^* = 0$ for all particles. Finally, it should be noted that both approximations breakdown when $\nu = \nu' = \nu'' = 0$. This happens when $V_{\|,0} = \delta\omega_{n0}/k_\| = v^*$. In this case, the time integral is proportional to $\Gamma(1/4)\tau_{eff}\sin(n\phi + 3\pi/8)$, where $\tau_{eff}^{-4} = \nu'''(t^*)/3$. Figure 2 summarizes these last two paragraphs. Here, the effective time in resonance is plotted for the particles shown in Figure 1. The oscillations shown in the figure result from retaining the phase information between two successive resonances for the Airy approximations. For some particles, the two interactions cancel. Both of the Airy approximations, at $s^* = 0$ and $v_\|^* = k_\|\omega_B^2 L_B^2/2n\omega_{c0}$ are included. The figure corresponds to $n\omega_{c0}/\omega_B \sim 600$ which corresponds to $4kev$ electrons in Constance II.

4. Conditions for Stochasticity .

As explained by Lieberman and Lichtenberg[8], when the phase, ϕ, at each resonance crossing is random and with $\Delta\mu \ll \mu$, then the magnetic moment under goes stochastic diffusion. $\langle\Delta\mu\rangle = 0$ and $D_\mu \sim \sum(\Delta\mu^*)^2/\tau_B$. The diffusion equation is

$$\frac{\partial f_0}{\partial t} = \frac{1}{2}\frac{\partial^2}{\partial\mu^2} D_\mu f_0(\mu, t) \tag{12}$$

and

$$D_\mu = \sum_{res} \frac{q^2}{m^2}|E|^2 \tau_{eff}^2 \frac{\omega_c^2\rho^2}{B^2} J_n'^2 \tag{13}$$

Note that, from Equation 6, the diffusion paths in (E, μ) phase space is given by $D_E \sim B^{*2} D_\mu$. When ϕ is not random, the electrons are superadiabatic[7], and no average energy is exchanged between the waves and the particles.

Three conditions may make ϕ random: (1) collisions, (2) the presence of many, uncorrelated waves, or (3) the overlap of the wave-particle bounce resonances. For electron temperatures greater than $\sim 200eV$, collisions will induce diffusion on a time scale $t > 10\tau_B$ which is usually the weakest of the three effects. Since the bandwidth of typical microwave sources is $\Delta\omega/\omega \sim 10^{-4}$, the statement "many uncorrelated waves" refers to a broad k-spectrum. However, since the RF is launched from a single horn, k(r) is fixed by geometry, and the k-spectrum is not broad if the power is absorbed on the first pass. Note, that even though the resulting resonances for each particle do not look like those in Figure 1 (since k(r) is a function of position) each particle still experiences a finite number of distinct resonances, and, in general, stochastic diffusion will not result. (Of course, density and temperature fluctuations change the ray geometry, but these effects are usually slow compared to a bounce time for moderately energetic electrons.) On the other hand, when the first pass absorption is poor, the microwaves will bounce several times within the over-moded vacuum chamber. Now, the k-spectrum would be very broad, and ϕ should be random. Figure 3 illustrates the many resonances for a weakly absorbant plasma.

The final condition for stochastic diffusion is the overlap of bounce resonances. For Constance II and other mirror ECRH experiments, this is the major justification for the use of the quasi-linear equation, since high, first-pass absorption is expected. To estimate the size of the electric field producing stochastic orbits, the particle motion in (μ, ϕ) space can be written by a set of coupled, difference equations, and KAM theory (Kolmogorov-Arnold-Moser), as summarized by Lichtenberg[12], can be applied. In general, the difference equation is fourth-order. However, when both $\delta\omega_{n0} \approx 0$ and $V_{\|,0} < \delta\omega_{n0}/k_\|$, then for most particles $\nu^* \sim \nu^* \sim 0$, and Equation 12 can be used for $\Delta\mu$ for each pass through the midplane. The effective

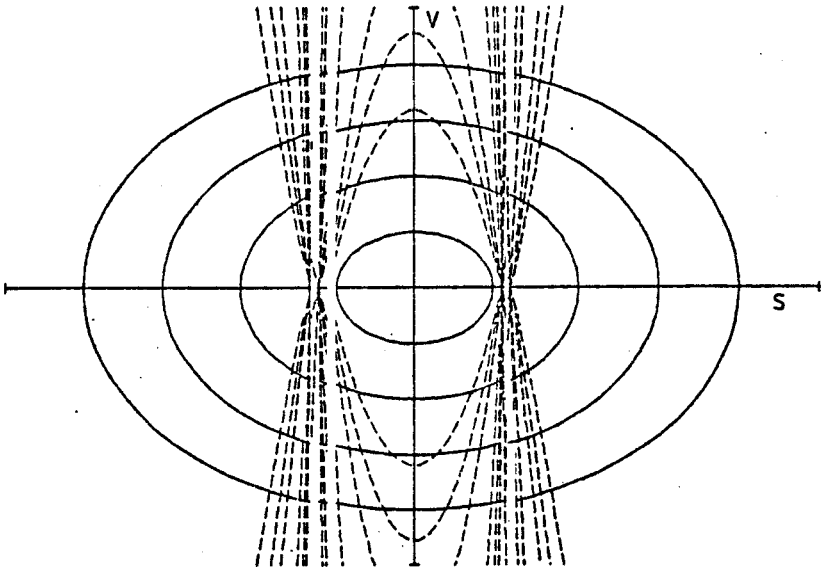

Figure 3. An example of a br oad k-spectrum . Each dashed line represents one of many waves propagati ng through a ve ry tenuous, weakly absorbant plasma. Axis same as Figure 1 .

interaction time is $\tau_{eff}^{-3} = n\omega_{c0}V_{\|}^2/L_B^2$, and $A_i(0) \approx 0.35$. A further simplification is to assume $\Delta\mu \ll \mu$, then the change in gyro-phase, ϕ, due to the resonant electric field can be ignored[7]. The approximate second order difference equation is

$$\begin{aligned} \mu_{n+1} &:= \mu_n + \Delta\mu(\mu_n, \phi_n) \\ \phi_{n+1} &:= \phi_n + \Delta\phi(\mu_{n+1}) \end{aligned} \tag{14}$$

where

$$\begin{aligned} \Delta\phi &\approx \oint_{\frac{1}{2}bounce} (\omega - \frac{d\phi}{dt})dt \\ &\approx -\frac{\pi}{2}\frac{n\omega_{c0}}{\omega_B}\frac{s_m^2}{L_B^2} \end{aligned}$$

In the last expression, the magnetic well was as sumed to be pa rabolic. (μ_n, ϕ_n) are the magnetic moment and phase before the nth resonance crossing.

