Lagrangian caps

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Abstract

We establish an h-principle for exact Lagrangian embeddings with concave Legendrian boundary. We prove, in particular, that in the complement of the unit ball B in the standard symplectic $\mathbb{R}^{2n}, 2n \geq 6$, there exists an embedded Lagrangian n-disc transversely attached to B along its Legendrian boundary.

1 Introduction

Question. Let B be the round ball in the standard symplectic $\mathbb{R}^{2n}$. *Is there an embedded Lagrangian disc* $\Delta \subset \mathbb{R}^{2n} \setminus \operatorname{Int} B$ *with* $\partial\Delta \subset \partial B$ *such that* $\partial\Delta$ *is a Legendrian submanifold and* Δ *transversely intersects* ∂B *along its boundary?*

If $n = 2$ then such a Lagrangian disc does not exist. Indeed, it is easy to check that the existence of such a Lagrangian disc implies that the Thurston-Bennequin invariant $\operatorname{tb}(\partial\Delta)$of the Legendrian knot $\partial\Delta \subset S^3$ is equal to $+1$. On the other hand, the knot $\partial\Delta$ is sliced, i.e its 4-dimensional genus is equal to 0. But then according to Lee Rudolph's slice Bennequin inequality [8] we should have $\operatorname{tb}(\partial\Delta) \leq -1$, which is a contradiction.

As far as we know no such Lagrangian discs have been previously constructed in higher dimensions either. We prove in this paper that if $n > 2$ such discs exist in abundance. In particular, we prove

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Theorem 1.1. *Let L be a smooth manifold of dimension $n > 2$ with non-empty boundary such that its complexified tangent bundle $T(L) \otimes \mathbb{C}$ is trivial. Then there exists an exact Lagrangian embedding $f : (L, \partial L) \to (\mathbb{R}^{2n} \setminus \operatorname{Int} B, \partial B)$ with $f(\partial\Delta) \subset \partial B$ such that $f(\partial\Delta) \subset \partial B$ is a Legendrian submanifold and f transverse to ∂B along the boundary ∂L.*

Note that the triviality of the bundle $T(L) \otimes \mathbb{C}$ is a necessary (and according to Gromov's h-principle for Lagrangian immersions, [6] sufficient) condition for existence of any Lagrangian *immersion* $L \to \mathbb{C}^n$.

In fact, we prove a very general h-principle type result for Lagrangian embeddings generalizing this claim, see Theorem 2.2 below. As corollaries of this theorem we get

- an h-principle for Lagrangian embeddings in any symplectic manifold with a unique conical singular point, see Corollary 6.1;
- a general h-principle for embeddings of flexible Weinstein domains, see Corollary 6.3;
- construction of Lagrangian immersions with minimal number of self-intersection points; this is explored in a joint paper of the authors with T. Ekholm and I. Smith, [2].

Theorem 2.2 together with the results from the book [1] yield new examples of rationally convex domains in $\mathbb{C}^n$, which will be discussed elsewhere. The authors are thankful to Stefan Nemirovski, whose questions concerning this circle of questions motivated the results of the current paper.

2 Main Theorem

Loose Legendrian submanifolds

Let (Y, ξ) be a $(2n-1)$-dimensional contact manifold. Let us recall that each contact plane ξ_y, $y \in Y$, carries a canonical linear symplectic structure defined up to a scaling factor. Thus, there is a well defined class of isotropic and, in particular, Lagrangian linear subspaces of ξ_y. Given a k-dimensional , $k \leq n-1$, manifold Λ, an injective homomorphism $\Phi : T\Lambda \to TY$ covering a map $\phi : \Lambda \to Y$ is called isotropic (or if $k = n-1$ Legendrian) if $\Phi(T\Lambda) \subset \xi$ and $\Phi(T_x\Lambda) \subset \xi_{\phi(x)}$ is isotropic for each $x \in \Lambda$.

Given a $(2n-1)$-dimensional contact manifold (Y, ξ), an embedding $f : \Lambda \to Y$ is called *isotropic* if it is tangent to ξ; if in addition $\dim \Lambda = n-1$ then it is called

Legendrian. The differential of an isotropic (resp. Legendrian) embedding is an isotropic (resp. Legendrian) homomorphism.

Two Legendrian embeddings $f_0, f_1 : \Lambda \to Y$ are called *formally Legendrian isotopic* if there exists a smooth isotopy $f_t : \Lambda \to Y$ connecting f_0 and f_1 and a 2-parametric family of injective homomorphisms $\Phi_t^s : T\Lambda \to TY$, such that $\Phi_t^0 = df_t, \Phi_0^s = df_0, \Phi_1^s = df_1$ and Φ_t^1 is a Legendrian homomorphism $(s, t \in [0,1])$.

The results of this paper essentially depend on the theory of *loose Legendrian* embeddings developed in [7]. This is a class of Legendrian embeddings into contact manifolds of dimension > 3 which satisfy a certain form of an h-principle. For the purposes of this paper we will not need a formal definition of loose Legendrian embeddings, but instead just describe their properties.

Let $\mathbb{R}^{2n-1}_{\mathrm{std}} := (\mathbb{R}^{2n-1}, \xi_{\mathrm{std}} = \{dz - \sum\limits_{1}^{n-1} y_i dx_i = 0\})$ be the standard contact $\mathbb{R}^{2n-1}$, $n > 2$, and $\Lambda_0 \subset \mathbb{R}^{2n-1}_{\mathrm{std}}$ be the Legendrian $\{z = 0, y_i = 0\}$. Note that a small neighborhood of any point on a Legendrian in a contact manifold is contactomorphic to the pair $(\mathbb{R}^{2n-1}_{\mathrm{std}}, \Lambda_0)$. There is another Legendrian $\tilde{\Lambda}$, called the *universal loose Legendrian*, which is equal to Λ_0 outside of a compact subset, and formally Legendrian isotopic to it. A picture of $\tilde{\Lambda}$ is given in Figure 2.1, though we do not use any properties of Λ besides those stated above. A *connected* Legendrian submanifold $\Lambda \subset Y$ is called *loose*, if there is a contact embedding $(\mathbb{R}^{2n-1}_{\mathrm{std}}, \widetilde{\Lambda}) \to (Y, \Lambda)$. We refer the interested readers to the paper [7] and the book [1] for more information. The following proposition summarizes the properties of loose Legendrian embeddings.

Proposition 2.1. *For any contact manifold (Y, ξ) of dimension $2n-1 > 3$ the set of connected loose Legendrians have the following properties:*

(i) For any Legendrian embedding $f : \Lambda \to Y$ there is a loose Legendrian embedding $\widetilde{f} : \Lambda \to Y$ which coincides with f outside an arbitrarily small neighborhood of a point $p \in \Lambda$ and which is formally isotopic to f via a formal Legendrian isotopy supported in this neighborhood.

(ii) Let $f_0, f_1 : \Lambda \to Y$ be two loose Legendrian embeddings of a connected Λ which coincide outside a compact set and which are formally Legendrian isotopic via a compactly supported isotopy. Then f_0, f_1 are Legendrian isotopic via a compactly supported Legendrian isotopy.

(iii) Let $f_t : \Lambda \to Y$, $t \in [0,1]$, be a smooth isotopy which begins with a lose Legendrian embedding f_0. Then it can be C^0-approximated by a Legendrian isotopy $\widetilde{f_t} : \Lambda \to Y$, $t \in [0,1]$, beginning with $\widetilde{f_0} = f_0$.

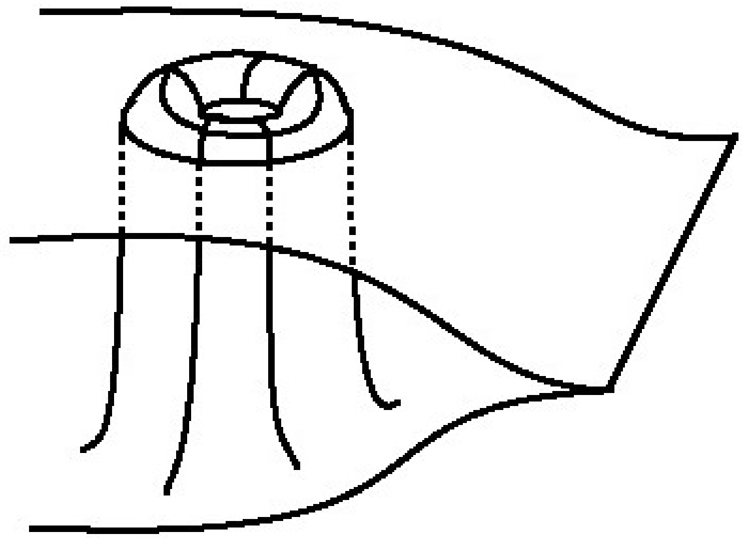

Fig. 2.1: The universal loose Legendrian, $\tilde{\Lambda}$. In the terminology of [7] and [1] $\tilde{\Lambda}$ is the stabilization of Λ_0 over a manifold of Euler characteristic 0.

Statement (i) is the *Legendrian stabilization* construction which replaces a small neighborhood of a point on a Legendrian submanifold by the model $(\mathbb{R}^{2n-1}_{\text{std}}, \widetilde{\Lambda})$. It was first described for $n > 2$ in [3]. The main part of Proposition 2.1, parts (ii) and (iii), are proven in [7]. Notice that (ii) implies that if a Legendrian is already loose that any further stabilizations do not change its Legendrian isotopy class.

Symplectic manifolds with negative Liouville ends

Throughout the paper we use the terms *closed submanifold* and *properly embedded submanifold* as synonyms, meaning a submanifold which is a closed subset, but not necessarily a closed manifold itself.

Let L be an n-dimensional smooth manifold. A *negative end* structure on L is a choice of

- a codimension 1 submanifold $\Lambda \subset L$ which divides L into two parts: $L = L_- \cup L_+$, $L_- \cap L_+ = \Lambda$, and
- a non-vanishing vector field S on $\mathcal{O}p\, L_- \subset L$ which is outward transverse to

the boundary $\Lambda = \partial L_-$, and such that the negative flow $S^{-t} : L_- \to L_-$ is defined for all t and all its trajectories intersect Λ.

In other words, there is a canonical diffeomorphism $L_- \to (-\infty, 0] \times \Lambda$ which is defined by sending the ray $(-\infty, 0] \times x$, $x \in \Lambda$, onto the trajectory of $-S$ originated at $x \in \Lambda$.

Alternatively, the negative end structure can be viewed as a *negative completion* of the manifold L_+ with boundary Λ:

$$L = L_+ \bigcup_{0\times\Lambda\ni(0,x)\sim x\in\Lambda} (-\infty, 0] \times \Lambda.$$

Negative end structures which differ by a choice of the cross-section Λ transversely intersecting all the negative trajectories of L will be viewed as equivalent.

Let (X, ω) be a $2n$-dimensional *symplectic* manifold. A properly embedded co-oriented hypersurface $Y \subset X$ is called a *contact slice* if it divides X into two domains $X = X_- \cup X_+$, $X_- \cap X_+ = Y$, and there exists a Liouville vector field Z in a neighborhood of Y which is transverse to Y, defines its given co-orientation and points into X_+. Such hypersurfaces are also called *symplectically convex* [4], or of *contact type* [9].

If the Liouville field extends to X_- as a non-vanishing Liouville field such that the negative flow Z^{-t} is defined for all $t \geq 0$ and all its trajectories in X_- intersect Y then X_- with a choice of such Z is called a *negative Liouville end* structure of the symplectic manifold (X, ω).

The restriction α of the Liouville form $\lambda = i(Z)\omega$ to Y is a contact form on Y and the diffeomorphism $(-\infty, 0] \times Y \to X_-$ which sends each ray $(-\infty, 0] \times x$ onto the trajectory of $-Z$ originated at $x \in \Lambda$ is a Liouville isomorphism between the negative symplectization $((-\infty, 0] \times Y, d(t\alpha))$ of the contact manifold $(Y, \{\alpha = 0\})$ and (X_-, λ). Hence alternatively the negative Liouville end structure can be viewed as a *negative completion* of the manifold X_+ with the negative contact boundary Y, i.e. as an attaching the negative symplectization $((-\infty, 0] \times Y, d(t\alpha))$ of the contact manifold $(Y, \{\alpha = 0\})$ to X_+ along Y.

A negative Liouville end structure which differs by another choice of the cross-section Y transversely intersecting all negative trajectories of X will be viewed as an equivalent one. Note that the holonomy along trajectories of X provides a contactomorphism between any two transverse sections. Any such transverse section will be called a *contact slice*.

If the symplectic form ω is exact and the Liouville form λ is extended as a Liouville form, still denoted by λ, to the whole manifold X, then we will call (X,λ) a *Liouville manifold with a negative end*.

Let L be an n-dimensional manifold with a negative end, and X a symplectic $2n$-manifold with a negative Liouville end. A proper Lagrangian immersion $f: L \to X$ is called *cylindrical at* $-\infty$ if it maps the negative end L_- of L into a negative end X_- of X, the restriction $f|_{L_-}$ is an embedding, and the differential $df|_{TL_-}$ sends the vector field S to Z. Composing the restriction of f to a transverse slice Λ with the projection of the negative Liouville end of X to Y along trajectories of Z we get a Legendrian embedding $f_{-\infty}: \Lambda \to Y$, which will be called the *asymptotic negative boundary* of the Lagrangian immersion f.

