THE LECTURE HALL PARALLELEPIPED

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ABSTRACT. The $\boldsymbol{s}$-lecture hall polytopes $P_{\boldsymbol{s}}$ are a class of integer polytopes defined by Savage and Schuster which are closely related to the lecture hall partitions of Eriksson and Bousquet-Mélou. We define a half-open parallelepiped $\mathrm{Par}_{\boldsymbol{s}}$ associated with $P_{\boldsymbol{s}}$ and give a simple description of its integer points. We use this description to recover earlier results of Savage et al. on the δ-vector (or h^*-vector) and to obtain the connections to $\boldsymbol{s}$-ascents and $\boldsymbol{s}$-descents, as well as some generalizations of these results.

1. Introduction

Suppose that P is an n-dimensional integral polytope, i.e., a (convex) polytope whose vertices have integer coordinates. Let $i(P,t)$ be the number of lattice points in the tth dilation tP of P. Then $i(P,t)$ is a polynomial in t of degree n, called the *Ehrhart polynomial* of P [4]. One way to study the Ehrhart polynomial of an integral polytope is to consider its generating function $\sum_{t\geq 0} i(P,t)z^t$. It is known that the generating function has the form

$$\sum_{t\geq 0} i(P,t)z^t = \frac{\delta_P(z)}{(1-z)^{n+1}},$$

where $\delta_P(z)$ is a polynomial of degree at most n with nonnegative integer coefficients [9]. We denote by $\delta_{P,i}$ the coefficient of z^i in $\delta_P(z)$, for $0 \leq i \leq n$. Thus $\delta_P(z) = \sum_{i=0}^n \delta_{P,i} z^i$. For an n-dimensional polytope P in $\mathbb{R}^n$, the *normalized volume* $\mathrm{nvol}(P)$ is given by $\mathrm{nvol}(P) = n! \cdot \mathrm{vol}(P)$, where $\mathrm{vol}(P)$ is the usual volume (Lebesgue measure). Another well-known result is that $\delta_P(1) = \sum_{i=0}^n \delta_{P,i}$ is the normalized volume of P. We call $(\delta_{P,0}, \delta_{P,1}, \dots, \delta_{P,n})$ the *δ-vector* or *h^*-vector* of P. In this paper, we will investigate the δ-vectors of $\boldsymbol{s}$-lecture hall polytopes, which were introduced by Savage and Schuster [7]. A basic idea we use is a result by the second author [11, Lemma 4.5.7]: one can determine the δ-vector of an integral simplex by counting the number of lattice points inside an associated parallelepiped.

Let $\boldsymbol{s} = (s_1, \dots, s_n)$ be a sequence of positive integers. An *$\boldsymbol{s}$-lecture hall partition* is an integer sequence $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$ satisfying

$$0 \leq \frac{\lambda_1}{s_1} \leq \frac{\lambda_2}{s_2} \leq \cdots \leq \frac{\lambda_n}{s_n}.$$

When $\boldsymbol{s} = (1, 2, \dots, n)$, this gives the original lecture hall partitions introduced by Bousquet-Mélou and Eriksson [1]. Savage and Schuster [7] define the *s-lecture hall polytope* to be the

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polytope, denoted $P_{\boldsymbol{s}}$, in $\mathbb{R}^n$ defined by the inequalities

$$0 \leq \frac{x_1}{s_1} \leq \frac{x_2}{s_2} \leq \cdots \leq \frac{x_n}{s_n} \leq 1.$$

They use the $\boldsymbol{s}$-lecture hall polytopes to establish a connection between $\boldsymbol{s}$-lecture hall partitions and their geometric setup. A further result, appearing in [2], is that the Ehrhart polynomial of the lecture hall polytope associated to $\boldsymbol{s} = (1, 2, \dots, n)$ and the anti-lecture hall polytope associated to $s = (n, n-1, \dots, 1)$ is the same as that of the n-dimensional unit cube. It is well known that components in the δ-vector of the n-dimensional unit cube are Eulerian numbers, which count the number of permutations in $\mathfrak{S}_d$ with a certain number of descents [10, Prop. 1.4.4]. Thus, the same is true for $P_{(1,2\dots,n)}$ and $P_{(n,n-1,\dots,1)}$.

It is easy to see that $P_{\boldsymbol{s}}$ has the vertex set

$$\{(0,0,0,\dots,0),(0,0,\dots,0,s_n),(0,0,\dots,0,s_{n-1},s_n),\dots,(s_1,s_2,\dots,s_n)\}.$$

Hence $P_{\boldsymbol{s}}$ is a simplex with normalized volume $\prod_{i=1}^n s_i$. In particular, when $\boldsymbol{s} \in \mathfrak{S}_n$, the normalized volume is $n!$, which is exactly the cardinality of $\mathfrak{S}_n$. Thus, the sum of the components in the δ-vector of $P_{\boldsymbol{s}}$ is $\prod_{i=1}^n s_i$ or $n!$. On the other hand, since $P_{\boldsymbol{s}}$ is a simplex, its δ-vector corresponds to gradings of the lattice points in a fundamental parallelepiped associated to it. (See Lemma 2.3 for details.)

The original motivation of this paper was to give a bijection between $\mathfrak{S}_n$ and lattice points in the fundamental parallelepipeds associated to $P_{(1,2,\dots,n)}$ and $P_{(n,n-1,\dots,1)}$ so that we can recover the result of Corteel-Lee-Savage [2] on the Ehrhart polynomials of these polytopes. In fact we can extend our original aim to the fundamental parallelopiped associated to $P_{\boldsymbol{s}}$ for any sequence $\boldsymbol{s}$ of positive integers. Our results are stated in terms of ascents and descents of certain sequences associated to $\boldsymbol{s}$ which generalize the notion of the inversion sequence of a permutation. We also consider the connection between descents and ascents of sequences associated to $\boldsymbol{s}$ and the reverse of $\boldsymbol{s}$.

The paper is organized as follows. In Section 2, we review basic results of δ-vectors that are relevant to our paper, introduce the $\boldsymbol{s}$-lecture hall parallelepiped $\mathrm{Par}_{\boldsymbol{s}}$, and establish in Lemma 2.3 the connection between the number lattice points in $\mathrm{Par}_{\boldsymbol{s}}$ and the δ-vector of $P_{\boldsymbol{s}}$. In Section 3, we give a bijection $\mathrm{REM}_{\boldsymbol{s}}$ from the lattice points in $\mathrm{Par}_{\boldsymbol{s}}$ to some simple set (which we call Ψ_n). By figuring out the inverse of $\mathrm{REM}_{\boldsymbol{s}}$, we are able to describe in Theorem 3.9 the δ-vector of $P_{\boldsymbol{s}^*}$, a polytope closely related to $P_{\boldsymbol{s}}$, using the language of descents. A special situation of this theorem agrees with results by Savage-Schuster [7] on the δ-vector of $P_{\boldsymbol{s}}$. In Section 4, we apply results from Section 3 to the case when $\boldsymbol{s} = (n, n-1, \dots, 1)$ and recover the result of Corteel-Lee-Savage on the Ehrhart polynomial of the anti-lecture hall polytope. In Sections 5, we consider $\boldsymbol{s}$ and its reversal $\boldsymbol{u}$, and their corresponding polytopes $P_{\boldsymbol{s}}$ and $P_{\boldsymbol{u}}$, and provide a bijection from the lattice points $\mathrm{Par}_{\boldsymbol{s}^*}$ to the lattice points in $\mathrm{Par}_{\boldsymbol{u}^*}$ through maps defined in Section 3. In Section 6, we show that the δ-vector of $P_{\boldsymbol{s}}$ can be described using ascents of elements in Φ_n, using which we give the desired bijective proof for Corteel-Lee-Savage's result on the Ehrhart polynomial of $P_{(1,2,\dots,n)}$.

2. Background

For any nonnegative integer N, we denote by $\langle N \rangle$ the set $\{0, 1, \dots, N\}$.

