

Gauge theory and Rasmussen's invariant

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1 Introduction

For a knot $K \subset S^3$, the (smooth) *slice-genus* $g_*(K)$ is the smallest genus of any properly embedded, smooth, oriented surface $\Sigma \subset B^4$ with boundary K . In [12], Rasmussen used a construction based on Khovanov homology to define a knot-invariant $s(K) \in 2\mathbb{Z}$ with the following properties:

- (a) s defines a homomorphism from the knot concordance group to $2\mathbb{Z}$;
- (b) $s(K)$ provides a lower bound for the slice-genus a knot K , in that

$$2g_*(K) \geq s(K);$$

- (c) the above inequality is sharp for positive knots (i.e. knots admitting a projection with only positive crossings).

These results from [12] were notable because they provided, for the first time, a purely combinatorial proof of an important fact of 4-dimensional topology, namely that the slice-genus of a knot arising as the link of a singularity in a complex plane curve is equal to the genus of its Milnor fiber. This result, the local Thom conjecture, had been first proved using gauge theory, as an extension of Donaldson's techniques adapted specifically to the embedded surface problem [8, 7].

The purpose of this paper is to exhibit a very close connection between the invariant s and the constructions from gauge theory that had been used earlier. Specifically, we shall use gauge theory to construct an invariant $s^\sharp(K)$ for knots K in S^3 in a manner that can be made to look structurally very

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similar to a definition of Rasmussen's $s(K)$. We shall then prove that these invariants are in fact equal:

$$s^\sharp(K) = s(K), \tag{1}$$

for all knots K . The construction of s^\sharp is based on the instanton homology group $I^\sharp(K)$ as defined in [9]. The equality of s^\sharp and s is derived from the relationship between instanton homology and Khovanov homology established there. A knot invariant very similar to s^\sharp appeared slightly earlier in [10], and its construction derives from an approach to the minimal-genus problem developed in [5].

Because it is defined using gauge theory, the invariant s^\sharp has some properties that are not otherwise manifest for s . We shall deduce:

Corollary 1.1. *Let K be a knot in S^3 , let X^4 be an oriented compact 4-manifold with boundary S^3 , and let $\Sigma \subset X^4$ be a properly embedded, connected oriented surface with boundary K . Suppose that X^4 is negative-definite and has $b_1(X^4) = 0$. Suppose also that the inclusion of Σ is homotopic relative to K to a map to the boundary S^3 . Then*

$$2g(\Sigma) \geq s(K).$$

In particular, this inequality holds for any such surface Σ in a homotopy 4-ball.

The smooth 4-dimensional Poincaré conjecture is equivalent to the statement that a smooth homotopy 4-ball with boundary S^3 must be a standard ball. A possible approach to showing that this conjecture is false is to seek a knot K in S^3 that bounds a smooth disk in some homotopy ball but not in the standard ball. To carry this through, one needs some way of showing that a suitably-chosen K does not bound a disk in the standard ball, by an argument that does not apply equally to homotopy balls. This approach was advanced in [2], with the idea of using the non-vanishing of $s(K)$ for this key step. The corollary above shows that Rasmussen's invariant s cannot be used for this purpose: a knot K cannot bound a smooth disk in any homotopy-ball if $s(K)$ is non-zero.

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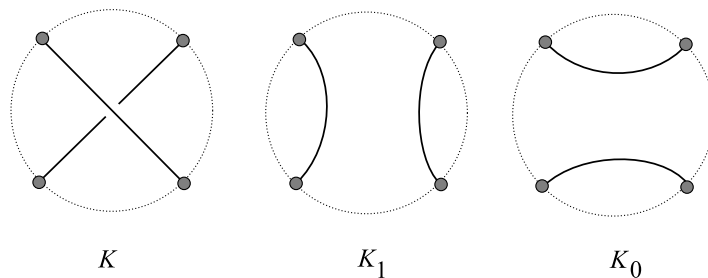


Figure 1: The 0- and 1- smoothings at a crossing of K .