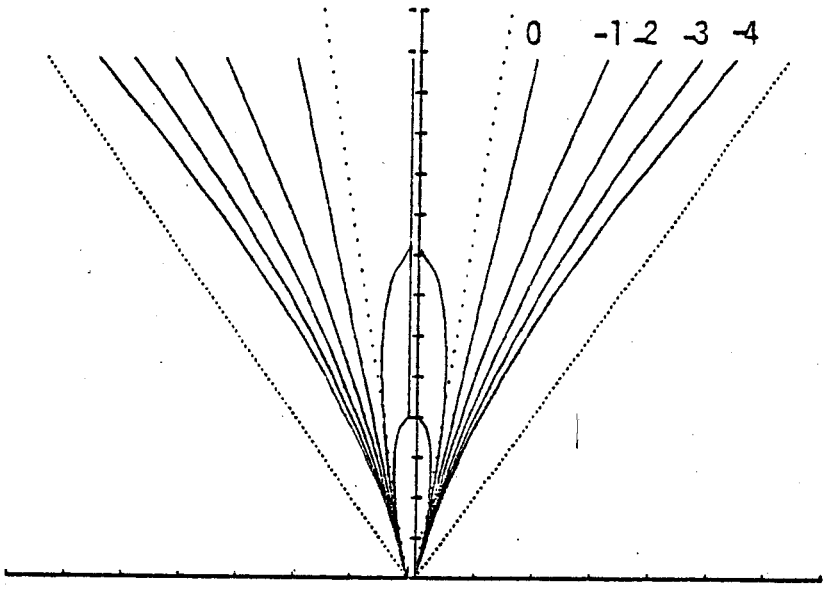

Figure 4. Primary bounce resona nces for a parabolic, magnetic well. Shown are $p = 1, 2$ and $p = 0, -1, -2, -3, -4$. Thc verticle axis is $V_{\perp,0}$, and the horizontal is $V_{\|,0}$. Th e inner, dotted lines are the turning-point resonances. The outer are the lo ss-cone boundaries.

The first-order fixed points, (μ_0, ϕ_0), of Equa tion 14 are the bounce resonances, They are the solutions of $\Delta\phi(\mu_0) = 2\pi p$ and $\cos(n\phi_0) = 0$. Figure 4 illustrates the bounce resonances. Note, that for a real trap, $\omega_B \to 0$ at the loss-cone boundary. Linearizing a bout the fixed points give

$$\begin{aligned} \tilde{\phi}_{n+1} &= \tilde{\phi}_n + 3\pi \frac{n\omega_{c0}}{\omega_B}\left(\frac{V_{\|,0}}{V_{\perp,0}}\right)^2 \frac{\tilde{\mu}_n}{\mu_0} \\ \tilde{\mu}_{n+1} &= \tilde{\mu}_n + K \sin(n\tilde{\phi}_n) \end{aligned} \tag{15}$$

where

$$K \approx 1.1 \frac{ql}{n} \frac{\tilde{z}_k}{n} \frac{\rho\omega_{c0}}{B} \tau_{eff}$$

and $\mu_n \approx \mu_0 + \tilde{\mu}_n$. From Lichtenberg[12], thc conditio n for stochasticity is

$$\frac{3\pi K}{\mu_0}\frac{n^2\omega_{c0}}{\omega_B}\left(\frac{V_{\|,0}}{V_{\perp,0}}\right)^2 > 1 \tag{16}$$

which was determined both numerically and analytically from solutions of the standard mapping of the Fermi accelerator. As explained by Lichtenberg, Equation 16 is actually a factor of two less severe than the condition of primary resonance overlap (given by $K_p + K_{p+1} > \mu_p - \mu_{p+1}$ for all $p \neq 0$) because of the overlap of higher-order bounce resonances. Equation 17 can be re-written as

$$\frac{q}{m}E_k\tau_{eff} > V_{\perp,0}\frac{\omega_B}{n^2\omega_c}\left(\frac{V_{\perp,0}}{V_{\|,0}}\right)^2 \tag{17}$$

which, for $V_{\perp,0}/V_{\|,0} \sim 4$ and $n = 1$, requires $E_k > 0.005V/cm$ for $T_\perp = 1keV$, and $E_k > 2.3V/cm$ for $T_\perp = 100keV$. Except for very small fields and very high temperatures, superadiabatic motion should not be expected in Constance II. The condition $\Delta\mu \ll \mu$ gives the upper bound on E_k as $50v/cm$ and $3kV/cm$, respectively. For ECRH in Constance II, the heating is initially highly non-linear ($T_e \lesssim 15ev$ before heating). In this case, the unperturbed orbits cannot be used to calculate $\Delta\mu$, and $\langle\Delta\mu\rangle$ no longer vanishes since the particles will change their phase and larmor radius as they accelerate. The non-linear heating rate is approximately given by $\langle\Delta\mu\rangle/\tau_B$ which scales roughly as $P_{rf}^{1/2}$ instead of the P_{rf} scaling for linear heating. In Constance II, the quasi-linear theory should become valid after a few microseconds for powers not greater than a few kilowatts.

5. Bounce-averaged quasi-linear theory .

The bounce-averaged quasi-linear equation is written symbolically as

$$\oint_{bounce}\frac{d\psi}{2\pi}\left\langle\frac{D}{Dt}f_0(E,\mu,\mathrm{R})\right\rangle_\phi = -\frac{q}{m}\oint_{bounce}\frac{d\psi}{2\pi}\left\langle\sum_{k,k'}\delta_{-k',k}\delta_{-\omega',\omega}\,\mathrm{M}^{ij}_{k'}E^j_{k'}\frac{\partial f_k}{\partial v^i}\right\rangle_\phi \tag{18}$$

where

$$M_k^{ij} = \delta^{ij}(1 - \frac{\mathbf{v}}{c} \cdot \mathrm{N}) + N^i \frac{v^j}{c} \tag{19}$$

$$E_k^j = \overline{E}_k^j(\mathrm{r}, t) \exp\left[-j\omega t + i\lambda(\mathrm{r})\right] \tag{20}$$

and

$$f_k = -\frac{q}{m} \oint_{-\infty}^{t} dt' \, M_k^{ij}(t') E_k^j(t') \frac{\partial f_0}{\partial v^i(t')} \tag{21}$$

The first equation above is the bounce and gyro-average of the electron response to the RF fields. Strictly speaking, the diffusion due to untrapped, streaming plasma must be added to the right-hand side of Equation 18, but this is ignored. The delta-function, $\delta_{-k',k}$ implies the random-phase approximation which is not exactly true in an inhomogeneous plasma. The various field components will couple within bandwidths of the order of $\Delta \mathrm{k} \sim \nabla(\ln[f_0(\mathrm{r})])$, but this effect is ignored in this treatment. In the time-integral, in Equation 21, the initial condition at $t \to -\infty$ has been ignored, and when evaluating the inverse Fourier transform, ω is assumed to have a small, positive imaginary part in the normal manner. The spatial phase, $\lambda(\mathrm{r})$, is the geometric optics approximation to the wave number of the waves, and $\mathrm{k}(\mathrm{r}) = \nabla \lambda(\mathrm{r})$. The index of refraction is $N^i = ck^i/\omega$. $\overline{E}_k^i$ is a slowly changing function of space and time, and ω is constant.