2.1. **Descents, the δ-vector of a unit cube, etc.** Write $\mathbb{N} = \{0, 1, 2. \dots\}$. Let $\boldsymbol{r} = (r_1, \dots, r_n) \in \mathbb{N}^n$. We say that i is a *(regular) descent* of $\boldsymbol{r}$ if $r_i > r_{i+1}$. Define the *descent set* $\mathrm{Des}(\boldsymbol{r})$ of $\boldsymbol{r}$ by

$$\mathrm{Des}(\boldsymbol{r}) = \{i \mid r_i > r_{i+1}\},$$

and define its size $\mathrm{des}(\boldsymbol{r}) = \# \mathrm{Des}(\boldsymbol{r})$.

The *Eulerian number* $A(n, i)$ is the number of permutations $\pi \in \mathfrak{S}_n$ with exactly $i - 1$ descents. Let $\square_n$ denote the n-dimensional unit cube. Then the δ-vector of $\square_n$ is given by

$$\delta_{\square_n, i} = A(n, i + 1) = \#\{\pi \in \mathfrak{S}_n \mid \mathrm{des}(\pi) = i\}.$$

By [7, Corollary 1] we have that for $\boldsymbol{s} = (1, 2, \dots, n)$,

$$\delta_{P_{\boldsymbol{s}}, i} = A(n, i + 1) = \#\{\pi \in \mathfrak{S}_n \mid \mathrm{des}(\pi) = i\}.$$

There are many other statistics related to permutations also counted by Eulerian numbers. Given a permutation $\pi = (\pi_1, \dots, \pi_n) \in \mathfrak{S}_n$, a pair (π_j, π_k) is an *inversion* of π, if $j < k$ and $\pi_j > \pi_k$. Define $I(\pi) = (a_1, \dots, a_n)$, where a_i is the number of inversions (π_j, π_k) of π that ends with $i = \pi_k$. The sequence $I(\pi)$ is known as the *inversion sequence* or *inversion table* of the permutation π. Clearly, $I(\pi) \in \langle n - 1\rangle \times \cdots \times \langle 1\rangle \times \langle 0\rangle$. In fact, $I : \mathfrak{S}_n \to \langle n - 1\rangle \times \cdots \times \langle 1\rangle \times \langle 0\rangle$ is a bijection [10, Prop. 1.3.12]. In this paper, it is more convenient to use inversion sequences to represent permutations. We give statistics of inversion sequences that are counted by Eulerian numbers.

Lemma 2.1. *The number of inversion sequences of length n with i descents is the Eulerian number $A(n, i + 1)$.*

Proof. Suppose that $\boldsymbol{r} = (r_1, \dots, r_n)$ is the inversion sequence of $\pi \in \mathfrak{S}_n$. Then

$$\begin{aligned} i \text{ is a descent of } \boldsymbol{r}, \text{ i.e., } r_i > r_{i+1} &\iff i + 1 \text{ precedes } i \text{ in } \pi \\ &\iff i \text{ is a descent of } \pi^{-1}. \end{aligned}$$

The lemma follows from the fact that $\pi \mapsto \pi^{-1}$ is a bijection on $\mathfrak{S}_n$. $\square$

2.2. **δ-vector of simplices.** When P is a simplex, it is easy to describe its δ-vector. We first give some related definitions and notation.

For a set of independent vectors $W = \{\boldsymbol{w}_1, \dots, \boldsymbol{w}_n\}$, we denote by $\mathrm{Par}(W) = \mathrm{Par}(\boldsymbol{w}_1, \dots, \boldsymbol{w}_n)$ the *fundamental (half-open) parallelepiped generated by W*:

$$\mathrm{Par}(\boldsymbol{w}_1, \dots, \boldsymbol{w}_n) := \left\{ \sum_{i=1}^{n} c_i \boldsymbol{w}_i \mid 0 \le c_i < 1 \right\}.$$

For any set $S \subset \mathbb{Z}^N$, we denote by $\mathcal{L}^i(S)$ the set of lattice points in S whose last coordinates are i :

$$\mathcal{L}^i(S) := \left\{ \boldsymbol{x} \in S \cap \mathbb{Z}^N \mid \text{last coordinate of } \boldsymbol{x} \text{ is } i \right\},$$

and let $\ell^i(S) := \#\mathcal{L}^i(S)$ be the cardinality of $\mathcal{L}^i(S)$.

For convenience, for any vector $\boldsymbol{v} \in \mathbb{R}^N$, we let $\boldsymbol{v}^* := (\boldsymbol{v}, 1)$ be the vector obtained by appending 1 to the end of $\boldsymbol{v}$.

Suppose P is an n-dimensional simplex with vertices $\boldsymbol{v}_0, \boldsymbol{v}_1, \dots, \boldsymbol{v}_n$. Then the δ-vector of P is determined by the grading of the fundamental parallelepiped $\mathrm{Par}(\boldsymbol{v}_0^*, \dots, \boldsymbol{v}_n^*)$. More precisely [11, Lemma 4.5.7],

$$\delta_{P,i} = \ell^i(\mathrm{Par}(\boldsymbol{v}_0^*, \dots, \boldsymbol{v}_n^*)), \quad 0 \le i \le n. \tag{2.1}$$

(Note that $\ell^i(\mathrm{Par}(\boldsymbol{v}_0^*, \dots, \boldsymbol{v}_n^*)) = 0$ for all $i > n$.)

2.3. $\boldsymbol{s}$-lecture hall parallelepiped.

Definition 2.2. Given a sequence $\boldsymbol{s} = (s_1, \dots, s_n)$ of positive integers, the *$\boldsymbol{s}$-lecture hall parallelepiped*, denoted by $\mathrm{Par}_{\boldsymbol{s}}$, is the fundamental parallelepiped generated by the non-origin vertices of the $\boldsymbol{s}$-lecture hall polytope $P_{\boldsymbol{s}}$:

$$\mathrm{Par}_{\boldsymbol{s}} := \mathrm{Par}((0,0,\dots,0,s_n),(0,0,\dots,0,s_{n-1},s_n),\dots,(s_1,s_2,\dots,s_n)).$$

Lemma 2.3. *Suppose that $\boldsymbol{s} = (s_1, \dots, s_n)$ is a sequence of positive integers. Then the δ-vector of $P_{\boldsymbol{s}}$ is given by*

$$\delta_{P_{\boldsymbol{s}},i} = \ell^i(\mathrm{Par}_{\boldsymbol{s}^*}), \quad 0 \le i \le n. \tag{2.2}$$

Furthermore, if $s_n = 1$, then the two fundamental parallelepipeds $\mathrm{Par}_{\boldsymbol{s}}$ and $\mathrm{Par}_{\boldsymbol{s}^*}$ have the same grading:

$$\ell^i(\mathrm{Par}_{\boldsymbol{s}}) = \ell^i(\mathrm{Par}_{\boldsymbol{s}^*}), \quad 0 \le i \le n. \tag{2.3}$$

Hence,

$$\delta_{P_{\boldsymbol{s}},i} = \begin{cases} \ell^i(\mathrm{Par}_{\boldsymbol{s}}), & 0 \le i \le n-1; \\ 0, & i = n. \end{cases} \tag{2.4}$$

Proof. Formula (2.2) follows from (2.1) and the observation that

$$\mathrm{Par}_{\boldsymbol{s}^*} = \mathrm{Par}(\boldsymbol{v}^* \mid \boldsymbol{v} \text{ is a vertex of } P).$$

Suppose that $s_n = 1$. We claim that for any $\boldsymbol{x} \in \mathrm{Par}_{\boldsymbol{s}^*} \cap \mathbb{Z}^{n+1}$, the last two coordinates of $\boldsymbol{x}$ are the same. Let $\boldsymbol{x} = (x_1, \dots, x_{n+1}) \in \mathrm{Par}_{\boldsymbol{s}^*} \cap \mathbb{Z}^{n+1}$. There exist (unique) $c_1, \dots, c_n, c_{n+1} \in [0,1)$ such that

$$\begin{aligned} \boldsymbol{x} &= c_{n+1}(0,\dots,0,0,0,1) + c_n(0,\dots,0,1,1) + c_{n-1}(0,\dots,s_{n-1},1,1) \\ &\quad + \dots + c_1(s_1,\dots,s_{n-1},1,1). \end{aligned}$$

Then

$$x_n = \sum_{i=1}^{n} c_i, \text{ and } x_{n+1} = \sum_{i=1}^{n+1} c_i.$$

Since both x_n and x_{n+1} are integers, the number $c_{n+1} = x_{n+1} - x_n$ is an integer. However, $0 \le c_{n+1} < 1$. We must have that $c_{n+1} = 0$. Therefore $x_n = x_{n+1}$, so the claim holds.