The action class

Given a proper Lagrangian immersion $f: L \to X$, we consider its mapping cylinder $C_f = L \times [0,1] \underset{(x,1)\sim f(x)}{\cup} X$, which is homotopy equivalent to X, and denote respectively by $H^2(X,f)$ and $H^2_\infty(X,f)$ the 2-dimensional cohomology groups $H^2(C_f, L\times 0)$ and $H^2_\infty(C_f, L\times 0) := \underset{K\subset C_f}{\overrightarrow{\lim}} H^2(C_f\setminus K, (L\times 0)\setminus K)$, where the direct limit is taken over all compact subsets $K \subset C_f$. We denote by r_∞ the restriction homomorphism $r_\infty: H^2(X,f) \to H^2_\infty(X,f)$. If f is an embedding then $H^2(X,f)$ and $H^2_\infty(X,f)$ are canonically isomorphic to $H^2(X,f(L))$ and $H^2_\infty(X,f(L)) := \underset{K\subset X}{\overrightarrow{\lim}} H^2(X\setminus K, f(L)\setminus K)$, respectively. We define the *relative action class* $A(f) \in H^2(X,f)$ of a proper Lagrangian immersion $f: L \to X$ as the class defined by the closed 2-form which is equal ω on X and to 0 on $L\times 0$. We say that f is *weakly exact* if $A(f) = 0$. The *relative action class at infinity* $A_\infty(f) \in H^2_\infty(X,f)$ is defined as $A_\infty(f) := r_\infty(A_\infty)$. We note we have $A_\infty(f) = A_\infty(g)$ if Lagrangian immersions f,g coincide outside a compact set.

Consider next a compactly supported Lagrangian regular homotopy, $f_t\colon L \to X$, $0 \le t \le 1$, and write $F\colon L\times[0,1] \to X$, for $F(x,t) = f_t(x)$. Let α denote the 1-form on $L\times[0,1]$ defined by the equation $\alpha := \iota_{\partial/\partial t}(F^*\omega)$, where t is the coordinate on the second factor of $L\times[0,1]$. Then the restrictions $\alpha_t := \alpha|_{L\times\{t\}}$ are closed for all $t \in [0,1]$. We call the Lagrangian regular homotopy f_t a *Hamiltonian regular homotopy* if the cohomology class $[\alpha_t] \in H^1(L)$ is independent of t. It is straightforward to verify that for a Hamiltonian regular homotopy f_t the action class $A(f_t)$ remains constant. Note, however, that the converse is not necessarily true.

If X is a Liouville manifold, then we define the *absolute action class* $a(f) \in H^1(L)$

as the class of the closed form $f^*\lambda$, and call a Lagrangian immersion f *exact* if $a(f)=0$. Note that in that case we have $\delta(a(f))=A(f)$, where δ is the boundary homomorphism $H^1(L)\to H^2(X,f)$ from the exact sequence of the pair $(C_f, L\times 0)$. We will also use the notation

$$H^1_\infty(L) := \underset{\substack{K\subset L\\ K \text{ is compact}}}{\overrightarrow{\lim}} H_1(L\setminus K),\ r_\infty : H^1(L)\to H^1_\infty(L),\ a_\infty(f)=r_\infty(a(f)).$$

If the the immersion f is cylindrical at $-\infty$ then the class $a_\infty(f)$ vanishes on L_-.

Statement of main theorems

We say that a symplectic manifold X has infinite Gromov width if an arbitrarily large ball in $\mathbb{R}^{2n}_{\mathrm{st}}$ admits a symplectic embedding into X. For instance, a complete Liouville manifold have infinite Gromov width.

Theorem 2.2. *Let $f : L\to X$ be a cylindrical at $-\infty$ proper embedding of an n-dimensional, $n\geq 3$, connected manifold L, such that its asymptotic negative Legendrian boundary has a component which is loose in the complement of the other components. Suppose that there exists a compactly supported homotopy of injective homomorphisms $\Psi_t : TL\to TX$ covering f and such that $\Psi_0 = df$ and Ψ_1 is a Lagrangian homomorphism. If $n=3$ assume, in addition, that the manifold $X\setminus f(L)$ has infinite Gromov width. Then given a cohomology class $A\in H^2(X, f(L))$ with $r_\infty(A)=A_\infty(f)$, there exists a compactly supported isotopy $f_t : L\to X$ such that*

- $f_0 = f$;
- f_1 *is Lagrangian;*
- $A(f_1)=A$ *and*
- $df_1 : TL\to TX$ *is homotopic to Φ_1 through Lagrangian homomorphisms.*

If X is a Liouville manifold with a negative contact end, then one can in addition prescribe any value $a\in H^1(L)$ to the absolute action class $a(f_1)$ provided that $r_\infty(a)=a_\infty$, and in particular make the Lagrangian embedding f_1 exact.

We do not know whether the infinite width condition when $n=3$ is really necessary, or it is just a result of deficiency of our method.

Suppose we are given a smooth proper immersion $f : L^n \to X^{2n}$ with only transverse double points and which is an embedding outside of a compact subset. If L is connected, L is orientable and X is oriented and n is even, we define the *relative self-intersection index* of f, denoted $I(f)$, to be the signed count of intersection points, where the sign of an intersection $f(p^0) = f(p^1)$ is $+1$ or -1 depending on whether the orientation defined by $(df_{p^0}(L), df_{p^1}(L))$ agrees or disagrees with the orientation on X. Because n is even, this sign does not depend on the ordering (p^0, p^1); if n is odd or L is non-orientable we instead define $I(f)$ as an element of $\mathbb{Z}_2$. If X is simply connected a theorem of Whitney [10] implies that f is regularly homotopic with compact support to an embedding if and only if $I(f) = 0$.

Theorem 2.2 will be deduced in Section 5 from the following

Theorem 2.3. *Let (X, λ) be a simply connected Liouville manifold with a negative end X_-, and $f : L \to X$ a cylindrical at $-\infty$ exact self-transverse Lagrangian immersion with finitely many self intersections. Suppose that $I(f) = 0$, and the asymptotic negative boundary Λ of f has a component which is loose in the complement of the others. If $n = 3$ suppose, in addition, that $X \setminus f(L)$ has infinite Gromov width. Then there exists a compactly supported Hamiltonian regular homotopy f_t, connecting $f_0 = f$ with an embedding f_1.*

Remark. If X is not simply connected the statement remains true if the self-intersection index $I(f)$ is understood as an element of the group ring of $\pi_1(X)$.

3 Weinstein recollections and other preliminaries

Weinstein cobordisms

We define below a slightly more general notion of a Weinstein cobordism than is usually done (comp. [1]), by allowing cobordisms between non-compact manifolds. Let W be a $2n$-dimensional smooth manifold with boundary. We allow W, as well as its boundary components to be non-compact. Suppose that the boundary ∂W is presented as the union of two disjoint subsets $\partial_\pm W$ which are open and closed in ∂W. A *Weinstein cobordism* structure on W is a triple (ω, Z, ϕ), where ω is a symplectic form on W, Z is a Liouville vector field, and $\phi : W \to [m, M]$ a Morse function with finitely many critical points, such that

- $\partial_- W = \{\phi = m\}$ and $\partial_+ W = \{\phi = M\}$ are regular level sets;
- the vector field Z is gradient like for ϕ, see [1], Section 9.3;

- outside a compact subset of W every trajectory of Z intersects both $\partial_- W$ and $\partial_+ W$.

The function ϕ is called a *Lyapunov function* for Z. The Liouville form $\lambda = i(Z)\omega$ induces contact structure on all regular levels of the function ϕ. All Z-stable manifolds of critical points of the function ϕ are isotropic for ω and, in particular, indices of all critical points are $\leq n = \frac{\dim W}{2}$. A Weinstein cobordism (W, ω, X, ϕ) is called *subcritical* if indices of all critical points are $< n$.

Extension of Weinstein structure

The following lemma is the standard handle attaching statement in the Weinstein category (see [9] and [1]). We provide a proof here because we need it in a slightly different than it is presented in [9] and [1].

Lemma 3.1. *Let (X, λ) be a Liouville manifold with boundary, Z the Liouville field corresponding to λ (i.e. $\iota_Z \omega = \lambda$ where $\omega = d\lambda$) and $Y \subset \partial X$ a (union of) boundary component(s) of X such that Z is inward transverse to Y. Let $(\Delta, \partial\Delta) \subset (X, Y)$ be a k-dimensional ($k \leq n$) isotropic disc, which is tangent to Z near $\partial\Delta$. If $k = 1$ suppose, in addition, that $\int\limits_\Delta \lambda = 0$, and if $k < n$ suppose, in addition, that Δ is extended to (a germ of) a Lagrangian submanifold $(L, \partial L) \subset (X, Y)$ which is also tangent to Z near ∂L. Then for any neighborhoods $U \supset \Delta$ and $\Omega \supset Y$ there exists a Weinstein cobordism $(W, \omega, \widetilde{Z}, \phi)$ with the following properties :*

- *$Y \cup \Delta \subset W \subset \Omega \cup U$;*
- *$\partial_- W = Y$;*
- *the function ϕ has a unique critical point p of index k at the center of the disc Δ;*
- *the disc Δ is contained in the $\widetilde{Z}$-stable manifold of the point p;*
- *the field $\widetilde{Z}|_{L \cap W}$ is tangent to L;*
- *the Liouville form $\widetilde{\lambda} = i(\widetilde{Z})\omega$ can be written as $\lambda + dH$ for a function H compactly supported in $U \setminus Y$.*

Proof. Let us set $L=\Delta$ if $k=n$. For a general case we can assume that $L=\Delta\times\mathbb{R}^{n-k}$. Let $\omega_{\rm st}$ denote the symplectic form on $T^*(L)=T^*L\times T^*\mathbb{R}^k=\Delta^k\times\mathbb{R}^k\times\mathbb{R}^{n-k}\times\mathbb{R}^{n-k}$ given by the formula

$$\omega_{\rm st}=\sum_1^k dp_i\wedge dq_i+\sum_1^{n-k}du_j\wedge dv_j$$

with respect to the coordinates $(q,p,v,u)\in\Delta^k\times\mathbb{R}^k\times\mathbb{R}^{n-k}\times\mathbb{R}^{n-k}$ which correspond to this splitting. Denote by λ_k the Liouville form $\sum\limits_1^k(2p_idq_i+q_idp_i)+\frac{1}{2}\sum\limits_1^{n-k}(v_idu_j-u_jdv_j)$, $d\lambda_k=\omega_{\rm st}$. Note that the Liouville field

$$Z_k:=\sum_1^k\left(-q_i\frac{\partial}{\partial q_i}+2p_i\frac{\partial}{\partial p_i}\right)+\frac{1}{2}\sum_1^{n-k}\left(v_i\frac{\partial}{\partial v_i}+u_j\frac{\partial}{\partial u_j}\right)$$

corresponding to the form λ_k is gradient like for the quadratic function

$$Q:=\sum_1^k(p_i^2-q_i^2)+\sum_i^{n-k}(u_j^2+v_j^2),$$

tangent to L, and the disc Δ serves as the Z_k-stable manifold of its critical point. Using the normal form for the Liouville form λ near ∂L (see [9], and also [1], Proposition 6.6) and the Weinstein symplectic normal form along the Lagrangian L we can find, possibly decreasing the neighborhoods Ω and U, a symplectomorphism $\Phi:U\to U'$, where U' is a neighborhood of Δ in T^*L, such that

- $\Phi(L\cap U)=L\cap U'$, $\Phi(\Delta\cap U)=\Delta\cap U'$;
- $\Phi^*\omega_{\rm st}=\omega$;
- $\Phi^*\lambda_k=\lambda$ on $\Omega\cap U$;
- $\Phi(Y\cap U)=\{Q=-1\}\cap U'$.

Thus the closed, and hence exact 1-form $\Phi_*\lambda-\lambda_k$ vanishes on $\Omega':=\Phi(\Omega\cap U)$, and therefore, using the condition $\int\limits_\Delta\lambda=0$ when $k=1$, we can conclude that $\lambda_k=\Phi_*\lambda+dH$ for a function $H:G\to\mathbb{R}$ vanishing on $\Omega'\supset\partial\Delta$. Let $\theta:U'\to[0,1]$ be a C^∞-cut-off function equal to 0 outside a neighborhood $U_1'\supset\Delta$, $U_1'\Subset U'$, and equal to 1 on a smaller neighborhood $U_2' \supset \Delta$, $U_2' \Subset U_1'$. Denote $\widehat{H} := \theta H$. Then the form $\widehat{\lambda} := \Phi_* \lambda + d\widehat{H}$ coincides with $\Phi^*\lambda$ on $\Omega' \cup (U' \setminus U_1')$, and equal to λ_k on U_2'.

Then, according to Corollary 9.21 from [1], for any sufficiently small $\varepsilon > 0$ and a neighborhood $U_3' \supset \Delta$, $U_3' \Subset U_2'$, there exists a Morse function $\widehat{Q} : U' \to \mathbb{R}$ such that

- $\widehat{Q}$ coincides with Q on $\{Q \le -1\} \cup (\{Q \le -1+\varepsilon\} \setminus U_2'$;
- $\widehat{Q}$ and Q are target equivalent over U_3', i.e. there exists a diffeomorphism $\sigma : \mathbb{R} \to \mathbb{R}$ such that over U_3' we have $\widehat{Q} = \sigma \circ Q$;
- $-1+\varepsilon$ is a regular value of $\widehat{Q}$ and $\{\widehat{Q} \le -1+\varepsilon\} \subset \Omega' \cup U_2'$;
- inside $\widehat{W} := \{-1 \le \widehat{Q} \le -1+\varepsilon\} \subset U'$ the function $\widehat{Q}$ has a unique critical point.