In Equation 21, the integral over t' is along the unperturbed particle orbits as in Equation 7. However, in Equation 18, this orbit integral is multiplied by the complex conjugate of the electric field at $t' = t$, and this phase-dependent product is then averaged over a full bounce. The resulting average is highly oscillatory unless the end-point corresponds to a stationary point, $\nu(t) \sim 0$. Said in another way, the orbit-integral is the sum of contributions from past stationary points ($t' < t$) and from those near the end point ($t' \sim t$). The phases of these terms are then added to the phase of the electric field which nearly cancels the phase of the end-point. Now, when the total phase of each term is bounce-averaged, then (1) the real part of the terms from past stationary points are zero, and (2) the real part of the end-point term is also zero except when the phase of the end-point and of the electric-field exactly cancel for a finite period of time. The time during which this cancellation takes place is τ_{eff}. Therefore, the major contributions to the bounce-averaged quasi-linear equation are when $\psi(t)$ is near a stationary point. Note, that in this theory, the history of the particle has been truncated. The original global equation has been reduced to a sum of local wave-particle resonances. This is the same premise used to justify Equation 12. In a linear, bounce-averaged theory, if the phase information of the past resonance crossings were retained, superadiabaticity would result.

Keeping the remarks of the last paragraph in mind, the averages and integrals in Equations 18 to 21 can be performed by expanding the field about $(\overline{\mathrm{X}}, s, t)$, the current guiding-center position and the current time. This gives

$$E_k^j(\mathrm{r}', t') = \exp\left[-i\omega t + i\lambda(\mathrm{R})\right]\left\{\overline{E}_k^j(\mathrm{R}, t) + (\mathrm{r}' - \mathrm{R})\cdot\nabla\,\overline{E}_k^j(\mathrm{R}, t) + (t' - t)\frac{\partial}{\partial t}\overline{E}_k^j(\mathrm{R}, t)\right\}$$
$$\times\left(1 + \frac{i}{2}(\mathrm{r}' - \mathrm{R})(\mathrm{r}' - \mathrm{R}) : \nabla\,\mathrm{k}\right)\exp\left[-i\omega(t' - t) + i\int_{\mathrm{R}}^{\mathrm{r}'} dt''\mathrm{v}''\cdot\mathrm{k}(\mathrm{R})\right] \tag{22}$$

Notice that the variation of k along the orbit is assumed to be slow enough such that $(\mathrm{r}'-\mathrm{R})\cdot\nabla(\ln[k(\mathrm{R})]) \ll 1$, and the exponential containing $\nabla\,\mathrm{k}$ can be expanded. The double dot-product is $(r^{i\prime} - R^i)(r^{j\prime} - R^j)(\partial k^j/\partial R^i)$ where the repeated indices are assumed to be summed. Equation 22 can be re-written as

$$E_k^j(\mathrm{r}', t') = \exp\left[-i\omega t + i\lambda(\mathrm{R})\right]\left\{\overline{E}_k^j - i\,\nabla\,\overline{E}_k^j\cdot\frac{\partial}{\partial\mathrm{k}} + i\frac{\partial\overline{E}_k^j}{\partial t}\frac{\partial}{\partial\omega}\right.$$
$$\left. - \frac{i}{2}\overline{E}_k^j\nabla\,\mathrm{k} : \frac{\partial}{\partial\mathrm{k}}\frac{\partial}{\partial\mathrm{k}}\right\}\exp\left[-i\omega(t' - t) + i(\mathrm{r}' - \mathrm{R})\cdot\mathrm{k}(\mathrm{R})\right] \tag{23}$$

The electric field, when $t' = t$, can be expanded similarly as

$$E_k^j(\mathrm{r}, t) = \exp\left[-i\omega t + i\lambda'(\mathrm{R})\right]\left\{\overline{E}_{k'}^j - i\,\nabla\,\overline{E}_{k'}^j\cdot\frac{\partial}{\partial\mathrm{k}'} - \frac{i}{2}\overline{E}_{k'}^j\nabla\,\mathrm{k}' : \frac{\partial}{\partial\mathrm{k}'}\frac{\partial}{\partial\mathrm{k}'}\right\}\exp\left[i(\mathrm{r} - \mathrm{R})\cdot\mathrm{k}'(\mathrm{R})\right] \tag{24}$$

Then, Equation 18 can now be written as

$$\oint\frac{d\psi}{2\pi}\left\langle\frac{\partial f_0}{\partial t} + v_\parallel\frac{\partial f_0}{\partial s}\right\rangle_\phi = \frac{q^2}{m^2}\oint\frac{d\psi}{2\pi}\left\langle\sum_{k,k'}\delta_{-k',k}\,\mathrm{M}_k^{ij}\right.$$
$$\times\left\{\overline{E}_{k'}^j - i\,\nabla\,\overline{E}_{k'}^j\cdot\frac{\partial}{\partial\mathrm{k}'} - \frac{i}{2}\overline{E}_{k'}^j\nabla\,\mathrm{k}' : \frac{\partial}{\partial\mathrm{k}'}\frac{\partial}{\partial\mathrm{k}'}\right\}$$
$$\cdot\frac{\partial}{\partial v^i}\int_{-\infty}^{t} dt'\,\mathrm{M}_k^{lm}(t') \tag{25}$$
$$\times\left\{\overline{E}_k^m - i\,\nabla\,\overline{E}_k^m\cdot\frac{\partial}{\partial\mathrm{k}} + i\frac{\partial\overline{E}_k^m}{\partial t}\frac{\partial}{\partial\omega} - \frac{i}{2}\overline{E}_k^m\nabla\,\mathrm{k} : \frac{\partial}{\partial\mathrm{k}}\frac{\partial}{\partial\mathrm{k}}\right\}$$
$$\left.\times\exp\left[-i\int_t^{t'}(\omega - \mathrm{v}(t'')\cdot\mathrm{k}(\mathrm{R}))dt''\right]\cdot\frac{\partial f_0}{\partial v^l(t')}\right\rangle_\phi$$

where the argument of the electric field is (R, t), and where $k = -k'$ and $\omega = -\omega'$ were used to express the phase in the last exponential. Using Equations 2 and 3, the phase-dependent exponential can be simplified, since

$$(\mathrm{r} - \mathrm{R}) \cdot \mathrm{k}_\perp = \rho k_\perp \sin\left(\int_0^{t'} \omega_c dt'' + \phi - \xi + \pi/2\right)$$

where the local wave-vector is $\mathrm{k} = k_\parallel \hat{s} + k_\perp(\cos\xi\hat{x} + \sin\xi\hat{y})$, then the standard bessel expansion allows

$$\begin{aligned}\exp\left[-i\int_t^{t'} (\omega - \mathrm{v}\cdot\mathrm{k})dt''\right] = \sum_{n,n'} J_n J_{n'} e^{i(\phi+\pi/2)(n-n')} e^{-i(\xi n - \xi' n')} \\ \times \exp\left[-i\int_t^{t'} \nu_n(t'')dt''\right]\end{aligned} \tag{26}$$

where $\nu_n(t)$ is given by Equation 8. The argument of the bessel function is $k_\perp \rho$.