By our claim, one sees that the map that drops the last coordinate of a point in $\mathbb{R}^{n+1}$ induces a bijection between $\mathcal{L}^i(\mathrm{Par}_{\boldsymbol{s}^*})$ and $\mathcal{L}^i(\mathrm{Par}_{\boldsymbol{s}})$ for every i. Hence equation (2.3) follows.

Finally, the last coordinate of any point in $\mathrm{Par}_{\boldsymbol{s}}$ is strictly smaller than $ns_n = n$. Hence, $\mathcal{L}^n(\mathrm{Par}_{\boldsymbol{s}}) = \emptyset$ and $\ell^n(\mathrm{Par}_{\boldsymbol{s}}) = 0$. Then formula (2.4) follows from (2.2) and (2.3). ☐

3. Bijections

Throughout this section, we assume that $\boldsymbol{s} = (s_1, \dots, s_n)$ is a sequence of positive integers. For brevity, for the rest of the paper, whenever $\boldsymbol{s}$ is a fixed sequence, we associate the following set to $\boldsymbol{s}$:

$$\Psi_n = \langle s_1 - 1 \rangle \times \dots \times \langle s_n - 1 \rangle.$$

This set coincides with the set I_n of $\boldsymbol{s}$-inversion sequences introduced in [7] and further investigated in [8, 5, 6].

Definition 3.1. We define a map

$$\mathrm{REM}_{\boldsymbol{s}} : \mathrm{Par}_{\boldsymbol{s}} \cap \mathbb{Z}^n \to \Psi_n$$

in the following way. Let $\boldsymbol{x} = (x_1, \ldots, x_n) \in \mathrm{Par}_{\boldsymbol{s}} \cap \mathbb{Z}^n$. For each x_i, let $k_i = \lfloor \frac{x_i}{s_i} \rfloor$ be the quotient of dividing x_i by s_i, and r_i be the remainder. Hence

$$x_i = k_i s_i + r_i,$$

where $k_i \in \langle n-1 \rangle$ and $r_i \in \langle s_i - 1 \rangle$. Let $\boldsymbol{k} = (k_1, \ldots, k_n)$ and $\boldsymbol{r} = (r_1, \ldots, r_n)$. Then we define $\mathrm{REM}_{\boldsymbol{s}}(\boldsymbol{x}) = \boldsymbol{r}$.

Lemma 3.2. $\mathrm{REM}_{\boldsymbol{s}}$ *is a bijection from* $\mathrm{Par}_{\boldsymbol{s}} \cap \mathbb{Z}^n$ *to* Ψ_n.

In order to prove Lemma 3.2, we write $\mathrm{REM}_{\boldsymbol{s}}$ as a composition of two maps. Let

$$f_{\boldsymbol{s}} : \mathrm{Par}_{\boldsymbol{s}} \cap \mathbb{Z}^n \to \langle n-1 \rangle^n \times \Psi_n, \quad \boldsymbol{x} \mapsto (\boldsymbol{k}, \boldsymbol{r}),$$

where $\boldsymbol{k}$ and $\boldsymbol{r}$ are defined as in Definition 3.1. We denote by $\mathrm{KR}_{\boldsymbol{s}}$ the image set of $\mathrm{Par}_{\boldsymbol{s}} \cap \mathbb{Z}^n$ under the map $f_{\boldsymbol{s}}$. It is clear that the map $f_{\boldsymbol{s}}$ is a bijection between $\mathrm{Par}_{\boldsymbol{s}} \cap \mathbb{Z}^n$ and $\mathrm{KR}_{\boldsymbol{s}}$.

Let

$$g_{\boldsymbol{s}} : \mathrm{KR}_{\boldsymbol{s}} \to \Psi_n, \quad (\boldsymbol{k}, \boldsymbol{r}) \mapsto \boldsymbol{r}.$$

Clearly, $\mathrm{REM}_{\boldsymbol{s}}$ is the composition of $f_{\boldsymbol{s}}$ and $g_{\boldsymbol{s}}$, and Lemma 3.2 follows from the following lemma.

Lemma 3.3. *The map* $g_{\boldsymbol{s}}$ *gives a bijection between* $\mathrm{KR}_{\boldsymbol{s}}$ *and* Ψ_n.

To prove Lemma 3.3, we will construct an inverse for $g_{\boldsymbol{s}}$; in other words, we will show how to recover the quotient vector $\boldsymbol{k}$ from the remainder vector $\boldsymbol{r}$. We give the following preliminary definition and lemma.

Definition 3.4. Let $\boldsymbol{r} \in \mathbb{N}^n$. We say that i is an $\boldsymbol{s}$*-descent of* $\boldsymbol{r}$ if $\dfrac{r_i}{s_i} > \dfrac{r_{i+1}}{s_{i+1}}$.

We denote by $\mathrm{Des}_{\boldsymbol{s}}(\boldsymbol{r})$ the set of $\boldsymbol{s}$-descents of $\boldsymbol{r}$, and let $\mathrm{des}_{\boldsymbol{s}}(\boldsymbol{r}) = \#\,\mathrm{Des}_{\boldsymbol{s}}(\boldsymbol{r})$ be its cardinality. For any $1 \leq i \leq n$, we let $\mathrm{Des}_{\boldsymbol{s}}^{<i}(\boldsymbol{r})$ be the set of $\boldsymbol{s}$-descents of $\boldsymbol{r}$ whose indices are strictly smaller than i :

$$\mathrm{Des}_{\boldsymbol{s}}^{<i}(\boldsymbol{r}) = \{ j < i \mid j \text{ is an } \boldsymbol{s}\text{-descent of } \boldsymbol{r} \},$$

and $\mathrm{des}_{\boldsymbol{s}}^{<i}(\boldsymbol{r}) = \#\,\mathrm{Des}_{\boldsymbol{s}}^{<i}(\boldsymbol{r})$ be its cardinality.

We similarly define $\mathrm{Des}^{<i}$ and $\mathrm{des}^{<i}$ for (regular) descents.

Lemma 3.5. a) *A point* $\boldsymbol{x} = (x_1, \ldots, x_n)$ *is in* $\mathrm{Par}_{\boldsymbol{s}}$ *if and only if*

$$0 \leq \frac{x_1}{s_1} < 1, \text{ and } 0 \leq \frac{x_{i+1}}{s_{i+1}} - \frac{x_i}{s_i} < 1, \quad 1 \leq i \leq n-1.$$

b) *Let* $\boldsymbol{r} \in \Psi_n$. *Then a point* $(\boldsymbol{k}, \boldsymbol{r}) = ((k_1, \ldots, k_n), \boldsymbol{r})$ *is in* $\mathrm{KR}_{\boldsymbol{s}}$ *if and only if* $k_1 = 0$ *and for any* $i \in \{1, \ldots, n-1\}$,

$$k_{i+1} - k_i = \begin{cases} 1, & \text{if } i \text{ is an } \boldsymbol{s}\text{-descent of } \boldsymbol{r}; \\ 0, & \text{otherwise.} \end{cases} \tag{3.1}$$

Proof. First, $\boldsymbol{x} \in \mathrm{Par}_{\boldsymbol{s}}$ if and only if there exists $c_1, \ldots, c_n$ in $[0, 1)$ such that

$$\begin{aligned} \boldsymbol{x} &= c_n(0, \ldots, 0, s_n) + c_{n-1}(0, \ldots, 0, s_{n-1}, s_n) + \cdots + c_1(s_1, \ldots, s_n) \\ &= (s_1c_1,\ s_2(c_1 + c_2),\ \ldots,\ s_n(c_1 + c_2 + \cdots + c_n)). \end{aligned}$$

This is equivalent to the existence of $c_1, \ldots, c_n \in [0, 1)$ such that

$$\begin{aligned} \frac{x_1}{s_1} &= c_1 \\ \frac{x_2}{s_2} &= c_1 + c_2 \\ &\vdots \\ \frac{x_n}{s_n} &= c_1 + c_2 + \cdots + c_n \end{aligned}$$

Solving the above equations for c_i's, we see that a) follows.