Denote $\widetilde{Q} := \widehat{Q} \circ \Phi : U \to \mathbb{R}$. Let us extend the function $\widetilde{Q}$ to the whole manifold X in such a way that

- $\{\widetilde{Q} = -1\} \setminus U = Y \setminus U$,
- $\{-1 \le \widetilde{Q} \le -1+\varepsilon\} \setminus U \subset \Omega \setminus U$,
- the function $\widetilde{Q}|_{X \setminus U}$ has no critical values in $[-1, -1+\varepsilon]$ and
- the Liouville vector field Z is gradient like for $\widehat{Q}$ on $\{-1 \le \widetilde{Q} \le -1+\varepsilon\} \setminus U$.

Let us define $W := \{-1 \le \widetilde{Q} \le -1+\varepsilon\} \subset X$,

$$\widetilde{\lambda} = \begin{cases} \Phi^*\widehat{\lambda} = \lambda + d\widehat{H} \circ \Phi, & \text{on } U, \\ \lambda, & \text{on } X \setminus U. \end{cases}$$

Let $\widetilde{Z}$ be the Liouville field ω-dual to the Liouville form $\widetilde{\lambda}$ Then the Weinstein cobordism $(W, \omega, \widetilde{Z}, \phi := \widehat{H} \circ \Phi)$ has the required properties. □

We will also need the following simple

Lemma 3.2. *Let (X, λ) be a Liouville manifold and $f : L \to X$ a Lagrangian immersion. Let $p \in X$ be a transverse self-intersection point. Then there exists a symplectic embedding $h : B \to X$ of a sufficiently small ball in $\mathbb{R}^{2n}_{\mathrm{st}}$ into X such that $h(0) = p$ and $h^{-1}(f(L)) = B \cap (\{x = 0\} \cup \{y = 0\})$.*

Proof. By the Weinstein neighborhood theorem, there exist coordinates in a symplectic ball near p so that $f(L)$ is given by $\{x=0\}\cup\{y=dg(x)\}$ for some function $g:\mathbb{R}^n\to\mathbb{R}$ so that $dg(0)=0$ (here we use natural coordinates on $T^*\mathbb{R}^n$). By transversaility the critical point of g at 0 is non-degenerate. Composing with the symplectomorphism $(x,y)\mapsto(x,y-dg(x))$ gives the desired coordinates. □

Cancellation of critical points in a Weinstein cobordism

The following proposition concerning cancellations of critical points in a Weinstein cobordism is proven in [1], see there Proposition 12.22.

Proposition 3.3. *Let (W,ω,Z_0,ϕ_0) be a Weinstein cobordism with exactly two critical points p,q of index k and $k-1$, respectively, which are connected by a unique Z-trajectory along which the stable and unstable manifolds intersect transversely. Let Δ be the closure of the stable manifold of the critical point p. Then there exists a Weinstein cobordism structure (ω,Z_1,ϕ_1) with the following properties:*

(i) $(Z_1,\phi_1)=(Z_0,\phi_0)$ *near ∂W and outside a neighborhood of Δ;*

(ii) ϕ_1 *has no critical points.*

From Legendrian isotopy to Lagrangian concordance

The following Lemma about Lagrangian realization of a Legendrain isotopy is proven in [5], see there Lemma 4.2.5.

Lemma 3.4. *Let $f_t:\Lambda\to(Y,\xi=\{\alpha=0\})$, $t\in[0,1]$, be a Legendrian isotopy connecting f_0,f_1. Let us extend it to $t\in\mathbb{R}$ as independent of t for $t\notin[0,1]$. Then there exists a Lagrangian embedding*

$$F:\mathbb{R}\times\Lambda\to\mathbb{R}\times Y,d(e^s\alpha)),$$

of the form $F(t,x)=(\widetilde{f}_t(x),h(t,x))$ such that

- $F(t,x)=(f_1(x),t)$ *and* $F(x,-t)=f_0(x)$ *for $t>C$, for a sufficiently large constant C;*
- $\widetilde{f}_t(x)$ C^∞*-approximate* $f_t(x)$.

4 Action-balanced Lagrangian immersions

Suppose we are given an exact proper Lagrangian immersion $f : L \to X$ of an orientable manifold L into a simply connected Liouville manifold (X, λ) with finitely many transverse self-intersection points. For each self-intersection point $p \in X$ we denote by $p^0, p^1 \in L$ its pre-images in L. The integral $a_{\mathrm{SI}}(p, f) = \int\limits_\gamma f^*\lambda$, where $\gamma : [0,1] \to L$ is any path connecting the points $\gamma(0) = p^0$ and $\gamma(1) = p^1$, will be called the *action* of the self-intersection point p. Of course, the sign of the action depends on the ordering of the pre-images p^0 and p^1. We will fix this ambiguity by requiring that $a_{\mathrm{SI}}(p, f) > 0$ (by a generic perturbation of f we can assume there are no points p with $a_{\mathrm{SI}}(p, f) = 0$).

A pair of self-intersection points (p, q) is called a *balanced Whitney pair* if $a_{\mathrm{SI}}(p, f) = a_{\mathrm{SI}}(q, f)$ and the intersection indices of $df(T_{p^0}L)$ with $df(T_{p^1}L)$ and of $df(T_{q^0}L)$ with $df(T_{q^1}L)$ have opposite signs. In the case where L is non-orientable we only require that p and q have the same action. A Lagrangian immersion f is called *balanced* if the set of its self-intersection points can be presented as the union of disjoint balanced Whiney pairs.

The goal of this section is the following

Proposition 4.1. *Let (X, λ) be a simply connected Liouville manifold with a negative end and $f : L \to X$ a proper exact and cylindrical at $-\infty$ Lagrangian immersion with finitely many transverse double points. If $n = 3$ suppose, in addition, that $X \setminus f(L)$ has infinite Gromov width. Then there exists an exact cylindrical at $-\infty$ Lagrangian regular homotopy $f_t : L \to X$, $t \in [0,1]$, which is compactly supported away from the negative end, and such that $f_0 = f$ and f_1 is balanced.*
If the asymptotic negative boundary of f has a component which is loose in the complement of the other components and $I(f) = 0$ then the Lagrangian regular homotopy f_t can be made fixed at $-\infty$.

Note that Proposition 4.1 is the only step in the proof of the main results of this paper where one need the infinite Gromov width condition when $n = 3$.

The following two lemmas will be used to reduce the action of our intersection points in the case where we only have a finite amount of space to work with, for example when X_+ is compact. In the case where X_+ contains a symplectic ball B_R of arbitrarily large radius, e.g. in the situation of Theorem 1.1, these lemmas are not needed.

Lemma 4.2. *Consider an annulus $A := [0,1] \times S^{n-1}$. Let x, z be coordinates corresponding to the splitting, and y, u the dual coordinates in the cotangent bundle T^*A, so that the canonical Liouville form λ on T^*A is equal to $ydx + udz$. Then for any integer $N > 0$ there exists a Lagrangian immersion $\Delta : A \to T^*A$ with the following properties:*

- $\Delta(A) \subset \{|y| \leq \frac{5}{N}, ||u|| \leq \frac{5}{N}\}$*;*
- Δ *coincides with the inclusion of the zero section* $j_A : A \hookrightarrow T^*A$ *near* ∂A*;*
- *there exists a fixed near* ∂A *Lagrangian regular homotopy connecting* j_A *and* Δ*;*
- $\int_\zeta \lambda = 1$, *where* ζ *is the* Δ*-image of any path connecting* $S^{n-1} \times 0$ *and* $S^{n-1} \times 1$ *in* A*;*
- *action of any self-intersection point of* Δ *is* $< \frac{1}{N}$*;*
- *the number of self-intersection points is* $< 8N^3$.

Proof. Consider in $\mathbb{R}^2$ with coordinates (x, y) the rectangulars

$$I_{j,N} = \left\{ \frac{j}{5N^2} \leq x \leq \frac{j}{5N^2} + \frac{1}{5N}, 0 \leq y \leq \frac{5}{N} \right\}, j = 0, \dots (N-1)N.$$

Consider a path γ in $\mathbb{R}^2$ which begins at the origin, travels counter-clockwise along the boundary of $I_{0,N}$, then moves along the x-axis to the point $(\frac{1}{5N^2}, 0)$, travels counter-clockwise along the boundary of $J_{1,N}$ etc., and ends at the point $(1, 0)$. Note that $\int_\gamma ydx = \frac{N-1}{N}$. We also observe that squares $I_{j,N}$ and $I_{i,N}$ intersect only when $|i-j| \leq N$, and hence for any self-intersection point p of γ its action is bounded by $N\frac{1}{N^2} = \frac{1}{N}$. Let us C^∞-approximate γ by an immersed curve γ_1 with transverse self-intersections and which coincides with γ near its end points. We can arrange that

- $\left| \int_{\gamma_1} ydx - 1 \right| < \frac{2}{N}$;
- action of any self-intersection point of γ_1 is $< \frac{1}{N}$;
- the number of self-intersection points is $< 2N^3$;

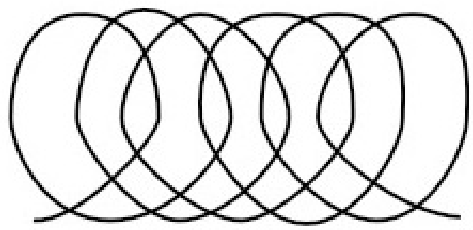

Fig. 4.1: The curve γ_1 when $N = 3$.

- the curve γ_1 is contained in the rectangular $\{0 \leq x \leq \frac{1}{5}, 0 \leq y \leq \frac{5}{N}\}$.

See Figure 4.1. The only non-trivial statement is the upper bound on the number of self-intersections. Notice that there are less than N^2 loops, and each loop intersects at most $2N$ other loops, in 2 points each. Thus the number of self intersections, double counted, is less than $4N^3$.

We will assume that γ_1 is parameterized by the interval $[0, \frac{1}{5}]$. Let r_N denote the affine map $(x, y) \mapsto (x + \frac{1}{5}, -\frac{y}{N})$. We define a path $\gamma_2 : [\frac{1}{5}, \frac{2}{5}] \to \mathbb{R}^2$ by the formula

$$\gamma_2(t) = r_N(\gamma_1(t - \frac{1}{5})).$$

Note that the immersion $\gamma_{12} : [0, \frac{2}{5}] \to \mathbb{R}^2$ which coincides with γ_1 on $[0, \frac{1}{5}]$ and with γ_2 on $[\frac{1}{5}, \frac{2}{5}]$ is regularly homotopic to the straight interval embedding via a homotopy which is fixed near the end of the interval, and which is inside $\{0 \leq x \leq \frac{2}{5}, -\frac{5}{N^2} \leq y \leq \frac{5}{N}\}$. We also note that $\left|\int\limits_{\gamma_{12}} y dx - 1\right| < \frac{3}{N}$. See Figure 4.2.

We further extend γ_{12} to an immersion $\gamma_{123} : [0, 1] \to \mathbb{R}^2$ by extending it to $[\frac{2}{5}, 1]$ as a graph of function $\theta : [\frac{2}{5}, 1] \to [-\frac{5}{N}, \frac{5}{N}]$ with

$$\int\limits_{2/5}^{1} \theta(x) dx = 1 - \int\limits_{\gamma_{12}} y dx,$$

which implies $\int\limits_{\gamma_{123}} y dx = 1$.

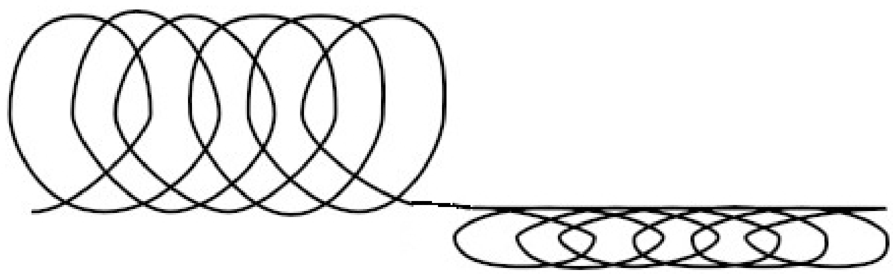

Fig. 4.2: The curve γ_{12}.

Let $j_{S^{n-1}}$ denote the inclusion $S^{n-1} \to T^*S^{n-1}$ as the 0-section. Consider a Lagrangian immersion $\Gamma : A \to T^*A$ given by the formula

$$\Gamma(x, z) = (\gamma_{123}(x), j_{S^{2n-1}}(z)) \in T^*[0,1] \times T^*S^{n-1} = T^*A.$$

The Lagrangian immersion Γ self-intersects along spheres of the form $p \times S^{n-1}$ where p is a self-intersection point of $\widetilde{\gamma}$. By a C^∞-perturbation of Γ we can construct a Lagrangian immersion $\Delta : A \to T^*A$ with transverse self-intersection points which have all the properties listed in Lemma 4.2. Indeed, for each of the $4N^3$ intersection points p of γ_{123}, the sphere $p \times S^{n-1}$ can be perturbed to have two self-intersections. The other required properties are straightforward from the construction. $\square$

Remark 4.3. Given any $a > 0$ we get, by scaling the Lagrangian immersion Δ with the dilatation $(y, u) \mapsto (ay, au)$, a Lagrangian immersion $\Delta_a : A \to T^*A$ which satisfy

- $\int_\zeta \lambda = a$, where ζ is the Δ_a-image of any path connecting the boundary $S^{n-1} \times 0$ and $S^{n-1} \times 1$ of A;
- action of any self-intersection point of Δ_a is $< \frac{a}{N}$;
- the number of self-intersection points is $< 8N^3$;
- $\Delta_a(A) \subset \{|y|, ||u|| \leq \frac{5a}{N}\}$;
- the immersion Δ_a is regularly homotopic relative its boundary to the inclusion $A \hookrightarrow T^*A$.