Before making any further progress, Equation 25 can be greatly simplified by transforming to the complex basis defined by $(x, y, z) \rightarrow (r, l, z)$ where $r = (x - iy)/\sqrt{2}$, $l = r^*$, and $z = z$. The symbol, * or "star", denotes the complex conjugate. In this basis, if $E_k^l = E_k^z = 0$, then the electric field is right-hand circularly polarized in the direction of the magnetic field. A dot-product in the real, cartesian coordinate system is re-written in the new, complex basis as $\mathrm{A}\cdot\mathrm{B} \rightarrow \mathrm{A}\cdot\mathrm{B}^*$. Then, suppressing the gradient terms containing the dot and double-dot products, then right-hand side of Equation 25 can be written as

$$\begin{aligned}\sum_{k,k'} \delta_{-k',k} \sum_{n,n'} \frac{q^2}{m^2} \oint \frac{d\psi}{2\pi} \left\langle \mathrm{M}_{k'}^{i^* j} \cdot \frac{\partial}{\partial v^i}\left\{\overline{E}_{k'}^{j^*} - \ldots\right\}\left\{\overline{E}_k^m - \ldots\right\}\right. \\ \left. \times \int_{-\infty}^0 dt' \, \mathrm{M}_k^{l m^*} J_n J_{n'} e^{i(\phi+\pi/2)(n-n')} e^{-i(\xi n - \xi' n')} \exp\left[-i\int_0^{t'} \nu_n dt''\right] \cdot \frac{\partial f_0}{\partial v^{l^*}}\right\rangle_\phi\end{aligned} \tag{27}$$

Note the location of the conjugated components. The only t' dependence besides the phase integral is the product $\mathrm{M}_k^{i j^*} \partial/\partial v^{i^*}$, but with some algebra and Equations 4 and 19, this is

$$M^{m*}_{op,n} \equiv \mathrm{M}^{lm^*}_{k} \frac{\partial}{\partial v^{l*}} \tag{28}$$

whose componets are

$$M^{r*}_{op,n} = \left\{ \frac{\rho\omega_c}{\sqrt{2}} \delta_{n,n-1} e^{i\xi} + \frac{V_D}{\sqrt{2}} e^{i(\overline{\Theta}+\pi/2)} \right\} \frac{\partial}{\partial \chi} - \beta_n \frac{V_D}{\sqrt{2}} e^{i(\overline{\Theta}+\pi/2)} \left\{ \frac{1}{B}\frac{\partial}{\partial \mu} - \frac{1}{V_D \omega_c} \frac{\partial}{\partial \overline{X}} \right\}$$

$$M^{l*}_{op,n} = M^{r}_{op,n}$$

$$M^{z*}_{op,n} = v_{\|} \frac{\partial}{\partial \chi} - v_{\|} \beta_n \frac{1}{B} \frac{\partial}{\partial \mu}$$

and where

$$\frac{\partial}{\partial \chi} = \frac{1}{B}\frac{\partial}{\partial \mu} + \frac{\partial}{\partial E} - \left(N_{\|} \frac{v_{\|}}{c} + \mathrm{N} \cdot \frac{\mathrm{V}_D}{c} \right) \frac{1}{B} \frac{\partial}{\partial \mu} + \mathrm{N} \cdot \hat{v}_D \frac{1}{c\omega_c} \frac{\partial}{\partial \overline{X}}$$

$$\beta_n = 1 - \frac{k_{\|} v_{\|}}{\omega} - \frac{\mathrm{k}_{\perp} \cdot \mathrm{V}_D}{\omega} - \frac{n\omega_c}{\omega}$$

The operator $\delta_{n,n\pm1}$ acts to raise or lower the order of the bessel function designated by n. This gives the identity $\delta_{n,n+1} + \delta_{n,n-1} = 2n/k_{\perp}\rho$ which was used to obtain these expressions. Note that the time dependence of $\mathrm{M}_{op,n}$ has been replaced by using these operators. This is because the operation of $\delta_{n,n-1}$ is equivalent to multiplication by $\exp[i(\int_0^{t'} \omega_c dt'' + \phi - \xi + \pi/2)]$ and then re-defining the sum over n. Therefore, the complex conjugate of $\delta_{n,n+1}$ is $\delta_{n,n-1}$. The same operators can be used to express $M^{i}_{op,n'} = \mathrm{M}^{ij^*}_{k} \partial/\partial v^{i}$ except, here, after complex conjugating the expression for M^{l*}_{op} in Equation 28, the direction of the $\delta_{n',n'\pm1}$ operators must be reversed due to the opposite sign of k'. Physically, the operators, M_{op}, give the diffusion paths of the electrons in $(E, \mu, \overline{X})$ phase-space.

Then, in the complex basis, and after replacing the time-dependances of the gradient operators by the delta-operators, Equation 25 is

$$\frac{\partial f_0}{\partial t} = \frac{q^2}{m^2} \oint \frac{d\psi}{2\pi} \sum_{n,n'} \sum_{k,k'} \delta_{-k',k} \left\langle M^{j}_{op,n'} \left\{ \overline{E}^{j^*}_{k'} - \ldots \right\} \left\{ \overline{E}^{m}_{k} - \ldots \right\} \Omega^{-1}_{n,n'} M^{m*}_{op,n} f_0 \right\rangle_{\phi} \tag{29}$$

where

$$\Omega^{-1}_{n,n'} = \int_{-\infty}^{0} dt' \, J_{n'} J_n \, e^{i(\phi+\pi/2)(n-n')} e^{-i(\xi n - \xi' n')} \exp\left[-i \int_0^{t'} \nu_n \, dt''\right] \tag{30}$$

Note, that the term containing $\partial/\partial\phi$ in $\partial/\partial v^i$ has been excluded since, when the averaged over ϕ, this term is highly phase dependent and therefore does not contribute to diffusion. Also, the bounce and gyro averages of Df_0/Dt leaves only the derivative of f_0 with respect to slow time changes since f_0 is independent of ψ and ϕ. Furthermore, since the only ϕ dependances in Equation 29 are contained in $\Omega_{n,n'}$, the gyro-average sets n to n'. (However, the conventions, explained in the last paragraph, between the raising and lowering operators, $\delta_{n,n\pm1}$, designated by M^j_{op} and $M^{m^*}_{op}$ are still maintained.) Finally, the average over ψ is calculated as in Equation 8. The real-part of the integral will be dominated by the rapidly varying phase in the exponential which will give non-vanishing contributions only when ψ corresponds to a stationary point. Thus, resonant energy exchange is the sum of local interactions. The imaginary part of the average is global, since it represents the average "sloshing" wave energy along the particle's bounce path.