To prove b), we let $x_i = k_is_i + r_i$ for each i. Note that $(\boldsymbol{k}, \boldsymbol{r}) \in \mathrm{KR}_{\boldsymbol{s}}$ if and only if $\boldsymbol{x} := (x_1, \ldots, x_n) \in \mathrm{Par}_{\boldsymbol{s}} \cap \mathbb{Z}^n$, which is equivalent to $\boldsymbol{k} \in \mathbb{Z}^n$ and $\boldsymbol{x} \in \mathrm{Par}_{\boldsymbol{s}}$. Applying part a) to $\boldsymbol{x}$, we get that $(\boldsymbol{k}, \boldsymbol{r}) \in \mathrm{KR}_{\boldsymbol{s}}$ if and only if $\boldsymbol{k} \in \mathbb{Z}^n$ and

$$0 \le \frac{k_1s_1 + r_1}{s_1} < 1, \text{ and } 0 \le \frac{k_{i+1}s_{i+1} + r_{i+1}}{s_{i+1}} - \frac{k_is_i + r_i}{s_i} < 1, \quad 1 \le i \le n - 1.$$

The above inequalities are equivalent to

$$0 \le k_1 + \frac{r_1}{s_1} < 1, \text{ and } 0 \le k_{i+1} - k_i + \frac{r_{i+1}}{s_{i+1}} - \frac{r_i}{s_i} < 1, \quad 1 \le i \le n - 1.$$

Note that $0 \le \frac{r_1}{s_1} < 1$ and $-1 < \frac{r_{i+1}}{s_{i+1}} - \frac{r_i}{s_i} < 1$. One checks that

$$k_1 \in \mathbb{Z} \text{ and } 0 \le k_1 + \frac{r_1}{s_1} < 1 \iff k_1 = 0,$$

and for any $1 \le i \le n - 1$ given $k_i \in \mathbb{Z}$,

$$k_{i+1} \in \mathbb{Z} \text{ and } 0 \le k_{i+1} - k_i + \frac{r_{i+1}}{s_{i+1}} - \frac{r_i}{s_i} < 1 \iff k_{i+1} - k_i \text{ is given as in (3.1).}$$

Therefore, we have b). □

Part b) of Lemma 3.5 provides us a way to construct the inverse of $g_{\boldsymbol{s}}$. For any $\boldsymbol{r} \in \Psi_n$, we define $h_{\boldsymbol{s}}(\boldsymbol{r}) = (\boldsymbol{k}, \boldsymbol{r}) = ((k_1, \ldots, k_n), \boldsymbol{r})$, where

$$k_i = \mathrm{des}_{\boldsymbol{s}}^{<i}(\boldsymbol{r}), \quad 1 \le i \le n.$$

By Lemma 3.5(b) we see that $h_{\boldsymbol{s}}$ is the inverse of $g_{\boldsymbol{s}}$. Hence, we have proved Lemma 3.3. Our discussion also gives us the inverse map of $\mathrm{REM}_{\boldsymbol{s}}$.

Theorem 3.6. *The inverse of the map* $\mathrm{REM}_{\boldsymbol{s}}$ *(defined in Definition 3.1) is:*

$$\begin{aligned} \mathrm{REM}_{\boldsymbol{s}}^{-1} : \Psi_n &\to \mathrm{Par}_{\boldsymbol{s}} \cap \mathbb{Z}^n \\ \boldsymbol{r} = (r_1, \ldots, r_n) &\mapsto (\mathrm{des}_{\boldsymbol{s}}^{<1}(\boldsymbol{r})s_1 + r_1, \ldots, \mathrm{des}_{\boldsymbol{s}}^{<n}(\boldsymbol{r})s_n + r_n) \end{aligned}$$

Note that $\mathrm{des}_{\boldsymbol{s}}^{<n}(\boldsymbol{r}) = \mathrm{des}_{\boldsymbol{s}}(\boldsymbol{r})$, and thus when s_n=1,

$$\mathrm{des}_{\boldsymbol{s}}^{<n}(\boldsymbol{r})s_n + r_n = \mathrm{des}_{\boldsymbol{s}}(\boldsymbol{r}).$$

Hence we have the following result.

Corollary 3.7. *Suppose* $s_n = 1$. *Then*

$$\begin{aligned}\mathcal{L}^i(\mathrm{Par}_{\boldsymbol{s}}) &= \{\boldsymbol{x} \in \mathrm{Par}_{\boldsymbol{s}} \cap \mathbb{Z}^n \mid \mathrm{des}_{\boldsymbol{s}}(\mathrm{REM}_{\boldsymbol{s}}(\boldsymbol{x})) = i\} \\ &= \{\mathrm{REM}_{\boldsymbol{s}}^{-1}(\boldsymbol{r}) \mid \mathrm{des}_{\boldsymbol{s}}(\boldsymbol{r}) = i,\ \boldsymbol{r} \in \Psi_n\},\end{aligned}$$

and

$$\ell^i(\mathrm{Par}_{\boldsymbol{s}}) = \#\{\boldsymbol{r} \in \Psi_n \mid \mathrm{des}_{\boldsymbol{s}}(\boldsymbol{r}) = i\}.$$

Applying this to s^* whose last coordinate is 1 by definition, we get the next corollary.

Corollary 3.8.

$$\begin{aligned}\mathcal{L}^i(\mathrm{Par}_{\boldsymbol{s}^*}) &= \{\boldsymbol{x} \in \mathrm{Par}_{\boldsymbol{s}^*} \cap \mathbb{Z}^{n+1} \mid \mathrm{des}_{\boldsymbol{s}^*}(\mathrm{REM}_{\boldsymbol{s}^*}(\boldsymbol{x})) = i\} \\ &= \{\mathrm{REM}_{\boldsymbol{s}^*}^{-1}(\boldsymbol{r}) \mid \mathrm{des}_{\boldsymbol{s}^*}(\boldsymbol{r}) = i,\ \boldsymbol{r} \in \Psi_n \times \langle 0 \rangle\},\end{aligned}$$

and

$$\ell^i(\mathrm{Par}_{\boldsymbol{s}^*}) = \#\{\boldsymbol{r} \in \Psi_n \times \langle 0 \rangle \mid \mathrm{des}_{\boldsymbol{s}^*}(\boldsymbol{r}) = i\}.$$

The above two corollaries, together with Lemma 2.3, give the following result on δ-vectors of the $\boldsymbol{s}$-lecture hall polytope.

Theorem 3.9. *Suppose that* $\boldsymbol{s} = (s_1, \ldots, s_n)$ *is a sequence of positive integers. Then the* δ*-vector of the* $\boldsymbol{s}$*-lecture hall polytope* $P_{\boldsymbol{s}}$ *is given by*

$$\delta_{P_{\boldsymbol{s}^*},i} = \#\{\boldsymbol{r} \in \Psi_n \times \langle 0 \rangle \mid \mathrm{des}_{\boldsymbol{s}^*}(\boldsymbol{r}) = i\}, \quad 0 \le i \le n. \tag{3.2}$$

Furthermore, if $s_n = 1$ *then*

$$\delta_{P_{\boldsymbol{s}},i} = \#\{\boldsymbol{r} \in \Psi_n \mid \mathrm{des}_{\boldsymbol{s}}(\boldsymbol{r}) = i\}, \quad 0 \le i \le n. \tag{3.3}$$

We note that equation (3.3) agrees with Corollary 4 in [7].