Given a proper Lagrangian immersion $f : L \to X$ with finitely many transverse self-intersection points, we denote the number of self-intersection points by $\mathrm{SI}(f)$. The action of a self-intersection point p of f is denoted by $a_{\mathrm{SI}}(p, f)$. We set $a_{\mathrm{SI}}(f) := \max_p |a_{\mathrm{SI}}(p, f)|$, where the maximum is taken over all self-intersection points of f.

Lemma 4.4. *Let $f_0 : L \to (X, \lambda)$ be a proper exact Lagrangian immersion into a simply connected Liouville manifold with finitely many transverse self-intersection points. Then for any sufficiently large integer $N > 0$ there exists a fixed at infinity C^0-small exact Lagrangian regular homotopy $f_t : L \to X$, $t \in [0, 1]$, such that f_1 has transverse self-intersections,*

$$a_{\mathrm{SI}}(f_1) \leq \frac{a_{\mathrm{SI}}(f)}{N}, \ \ \mathrm{SI}(f_1) \leq 9N^3 \mathrm{SI}(f_0).$$

Proof. Let $p_1, \dots, p_k$ be the self-intersection points of f_0 and $p_1^0, p_1^1, \dots, p_k^0, p_k^1$ their pre-images, $k = \mathrm{SI}(f_0)$. Let us recall that we order the pre-images in such a way that $a_{\mathrm{SI}}(f_0)(p_i) > 0$, $i = 1, \dots, k$. Choose

- disjoint embedded n-discs $D_i \ni p_i^1$, $i = 1, \dots, k$, which do not contain any other pre-images of double points, and
- annuli $A_i \subset D_i$ bounded by two concentric spheres in D_i.

For a sufficiently large $N > 0$ there exist disjoint symplectic embeddings h_i of the domains $U_i := \{|y|, ||u|| \leq \frac{5a_{\mathrm{SI}}(p, f_0)}{N}\} \subset T^*A$ in X, $i = 1, \dots, k$, such that $h_i^{-1}(f_0(L)) = h_i^{-1}(A_i) = A$. Then, using Remark 4.3, we find a Lagrangian regular homotopy f_t supported in $\bigcup_1^k h_i(U_i)$ which annihilates the action of points p_i, i.e. $a_{\mathrm{SI}}(p_i, f_1) = 0$, $i = 1, \dots k$, and which creates no more than $8kN^3$ new self-intersection points of action $< \frac{a_{\mathrm{SI}}(f_0)}{N}$. Hence, the total number of self-intersection points of f_1 satisfies the inequality $\mathrm{SI}(f_1) < 9\mathrm{SI}(f_0)N^3$.

□

The next lemma is a local model which will allow us to match the action of a given intersection point, during our balancing process. For a positive C we denote by Q_C the parallelepiped

$$\{|z| \leq C, |x_i| \leq 1, |y_i| \leq C, \ i = 1, \dots, n-1\}$$

in the standard contact space $\mathbb{R}^{2n-1}_{\mathrm{st}} = (\mathbb{R}^{2n-1}, \xi = \{\alpha_{\mathrm{st}} := dz - \sum_1^{n-1} y_i dx_i = 0\})$. Let SQ_C denote the domain $[\frac{1}{2}, 1] \times Q_C$ in the symplectization $(0, \infty) \times Q_C$ of Q_C endowed with the Liouville form $\lambda_0 := s\alpha_{\mathrm{st}}$. We furthermore denote by L^t the Lagrangian rectangular $\{z = t, y = 0; j = 1, \ldots, n-1\} \cap SQ_C \subset SQ_C$, $t \in [-C, C]$.

Lemma 4.5. *For any positive $b_0, b_1, \ldots, b_k \in (0, \infty)$, $k \geq 0$, such that*

$$\frac{C}{4k+4} > b_0 > \max(b_1, \ldots, b_k),$$

and a sufficiently small $\varepsilon > 0$ there exists a Lagrangian isotopy which starts at $L^{-\varepsilon}$, fixed near $1 \times Q_C$ and $[\frac{1}{2}, 1] \times \partial Q_C$, cylindrical near $\frac{1}{2} \times Q_C$, and which ends at a Lagrangian submanifold $\widetilde{L}^{-\varepsilon}$ with the following properties:

- *$\widetilde{L}^{-\varepsilon}$ intersects L^0 transversely at $k+1$ points $B_0, B_1, \ldots, B_k$;*
- *if $\gamma_{B_j}, j = 0. \ldots, k$, is a path in $\widetilde{L}^{-\varepsilon}$ connecting the point B_j with a point on the boundary ∂Q_C, then*

$$\int_{\gamma_{B_j}} \lambda_0 = b_j, \ j = 0, \ldots, k;$$

- *the intersection indices of L^0 and $\widetilde{L}^{-\varepsilon}$ at the points $B_0, B_1, \ldots, B_k$ are equal to $1, -1, \ldots, -1$, respectively.*
- *$\widetilde{L}^{-\varepsilon} \cap \{s = \frac{1}{2}\}$ is a Legendrian submanifold in Q_C defined by a generating function which is equal to $-\varepsilon$ near ∂Q_C and positive over a domain in Q_C of Euler characteristic $1 - k$.*

Proof. We have

$$\omega := d\lambda_0 = ds \wedge dz - \sum_1^{n-1} dx_i \wedge d(sy_i) = -d(zds + \sum_1^{n-1} v_i dq_i),$$

we denoted $v_i := sy_i, \ i = 1, \ldots, n-1$. Let $I^{n-1} \subset \mathbb{R}^{n-1}$ be the cube $\{\max_{i=1,\ldots,n-1} |q_i| \leq 1\}$. Choose a smooth non-negative function $\theta : [\frac{1}{2}, 1] \to \mathbb{R}$ such that

- $\theta(s) = s$ for $s \in [\frac{1}{2}, \frac{5}{8}]$;

- θ has a unique local maximum at a point $\frac{3}{4}$;
- $\theta(s) = 0$ for s near 1;
- the derivative θ' is monotone decreasing on $[\frac{5}{8}, \frac{3}{4}]$.

For any $\widetilde{b}_0, \dots, \widetilde{b}_k \in (0, \frac{C}{2k+2})$ which satisfy $\widetilde{b}_0 > \max(\widetilde{b}_1, \dots, \widetilde{b}_k)$ one can construct a smooth non-negative function $\phi : I^{n-1} \to \mathbb{R}$. with the following properties:

- $\phi = 0$ near ∂I^{n-1};
- $\max\limits_{i=1,\dots,n-1} \left|\frac{\partial \phi}{\partial q_i}\right| < \frac{C}{2}$;
- besides degenerate critical points corresponding to the critical value 0, the function ϕ has $k+1$ positive non-degenerate critical points: 1 local maximum $\widetilde{B}_0$ and k critical points $\widetilde{B}_1, \dots, \widetilde{B}_k$ of index $n-2$ with critical values $\widetilde{b}_0, \widetilde{b}_1, \dots, \widetilde{b}_k$ respectively.

Take a positive $\varepsilon < \min(\widetilde{b}_1, \dots, \widetilde{b}_k, \frac{C}{8k+8})$ and define a function $h : [\frac{1}{2}, 1] \times I^{n-1} \to \mathbb{R}$ by the formula

$$h(s, q) = -\varepsilon s + \theta(s)\phi(q),\ s \in \left[\frac{1}{2}, 1\right], q \in I^{n-1}.$$

Thus the function h is equal to $s(-\varepsilon + \phi(q))$ for $s \in [\frac{1}{2}, \frac{5}{8}]$ and equal to $-\varepsilon s$ near the rest of the boundary of $[\frac{1}{2}, 1] \times I^{n-1}$. The function h has one local maximum at a point $(s_0, \widetilde{B}_0)$ and k index $n-1$ critical points with coordinates $(s_j, \widetilde{B}_j)$, $j = 1, \dots, k$. Here the values $s_j \in [\frac{5}{8}, \frac{3}{4}]$ are determined from the equations $\widetilde{b}_j \theta'(s_j) = \varepsilon$, $j = 0, \dots, k$. Respectively, the critical values are equal to $\widehat{b}_k := -\varepsilon s_j + \theta(s_j)\widetilde{b}_j$, For $\widetilde{b}_j$ near ε we have $\widehat{b}_j < \varepsilon$, while for $\widetilde{b}_j$ close to $\frac{C}{2k+2}$ we have $\widehat{b}_j > \frac{C}{4k+4}$. Hence, by continuity, any critical values $b_0, b_1, \dots b_k \in \left(\varepsilon, \frac{C}{4k+4}\right)$ which satisfy the inequality $b_0 > \max(b_1, \dots, b_k)$ can be realized.

The required Lagrangian manifold $\widetilde{L}^{-\varepsilon}$ can be now defined by the equations

$$z = \frac{\partial h}{\partial s},\ x_j = q_j,\ v_j = \frac{\partial h}{\partial p_j}, j = 1, \dots, n-1,\ s \in \left[\frac{1}{2}, 1\right],\ q \in I^{n-1},$$

or returning to x, y, z, s coordinates by the equations

$$\widetilde{L}^{-\varepsilon} = \left\{ z = \frac{\partial h}{\partial s}, y_j = \frac{1}{s}\frac{\partial h}{\partial q_j} \right\}.$$

It is straightforward to check that $\widetilde{L}^{-\varepsilon}$ has the required properties. □

After using Lemma 4.4 to shrink the action of an intersection point, Lemma 4.5, applied with $k=0$, will allow us to balance any negative intersection point. Positive intersection points still provide a challenge though, because the intersection point with the largest action created by Lemma 4.5 is always positive. The following lemma solves this issue.

Lemma 4.6. *Let $f : L \to (X, \lambda)$ be a proper exact Lagrangian immersion into a simply connected X and $D \subset L$ an n-disc which contains no double points of the immersion f. Then for any $A > 0$ and a sufficiently small $\sigma > 0$ there exists a supported in D Hamiltonian regular homotopy of f to $\widetilde{f}$ which creates a pair p_+, p_- of additional self-intersection points such that $a_{\rm SI}(p_\pm, \widetilde{f}) = A \pm \sigma$, the self-intersection indices of $p_\pm$ have opposite signs and can be chosen at our will.*

Let us introduce some notation. Consider a domain

$$U_\varepsilon := \{-2\varepsilon < p_1 < 1 + 2\varepsilon, \max_{1\le i\le n} |q_i| < 2\varepsilon, \max_{1\le j\le n} |p_j| < 2\varepsilon\}$$

in the standard symplectic $\mathbb{R}^{2n}_{\rm st} = (\mathbb{R}^{2n}, \sum\limits_1^n dp_i \wedge dq_i)$. Let L^t be the Lagrangian plane $\{p_1 = t, p_j = 0 \text{ for } j = 2, \dots, n\} \cap U_\varepsilon \subset U_\varepsilon,\ t \in \{0, 1\}$. Note that $pdq|_{L^t} = tdq_1$. We will also use the following notation associated with U_ε:

$u_\pm \in L^1$ denote the points with coordinates $p = (1, 0, \dots, 0), q = (\pm\delta_1, 0, \dots, 0)$;

$z_\pm \in L^0$ denote the points with coordinates $p = (0, 0, \dots, 0), q = (\pm\delta_1, 0, \dots, 0)$

c^0 denote the point with coordinates $p = (0, 0, \dots, 0), q = (-\varepsilon, 0, \dots, 0)$;

c^1 denote the point with coordinates $p = (1, 0, \dots, 0), q = (-\varepsilon, 0, \dots, 0)$;

$J^1_\pm$ denote the intervals connecting c^1 and $u_\pm$;

$J^0_\pm$ denote the intervals connecting c^0 and $z_\pm$.

We will use in the proof of 4.6 the following

Lemma 4.7. *There exists a Lagrangian isotopy $f_t : L^1 \to U_\varepsilon$ fixed near ∂L^1 and starting at the inclusion $f_0 : L^1 \hookrightarrow U_\varepsilon$ such that $\widetilde{L}^1 = f_1(L^1)$ transversely intersects L^0 at two points $z_\pm$ with the following properties:*

- *$f_1^*(pdq) = q_1 + d\theta$, where $\theta : L^1 \to \mathbb{R}$ is a compactly supported in $\operatorname{Int} L^1$ function such that $\theta(z_\pm) = \mp\delta$ for a sufficiently small $\delta > 0$;*

- *the intersection indices of $\widetilde{L}^1$ and L^0 at z_+ and z_- have opposite signs and can be chosen at our will.*

Proof. For sufficiently small δ_1, δ_2, $0 < \delta_1 \ll \delta_2 \ll \varepsilon$, there exists a C^∞-function $\alpha : [-\varepsilon, \varepsilon] \to \mathbb{R}$ with the following properties:

- $\alpha(t) = t$ for $\delta_2 \leq |t| \leq \varepsilon$;
- $\alpha(t) = t^3 - 3\delta_1^2 t$ for $|t| \leq \delta_1$;
- the function α has no critical points, other than $\pm\delta_1$;
- $-\frac{\varepsilon}{2} < \alpha'(t) < 1 + \frac{\varepsilon}{2}$.