Following Berk[1], the gradient terms can now be inserted and the sum over k and k' completed. Note that

$$\frac{\partial}{\partial \mathbf{k}} = \hat{s}\frac{\partial}{\partial v_\parallel} + \hat{x}\left(\frac{k_x}{k_\perp}\frac{\partial}{\partial k_\perp} - \frac{k_y}{k_\perp^2}\frac{\partial}{\partial \xi}\right) + \hat{y}\left(\frac{k_y}{k_\perp}\frac{\partial}{\partial k_\perp} + \frac{k_x}{k_\perp^2}\frac{\partial}{\partial \xi}\right) \tag{31}$$

since $k_\perp^2 = k_x^2 + k_y^2$ and $\xi = \tan^{-1}(k_y/k_x)$. Remember, also, that the derivatives with respect to ω and k act only upon Ω_n^{-1}. Then, since the terms proportional to $\overline{E}^{j^*}_{-k} \nabla \overline{E}^m_k - \overline{E}^m_k \nabla \overline{E}^{j^*}_{-k}$ sum to zero, the only first-order contributions come from the derivatives with respect to ξ and ξ'. In other words, since

$$\frac{1}{k_\perp}\frac{\partial}{\partial \xi} = -\frac{1}{k_\perp}\frac{\partial}{\partial \xi'} = -\frac{in}{k_\perp}$$

the final form of the quasi-linear equation is a sum of resonant interactions and a gradient term which acts on the electric field and the resonance function. The result is

$$\frac{\partial}{\partial t} F_0(E, \mu, \overline{X}, t) = \frac{q^2}{m^2} \sum_{res} \sum_{k,n} M^j_{op} \{1 - \delta \cdot (\frac{n\mathbf{k}}{k_\perp^2} \times \nabla)\} |\overline{E}^{j^*} \overline{E}^m|_k Re\{\overline{\Omega_n}^{-1}\} M^{m^*}_{op} F_0 \tag{32}$$

The gradient term can be considered as the correction to the field intensity and its interaction which results from expressing the field in guiding-center coordinates. As in Berk[1], F_0 is the average particle distribution after subtracting the "sloshing" energy due to the non-resonant wave-particle interactions, and $\overline{\Omega_n}^{-1}$ is the

bounce-averaged resonance function. Finally, the sum over resonances does not necessarily refer to a sum over definite resonant layers in space. In general, $k_\parallel \neq 0$, and the resonances for each region of velocity-space will occur in different regions of coordinate-space.

F_0 is given by

$$F_0 = f_0 + \frac{1}{2}\frac{q^2}{m^2}\sum_{k,n}\oint \frac{d\psi}{2\pi} M^j_{op'}|\overline{E}^{j^*}\overline{E}^m|_k \frac{\partial}{\partial\omega} Im\{\Omega_n^{-1}\} M^{m^*}_{op} F_0 \tag{33}$$

In Section 8, F_0 will be shown to represent the particle kinetic energy after the "sloshing" wave energy is subtracted.

It should also be noticed that only the hermitian part of the matrix operators $M^i_{op}...M^{j^*}_{op}$ enters Equation 32, since all of the anti-hermitian terms contain β_n and are not resonant. Therefore, Equation 32 is real. The terms containing β_n are non-resonant because

$$-i\omega\beta_n \exp\left[-i\int_0^{t'} \nu_n dt''\right] = \frac{d}{dt'}\exp\left[-i\int_0^{t'}\nu_n dt''\right]$$

and the initial conditions at $t \to -\infty$ have been ignored.

Finally, for the simple example discussed in Section 3, $V_D = 0$, $\overline{\mathrm{E}} = \hat{y}\overline{E}$, $\xi = 0$, and $\partial/\partial\overline{X} = 0$. Then, the sum of the terms containing $|\overline{E}^{r^*}\overline{E}^r|$, $|\overline{E}^{l^*}\overline{E}^l|$, $|\overline{E}^{r^*}\overline{E}^l|$, and $|\overline{E}^{l^*}\overline{E}^r|$ give

$$\frac{\partial F_0}{\partial t} = \frac{q^2}{m^2}\sum_{res}\rho\omega_c\left(\frac{1}{B}\frac{\partial}{\partial\mu} + \frac{\partial}{\partial E}\right)|\overline{E}^2|_k J_1'^2 Re\{\overline{\Omega_1}^{-1}\}\rho\omega_c\left(\frac{1}{B}\frac{\partial}{\partial\mu} + \frac{\partial}{\partial E}\right)F_0 \tag{34}$$

where terms of the order of $v/c \ll 1$ have been ignored, and the slowly varying quantities which define the diffusion paths are evaluated at the resonances. The identity $\delta_{n,n-1} - \delta_{n,n+1} = 2J'_n$ was used. Note that Equation 34 is the equivalent of Equation 12 derived from via the Maxwell-Vlasov equations.

6. Bounce-averaged Resonance Function .

To calculate the bounce-averaged resonance function, $\overline{\Omega_n}^{-1}$, the techniques used in Section 3 are used again. First, $\nu_n(\psi, t'')$ is expanded about $t'' = 0$ and $\psi = \psi^*$, such that $\nu(\psi^*, 0) = 0$. Then,

$$-i\int_0^{t'}\nu_n\,dt'' \approx -i\int_0^{t'}[\nu_n(\psi^*)(\psi-\psi^*)/\omega_B + t\nu(\psi^*)]$$
$$\approx -i\tau_{eff}^2[t+(\psi-\psi^*)/\omega_B]^2 + i\tau_{eff}^2(\psi-\psi^*)^2/\omega_B^2 \tag{35}$$

and, for cases when $\nu_n \approx 0$, then

$$\approx -i[\nu_n(\psi^*)t + \frac{t}{\tau_{eff}^3\omega_B^2}(\psi-\psi^*+t\omega_B/2)^2 + \tau_{eff}^{-3}t^3/12] \tag{36}$$

In the above equation, the identity $\omega_B\partial/\partial\psi = \partial/\partial t$ was used. In the first case, integrating first over time and then over bounce angle gives

$$Re\{\overline{\Omega_n^{-1}}\} = \sum_{res} J_n^2\frac{\omega_B}{2\pi}\tau_{eff}^2 \tag{37}$$

while, in the second case, the first integration is over ψ, and the second over t', which gives

$$Re\{\overline{\Omega_n^{-1}}\} \approx \sum_{res} J_n^2\frac{\omega_B}{2\sqrt{2}}\tau_{eff}e^{-i\pi/4}\int_{-\infty}^{0}\frac{dt'}{\sqrt{\tau_{eff}^{-1}t'}}\exp -i[\nu_n^*t' + \tau_{eff}^{-3}t'^3/12]$$
$$\approx \sum_{res}\pi\omega_B\tau_{eff}^2[Ai^2(\nu_n^*\tau_{eff}) - iAi(\nu_n^*\tau_{eff})Bi(\nu_n^*\tau_{eff})] \tag{38}$$

using the Pearlstein identity[1]. The value of the resonant interaction is the same as that calculated in Equations 10 and 11.