The bijection REM is not always the most convenient one to use. Fortunately, there are many bijections between $\mathrm{Par}_{\boldsymbol{s}} \cap \mathbb{Z}^n$ and Ψ_n that can be constructed from REM: for any bijection

$$b : \Psi_n \to \Psi_n$$

the composition of REM and b is another bijection from $\mathrm{Par}_{\boldsymbol{s}} \cap \mathbb{Z}^n$ to Ψ_n

Definition 3.10. Let $\boldsymbol{q} = (q_1, \ldots, q_n) \in \Psi_n$.

a) We define

$$\begin{aligned}\mathrm{REM}_{\boldsymbol{s}}^{\boldsymbol{q}} : \mathrm{Par}_{\boldsymbol{s}} \cap \mathbb{Z}^n &\to \Psi_n \\ \boldsymbol{x} = (x_1, \ldots, x_n) &\mapsto \boldsymbol{y} = (y_1, \ldots, y_n),\end{aligned}$$

where

$$y_i = x_i + q_i \mod s_i.$$

Note that when $\boldsymbol{q} = (0, \ldots, 0)$, the map $\mathrm{REM}_{\boldsymbol{s}}^{\boldsymbol{q}}$ is the same as $\mathrm{REM}_{\boldsymbol{s}}$.

b) We define

$$\begin{aligned}\overline{\mathrm{REM}}_{\boldsymbol{s}}^{\boldsymbol{q}} : \mathrm{Par}_{\boldsymbol{s}} \cap \mathbb{Z}^n &\to \Psi_n \\ \boldsymbol{x} = (x_1, \ldots, x_n) &\mapsto \boldsymbol{z} = (z_1, \ldots, z_n),\end{aligned}$$

where

$$x_i + z_i = q_i \mod s_i.$$

When $\boldsymbol{q} = (0, \ldots, 0)$, we abbreviate $\overline{\mathrm{REM}}_{\boldsymbol{s}}^{\boldsymbol{q}}$ to $\overline{\mathrm{REM}}_{\boldsymbol{s}}$.

Lemma 3.11. *Let $\boldsymbol{q} = (q_1, \ldots, q_n) \in \Psi_n$. Then both* $\mathrm{REM}_{\boldsymbol{s}}^{\boldsymbol{q}}$ *and* $\overline{\mathrm{REM}}_{\boldsymbol{s}}^{\boldsymbol{q}}$ *are bijections from* $\mathrm{Par}_{\boldsymbol{s}} \cap \mathbb{Z}^n$ *to* Ψ_n.

Proof. Let

$$\begin{aligned} \Phi_{\boldsymbol{s}}^{\boldsymbol{q}} : \Psi_n &\to \Psi_n \\ \boldsymbol{r} = (r_1, \ldots, r_n) &\mapsto \boldsymbol{z} = (z_1, \ldots, z_n), \end{aligned} \tag{3.4}$$

where

$$r_i + z_i = q_i \mod s_i.$$

Clearly, $\Phi_{\boldsymbol{s}}^{\boldsymbol{q}}$ is a bijection and $\overline{\mathrm{REM}}_{\boldsymbol{s}}^{\boldsymbol{q}} = \Phi_{\boldsymbol{s}}^{\boldsymbol{q}} \circ \mathrm{REM}$. Hence, $\overline{\mathrm{REM}}_{\boldsymbol{s}}^{\boldsymbol{q}}$ is a bijection. The proof is similar for $\mathrm{REM}_{\boldsymbol{s}}^{\boldsymbol{q}}$. □

4. The anti-lecture hall parallelepiped

In this section, we will focus on the case when $\boldsymbol{s} = (n, n-1, \ldots, 1)$. For consistency with the terminology in [3], we call the associated parallelepiped the *anti-lecture hall parallelepiped*. The following theorem is the main result of this section, originally proved by Corteel-Lee-Savage [2, Corollary 4].

Theorem 4.1. *The Ehrhart polynomial of the anti-lecture hall polytope $P_{(n,n-1,\ldots,2,1)}$ is the same as that of the n-dimensional cube:*

$$i(P_{(n,n-1,\ldots,2,1)}, t) = (t+1)^n;$$

or equivalently, the δ-vector of $P_{(n,n-1,\ldots,2,1)}$ is given by

$$\delta_{P_{(n,n-1,\ldots,2,1)},i} = A(n, i+1).$$

The following lemma is the key ingredient for proving Theorem 4.1.

Lemma 4.2. *Suppose that s, s' are positive integers and $s - s' = 1$. Let $r \in \langle s-1 \rangle$ and $r' \in \langle s'-1 \rangle$. Then*

$$\frac{r}{s} > \frac{r'}{s'} \iff r > r'.$$

Proof. First,

$$\frac{r}{s} > \frac{r'}{s'} \iff rs' > r's \iff (r-r')s' > r'(s-s') = r'.$$

We then show $r > r'$ if and only if $(r-r')s' > r'$. Suppose $r > r'$, we have $r - r' \geq 1$. So $(r-r')s' \geq s' > r'$. Conversely, suppose $r \leq r'$. Then $r - r' \leq 0$. Thus, $(r-r')s' \leq 0 \leq r'$. □

By Lemma 4.2, one sees that if $\boldsymbol{s} = (n, n-1, \ldots, 2, 1)$, then for any $\boldsymbol{r} \in \langle n-1 \rangle \times \cdots \times \langle 0 \rangle$, $\boldsymbol{s}$-descents of $\boldsymbol{r}$ are the same as regular descents of $\boldsymbol{r}$. Hence, we get the following two corollaries as special cases of Theorem 3.6 and Corollary 3.7.

Corollary 4.3. *Let $\boldsymbol{s} = (n, n-1, \ldots, 2, 1)$. Then the map* $\mathrm{REM}_{\boldsymbol{s}}$ *give a bijection between* $\mathrm{Par}_{\boldsymbol{s}} \cap \mathbb{Z}^n$ *and inversion sequences of length n.*

Moreover, the inverse of $\mathrm{REM}_{\boldsymbol{s}}$ *is given by*

$$\begin{aligned} \mathrm{REM}_{\boldsymbol{s}}^{-1} : \langle n-1 \rangle \times \cdots \times \langle 0 \rangle &\to \mathrm{Par}_{\boldsymbol{s}} \cap \mathbb{Z}^n \\ \boldsymbol{r} = (r_1, \ldots, r_n) &\mapsto (\mathrm{des}^{<1}(\boldsymbol{r}) s_1 + r_1, \ldots, \mathrm{des}^{<n}(\boldsymbol{r}) s_n + r_n). \end{aligned}$$

Corollary 4.4. *If* $\boldsymbol{s} = (n, n-1, \dots, 2, 1)$, *we have that*

$$\begin{aligned} \mathcal{L}^i(\mathrm{Par}_{\boldsymbol{s}}) &= \{\boldsymbol{x} \in \mathrm{Par}_{\boldsymbol{s}} \cap \mathbb{Z}^n \mid \mathrm{des}(\mathrm{REM}_{\boldsymbol{s}}(\boldsymbol{x})) = i\} \\ &= \{\mathrm{REM}_{\boldsymbol{s}}^{-1}(\boldsymbol{r}) \mid \mathrm{des}(\boldsymbol{r}) = i, \ \boldsymbol{r} \in \langle n-1 \rangle \times \cdots \times \langle 0 \rangle\}, \end{aligned}$$

and

$$\begin{aligned} \ell^i(\mathrm{Par}_{\boldsymbol{s}}) &= \#\{\boldsymbol{r} \in \langle n-1 \rangle \times \cdots \times \langle 0 \rangle \mid \mathrm{des}(\boldsymbol{r}) = i\} \\ &= \#\,\text{inversion sequences of length } n \text{ that have } i \text{ descents.} \end{aligned}$$

Proof of Theorem 4.1. The theorem follows from Lemma 2.1, formula (2.4), and Corollary 4.4. □

5. The reversal of the sequence $\boldsymbol{s}$

In this section, we assume that $\boldsymbol{s} = (s_1, \dots, s_n)$ is a sequence of positive integers and $\boldsymbol{u} = (u_1, \dots, u_n) = (s_n, \dots, s_1)$ is the reverse of $\boldsymbol{s}$. Recall we associate the following set to $\boldsymbol{s}$:

$$\Psi_n = \langle s_1 - 1 \rangle \times \cdots \times \langle s_n - 1 \rangle.$$

Similarly, we associate a set to $\boldsymbol{u}$:

$$\bar{\Psi}_n = \langle u_1 - 1 \rangle \times \cdots \times \langle u_n - 1 \rangle = \langle s_n - 1 \rangle \times \cdots \times \langle s_1 - 1 \rangle.$$

As usual, we let

$$\boldsymbol{s}^* = (s_1, \dots, s_n, s_{n+1} = 1), \text{ and } \boldsymbol{u}^* = (u_1, \dots, u_n, u_{n+1} = 1).$$

The following lemma suggests a question (Question 5.4), which is the primary motivation for this section.