Let us also take a cut-off function $\beta : [0,1] \to [0,1]$ which is equal to 0 near 1 and equal to 1 near 0. Take a quadratic form Q_j of index $j-1$:

$$Q_j(q_2, \ldots, q_n) = -\sum_{i=2}^{j} q_i^2 + \sum_{j+1}^{n} q_i^2,\ j = 1, \ldots, n,$$

and define a function $\sigma : \{|q_i| \leq \varepsilon; i = 1, \ldots, n\} \to \mathbb{R}$ by the formula

$$\sigma_j(q_1, q_2, \ldots, q_n) = q_1 + \delta_2 Q_j(q_2, \ldots, q_n)\beta\left(\frac{\rho}{\varepsilon}\right)\beta\left(\frac{|q_1|}{\varepsilon}\right) + (\alpha(q_1) - q_1)\beta\left(\frac{\rho}{\varepsilon}\right),$$

where we denoted $\rho := \max_{2 \leq i \leq n} |q_i|$. The function σ_j has two critical points $(-\delta_1, 0, \ldots, 0)$ and $(\delta_1, 0, \ldots, 0)$ of index j and $j-1$, respectively. We note that

$$-\frac{\varepsilon}{2} - Cn\delta_2\varepsilon \leq \frac{\partial \sigma_j}{\partial q_1} < 1 + \frac{\varepsilon}{2} + Cn\delta_2\varepsilon$$

and

$$\left|\frac{\partial \sigma_j}{\partial q_i}\right| \leq 2\delta_2\varepsilon + Cn\delta_2\varepsilon + \frac{C\delta_2}{\varepsilon}$$

for $i > 1$, where $C = ||\beta||_{C^1}$. In particular, if δ_2 is chosen small enough we get $-\varepsilon < \frac{\partial \sigma_j}{\partial q_1} < 1 + \varepsilon$ and $\left|\frac{\partial \sigma_j}{\partial q_i}\right| < \varepsilon$ for $i = 2, \ldots, n$.

Assuming that L^1 is parameterized by the q-coordinates we define the required Lagrangian isotopy $f_t : L^1 \to U_\varepsilon$ by the formula:

$$f_t(q) = \left(q, 1 + t\left(\frac{\partial \sigma_j}{\partial q_1} - 1\right), t\frac{\partial \sigma_j}{\partial q_2}, \ldots, t\frac{\partial \sigma_j}{\partial q_n}\right)), |q_i| < 2\varepsilon;\ i = 1, \ldots, n.$$

The Lagrangian manifold $\widetilde{L}^1 = f_1(L^1)$ intersects L^0 at two points $z_\pm$ with coordinates $p = 0, q_1 = \pm\delta_1, q_2 = 0, \dots, q_n = 0$. The intersection index of $\widetilde{L}^1$ and L^0 at z_- is equal to $(-1)^j$, and to $(-1)^{j-1}$ at z_+. Thus by choosing j even or odd we can arrange the intersection to be positive at z_+ and negative at z_-, or the other way around. The compactly supported function θ determined from the equation $f_1^*(pdq) = dq_1 + d\theta$ is equal to $\sigma_j - q_1$. In particular, $\theta(z_\pm) = \mp 2\delta_1^3$. □

Proof of Lemma 4.6. We denote $\widetilde{J}^1_\pm := f_1(J^1_\pm)$, where f_t is the isotopy constructed in Lemma 4.7. Take any two points $a, b \in D \subset \widetilde{D} := f(D) \subset \widetilde{L} := f(L)$ and connect them by a path $\eta : [0,1] \to \widetilde{D}$ such that $\eta(0) = \widetilde{b} := f(b)$ and $\eta(1) = \widetilde{a} := f(a)$. Denote $B := \int\limits_\eta \lambda$.

For any real R there exists an embedded path $\gamma : [0,1] \to X$ connecting the points $\gamma(0) = \widetilde{a}$ and $\gamma(1) = \widetilde{b}$ in the complement of $\widetilde{L}$, homotopic to a path in $\widetilde{L}$ with fixed ends, and such that $\int\limits_\gamma \lambda = R$. For a sufficiently small $\varepsilon > 0$ the embedding γ can be extended to a symplectic embedding $\Gamma : U_\varepsilon \to X$ such that $\Gamma^{-1}(\widetilde{L}) = L^0 \cup L^1$. Here we identify the domain $[0,1]$ of the path γ with the interval

$$I = \{q_1 = -\varepsilon, q_j = 0, j = 2, \dots, n; 0 \le p_1 \le 1, p_j = 0, j = 2, \dots, n\} \subset \partial U_\varepsilon,$$

so that we have $\Gamma(c^0) = \widetilde{a}$ and $\Gamma(c^1) = \widetilde{b}$.

The Lagrangian isotopy $\widetilde{f}_t := \Gamma \circ f_t : L^1 \to X$, where $f_t : L^1 \to U_\varepsilon$ is the isotopy constructed in Lemma 4.7, extends as a constant homotopy to the rest of L and provides us with a regular Lagrangian homotopy connecting the immersion f with a Lagrangian immersion $L \to X$ which has two more transverse intersection points $p_\pm := \Gamma(z_\pm)$ of opposite intersection index sign. See Figure 4.3. Consider the following loops $\zeta_\pm$ in $\widetilde{L} \subset X$ based at the points $p_\pm$. We start from the point $p_\pm$ along the Γ-image of the oppositely oriented interval $\widetilde{J}^1_\pm$ to the point $\widetilde{b}$, then follow the path η to the point $\widetilde{a}$, and finally follow along the Γ-image of the path J_0 back to $p_\pm$.

Then we have

$$\begin{aligned}\int\limits_{\zeta_\pm} \lambda &= -\int\limits_{\widetilde{J}^1_\pm} \Gamma^*\lambda + \int\limits_\eta \lambda + \int\limits_{J^0_\pm} \Gamma^*\lambda \\ &= \left(-\int\limits_{\widetilde{J}^1_\pm} \Gamma^*\lambda + \int\limits_\gamma \lambda + \int\limits_{J^0_\pm} \Gamma^*\lambda \right) + \left(\int\limits_\eta \lambda - \int\limits_\gamma \lambda \right)\end{aligned}$$

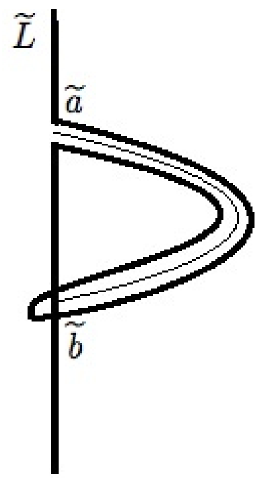

Fig. 4.3: The Lagrangian $f_1(L)$. The light curve represents γ.

$$= \left(-\int_{\widetilde{J}^1_\pm} pdq - \int_I pdq + \int_{J^0_\pm} pdq \right) + (B+R) = -\varepsilon + B + R \mp 2\delta_1^3.$$

It remains to observe that there exists a sufficiently small $\varepsilon_0 > 0$ which can be chosen for any $R \in [A-C-1, A-C+1]$. Hence, by setting $R = A-C-\varepsilon_0$ and $\varepsilon = \varepsilon_0$ we arrange that the action of the intersection points $p_\pm$ is equal to $A \mp 2\delta_1^3$ while their intersection indices have opposite sign which could be chosen at our will. □

Lemma 4.8. *Let $((0,\infty) \times Y, d(t\alpha))$ be the symplectization of a manifold Y with a contact form α. Let Λ be a Legendrian submanifold and $L = (0,\infty) \times \Lambda$ the Lagrangian cylinder over it. Suppose that there exists a contact form preserving embedding $\Phi : (Q_C, \alpha_{\mathrm{st}}) \to (Y, \alpha)$ and $\Gamma \subset Y$ an embedded isotropic arc connecting a point $b \in \Lambda$ with a point*

$$\Phi(x_1 = 1, x_2 = 0, \ldots, x_{n-1} = 0, y_1 = 0, \ldots, y_n = 0, z = 0) \in \partial\Phi(Q_C).$$

Then there exists a Lagrangian isotopy $L_t \subset \mathbb{R} \times \Lambda$ supported in a neighborhood of $1 \times \Gamma \cup \Phi(Q_C)$, $t \in [0,1]$, which begins at $L_0 = L$ such that

- L_t *transversely intersects* $1 \times Y$ *along a Legendrian submanifold* Λ_t*;*
- $\Phi^{-1}(\Lambda_1) = \Lambda^0 \cup \Lambda^{-\varepsilon}$ *for a sufficiently small* $\varepsilon > 0$*.*

Proof. We use below the notation I^k_a, $a > 0$ for the cube $\{|x_i| \le a, i = 1, \dots, k\} \subset \mathbb{R}^k$. The embedding Φ can be extended to a slightly bigger domain $\widehat{Q} = \{|x_i| \le 1 + \sigma, |y_i| \le C, i = 1, \dots, n-1, |z| \le C + \sigma\} \subset \mathbb{R}^{2n-1}_{\mathrm{st}}$ for a sufficiently small $\sigma > 0$. The intersection $\widehat{Q} \cap (\mathbb{R}^{n-1} = \{y = 0, z = 0\})$ is the cube $I^{n-1}_{1+\sigma} \subset \mathbb{R}^{n-1}$. We can assume that the intersection of the path Γ with $\widehat{Q}$ coincides with the interval $\{1 \le x_1 \le 1 + \sigma, x_j = 0, j = 2, \dots, n-1\} \subset I^{n-1}_{1+\sigma}$. The Legendrian embedding $\Psi := \Phi|_{I^{n-1}_{1+\sigma}} : I^{n-1}_{1+\sigma} \to Y$ can be extended to a bigger parallelepiped

$$\Sigma = \{-1 - \sigma \le x_1 \le 2 + \sigma, |x_j| \le 1 + \sigma, j = 2, \dots, n-1\} \subset \mathbb{R}^{n-1}$$

such that the extended Legendrian embedding, still denoted by Ψ, has the following properties:

- $\Psi(\{1 \le x_1 \le 2, x_j = 0, j = 2, \dots, n-1\}) = \Gamma$;
- $\Psi(\{x_1 = 2\}) \subset \Lambda$.

For a sufficiently small positive $\delta < C$ the Legendrian embedding can be further extended as a contact form preserving embedding

$$\widehat{\Psi} : (\widehat{P} := \{(x, y, z) \in \mathbb{R}^{2n-1}_{\mathrm{st}}; x \in \Sigma, |y_i| \le \delta, i = 1, \dots, n-1, |z| \le \delta\}, \alpha_{\mathrm{st}}) \to (Y, \alpha),$$

such that

- $\widehat{\Psi}|_{\widehat{P} \cap \widehat{Q}} = \Phi|_{\widehat{P} \cap \widehat{Q}}$;
- the Legendrian manifold $\widehat{\Lambda} := \widehat{\Psi}^{-1}(\Lambda)$ is given by the formulas

$$\widehat{\Lambda} := \{z = \pm(x_1 - 2)^{\frac{3}{2}}, y_1 = \pm\frac{3}{2}\sqrt{x_1 - 2}, x_1 \ge 2, y_j = 0, j = 2, \dots, n-1\}$$

(note that any point on any Legendrian admits coordinates describing $\widehat{\Lambda}$ as above). Consider a cut-off C^∞-function $\theta : [0, 1 + \sigma] \to [0, 1]$ such that $\theta(u) = 1$ if $u \le 1$, $\theta(u) = 0$ if $u > 1 + \frac{\sigma}{2}, \theta' \le 0$, and denote

$$\Theta(u_1, \dots, u_{n-2}) := (3 + \sigma) \prod_{1}^{n-2} \theta(u_i),\ u_1, \dots, u_{n-2} \in [0, 1 + \sigma].$$

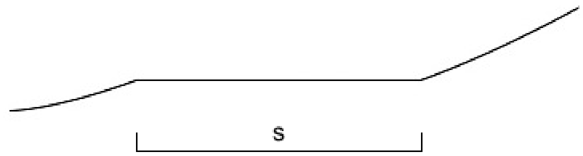

Fig. 4.4: The function g_s.

For $s \in [0,1]$ denote

$$\Omega_s := \{2 - s\Theta(|x_2|, \dots, |x_{n-1}|) \le x_1 \le 2 + \sigma\} \cap \Sigma \subset \mathbb{R}^{n-1}.$$

We have $\Omega_1 \supset \{-1 - \sigma \le x_1 \le 2, |x_2|, \dots, |x_{n-1}| \le 1\} \supset I_1^{n-1}$ and $\Omega_0 = \{2 \le x_1 \le 2 + \sigma\} \cap \Sigma$.

For a sufficiently small positive $\varepsilon < \frac{\sigma^{\frac{3}{2}}}{2}$ consider a family of piecewise smooth continuous functions $g_s : [2 - s, 2 + \sigma] \to [0, \sigma^{\frac{3}{2}}]$, $s \in [0, 3 + \sigma]$ defined by the formulas

$$g_s(u) = \begin{cases} (u - 2 + s)^{\frac{3}{2}}, & u \le 2 - s + \varepsilon^{\frac{2}{3}}; \\ \varepsilon, & 2 - s + \varepsilon^{\frac{2}{3}} < u < 2 + \varepsilon^{\frac{2}{3}}; \\ (u - 2)^{\frac{3}{2}}, & u \ge 2 + \varepsilon^{\frac{2}{3}}. \end{cases}$$

See Figure 4.4. We can smooth g_s near the points $2 + \varepsilon^{\frac{2}{3}}$ and $2 - s + \varepsilon^{\frac{2}{3}}$ in such away that the derivative is monotone near these points (i.e. decreasing near $2 - s + \varepsilon^{\frac{2}{3}}$ and increasing near $2 + \varepsilon^{\frac{2}{3}}$). We continue to denote the smoothened by g_s.