It is also informative to calculate the resonance function in a manner which illustrates the points of Section 4. If we take a simple example, with $k_\parallel = V_D = 0$, then the exact orbits for electrons deeply-trapped in a magnetic well give

$$\overline{\Omega_n^{-1}} = \sum_p J_n^2J_p^2(\tau_{eff}^{-3}\omega_B^{-3}/4)\frac{i}{\delta\omega_{0n} - \tau_{eff}^{-3}\omega_B^{-2}/2 - 2p\omega_B} \tag{39}$$

where $\tau_{eff}^{-3} = n\omega_{c0}V_{\parallel,0}^2/L_B^2$. The bounce resonances are those shown in Figure 4. The resonance function can be considered to represent the wave-particle interaction in the limit that $\overline{E} \to 0$ and $t \to \infty$. Those

particles which do not have exactly the same phase during each pass through resonance cannot gain energy as $t \rightarrow \infty$. Of course, this condition is also the condition which defined the fixed-points of Equation 14.

However, when the electric field is finite, then the resonances overlap, and Equation 39 should be equivalent to Equation 38. To show this, a broadening term is added to the resonant denominator, so that the real part of Equation 30 is

$$Re\{\overline{\Omega_n}^{-1}\} = J_n^2 \sum_p J_p^2(\tau_{eff}^{-3}\omega_B^{-3}/4)\frac{\eta_k}{(\delta\omega_{0n} - \tau_{eff}^{-3}\omega_B^{-2}/2 - 2p\omega_B)^2 + \eta_k^2} \tag{40}$$

where η_k can be considered to be defined from

$$\eta_k f_k(E,\mu) \approx \frac{1}{2}\left[\frac{\partial^2}{\partial\mu^2} + B^2\frac{\partial^2}{\partial E^2}\right]\sum_{res}\frac{\langle\Delta\mu\rangle_k^2}{\tau_B} f_k(E,\mu)$$

as in Equation 12. Then, since $J_p[p(1 + z\,p^{-2/3})] \sim (2/p)^{1/3} Ai(-2^{1/3}z)$ as $p \rightarrow \infty$, Equation 40 becomes

$$Re\{\overline{\Omega_n}^{-1}\} \approx \sum_{p\ll 0} J_n^2\, 4\tau_{eff}^2\omega_B^2 Ai^2(-\delta\omega_{0n}\tau_{eff}) \tag{41}$$

for p less than zero, and

$$\begin{aligned} Re\{\overline{\Omega_n}^{-1}\} \approx \sum_{p\gg 0} J_n^2\,\tau_{eff}^2\omega_B^2 Ai^2[(\delta\omega_{0n} - \omega_B^{-2}\tau_{eff}^{-3})\tau_{eff}] \\ \times \frac{\eta_k}{\omega_B^2(p - \delta\omega_{0n}/2\omega_B + 1/4\tau_{eff}^3\omega_B^3)^2 + \eta_k^2} \end{aligned} \tag{42}$$

for p greater than zero. To obtain these equations, the argument of the bessel function was evaluated at resonance, or when

$$\frac{1}{4p\tau_{eff}^3\omega_B^2} = -1 + \frac{\delta\omega_{0n}}{2p\omega_B} \tag{43}$$

Then, as $p \rightarrow \pm\infty$, $4\tau_{eff}^3\omega_B^3 \sim -1/p$. Also needed is the cube root of p which, when evaluated,

the real branch is used. For large $|p|$, the sum over p is converted into an integral, and assuming that $\eta_k^2 > (\delta\omega_{0n}/2\omega_B - 1/4\tau_{eff}^3\omega_B^3)^2$, then the bounce-averaged resonance function is approximately

$$Re\{\overline{\Omega_n^{-1}}\} \approx J_n^2 \pi\tau_{eff}^2\omega_B Ai^2(-\delta\omega_{0n}\tau_{eff}) \tag{44}$$

for those particles with $V_{\|,0} \gg V_{\perp,0}(2\delta\omega_{0n}/n\omega_{c0})^{1/2}$, and

$$\approx J_n^2 \pi\tau_{eff}^2\omega_B Ai^2[(\delta\omega_{0n} - \omega_B^{-2}\tau_{eff}^{-3})\tau_{eff}] \tag{45}$$

for those particles resonant near their turning points. Equations 44 and 45 are independent of η_k. Note, that these results are the same as those obtained from Equation 38. Finally, when $p \sim 0$, the particles are resonant far from the bounce phases when either $s^* \sim 0$ or $v^* \sim 0$. In this case, no simple expression for $\overline{\Omega_n^{-1}}$ can be found independent of η_k. For these particles, Equation 37 is the only simple way to calculate the resonance function.

7. WKB Theory

In this section, the WKB theory for the wave propagation from the launching horn through the plasma is discussed. The fundamentals of the theory of electromagnetic waves in an inhomogeneous plasma are well known (see, for example, Budden, 1961). When combined with the quasi-linear equation of Section 4, the theory presented in this report is the simplest, self-consistent model of non-relativistic electron-cyclotron heating in a mirror that conserves energy.

Two types of equations are needed. First, the local equation of energy conservation is derived which determines the geometric and physical optics solutions for the propagation of electromagnetic waves. The second is the bounce average of this equation, which gives the energy conservation equation for the trapped particles. Regenerative effects due to the "phase-memory" of bouncing particles is ignored (see Berk and Book, 1969).