Lemma 5.1. *The Ehrhart polynomial of the* $\boldsymbol{s}$*-lecture hall polytope* $P_{\boldsymbol{s}}$ *is the same as the Ehrhart polynomial of the* $\boldsymbol{u}$*-lecture hall polytope* $P_{\boldsymbol{u}}$*; or equivalently,* $P_{\boldsymbol{s}}$ *and* $P_{\boldsymbol{u}}$ *have the same* δ*-vectors.*

Remark 5.2. Note that Theorem 4.1 and Lemma 5.1 recover the result on the Ehrhart polynomial of the lecture hall polytope $P_{\boldsymbol{s}}$, where $\boldsymbol{s} = (1, 2, \dots, n)$, given in [2, Corollary 2(i)] and [7, Corollary 1]. However, we want to describe a bijection from the lattice points in the fundamental parallelepiped associated to $P_{(1,2,\dots,n)}$ to inversion sequences. We will give such a bijection in Proposition 6.4 in the next section.

The proof of Lemma 5.1 is straightforward and is also proved in [2]. We defer it to the end of the section.

The following result follows immediately from Theorem 3.9 and Lemma 5.1.

Corollary 5.3. *For each* $i : 0 \le i \le n$, *the two sets*

$$\{\boldsymbol{r} \in \Psi_n \times \langle 0 \rangle \mid \mathrm{des}_{\boldsymbol{s}^*}(\boldsymbol{r}) = i\} \tag{5.1}$$

and

$$\{\boldsymbol{r} \in \bar{\Psi}_n \times \langle 0 \rangle \mid \mathrm{des}_{\boldsymbol{u}^*}(\boldsymbol{r}) = i\} \tag{5.2}$$

have the same cardinality.

One natural question arises: can we give a simple bijection from $\Psi_n \times \langle 0 \rangle$ to $\bar{\Psi}_n \times \langle 0 \rangle$ such that it induces a bijection from the set (5.1) to the set (5.2) for each i. Note that the last coordinates of any vector in $\Psi_n \times \langle 0 \rangle$ or $\bar{\Psi}_n \times \langle 0 \rangle$ is 0, which does not carry any information. For convenience, we drop the last coordinate when describe the bijection. Hence, we rephrase the question as follows:

Question 5.4. Can we give a simple bijection b from Ψ_n to $\bar{\Psi}_n$ such that the map $(\boldsymbol{r}, 0) \mapsto (b(\boldsymbol{r}), 0)$ induces a bijection from the set (5.1) to the set (5.2) for each i?

Before discussing Question 5.4, we define a simple function and fix some notation related to $\boldsymbol{s}$ and $\boldsymbol{u}$.

Definition 5.5. For any sequence/vector $\boldsymbol{r}$, we denote by $\operatorname{reverse}(\boldsymbol{r})$ the reverse of $\boldsymbol{r}$.

Notation 5.6. In addition to the usual notation $\boldsymbol{s}^*$ and $\boldsymbol{u}^*$, we also define the following vectors related to $\boldsymbol{s}$ and $\boldsymbol{u}$:

$$\begin{aligned} \tilde{\boldsymbol{s}} &:= (s_0 = 1, s_1, \ldots, s_n, s_{n+1} = 1) \\ \tilde{\boldsymbol{u}} &:= (u_0 = 1, u_1, \ldots, u_n, u_{n+1} = 1). \end{aligned}$$

Hence, $\tilde{\boldsymbol{u}}$ is the reverse of $\tilde{\boldsymbol{s}}$.

In order to describe a bijection asked by Question 5.4, we recall the bijection $\Phi_{\boldsymbol{s}}^{\boldsymbol{q}}$ defined in (3.4). The map $\Phi_{\boldsymbol{s}}^{\boldsymbol{0}}$ is important for this section, so we repeat its definition here.

Definition 5.7. Let $\boldsymbol{s} = (s_1, \ldots, s_n)$ be a sequence of positive integers and $\boldsymbol{0} = (0, \ldots, 0)$. Define

$$\begin{aligned} \Phi_{\boldsymbol{s}}^{\boldsymbol{0}} : \Psi_n &\to \Psi_n \\ \boldsymbol{r} = (r_1, \ldots, r_n) &\mapsto \boldsymbol{z} = (z_1, \ldots, z_n), \end{aligned}$$

where

$$r_i + z_i = 0 \mod s_i.$$

For convenience, we abbreviate $\Phi_{\boldsymbol{s}}^{\boldsymbol{0}}$ to $\Phi_{\boldsymbol{s}}$.

The following theorem is the main result of this section, which provides a desired bijection to Question 5.4.

Theorem 5.8. *For any $\boldsymbol{r} \in \Psi_n$, we have*

$$\operatorname{des}_{\boldsymbol{s}^*}(\boldsymbol{r}, 0) = \operatorname{des}_{\boldsymbol{u}^*}(\operatorname{reverse}(\Phi_{\boldsymbol{s}}(\boldsymbol{r})), 0). \tag{5.3}$$

By (5.3), one sees that the map $(\operatorname{reverse} \circ \Phi_{\boldsymbol{s}})$ is an answer to Question 5.4. If we put all the maps together, we have the following diagram, denoting by π the map that drops the last coordinate of a vector.

$$\begin{array}{ccccc} \operatorname{Par}_{\boldsymbol{s}^*} \cap \mathbb{Z}^{n+1} & \xrightarrow{\operatorname{REM}_{\boldsymbol{s}^*}} & \Psi_n \times \langle 0 \rangle & \xrightarrow{\pi} & \Psi_n \\ & & & & \downarrow \Phi_{\boldsymbol{s}} \\ & & & & \Psi_n \\ & & & & \downarrow \text{reverse} \\ \operatorname{Par}_{\boldsymbol{u}^*} \cap \mathbb{Z}^{n+1} & \xrightarrow{\operatorname{REM}_{\boldsymbol{u}^*}} & \bar{\Psi}_n \times \langle 0 \rangle & \xrightarrow{\pi} & \bar{\Psi}_n. \end{array}$$

Note that all the maps in the above diagram are bijections. Going around the diagram from $\mathrm{Par}_{\boldsymbol{s}^*} \cap \mathbb{Z}^{n+1}$ to $\mathrm{Par}_{\boldsymbol{u}^*} \cap \mathbb{Z}^{n+1}$, we obtain a bijection $\Gamma : \mathrm{Par}_{\boldsymbol{s}^*} \cap \mathbb{Z}^{n+1} \to \mathrm{Par}_{\boldsymbol{u}^*} \cap \mathbb{Z}^{n+1}$. By Theorem 5.8 and Corollary 3.8, we have that Γ induces a bijection from $\mathcal{L}^i(\mathrm{Par}_{\boldsymbol{s}^*})$ to $\mathcal{L}^i(\mathrm{Par}_{\boldsymbol{u}^*})$ for each i.

We can also simplify the above diagram slightly. One checks that

$$\Phi_{\boldsymbol{s}} \circ \pi \circ \mathrm{REM}_{\boldsymbol{s}^*} = \pi \circ \Phi_{\boldsymbol{s}^*} \circ \mathrm{REM}_{\boldsymbol{s}*} = \pi \circ \overline{\mathrm{REM}}_{\boldsymbol{s}^*},$$

where $\overline{\mathrm{REM}}_{\boldsymbol{s}^*}$ is defined in Definition 3.10. Then we redraw the diagram:

$$\begin{array}{ccccc}
\mathrm{Par}_{\boldsymbol{s}^*} \cap \mathbb{Z}^{n+1} & \xrightarrow{\overline{\mathrm{REM}}_{\boldsymbol{s}^*}} & \Psi_n \times \langle 0 \rangle & \xrightarrow{\pi} & \Psi_n \\
 & & & & \big\downarrow \text{reverse} \\
\mathrm{Par}_{\boldsymbol{u}^*} \cap \mathbb{Z}^{n+1} & \xrightarrow{\mathrm{REM}_{\boldsymbol{u}^*}} & \bar{\Psi}_n \times \langle 0 \rangle & \xrightarrow{\pi} & \bar{\Psi}_n .
\end{array}$$

This illustrates that if we use $\overline{\mathrm{REM}}_{\boldsymbol{s}^*}$ for $\mathrm{Par}_{\boldsymbol{s}^*} \cap \mathbb{Z}^{n+1}$ and $\mathrm{REM}_{\boldsymbol{u}^*}$ for $\mathrm{Par}_{\boldsymbol{u}^*} \cap \mathbb{Z}^{n+1}$, their image sets have very simple correspondence.