2 The Rasmussen invariant

2.1 Conventions for Khovanov homology

Let λ be a formal variable, and let \mathcal{R} denote the ring $\mathbb{Q}[[\lambda]]$ of formal power series. Let V be the free rank-2 module over \mathcal{R} with a basis \mathbf{v}_+ and \mathbf{v}_- , and let ∇ and Δ be defined by,

$$\begin{aligned} \nabla : \mathbf{v}_+ \otimes \mathbf{v}_+ &\mapsto \mathbf{v}_+ \\ \nabla : \mathbf{v}_+ \otimes \mathbf{v}_- &\mapsto \mathbf{v}_- \\ \nabla : \mathbf{v}_- \otimes \mathbf{v}_+ &\mapsto \mathbf{v}_- \\ \nabla : \mathbf{v}_- \otimes \mathbf{v}_- &\mapsto \lambda^2 \mathbf{v}_+, \end{aligned}$$

$$\begin{aligned} \Delta : \mathbf{v}_+ &\mapsto \mathbf{v}_+ \otimes \mathbf{v}_- + \mathbf{v}_- \otimes \mathbf{v}_+ \\ \Delta : \mathbf{v}_- &\mapsto \mathbf{v}_- \otimes \mathbf{v}_- + \lambda^2 \mathbf{v}_+ \otimes \mathbf{v}_+. \end{aligned}$$

These definitions coincide with those of \mathcal{F}_3 from [4], except that the variable t from that paper is now λ^2 and we have passed to the completion by taking formal power series.

In order to match the conventions which arise naturally from gauge theory, we will adopt for the duration of this paper a non-standard orientation convention for Khovanov homology, so that what we call $Kh(K)$ here will coincide with the Khovanov homology of the mirror image of K as usually defined. To be specific, let K be an oriented link with a plane diagram D , having n_+ positive crossings and n_- negative crossings. Write $N = n_+ + n_-$. For each vertex v of the cube $\{0, 1\}^N$, let K_v be the corresponding smoothing, with the standard convention about the “0” and “1” smoothings, as illustrated in Figure 1. For each vertex v of the cube let $C_v = C(K_v)$ be the tensor product $V^{\otimes p(v)}$, where the factors are indexed by the components of the unlink K_v and the tensor product is over \mathcal{R} . Suppose that $v > u$ and that

(v, u) is an edge of the cube: so the coordinates of v and u differ in exactly one spot at which v is 1 and u is 0. Define $d_{vu} : C_v \rightarrow C_u$ to be the map

$$d_{vu} = \begin{cases} \epsilon_{vu} \nabla, & p(v) = p(u) + 1, \\ \epsilon_{vu} \Delta, & p(v) = p(u) - 1, \end{cases}$$

where ϵ_{vu} is ± 1 and is defined by the conventions of [3]. Set

$$\mathbf{C}(D) = \bigoplus_v C_v$$

$$d = \bigoplus_{v>u} d_{vu}$$

where the second sum is over the edges of the cube. The essential difference here from the standard conventions is that our differential d runs from “1” to “0” along each edge of the cube, rather than the other way.

We define a quantum grading and a cohomological grading on $\mathbf{C}(D)$ as follows. We assign \mathbf{v}_+ and \mathbf{v}_- quantum gradings $+1$ and -1 respectively, we assign λ a quantum grading -2 , and for each vertex v we use these to define a grading Q on the tensor product $C_v = V^{\otimes p(v)}$ as usual. We then define the quantum grading q on C_v by

$$q = Q - |v| - n_+ + 2n_-$$

where $|v|$ denotes the sum of the coordinates. We define the cohomological grading on C_v by

$$h = -|v| + n_-.$$

With these conventions, $d : \mathbf{C}(D) \rightarrow \mathbf{C}(D)$ has bidegree $(1, 0)$ with respect to (h, q) . We define

$$Kh(K) = H_*(\mathbf{C}(D), d)$$

with the resulting bigrading. If one were to replace λ^2 by 1 in the above definitions of m and Δ , one would recover the Lee theory [11].