From Maxwell's equations, Poynting's theorem is

$$\nabla\cdot(\overline{\mathbf{E}}_{-k} \times \overline{\mathbf{B}}_k) + \frac{1}{2}\frac{\partial}{\partial t}(|\overline{E}|_k^2 + |\overline{B}|_k^2) + \frac{4\pi}{c}Re\{\overline{\mathbf{E}}_{-k}\cdot\overline{\mathbf{J}}_k\} = 0 \tag{46}$$

which can be written as

$$\left[\frac{\partial}{\partial t}\frac{\partial}{\partial\omega} - \nabla\cdot\frac{\partial}{\partial k}\right]\{\omega(1-N^2)\delta_{ij} - \omega N^i N^j\}\,\frac{1}{2}|\overline{E}^i\overline{E}^j|_k = 4\pi Re\{\overline{\mathbf{E}}_{-k}\cdot\overline{\mathbf{J}}_k\} \tag{47}$$

The last term is

$$4\pi Re\{\overline{\mathrm{E}}_{-k}\cdot\overline{\mathrm{J}}_k\} = -\frac{4\pi q^2}{m}Re\sum_{\pm}\left\langle\int\frac{dEd\mu B}{|v_\parallel|}v^i\overline{E}^i_{-k}\right.$$
$$\left.\times\int_{-\infty}^{0}dt'M_k^{lm}(t')\overline{E}_k^m(t')\frac{\partial}{\partial v^l(t')}f_0(E,\mu,\mathrm{R})\exp\left[-i\int_0^{t'}dt''(\omega-\mathbf{k}\cdot\mathbf{v})\right]\right\rangle_\phi \tag{48}$$

The sum over $\pm$ refers to the direction of motion along the field lines. The time dependence of $M_k^{lm}\partial/\partial v^l$ can be treated as those in Equation 27 by transforming to the complex basis. $\overline{E}_k^m$ can be expanded as in Equations 23 and 24, except in this case, the field is expanded about (r,t) instead of the guiding-center coordinates since the local currents are to be found. In addition, $f_0(\overline{X},s)$ can be expressed in terms of r as in Equation 32, which gives

$$f_0(E,\mu,\overline{X},s)\approx\{1+\delta\cdot(\frac{n\mathbf{k}}{k_\perp^2}\times\nabla)\}f_0(E,\mu,\mathrm{r})$$

Then, with $v_i=(\delta_{n',n'-1}\rho\omega_c e^{-i\xi}/\sqrt{2}+V_D e^{-i(\overline{\Theta}+\pi/2)}/\sqrt{2},\ \delta_{n',n'+1}\rho\omega_c e^{i\xi}/\sqrt{2}+V_D e^{i(\overline{\Theta}+\pi/2)}/\sqrt{2},\ v_\parallel)$, Equation 48 becomes

$$4\pi Re\{\overline{\mathrm{E}}_{-k}\cdot\overline{\mathrm{J}}_k\} = Re\left\{\left[1-\frac{i}{2}\frac{\partial}{\partial\mathbf{k}}\cdot\nabla+\frac{i}{2}\frac{\partial}{\partial\omega}\frac{\partial}{\partial t}\right]|\overline{E}^{i^*}\overline{E}^m|_k\right.$$
$$\left.\times\frac{4\pi q^2}{m}\sum_n\sum_{\pm}\int\frac{dEd\mu B}{|v_\parallel|}v^i\Omega_n^{-1}M_{op}^{m^*}\{1+\delta\cdot(\frac{n\mathbf{k}}{k_\perp^2}\times\nabla)\}f_0\right\} \tag{49}$$

Which, when combined with Equation 47, gives the local, energy conservation equation

$$\nabla\cdot\left(-\frac{\partial\mathrm{D}_R^{ij^*}}{\partial\mathbf{k}}\frac{1}{2}|\overline{E}^{i^*}\overline{E}^j|_k\right)+\frac{\partial}{\partial t}\left(\frac{\partial\mathrm{D}_R^{ij^*}}{\partial\omega}\frac{1}{2}|\overline{E}^{i^*}\overline{E}^j|_k\right)+\mathrm{D}_I^{ij^*}|\overline{E}^{i^*}\overline{E}^j|_k=0 \tag{50}$$

where $\mathrm{D}^{ij^*}=\mathrm{D}_R^{ij^*}+i\mathrm{D}_I^{ij^*}$ is the local dispersion tensor

$$
\begin{aligned}
\mathsf{D}^{ij^*} &= \omega\,\epsilon^{ij^*} \\
\epsilon^{ij^*} &= (1-N^2)\delta^{ij} + N^i N^{j^*} \\
&\quad - i\frac{4\pi q^2}{\omega\, m}\sum_n \sum_{\pm} \int \frac{dE d\mu B}{|v_{\|}|} v_i M^{j^*}_{op} \Omega_n^{-1}\{1+\delta\cdot(\frac{n\mathbf{k}}{k_{\perp}^2}\times\nabla)\} f_0
\end{aligned}
\tag{51}
$$

Equation 50 contains the first terms of a WKB theory for electromagnetic wave propagation in an inhomogeneous plasma. This is a generalization of the one-dimensional, electrostatic WKB theory derived by Berk and Book, 1969. For electromagnetic waves, the dispersion relation to all orders is

$$
\int d^3r'\, \mathsf{D}^{ij}(\mathbf{r}'-\mathbf{r},(\mathbf{r}'+\mathbf{r})/2)\,\overline{E}^j(\mathbf{r}')e^{-i\lambda(\mathbf{r}'-\mathbf{r})} = 0
$$

or, if D^{ij} and $\overline{E}$ vary slowly over a wavelength,

$$
\sum_{q,r} \frac{(-1)^{q+r}}{2^r\, q!\, r!}\left(\frac{\partial}{\partial \mathbf{k}^1}\cdots\frac{\partial}{\partial \mathbf{k}^q}\right)\left(\frac{\partial}{\partial \mathbf{k}^1}\cdots\frac{\partial}{\partial \mathbf{k}^r}\right)\left(\frac{\partial}{\partial \mathbf{r}^1}\cdots\frac{\partial}{\partial \mathbf{r}^r}\,\mathsf{D}^{ij}(\mathbf{k}(\mathbf{r}),\mathbf{r})\right)\left(\frac{\partial}{\partial \mathbf{r}^1}\cdots\frac{\partial}{\partial \mathbf{r}^q}\,\overline{E}^j(\mathbf{r})\right) = 0 \tag{52}
$$

The zeroth-order equation, $\mathsf{D}_R^{ij^*}|\overline{E}^{i^*}\overline{E}^j|_k = D^i_{R,mode}(\mathbf{k}_{mode},\omega)|\overline{E}^i|_k^2 = 0$, is an eigenvalue equation. The matrices $\mathsf{D}_R^{ij^*}$ and $\mathsf{D}_I^{ij^*}$ are hermitian for the same reason that Equation 33 is real. Furthermore, it can be shown that, when mode-coupling due to the plasma geometry is ignored, $\mathsf{D}_R^{ij^*}$ and $\mathsf{D}_I^{ij^*}$ commute so that they can be simultaneously diagonlized. The solutions to this equation give geometric optics. This is used to determine the path of mode-energy flow. However, the first-order terms are needed to show energy conservation. The eigenvectors are the polarizations of the local modes, and the eigenvalues are the solutions to $D^i_{R,mode} = 0$, or $\mathbf{k}_{mode} = \nabla\lambda_{mode}(\mathbf{r})$.