Next, define for $s \in [0,1]$ a function $G_s : \Omega_s \to \mathbb{R}$ by the formula

$$G_s(x_1, x_2 \dots, x_{n-1}) = g_{s\Theta(x_2, \dots, x_{n-1})}(x_1).$$

Note that by decreasing ε and σ we can arrange that $\frac{\partial G_s}{\partial s}(x), \left|\frac{\partial G_s}{\partial x_i}(x)\right| < \delta$, $i = 1, \dots, n-1$, for all $s \in [0,1]$ and $x \in \Omega_s$. We also observe that if $\frac{\partial G_s}{\partial x_1}(x) = 0$ then $G_s(x) = \varepsilon$. Choose a cut-off function $\mu : [1 - \delta, 1 + \delta] \to [0, 1]$ which is equal to 1 near 1 and equal to 0 near $1 \pm \delta$ and consider a family of Lagrangian submanifolds N_s, $s \in [0,1]$, defined in the domain $([1 - \delta, 1 + \delta] \times \widehat{P}, d(t\alpha_{\mathrm{st}}))$ in the symplectization of $\widehat{P}$ defined by the formulas

$$\begin{aligned} &z = \pm G_{s\mu(t)}(x) \pm t\frac{\partial G_{s\mu(t)}}{\partial t}(x), y_i = \pm \frac{\partial G_{s\mu(t)}}{\partial x_i}(x), \\ &x \in \Omega_{s\mu(t)},\ i = 1, \dots, n-1,\ t \in [1 - \delta, 1 + \delta]. \end{aligned}$$

First, let us check that N_s is Lagrangian for all $s \in [0,1]$. Indeed, we have $d(t\alpha_{\mathrm{st}}) = -d\left(zdt + \sum\limits_1^{n-1}(ty_i)dx_i\right)$, and hence

$$d(t\alpha_{\mathrm{st}})|_N = \pm d\left(\left(G_{s\mu(t)} + t\frac{\partial G_{s\mu(t)}}{\partial t}\right)dt + \sum_1^{n-1} t\frac{\partial G_{s\mu(t)}}{\partial x_i}dx_i\right) = \pm d(d(tG_{s\mu(t)})) = 0.$$

Next, we check that N_s is embedded. The only possible pairs of double points may be of the form (x,y,z) and $(x,-y,-z)$, that is $z=0$ and $y=0$. But then $\frac{\partial G_{s\mu(t)}}{\partial x_1} = 0$, and hence $G_{s\mu(t)}(x) = \varepsilon$ and $\frac{\partial G_{s\mu(t)}}{\partial t}(x) = 0$, which shows $z = G_{s\mu(t)}(x) + t\frac{\partial G_{s\mu(t)}}{\partial t}(x) \neq 0$.

We also note that $N_s \cap \{t=1\}$ is a Legendrian submanifold $\{z = \pm G_{s\mu(t)}(x), y_i = \pm\frac{\partial G_{s\mu(t)}}{\partial x_i}(x), i = 1,\dots,n-1\} \subset \widehat{P}$ and N_1 intersects Q_C along $\Lambda^{-\varepsilon} \cup \Lambda^{\varepsilon}$. Near $t = 1 \pm \delta$ the submanifold N_s coincides with the symplectization of the Legendrian submanifold $\widehat{\Lambda}$ for all $s \in [0,1]$.

Let us remove from the Lagrangian cylinder $L = (0,\infty)\times\Lambda \subset ((0,\infty)\times Y, t\alpha)$ the domain $[1-\delta, 1+\delta]\times\Lambda$ and replace it by $\Psi(N_s)$. The resulted Lagrangian isotopy L_s has the following properties: $L_0 = L$, L_1 intersects the contact slice $1\times Y$ along a Legendrian submanifold Λ_1 and $\Phi^{-1}(\Lambda_1) = \Lambda^{-\varepsilon}\cup\Lambda^{\varepsilon}$. Note that if we modify the embedding Φ as $\widetilde{\Phi}(x,y,z) = \Phi(x,y,z-\varepsilon)$ we still get a contact form preserving embedding $\widetilde{\Phi} : (Q_C, \alpha_{\mathrm{st}}) \to (Y,\alpha)$ for which $\widetilde{\Phi}^{-1}(\Lambda_1) = \Lambda^{-2\varepsilon}\cup\Lambda^0$. □

Proof of Proposition 4.1 for $n > 3$. Let X_- be a negative Liouville end of X bounded by a contact slice $Y \subset X$ such that f is cylindrical below it. Denote $\Lambda := f^{-1}(Y)$. According to Lemma 4.4 for any ε there exists a Hamiltonian regular homotopy of f into a Lagrangian immersion with transverse self-intersection points of action $< \varepsilon$. Moreover, the number of self-intersection points grows proportionally to $\frac{1}{\varepsilon^3}$ when $\varepsilon \to 0$. For a sufficiently small $C > 0$ there exists a contact form preserving embedding $(Q_C, \alpha_{\mathrm{st}}) \to (Y \setminus \Lambda, \alpha := \lambda|_Y)$. Note that given an integer $N > 0$ and a positive $\varepsilon < \frac{C}{N}$ there exists contact form preserving embeddings of N^n disjoint copies of $(Q_\varepsilon, \alpha_{\mathrm{st}})$ into $(Q_C, \alpha_{\mathrm{st}})$, i.e. when decreasing ε the number of domains $(Q_\varepsilon, \alpha_{\mathrm{st}})$ which can be packed into $(Y\setminus\Lambda, \alpha)$ grows proportionally to ε^{-n}, which is greater than ε^{-3} by assumption. Hence for a sufficiently small ε we can modify the Lagrangian immersion f, so that the action of all its self-intersection points are $< \varepsilon$, and at least $\mathrm{SI}(f)$ disjoint Darboux neighborhoods isomorphic to $Q_{12\varepsilon}$ which do not intersect Λ can be packed into (Y,α). We will denote the number of self-intersection points by N and the corresponding $Q_{12\varepsilon}$-neighborhoods by $U_1,\dots,U_N$. Notice that for a sufficiently small $\theta > 0$ there exists a Liouville form preserving embedding $((0, 1+\theta) \times Y, t\alpha) \to (X, \lambda)$ which sends $Y \times 1$ onto Y.

For each intersection point $p_i \in f(L)$, $i = 1, \dots N$, we will find a compactly supported Hamiltonian regular homotopy to balance each intersection point p_i without changing the action of the other intersection points. Recall $0 < a_{\mathrm{SI}}(p_1, f) < \varepsilon$. Using Lemma 4.8 we isotope the Lagrangian cylinder $(0, 1+\theta) \times \Lambda)$ via a Lagrangian isotopy supported in a neighborhood of $Y \times 1$ so that:

- the deformed cylinder $\widetilde{\Lambda}$ intersects Y transversely along a Legendrian submanifold $\widetilde{\Lambda}$;
- for a sufficiently small $\sigma > 0$ and each $i = 1, \dots, N$, the cylinder $\widetilde{\Lambda}$ intersects $U_i = Q_{12\varepsilon}$ along Legendrian planes $\Lambda^0 = \{y = 0, z = 0\}$ and $\Lambda^{-\sigma} = \{z = -\sigma, y = 0\}$.

We can further deform the Lagrangian $\widetilde{L}$ to make it cylindrical in $[\frac{1}{2}, 1] \times Y$, and hence, we get embeddings $(\left[\frac{1}{2}, 1\right] \times Q_{12\varepsilon}, t\alpha_{\mathrm{st}}) \to ((0, 1] \times Y, t\alpha)$ such that the intersections $(\left[\frac{1}{2}, 1\right] \times U_i, \alpha_{\mathrm{st}})$ with $\widetilde{L}$ coincide with the Lagrangians L^0 and $L^{-\delta}$ from Lemma 4.5.

There are two cases, depending on the sign of the intersection; suppose first that the self-intersection index at the point p_i is negative. Then we apply Lemma 4.5 with $k = 0$ and construct a cylindrical at $-\infty$ and fixed everywhere except $L^{-\delta}$ and $\Lambda^{-\delta} \times \left(0, \frac{1}{2}\right]$ Hamiltonian regular homotopy of the immersion f which deforms $L^{-\delta}$ to $\widetilde{L}^{-\delta}$ such that L^0 and $\widetilde{L}^{-\delta}$ positively intersect at 1 point B_0 of action $a_{\mathrm{SI}}(B_0, f) = a_{\mathrm{SI}}(p_i, f)$. Hence, the point B_0 balances p_i. Notice that this homotopes Λ to another Legendrian $\widetilde{\Lambda}$, and in fact $\widetilde{\Lambda}$ will never be Legendrian isotopic to Λ (after a balancing of a sigle intersection point; we show below that it will be isotopic after all intersection points are balaned).

If the self-intersection index of p_i is positive we first apply Lemma 4.6 to create two new intersection points p_+ and p_- of index 1 and -1 and action equal to $A - \sigma$ and $A + \sigma$ respectively, for some $A \in (a_{\mathrm{SI}}(p_i, f), a_{\mathrm{SI}}(p_i, f) + 4\epsilon)$ and sufficiently small $\sigma > 0$. We then apply Lemma 4.5 with $k = 2$ and create 3 new intersection points B_0, B_1, B_2 of indices $1, -1, -1$ and of action $A + \sigma$, $A - \sigma$ and $a_{\mathrm{SI}}(p_i, f)$, respectively. Then (p_i, B_2), (p_+, B_1) and (p_-, B_0) are balanced Whitney pairs.

In the course of the above proof, Λ is homotoped to the Legendrian $\tilde{\Lambda}$ at $-\infty$. In order to make the constructed Hamiltonian homotopy of our Lagrangian fixed at $-\infty$, it suffices to show that Λ is Legendrian isotopic to $\tilde{\Lambda}$, because we can then apply Lemma 3.4 to undo this homotopy near $-\infty$. Assume that Λ has a loose component and $I(f) = 0$. In the course of the above proof we only need to homotope a single component of Λ of our choosing; we choose the component of Λ which is loose. Obviously we can also fix a universal loose Legendrian embedded in this component of Λ, thus the corresponding component of $\tilde{\Lambda}$ is also loose. Using part (ii) of Proposition 2.1, it only remains to show that Λ is formally Legendrian isotopic to $\tilde{\Lambda}$. Because the algebraic count of self intersections of f is zero the homotopy from Λ to $\tilde{\Lambda}$ also has an algebraic count of zero self-intersections. This implies that they are formally isotopic; see Proposition 2.6 in [7]. □

To deal with the case $n = 3$ we will need an additional lemma. Let us denote by $P(C)$ the polydisc $\{p_i^2 + q_i^2 \leq \frac{C}{\pi},\ i = 1, \dots, n\} \subset \mathbb{R}^{2n}_{\mathrm{st}}$.

Lemma 4.9. *Let (X, ω) be a symplectic manifold with a negative Liouville end, $Y \subset X$ a contact slice, and λ is the corresponding Liouville form on a neighborhood $\Omega \supset X_-$ in X. Suppose that there exists a symplectic embedding $\Phi : P(C) \to X_+ \setminus Y$. Let Γ be an embedded path in X_+ connecting a point $a \in Y$ with a point in $b \in \partial\widetilde{P}$, $\widetilde{P} := \Phi(P(C))$. Then for any neighborhoods $U \supset (\Gamma \cup \widetilde{P})$ in X_+ there exists a Weinstein cobordism $(W, \omega, \widetilde{X}, \phi)$ such that*

(i) $W \subset X_+ \cap (U \cup \Omega)$, $\partial_- W = Y$;

(ii) *the Liouville form $\widetilde{\lambda} = \iota(\widetilde{X})\omega$ coincides with λ near Y and on $\Omega \setminus U$;*

(iii) *ϕ has no critical points;*

(iv) *the contact manifold $(\widetilde{Y} := \partial_+ W, \widetilde{\alpha} := \widetilde{\lambda}|_{\widetilde{Y}})$ admits a contact form preserving embedding $(Q_a, \alpha_{\mathrm{st}}) \to (\widetilde{Y}, \widetilde{\alpha})$ for any $a < \frac{C}{2}$.*

Proof. For any $b \in (a, \frac{C}{2})$ the domain $U_b := \{|q_i| \leq 1, |p_i| < b;\ i = 1, \dots, n\} \subset \mathbb{R}^{2n}_{\mathrm{st}}$ admits a symplectic embedding $H : U_b \to \operatorname{Int} P(C)$. Denote $\partial_n U_b := \{p_n = b\} \cap \partial \overline{U}_b$. Consider a Liouville form $\mu = \sum_1^n (1-\sigma) p_i dq_i - \sigma q_i dp_i = \sum_1^n p_i dq_i - \sigma d\left(\sum_1^n p_i q_i\right)$, where a sufficiently small $\sigma > 0$ will be chosen later. Then

$$\beta := \mu|_{\partial_n U_b} = d\left((b-\sigma) q_n - \sigma \sum_1^{n-1} p_i q_i\right) + \sum_1^{n-1} p_i dq_i.$$

Let us verify that for a sufficiently small $\sigma > 0$ there exists a contact form preserving embedding $(Q_a, \alpha_{\mathrm{st}}) \to (\partial_n U_b, \beta)$. Consider the map $\Psi : Q_a \to \mathbb{R}^{2n}_{\mathrm{st}}$ given by the formulas

$$p_i = -y_i, q_i = x_i,\ i = 1, \dots, n-1, p_n = b, q_n = \frac{z}{b-\sigma} - \frac{\sigma}{b-\sigma}\sum_1^{n-1} x_i y_i.$$

Note that $|q_n| \le \frac{a+a\sigma(n-1)}{b-\sigma} < 1$ if $\sigma < \frac{b-a}{n}$. Hence, if $(x, y, z) \in Q_a$ we have

$$|p_i| \le a < b, |q_i| \le 1 \text{ for } i = 1, \dots, n-1, p_n = b, |q_n| < 1,$$

i.e. $\Psi(Q_a) \subset \partial_n U_b$. On the other hand

$$\Psi^*\mu = \Psi^*\beta = d\left(z + \sigma\sum_1^{n-1} x_i y_i - \sigma\sum_1^{n-1} x_i y_i\right) - \sum_1^{n-1} y_i dx_i = \alpha_{\rm st}.$$

There exists a domain $\widehat{U}_b$, diffeomorphic to a ball with smooth boundary, such that

- $U_b \subset \widehat{U}_b \subset U_{b'}$ for some $b' \in (b, \frac{C}{2})$;
- $\partial\widehat{U}_b \supset \partial_n U_b$;
- $\widehat{U}_b$ is transverse to the Liouville field T, ω-dual to the Liouville form μ.