Corollary 5.9. *Let $\boldsymbol{r} \in \Psi_n$. Then for each i,*

$$\overline{\mathrm{REM}}_{\boldsymbol{s}^*}^{-1}(\boldsymbol{r}, 0) \in \mathcal{L}^i(\mathrm{Par}_{\boldsymbol{s}^*}) \iff \mathrm{REM}_{\boldsymbol{u}^*}^{-1}(\mathrm{reverse}(\boldsymbol{r}), 0) \in \mathcal{L}^i(\mathrm{Par}_{\boldsymbol{u}^*}).$$

One sees that the bijection $\overline{\mathrm{REM}}_{\boldsymbol{s}^*}$ is useful sometimes. Despite this, in general we do not have similar results for $\overline{\mathrm{REM}}_{\boldsymbol{s}*}$ as those for $\mathrm{REM}_{\boldsymbol{s}}$ or $\mathrm{REM}_{\boldsymbol{s}^*}$ described in Theorem 3.6 and Corollary 3.8. However, we will show in the next section that $\overline{\mathrm{REM}}_{\boldsymbol{s}^*}$ has a comparable result for the special cases when $s_1 = 1$.

We need a preliminary lemma before proving Theorem 5.8. The statement of this lemma involves *ascents*.

Definition 5.10. Let $\boldsymbol{s} = (s_1, \dots, s_n)$ be a sequence of positive integers and $\boldsymbol{r} = (r_1, \dots, r_n) \in \mathbb{N}^n$. We say that i is an *$\boldsymbol{s}$-ascent of $\boldsymbol{r}$* if $\dfrac{r_i}{s_i} < \dfrac{r_{i+1}}{s_{i+1}}$.

We denote by $\mathrm{Asc}_{\boldsymbol{s}}(\boldsymbol{r})$ the set of $\boldsymbol{s}$-ascents of $\boldsymbol{r}$, and let $\mathrm{asc}_{\boldsymbol{s}}(\boldsymbol{r}) = \#\,\mathrm{Asc}_{\boldsymbol{s}}(\boldsymbol{r})$ be its cardinality.

When $\boldsymbol{s} = (1, 1, \dots, 1)$, we get the *(regular) ascents*. We use notation $\mathrm{Asc}(\boldsymbol{r})$ and $\mathrm{asc}(\boldsymbol{r})$ for this case.

Lemma 5.11. *Recall that $\tilde{\boldsymbol{s}}$ and $\tilde{\boldsymbol{u}}$ are defined in Notation 5.6. For any $\boldsymbol{r} \in \langle 0 \rangle \times \Psi_n \times \langle 0 \rangle$, we have*

$$\mathrm{des}_{\tilde{\boldsymbol{s}}}(\boldsymbol{r}) = \mathrm{asc}_{\tilde{\boldsymbol{s}}}(\Phi_{\tilde{\boldsymbol{s}}}(\boldsymbol{r})) = \mathrm{des}_{\tilde{\boldsymbol{u}}}(\mathrm{reverse}(\Phi_{\tilde{\boldsymbol{s}}}(\boldsymbol{r}))). \tag{5.4}$$

Proof. Note apply Definition 5.7 to $\tilde{\boldsymbol{s}}$, we have

$$\begin{aligned}
\Phi_{\tilde{\boldsymbol{s}}} : \langle 0 \rangle \times \Psi_n \times \langle 0 \rangle &\to \langle 0 \rangle \times \Psi_n \times \langle 0 \rangle \\
\boldsymbol{r} = (r_0 = 0, r_1, \dots, r_n, r_{n+1} = 0) &\mapsto \boldsymbol{z} = (z_0 = 0, z_1, \dots, z_n, z_{n+1} = 0),
\end{aligned}$$

where

$$r_i + z_i = 0 \mod s_i.$$

Let $\boldsymbol{r} = (r_0, r_1, \dots, r_n, r_{n+1}) \in \langle 0 \rangle \times \Psi_n \times \langle 0 \rangle$ and $\boldsymbol{z} = (z_0, z_1, \dots, z_n, z_{n+1}) = \Phi_{\tilde{\boldsymbol{s}}}(\boldsymbol{r})$. By the definition of $\Phi_{\tilde{\boldsymbol{s}}}$, we have that

$$z_i = \begin{cases} s_i - r_i, & \text{if } r_i \neq 0; \\ 0, & \text{if } r_i = 0. \end{cases} \tag{5.5}$$

One can verify that the following four statements are true for $i : 0 \le i \le n$, by using (5.5).

(i) Suppose $r_i \neq 0$ and $r_{i+1} \neq 0$. Then i is an $\tilde{\boldsymbol{s}}$-descent of $\boldsymbol{r}$ if and only if i is an $\tilde{\boldsymbol{s}}$-ascent of $\boldsymbol{z}$.

(ii) Suppose $r_i \neq 0$ and $r_{i+1} = 0$. Then i is an $\tilde{\boldsymbol{s}}$-descent of $\boldsymbol{r}$ and i is not an $\tilde{\boldsymbol{s}}$-ascent of $\boldsymbol{z}$.

(iii) Suppose $r_i = 0$ and $r_{i+1} \neq 0$. Then i is not an $\tilde{\boldsymbol{s}}$-descent of $\boldsymbol{r}$ and i is an $\tilde{\boldsymbol{s}}$-ascent of $\boldsymbol{z}$.

(iv) Suppose $r_i = 0$ and $r_{i+1} = 0$. Then i is not an $\tilde{\boldsymbol{s}}$-descent of $\boldsymbol{r}$ and i is not an $\tilde{\boldsymbol{s}}$-ascent of $\boldsymbol{z}$.

However, since $r_0 = r_{n+1} = 0$, we see that the number of occurrences of situation (ii) and the number of occurrences of situation (iii) are the same. Therefore, the first equality in (5.4) follows. The second equality in (5.4) follows from the first one trivially. □

Proof of Theorem 5.8. One verifies that

$$\begin{aligned} \mathrm{des}_{\boldsymbol{s}^*}(\boldsymbol{r}, 0) &= \mathrm{des}_{\tilde{\boldsymbol{s}}}(0, \boldsymbol{r}, 0) = \mathrm{des}_{\tilde{\boldsymbol{u}}}(\mathrm{reverse}(\Phi_{\tilde{\boldsymbol{s}}}(0, \boldsymbol{r}, 0))) \\ &= \mathrm{des}_{\tilde{\boldsymbol{u}}}(0, \mathrm{reverse}(\Phi_{\boldsymbol{s}}(\boldsymbol{r})), 0) = \mathrm{des}_{\boldsymbol{u}^*}(\mathrm{reverse}(\Phi_{\boldsymbol{s}}(\boldsymbol{r})), 0), \end{aligned}$$

where the first and last equalities follow from the fact that appending 0's at the beginning of a nonnegative-entry vector does not create descents, the second equality follows from (5.4), and the third equality follows from the definitions of $\Phi_{\tilde{\boldsymbol{s}}}$ and $\Phi_{\boldsymbol{s}}$. □

Finally, We prove Lemma 5.1.

Proof of Lemma 5.1. Note that

$$\begin{aligned} & 0 \le \frac{x_1}{s_1} \le \frac{x_2}{s_2} \le \cdots \le \frac{x_n}{s_n} \le 1 \\ \iff & 0 \le \frac{s_n - x_n}{s_n} \le \frac{s_{n-1} - x_{n-1}}{s_{n-1}} \le \cdots \le \frac{s_1 - x_1}{s_1} \le 1. \end{aligned}$$

Hence, one see that the map $\boldsymbol{x} \mapsto \mathrm{reverse}(\boldsymbol{s} - \boldsymbol{x})$ gives a affine transformation from $P_{\boldsymbol{s}}$ to $P_{\boldsymbol{u}}$. Moreover, it is easy to see the transformation is unimodular. The desired result follows. □

6. The case when $s_1 = 1$

In this section, we focus on the special case when the first entry of $\boldsymbol{s}$ is 1.