Briefly, the procedure for computing the ray path is as follows. First, the ray path is considered to be sub-divided into many small segments $\Delta\mathbf{r}$. Within each segment, the dispersion tensor is diagonalized, and the dispersion relation and polarization for each of the eigenvectors, or modes, is found. The electric field at the back-side of $\Delta\mathbf{r}$ is then expressed in the basis formed by the mode polarizations. Finally, each mode then propagates at its group velocity to the front-side of $\Delta\mathbf{r}$, and the process is repeated. The group velocity is

$$
\mathbf{v}_g^i = -\frac{\partial D_R^i}{\partial \mathbf{k}}\left(\frac{\partial D_R^i}{\partial \omega}\right)^{-1} \tag{53}
$$

and, from its role in Equation 50, the group velocity is the velocity of energy flow. Furthermore, since $D_R^i = 0$ for each mode, the change in k after crossing Δr is given by

$$\frac{\partial \mathbf{k}}{\partial t} = \nabla D_R^i \left(\frac{\partial D_R^i}{\partial \omega} \right)^{-1} \tag{54}$$

Together, Equations 53 and 54 can be considered as the velocity of the mode-energy in (r, k) phase-space. Note that Equation 54 incorporates Snell's law, since k only changes in the direction of the gradient of D_R^i.

To obtain the physical optics solution to the problem of the wave propagation, the next-order terms of the WKB dispersion equation are used. These are equivalent to the energy conservation equation already derived (*ie.* Equation 50). This equation can be put into the familiar form if D_R^{ij} is diagonalized as before and if the total energy per mode is defined as

$$W_k^i = \frac{1}{8\pi} |E^i|^2 \frac{\partial D_R^i}{\partial \omega} \tag{55}$$

This gives

$$\nabla \cdot (\mathbf{v}_g^i W_k^i) + \frac{\partial W_k^i}{\partial t} - 2\mathbf{k}_I \cdot \mathbf{v}_g^i W_k^i = 0 \tag{56}$$

where $\mathbf{k}_I$ is the imaginary part of k given approximately by

$$\mathbf{k}_I = -D_I^i \left(\frac{\partial D_R^i}{\partial \mathbf{k}} \right)^{-1}$$

The solution to Equation 56 for each mode gives the physical optics solution to wave propagation. If the medium is loss-free, then the field intensity increases as $\sim 1/v_g^i$ along its ray path. When $v_g^i \to 0$, higher order derivatives of the field must be added to Equation 50. If the turning point is linear, then the D_R^i can be expanded about $\mathbf{r} \approx r_0$, giving

$$\frac{1}{2} \frac{\partial}{\partial \mathbf{k}} \frac{\partial}{\partial \mathbf{k}} D_R^i : \nabla \nabla \overline{E}^i - (\mathbf{r} - \mathbf{r}_0) \cdot (\nabla D_R^i) \overline{E}^i \approx 0 \tag{57}$$

Then, assuming that the spatial dependances are locally separable, then this equation is an Airy equation for that component of propagation along the gradient D_R^i. In this way, the standard WKB connection formulas and reflection coefficients can be calculated[13].

8. The local resonance function .

The local resonance function used in the WKB theory function differs from the bounce averaged version used to determine the electron energy evolution. The local resonance function includes both the reactive, induced plasma currents and the local, resonant dissipation. The induced currents determine the real part of the dispersion relation which is used to calculate $k = \nabla \lambda(r)$.

The local resonance function has three forms. For particles far from a stationary point, the resonance function is

$$\Omega_n^{-1} \approx J_n^2 \frac{i}{\nu(t)} \tag{58}$$

At this location, these particles are purely reactive.

When, a particle is near a stationary point, then

$$\Omega_n^{-1} \approx J_n^2 e^{-\pi/4} \sqrt{\pi} \tau_{eff} \tag{59}$$

And, when $\nu \approx \nu' \approx 0$, then $Re\{\Omega_n^{-1}\} \approx J_n^2 \pi \tau_{eff} Ai(\nu_n \tau_{eff})$, and $Im\{\Omega_n^{-1}\} \approx J_n^2 \pi \tau_{eff} Gi(\nu_n \tau_{eff})$ where $Gi(x) \sim 1/\pi x$ and $Gi(-x) \sim \pi^{-1/2} x^{-1/3} \cos(2x^{2/3}/3 + \pi/4)$ for large x.

9. Bounce-averaged energy conservation .

The local, energy conservation equation can be bounce-averaged to show the self-consistency of the approach used in the report. The total loss of wave energy averaged over the bounce motion of the trapped particles is equal to the bounce-averaged change in particle kinetic-energy.

The bounce-average of Equation 50 is

$$\oint \frac{ds\,\omega_B}{v_\parallel} \left\{ \nabla \cdot \mathbf{v}_g^i W_k^i + \frac{\partial}{\partial t} W_k^i \right\} = -\frac{1}{4\pi} \sum_{res} \overline{D_I^{ij}} |\overline{E^{i*} E^j}| \tag{60}$$

What is meant by the bounce average of the left-hand side is that the integral over velocity-space within each term is to be carried out after the bounce average. The equation states that the average of the divergence of the Poynting's flux and the time rate of change of the electric energy along the particle's orbit is equal to the loss of particle kinetic energy due to the local resonances. The integral on the left-hand side will not

be evaluated. However, the right-hand side is consistent with the bounce-averaged quasi-linear equation, previously derived. To show this, the time rate of change of the electron kinetic energy is

$$\begin{aligned}\frac{\partial}{\partial t}\sum_{\pm}\int\frac{dEd\mu B}{|v_{\|}|}EF_0 &= \int d^3v\frac{1}{2}m\,\mathbf{v}\cdot\mathbf{v}\frac{\partial F_0}{\partial t}\\ &= \frac{q^2}{m}\sum_{res}\int d^3v\,\frac{1}{2}\mathbf{v}\cdot\mathbf{v}M^{ij}\frac{\partial}{\partial v^i}Re\{\overline{\Omega_n^{-1}}\}|\overline{E}^i\overline{E}^j|_k M_{op}^{j*}F_0\end{aligned} \tag{61}$$

Equation 61 is now integrated by parts which is performed most easily when the left-most diffusion operator has been re-expressed in terms of a real, cartesian coordinate system. This gives

$$\begin{aligned}\frac{\partial}{\partial t}\sum_{\pm}\int\frac{dEd\mu B}{|v_{\|}|}EF_0 &= -\sum_{res}\frac{q^2}{m}\int\frac{dEd\mu B}{|v_{\|}|}v^i Re\{\overline{\Omega_n^{-1}}\}|\overline{E}^{i*}\overline{E}^j|_k M_{op}^{j*}F_0\\ &= \frac{1}{4\pi}\sum_{res}D_l^{ij*}|\overline{E}^{i*}\overline{E}^j|_k\end{aligned} \tag{62}$$

Therefore, the increase in trapped particle energy is equal to the loss of wave energy. For each region in velocity-space, the energy is exchanged at local, resonant interactions.

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