Note that there exists a Lyapunov function $\psi : \widehat{U}_b \to \mathbb{R}$ for T such that $(\widehat{U}_b, \omega, T, \phi)$ is a Weinstein domain.

Denote $\widetilde{U}_b := \Phi(H(U_a)) \Subset X_+$. We can assume that the path Γ connects a point on Y with a point on $\partial\widetilde{U}_b \setminus \Phi(H((\partial_n U_b))$.

We modify the Liouville form λ, making it equal to 0 on the path Γ and equal to $\Phi_* H_*\mu$ on $\widetilde{U}_b$. Next, we use Lemma 3.1 to construct the required cobordism $(W, \omega, \widetilde{X}, \phi)$ by connecting X_- and $\widehat{U}_b$ via a Weinstein surgery along Γ, and then apply Proposition 3.3 to cancel the zeroes of the Liouville field $\widetilde{X}$. As a result we ensure properties (i)–(iii). In fact, property (iv) also holds. Indeed, by construction $\partial_+ W \supset \Phi(H(\partial_n U_b))$, and hence there exists a contact form preserving embedding $(Q_a, \alpha_{\rm st}) \to (\partial_+ W, \widetilde{\alpha} := \iota(\widetilde{X})\omega|_{\partial_+ W})$. □

Proof of Proposition 4.1 for $n = 3$. The problem in the case $n = 3$ is that we cannot get sufficiently many disjoint contact neighborhoods Q_C embedded into Y to balance all the intersection points. Indeed, both the number of intersection of action $< \varepsilon$ and the number of $Q_{12\varepsilon}$-neighborhoods one can pack into contact slice Y grow as

ε^{-3} when $\varepsilon \to 0$. However, using the inifinite Gromov width assumption we can cite Lemma 4.9 to modify Y so that it would contain a sufficient number of disjoint neighborhoods isomorphic to $Q_{12\varepsilon}$. Indeed, suppose that there are N double points af action $< \varepsilon$. By the infinite Gromov width assumption there exists N disjoint embeddings of polydiscs $P(24\varepsilon)$ into $X_+ \setminus f(L)$.

Using Lemma 4.9, we modify the Liouville form λ into $\widetilde{\lambda}$ away from $f(L)$, so that $(X, \widetilde{\lambda})$ admits a negative end bounded by a contact slice $\widetilde{Y}$ such that there exists N disjoint embeddings $(Q_{12\varepsilon}, \alpha_{\text{st}}) \to (\widetilde{Y}, \widetilde{\alpha})$ preserving the contact form. The rest of the proof is identical to the case $n > 3$. □

5 Proof of main theorems

Proof of Theorem 2.3. We first use Proposition 4.1 to make the Lagrangian immersion f balanced and then use the following modified Whitney trick to eliminate each balanced Whitney pair.

Let $p, q \in X$ be a balanced Whitney pair, $p^0, p^1 \in L$ and $q^0, q^1 \in L$ the pre-images of the self-intersection points p, q, and $\gamma^0, \gamma^1 : [0,1] \to L$ are the corresponding paths such that $\gamma^j(0) = p^j, \gamma^j(1) = q^j$ for $j = 0, 1$, the intersection index of $df(T_{p^0}L)$ and $df(T_{p^1}L)$ is equal to 1 and the intersection index of $df(T_{q^0}L)$ and $df(T_{q^1}L)$ is equal to -1. Recall that according to our convention we are always ordering the pre-images of double points in such a way that their action is positive.

Choose a contact slice Y, and consider a path $\eta : [0,1] \to L$ connecting a point in the loose component Λ of ∂L_+ with p^0 such that $\overline{\eta} := f \circ \eta$ coincides with a trajectory of Z near the point $\overline{\eta}(0)$, and then modify the Liouville form λ, keeping it fixed on X_-, to make it equal to 0 on $\overline{\eta}$. We further modify λ in a neighborhood of $\overline{\gamma}^0$ making it 0 on $\overline{\gamma}^0$, where we use the notation $\overline{\gamma}^0 := f \circ \gamma^0$, $\overline{\gamma}^1 := f \circ \gamma^1$. Note that this is possible because $Y \cup \overline{\eta} \cup \overline{\gamma}^0$ deformation retracts to Y. Assuming that this is done, we observe that $\int\limits_{\overline{\gamma}^1} \lambda = \int\limits_{\overline{\gamma}^0} \lambda = 0$.

Next, we use Lemma 3.2 to construct Darboux charts B_p and B_q centered at the points p and q such that the the intersecting branches in these coordinates look like coordinate Lagrangian planes $\{q = 0\}$ and $\{p = 0\}$ in the standard $\mathbb{R}^{2n}$. Set $\lambda_{\text{st}} := \frac{1}{2} \sum\limits_1^n p_i dq_i - q_i dp_i$. Then the corresponding to it Liouville vector field $Z_{\text{st}} = \frac{1}{2} \sum\limits_1^n q_i \frac{\partial}{\partial q_i} + p_i \frac{\partial}{\partial p_i}$ is tangent to the Lagrangian planes through the origin.

We have $\lambda_{\mathrm{st}} - \lambda = dH$ in $B_p \cup B_q$. Choosing a cut-off function α on $B_p \cup B_q$ which is equal to 1 near p and q and equal to 0 near $\partial B_p \cup \partial B_q$ we define $\lambda_1 := \lambda + d(\alpha H)$. The Liouville structure λ_1 coincides with the standard structure λ_{st} in smaller balls around the points p and q, and with λ near $\partial B_p \cup \partial B_q$.

Next, we use Lemma 3.1 to modify the Liouville structure λ_1 in neighborhoods of paths $\overline{\gamma}^0$ and $\overline{\gamma}^1$ and create Weinstein domain C by attaching handles of index 1 with $\overline{\gamma}^0$ and $\overline{\gamma}^1$ as their cores. The corresponding Lyapunov function on C has two critical points of index 0, at p and q, and two critical points of index 1, at the centers of paths $\overline{\gamma}^0$ and $\overline{\gamma}^1$. Note that the property $\int\limits_{\overline{\gamma}^j} \lambda_1 = 0$, $j = 0, 1$, is crucial in order to apply Lemma 3.1.

Next, we choose an embedded isotropic disc $\Delta \subset X_+ \setminus \operatorname{Int} C$ with boundary in ∂C, tangent to Z along the boundary $\partial\Delta$, and such that $\partial\Delta$ is isotropic, and homotopic in C to the loop $\overline{\gamma}^0 \cup \overline{\gamma}^1$. We then again use Lemma 3.1 to attach to C a handle of index 2 with the core Δ. The resulted Liouville domain $\widetilde{C}$ is diffeomorphic to the $2n$-ball. Moreover, according to Proposition 3.3 the Weinstein structure on $\widetilde{C}$ is homotopic to the standard one via a homotopy fixed on $\partial\widetilde{C}$. In particular, the contact structure induced on the sphere $\partial\widetilde{C}$ is the standard one. The immersed Lagrangian manifold $f(L)$ intersects $\partial\widetilde{C}$ along two Legendrian spheres Λ^0 and Λ^1, each of which is the standard Legendrian unknot which bounds an embedded Lagrangian disc inside $\widetilde{C}$. These two discs intersect at two points, p and q. Note that the Whitney trick allows us to disjoint these discs by a smooth (non-Lagrangian) isotopy fixed on their boundaries. In particular, the spheres Λ^0 and Λ^1 are smoothly unlinked. If they were unlinked as Legendrians we would be done. Indeed, the Legendrian unlink in S^{2n-1}_{std} bounds two disjoint exact Lagrangian disks in B^{2n}_{std}. Unfortunately (or fortunately, because this would kill Symplectic Topology as a subject!), one can show that it is impossible to unlink Λ^0 and Λ^1 via a Legendrian isotopy.

The path $\overline{\eta}$ intersects $\partial\widetilde{C}$ at a point in Λ^0. Slightly abusing the notation we will continue using the notation $\overline{\eta}$ for the part of $\overline{\eta}$ outside the ball $\widetilde{C}$. We then use Lemma 3.1 one more time to modify λ_1 by attaching a handle of index 1 to $X_- \cup \widetilde{C}$ along $\overline{\eta}$. As a result, we create inside X_+ a Weinstein cobordism W which contains $\widetilde{C}$, so that $\partial_- W = Y$ and $\widetilde{Y} := \partial_+ W$ intersects $f(L)$ along a 2-component Legendrian link. One of its components is Λ^1, and the other one is the connected sum of the loose Legendrian Λ and the Legendrian sphere Λ^0, which we denote by $\widetilde{\Lambda}$. Again applying Proposition 3.3 we can deform the Weinstein structure on W keeping it fixed on ∂W to kill both critical points inside W. Hence all trajectories of the (new) Liouville vector field Z inside W begin at Y and end at $\widetilde{Y}$, and thus W is Liouville isomorphic to $\widetilde{Y} \times [0,T]$ for some T (with Liouville form $e^t\lambda_1$, $t \in [0,T]$). We also note that the intersection of $f(L)$ with W consists of two embedded Lagrangian submanifolds A and B transversely intersecting in the points p, q, where

- A is diffeomorphic to the cylinder $\Lambda \times [0,1]$, $A \cap Y = \Lambda$ and $A \cap \widetilde{Y} = \widetilde{\Lambda}$;
- B is a disc bounded by the Legendrian sphere $\Lambda^1 = B \cap \widetilde{Y}$.

The Legendrian $\widetilde{\Lambda}$ is smoothly unlinked with Λ^1. Since $\widetilde{\Lambda}$ is loose, Proposition 2.1 implies that there is a Legendrian isotopy of $\widetilde{\Lambda}$ to $\widehat{\Lambda}$ which is disjoint from a Darboux ball containing Λ^1. We realize this isotopy by a Lagrangian cobordism A_1 from $\widetilde{\Lambda}$ to $\widehat{\Lambda}$ using Lemma 3.4, and also realize the inverse isotopy by a Lagrangian cobordism A_2 from $\widehat{\Lambda}$ to $\widetilde{\Lambda}$. For some $\widetilde{T}$, these cobordisms embed into $\widetilde{Y} \times [0,\widetilde{T}]$. Inside $\widetilde{Y} \times [0, 2\widetilde{T}+2T]$, we define a cobordism $\widetilde{A}$ from Λ to $\widetilde{\Lambda}$, built from the following pieces.

- $\widetilde{A} \cap \widetilde{Y} \times [0,T] = A$,
- $\widetilde{A} \cap \widetilde{Y} \times [T, \widetilde{T}+T] = A_1$,
- $\widetilde{A} \cap \widetilde{Y} \times [\widetilde{T}+T, \widetilde{T}+2T] = \widehat{\Lambda} \times [\widetilde{T}+T, \widetilde{T}+2T]$,
- $\widetilde{A} \cap \widetilde{Y} \times [\widetilde{T}+2T, 2\widetilde{T}+2T] = A_2$.

We then define $\widetilde{B}$ by

- $\widetilde{B} \cap \widetilde{Y} \times [0, \widetilde{T}+T] = \varnothing$,
- $\widetilde{B} \cap \widetilde{Y} \times [\widetilde{T}+T, \widetilde{T}+2T] = B$,
- $\widetilde{B} \cap \widetilde{Y} \times [\widetilde{T}+2T, 2\widetilde{T}+2T] = \Lambda^1 \times [\widetilde{T}+2T, 2\widetilde{T}+2T]$.

A schematic of these cobordisms is given in Figure 5.1. After elongating W (which can be achieved by choosing a contact slice closer to $-\infty$), $\widetilde{A} \cup \widetilde{B}$ can be deformed to $A \cup B$ via a Hamiltonian compactly supported regular homotopy fixed on the boundary. We then define $\widetilde{f} : L \to X$ to be equal to f everywhere, except the portions of L which are mapped to A and B are instead mapped to $\widetilde{A}$ and $\widetilde{B}$, respectively. □

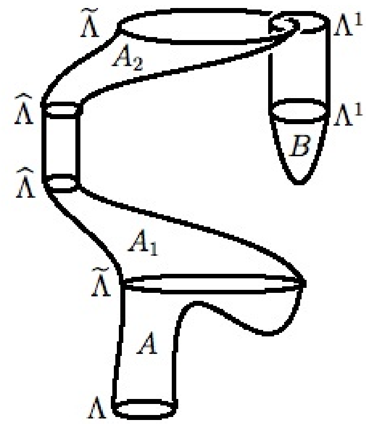

Fig. 5.1: The cobordisms $\widetilde{A}$ and $\widetilde{B}$.