Lemma 6.1. *Let $\boldsymbol{r} \in \Psi_n$. If $s_1 = 1$, we have*

$$\mathrm{des}_{\boldsymbol{s}^*}(\boldsymbol{r}, 0) = \mathrm{asc}_{\boldsymbol{s}^*}(\Phi_{\boldsymbol{s}}(\boldsymbol{r}), 0) = \mathrm{asc}_{\boldsymbol{s}}(\Phi_{\boldsymbol{s}}(\boldsymbol{r})). \tag{6.1}$$

Proof. Similarly to the proof of Theorem 5.8, we have by (5.4) that

$$\mathrm{des}_{\boldsymbol{s}^*}(\boldsymbol{r}, 0) = \mathrm{des}_{\tilde{\boldsymbol{s}}}(0, \boldsymbol{r}, 0) = \mathrm{asc}_{\tilde{\boldsymbol{s}}}(\Phi_{\tilde{\boldsymbol{s}}}(0, \boldsymbol{r}, 0)) = \mathrm{asc}_{\tilde{\boldsymbol{s}}}(0, \Phi_{\boldsymbol{s}}(\boldsymbol{r}), 0), \tag{6.2}$$

where $\tilde{\boldsymbol{s}}$ is defined in Notation 5.6.

Suppose $s_n = 1$. Then the first two entries of the vector $(0, \Phi_{\boldsymbol{s}}(\boldsymbol{r}), 0)$ are $(0, 0)$, which is not an ascent. Hence,

$$\operatorname{asc}_{\tilde{\boldsymbol{s}}}(0, \Phi_{\boldsymbol{s}}(\boldsymbol{r}), 0) = \operatorname{asc}_{\boldsymbol{s}^*}(\Phi_{\boldsymbol{s}}(\boldsymbol{r}), 0) = \operatorname{asc}_{\boldsymbol{s}}(\Phi_{\boldsymbol{s}}(\boldsymbol{r})).$$

Therefore equation (6.1) follows. □

Corollary 6.2. *Suppose $s_1 = 1$. (Recall $\overline{\mathrm{REM}}_{\boldsymbol{s}}$ is defined in part b) of Definition 3.10.) Then*

$$\begin{aligned} \mathcal{L}^i(\mathrm{Par}_{\boldsymbol{s}^*}) &= \{\boldsymbol{x} \in \mathrm{Par}_{\boldsymbol{s}^*} \cap \mathbb{Z}^{n+1} \mid \operatorname{asc}_{\boldsymbol{s}^*}(\overline{\mathrm{REM}}_{\boldsymbol{s}^*}(\boldsymbol{x})) = i\} \\ &= \{\overline{\mathrm{REM}}_{\boldsymbol{s}^*}^{-1}(\boldsymbol{r}) \mid \operatorname{asc}_{\boldsymbol{s}^*}(\boldsymbol{r}) = i,\ \boldsymbol{r} \in \Psi_n \times \langle 0 \rangle\} \\ &= \{\overline{\mathrm{REM}}_{\boldsymbol{s}^*}^{-1}(\boldsymbol{r}, 0) \mid \operatorname{asc}_{\boldsymbol{s}}(\boldsymbol{r}) = i,\ \boldsymbol{r} \in \Psi_n\}, \end{aligned}$$

and

$$\begin{aligned} \ell^i(\mathrm{Par}_{\boldsymbol{s}^*}) &= \#\{\boldsymbol{r} \in \Psi_n \times \langle 0 \rangle \mid \operatorname{asc}_{\boldsymbol{s}^*}(\boldsymbol{r}) = i\} \\ &= \#\{\boldsymbol{r} \in \Psi_n \mid \operatorname{asc}_{\boldsymbol{s}}(\boldsymbol{r}) = i\}. \end{aligned}$$

Hence, if let π be the map that drops the last coordinate of a vector, we have that $\pi \circ \overline{\mathrm{REM}}_{\boldsymbol{s}^*}$ gives a bijection between $\mathrm{Par}_{\boldsymbol{s}*} \cap \mathbb{Z}^{n+1}$ and Ψ_n such that

$$\boldsymbol{x} \in \mathcal{L}^i(\mathrm{Par}_{\boldsymbol{s}^*}) \iff \operatorname{asc}_{\boldsymbol{s}}(\pi(\overline{\mathrm{REM}}_{\boldsymbol{s}^*}(\boldsymbol{x}))) = i.$$

Proof. It follows from (6.1) and Corollary 3.8. □

Therefore we can describe the δ-vector of $\boldsymbol{s}$-lecture hall polytope with $\boldsymbol{s}$-ascents when $s_1 = 1$.

Corollary 6.3. *Suppose $s_1 = 1$. Then the δ-vector of $P_{\boldsymbol{s}}$ is given by*

$$\delta_{P_{\boldsymbol{s}},i} = \#\{\boldsymbol{r} \in \Psi_n \mid \operatorname{asc}_{\boldsymbol{s}}(\boldsymbol{r}) = i\},\ 0 \le i \le n. \tag{6.3}$$

Corollary 6.3 extends easily to arbitrary s_1 using equation (6.2). This result appears in [7] (a special case of their Theorem 5), but we have no need to state it here.

We find it is interesting to compare Corollary 3.7 and Corollary 6.2, and equations (3.3) and (6.3). These are parallel results for the cases $s_n = 1$ and $s_1 = 1$. One sees that the result of the case $s_n = 1$ is much easier to obtain than that of the case $s_1 = 1$. The above two corollaries also tell us that when $s_1 = 1$, it is better to use the map $\overline{\mathrm{REM}}_{\boldsymbol{s}^*}$ than $\mathrm{REM}_{\boldsymbol{s}^*}$.

Finally, applying the above results to $\boldsymbol{s} = (1, 2, \ldots, n)$, we obtain a bijection from the lattice points in the fundamental parallelepiped $\mathrm{Par}_{\boldsymbol{s}*}$ associated to $P_{(1,2,\ldots,n)}$ to inversion sequences.

Proposition 6.4. *Suppose $\boldsymbol{s} = (1, 2, \ldots, n)$. Let π be the map that drops the last coordinate of a vector. Then the composition map*

$$\begin{aligned} \mathcal{L}^i(\mathrm{Par}_{\boldsymbol{s}^*}) &\xrightarrow{\overline{\mathrm{REM}}_{\boldsymbol{s}^*}} \Psi_n \times \langle 0 \rangle \\ &\xrightarrow{\pi} \Psi_n \\ &\xrightarrow{\text{reverse}} \bar{\Psi}_n = \langle n-1 \rangle \times \cdots \times \langle 0 \rangle \end{aligned}$$

gives a bijection from $\mathrm{Par}_{\boldsymbol{s}^*} \cap \mathbb{Z}^{n+1}$ *to the inversion sequences of length* n*. Furthermore, for any* $\boldsymbol{x} \in \mathrm{Par}_{\boldsymbol{s}} \cap \mathbb{Z}^{n+1}$*, we have*

$$\text{the last coordinate of } \boldsymbol{x} = \mathrm{asc}(\pi(\overline{\mathrm{REM}}_{\boldsymbol{s}^*}(\boldsymbol{x}))) \tag{6.4}$$

$$= \mathrm{des}(\mathrm{reverse}(\pi(\overline{\mathrm{REM}}_{\boldsymbol{s}^*}(\boldsymbol{x})))). \tag{6.5}$$

Proof. The only thing we need to verify is the equality (6.4). (Note that the equality (6.5) follows from the equality (6.4) easily.) By Corollary 6.2, we have

$$\text{the last coordinate of } \boldsymbol{x} = \mathrm{asc}_{\boldsymbol{s}}(\pi(\overline{\mathrm{REM}}_{\boldsymbol{s}^*}(\boldsymbol{x}))).$$

However, since $\boldsymbol{s} = (1, 2, \dots, n)$, we have by Lemma 4.2 that for any $\boldsymbol{r} \in \langle 0\rangle \times \langle 1\rangle \times \cdots \times \langle n-1\rangle$, an $\boldsymbol{s}$-ascent of $\boldsymbol{r}$ is the same as a regular ascent of $\boldsymbol{r}$, and vice versa. Hence equation (6.4) follows. □

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