Proof of Theorem 2.2. We first use Gromov's h-principle for Lagrangian immersions [6] to find a compactly supported regular homotopy starting at f and ending at a Lagrangian immersion $\widetilde{f}$ with the prescribed action class $A(f)$ (or the action class $a(f)$ in the Liouville case). More precisely, let us choose a triangulation of L. There are finitely many simplices of the triangulation which cover the compact part of L where the embedding f is not yet Lagrangian. Let K be the polyhedron which is formed by these simplices. Using the h-principle for open Lagrangian immersions, we first isotope f to an embedding which is Lagrangian near the $(n-1)$-skeleton of K, realizing the given (relative) action class. Let us inscribe an n-disc D_i in each of the n-simplices of K, such that the embedding f is already Lagrangian near ∂D_i. Next, we thicken D_i to *disjoint* $2n$-balls $B_i \subset X$ intersecting $f(L)$ along D_i. We then apply Gromov's h-principle for Lagrangian immersions in a relative form to find for each i a fixed near the boundary regular homotopy $D_i \to B_i$ of D_i into a Lagrangian immersion. Note that all the self-intersection points of the resulted Lagrangian immersion $\widetilde{f}$ are localized inside the ball B_i and images of different discs D_i and D_j do not intersect.

Let us choose a negative end X_-, bounded by a contact slice Y in such a way that the immersion $\widetilde{f}$ is cylindrical in it and $X_- \cap \bigcup B_i = \varnothing$. Denote $L_- := \widetilde{f}^{-1}(X_-), \Lambda_- =$

∂L_-. Let us choose a universal loose Legendrian $U \subset Y$ for the Legendrian submanifold $\Lambda_- \subseteq Y$. Denote $\widetilde{\Lambda}_- = \Lambda_- \cap U$. Let $V_- := \bigcup\limits_0^\infty Z^{-s}(U) \subset X_-$ be the domain in X_- formed by all negative trajectories of Z intersecting U. Let us choose disjoint paths Γ_i in $L \setminus \mathrm{Int}\,(L_- \cup \bigcup_i D_i)$ connecting some points in $\widetilde{\Lambda}_-$ with points $z_i \in \partial D_i$ for each n-simplex in K. Choose small tubular neighborhoods U_i of $\widetilde{f}(\Gamma_i)$ in X

Set

$$\widetilde{X} := V_- \cup \bigcup_i (B_i \cup U_i) \text{ and } \widetilde{L} := \widetilde{f}^{-1}(\widetilde{X}).$$

The manifold $\widetilde{X}$ deformationaly retracts to V_- and hence $\widetilde{X}$ is contractible and the Liouville form $\lambda|_{V_-}$ extends as a Liouville form for ω on the whole manifold $\widetilde{X}$. We will keep the notation λ for the extended form. Thus $\widetilde{L}$ is an exact Lagrangian immersion into the contractible Liouville manifold $\widetilde{X}$, cylindrical at $-\infty$ over a loose Legendrian submanifold of U. Moreover, L is diffeomorphic to $\mathbb{R}^n$, and outside a compact set the immersion is equivalent to the standard inclusion $\mathbb{R}^n \hookrightarrow \mathbb{R}^{2n}$. We also note that $I(\widetilde{f}|_{\widetilde{L}} : \widetilde{L} \to \widetilde{X}) = 0$ since this immersion is regularly homotopic to the smooth embedding $f|_{\widetilde{L}} : \widetilde{L} \to \widetilde{X}$.

Applying Theorem 2.3 to $\widetilde{f}|_{\widetilde{L}}$ we find an exact Lagrangian embedding $\widehat{f}$ which is regularly Hamiltonian homotopic to $\widetilde{f}|_{\widetilde{L}}$ via a regular homotopy compactly supported in $\widetilde{X}$. We further note that the embeddings $\widehat{f}$ and $f : \widetilde{L} \to \widetilde{X}$ are isotopic relative the boundary. Indeed, it follows from the h-cobordism theorem that an embedding $\mathbb{R}^n \to \mathbb{R}^{2n}$ which coincides with the inclusion outside a compact set and which is regularly homotopic to it via a compactly supported homotopy is isotopic to the inclusion relative infinity.

Slightly abusing notation we define $\widehat{f} : L \to X$ to be equal to $\widetilde{f}$ on $L \setminus \widehat{L}$. This Lagrangian embedding is isotopic to f via an isotopy fixed outside a compact set. Finally we note that $d\widetilde{f} : TL \to TX$ is homotopic to Φ_1 since it is constructed with the h-principle for Lagrangian immersions, and $d\widehat{f}$ is homotopic to $d\widetilde{f}$ since they are regularly Lagrangian homotopic. □

Next, we deduce Theorem 1.1 from Theorem 2.2.

Proof of Theorem 1.1. Let B be the unit ball in $\mathbb{R}^{2n}$. The triviality of the bundle $T(L) \otimes \mathbb{C}$ is equivalent to existence of a Lagrangian homomorphism $\Phi : TL \to T\mathbb{C}^n$. We can assume that Φ covers a map $\phi : L \to \mathbb{C}^n \setminus \mathrm{Int}\, B$ such that $\phi(\partial L) \subset \partial B$. Let $v \in TL|_{\partial L}$ be the inward normal vector field to ∂L in L, and ν an outward normal to the boundary ∂B of the ball $B \subset \mathbb{C}^n$. Homomorphism Φ is homotopic to a

Lagrangian homomorphism, which will still be denoted by Φ, sending v to ν. Indeed, the obstructions to that lie in trivial homotopy group $\pi_j(S^{2n-1})$, $j \leq n-1$. Then $\Phi|_{T\partial L}$ is a Legendrian homomorphism $T\partial L \to \xi$, where ξ is the standard contact structure on the sphere ∂B formed by its complex tangencies. Using Gromov's h-principle for Legendrian embeddings we can, therefore, assume that $\phi|_{\partial L} : \partial L \to \partial B$ is a Legendrian embedding, and then, using Gromov's h-principle for Lagrangian immersions deform ϕ to an exact Lagrangian immersion $\phi : L \to \mathbb{C}^n \setminus \operatorname{Int} B$ with Legendrian boundary in ∂B and tangent to ν along the boundary. Finally, we use Theorem 2.2 to make ϕ a Legrangian embedding. □

6 Applications

Lagrangian embeddings with a conical singular point

Given a symplectic manifold (X, ω) we say that $L \subset M$ is a *Lagrangian submanifold with an isolated conical point* if it is a Lagrangian submanifold away from a point $p \in L$, and there exists a symplectic embedding $f : B_\varepsilon \to X$ such that $f(0) = p$ and $f^{-1}(L) \subset B_\varepsilon$ is a Lagrangian cone. Here B_ε is the ball of radius ε in the standard symplectic $\mathbb{R}^{2n}$. Note that this cone is automatically a cone over a Legendrian sphere in the sphere ∂B_ε endowed with the standard contact structure given by the restriction to ∂B_ε of the Liouville form $\lambda_{\mathrm{st}} = \frac{1}{2}\sum\limits_1^n (p_i dq_i - q_i dp_i)$.

As a special case of Theorem 1.1 (when ∂L is a sphere) we get

Corollary 6.1. *Let L be an n-dimensional, $n > 2$, closed manifold such that the complexified tangent bundle $T^*(L \setminus p) \otimes \mathbb{C}$ is trivial. Then L admits an exact Lagrangian embedding into $\mathbb{R}^{2n}$ with exactly one conical point. In particularly a sphere admits a Lagrangian embedding to $\mathbb{R}^{2n}$ with one conical point for each $n > 2$.*

Flexible Weinstein cobordisms

The following notion of a flexible Weinstein cobordism is introduced in [1].

A Weinstein cobordism (W, ω, Z, ϕ) is called *elementary* if there are no Z-trajectories connecting critical points. In this case stable manifolds of critical points intersect $\partial_- W$ along isotropic in the contact sense submanifolds. For each critical point p we call the intersection S_p of its stable manifold with $\partial_- W$ the *attaching sphere*. The attaching spheres for index n critical points are Legendrian.

An elementary Weinstein cobordism (W, ω, Z, ϕ) is called *flexible* if the attaching spheres for all index n critical points in W form a loose Legendrian link in $\partial_- W$.

A Weinstein cobordism (W, ω, Z, ϕ) is called *flexible* if it can be partitioned into elementary Weinstein cobordisms: $W = W_1 \cup \cdots \cup W_N$, $W_j := \{c_{j-1} \le \phi \le c_j\}, j = 1, \dots, N$, $m = c_0 < c_1 < \cdots < c_N = M$. Any subcritical Weinstein cobordism is by definition flexible.

Theorem 6.2. *Let (W, ω, Z, ϕ) be a flexible Weinstein domain. Let λ be the Liouville form ω-dual to Z, and Λ any other Liouville form such that the symplectic structures ω and $\Omega := d\Lambda$ are homotopic as non-degenerate (not necessarily closed) 2-forms. Then there exists an isotopy $h_t : W \to W$ such that $h_0 = \mathrm{Id}$ and $h_1^*\Lambda = \varepsilon\lambda + dH$ for a sufficiently small $\varepsilon > 0$ and a smooth function $H : W \to \mathbb{R}$. In particular, h_1 is a symplectic embedding $(W, \varepsilon\omega) \to (W, \Omega)$.*

Recall that a Weinstein cobordism (W, ω, Z, ϕ) is called a *Weinstein domain* if $\partial_- W = \varnothing$.

Corollary 6.3. *Let (W, ω, Z, ϕ) be a flexible Weinstein domain, and (X, Ω) any symplectic manifold of the same dimension. If this dimension is 3 we further assume that X has infinite Gromov width. Then any smooth embedding $f_0 : W \to X$, such that the form $f_0^*\Omega$ is exact and the differential $df : TW \to TX$ is homotopic to a symplectic homomorphism, is isotopic to a symplectic embedding $f_1 : (W, \varepsilon\omega) \to (X, \Omega)$ for a sufficiently small $\varepsilon > 0$. Moreover, if $\Omega = d\Theta$ then the embedding f_1 can be chosen in such a way that the 1-form $f_1^*\Theta - i(Z)\omega$ is exact. If, moreover, the Ω-dual to Θ Liouville vector field is complete then the embedding f_1 exists for an arbitrarily large constant ε.*

Proof of Theorem 6.2. Let us decompose W into flexible elementary cobordisms: $W = W_1 \cup \cdots \cup W_k$, where $W_j = \{c_{j-1} \le \phi \le c_j\}$, $j = 1, \dots, k$ for a sequence of regular values $c_0 < \min\phi < c_1 < \cdots < c_k = \max\phi$ of the function ϕ. Set $V_j = \bigcup\limits_1^j W_i$ for $j \ge 1$ and $V_0 = \varnothing$.

We will construct an isotopy $h_t : W \to W$ beginning from $h_0 = \mathrm{Id}$ inductively over cobordisms W_j, $j = 1, \dots, k$. It will be convenient to parameterize the required isotopy by the interval $[0, 2k]$. Suppose that for some $j = 1, \dots, k$ we already constructed an isotopy $h_t : W \to W$, $t \in [0, j-1]$ such that $h_{j-1}^*\Lambda = \varepsilon_{j-1}\lambda + dH$ on V_{j-1}. Our goal is to extend it $[j-1, j]$ to ensure that h_j satisfies this condition on V_j. Without loss of generality we can assume that there exists only 1 critical point p of ϕ in W_j. Let Δ be the stable disc of p in W_j and $S := \partial\Delta \subset \partial_- W_j$

the corresponding attaching sphere. By assumption, S is subcritical or loose. The homotopical condition implies that there is a family of injective homomorphisms $\Phi_t : T\Delta \to TW$, $t \in [j-1, j]$, such that $\Phi_{j-1} = dh_{j-1}|_{\Delta_j}$, and $\Phi_j : T\Delta_j \to (TW, \Omega)$ is an isotropic homomorphism. We also note that the cohomological condition implies that $\int\limits_{\Delta} \Omega = 0$ when $\dim \Delta = 2$. Then, using Theorem 2.2 when $\dim \Delta = n$ and Gromov's h-principle, [6], for isotropic embeddings in the subcritical case, we can construct an isotopy $g_t : \Delta \to W_j$, $t \in [j-1, j]$, fixed at $\partial\Delta$, such that $g_{j-1} = h_{j-1}|_\Delta$ is the inclusion and the embedding $g_j : \Delta \to (W, \Omega)$ is isotropic. Furthermore, there exists a neighborhood $U \supset \Delta$ in W_j such that the isotopy g_t extends as a fixed on W_{j-1} isotopy $G_t : W_{j-1} \cup U \to W$ such that $G_t|_\Delta = g_t$, $G_t|_{W_j} = h_{j-1}|_{W_{j-1}}$ for $t \in [j-1, j]$, $G_{j-1}|_U = h_{j-1}|_U$ and $h_j : (W_{j-1} \cup U, \varepsilon_{j-1}\omega) \to (W, \Omega)$ is a symplectic embedding. Choose a sufficiently large $T > 0$ we have $Z^{-T}(W_j) \subset W_{j-1} \bigcup U_j$, and hence $h_j \circ e^{-T}|_{V_j}$ is a symplectic embedding $(W_j, \varepsilon_j\omega) \to (W, \Omega)$, where we set $\varepsilon_j := e^{-T}\varepsilon_{j-1}$. Then we can define the required isotopy $h_t : W \to W$, $t \in [j-1, j]$, which satisfy the property that $h_j|_{V_j}$ is a symplectic embedding $(V_j, \varepsilon_j\omega) \to (W, \omega)$ by setting

$$h_t = \begin{cases} h_{j-1} \circ Z^{-2T(t-j+1)} & \text{for } t \in [j-1, j-\frac{1}{2}], \\ G_t \circ Z^{-T} & \text{for } t \in [j-\frac{1}{2}, j]. \end{cases}$$

□

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