We will also make use of a mod 4 grading on the complex $\mathbf{C}(D)$. Define q_0 in the same manner as q above, but assigning grading 0 to the formal variable λ . Then define

$$\text{gr} = q_0 - h.$$

With respect to this grading, d has degree -1 and the homology of the unknot is a free module of rank 2 of the form $\mathcal{R}\mathbf{u}_+ \oplus \mathcal{R}\mathbf{u}_-$, with the two summands appearing in gradings 1 and -1 respectively.

2.2 Equivariant Khovanov homology and the s -invariant

The s -invariant of a knot K is defined in [12] using the Lee variant of Khovanov homology. An alternative viewpoint on the same construction is given in [4]. We give here a definition of $s(K)$ modelled on that second approach.

Let K be a knot. As a finitely generated module over the ring \mathcal{R} , the homology $Kh(K)$ is the direct sum of a free module and a torsion module. Write $Kh'(K)$ for the quotient

$$Kh'(K) = Kh(K)/\text{torsion}.$$

The first key fact, due essentially to Lee [11], is that $Kh'(K)$ has rank 2. More specifically, it is supported in h -grading 0 and has a single summand in the mod-4 gradings $\text{gr} = 1$ and $\text{gr} = -1$. We write \mathbf{z}_+ and \mathbf{z}_- for corresponding generators in these mod-4 gradings.

Choose now an oriented cobordism Σ from the unknot U to K . After breaking up Σ into a sequence of elementary cobordisms, one arrives at a map of \mathcal{R} -modules

$$\psi(\Sigma) : Kh(U) \rightarrow Kh(K)$$

and hence also a map of torsion-free \mathcal{R} -modules

$$\psi'(\Sigma) : Kh'(U) \rightarrow Kh'(K).$$

This map has degree 0 or 2 with respect to the mod 4 grading, according as the genus of Σ is even or odd respectively. There is therefore a unique pair of non-negative integers m_+ , m_- such that either

$$\begin{aligned} \psi'(\Sigma)(\mathbf{u}_+) &\sim \lambda^{m_+ \mathbf{z}_+} \\ \psi'(\Sigma)(\mathbf{u}_-) &\sim \lambda^{m_- \mathbf{z}_-} \end{aligned}$$

or

$$\begin{aligned} \psi'(\Sigma)(\mathbf{u}_+) &\sim \lambda^{m_+ \mathbf{z}_-} \\ \psi'(\Sigma)(\mathbf{u}_-) &\sim \lambda^{m_- \mathbf{z}_+}, \end{aligned}$$

according to the parity of the genus $g(\Sigma)$. Here \sim means that these elements are associates in the ring. Set

$$m(\Sigma) = \frac{1}{2}(m_+ + m_-).$$

Rasmussen's s -invariant can then be defined by

$$s(K) = 2(g(\Sigma) - m(\Sigma)).$$

Although our orientation convention for $Kh(K)$ differs from the usual one, the above definition of s brings us back into line with the standard signs, so that $s(K) \geq 0$ for positive knots (see [12]).

We have phrased the definition of $s(K)$ so that there is no up-front reference to the q -grading. If one grants that $s(K)$ is well-defined (i.e. that it is independent of the choice of Σ), then it is clear from the definition that $s(K)$ is a lower bound for $2g_*(K)$.

3 The gauge theory version

3.1 Local systems and the definition of s^\sharp

Given an oriented link K in \mathbb{R}^3 , let us form the link K^\sharp in S^3 by adding an extra Hopf link near the point at infinity. Let w be an arc joining the two components of the Hopf link. In [9], we defined

$$I^\sharp(K) = I^w(S^3, K^\sharp).$$

This is a Floer homology group defined using certain orbifold $SO(3)$ connections on S^3 (regarded as an orbifold with cone-angle π along K^\sharp). The arc w is a representative for w_2 of these connections. As defined in [9] the ring of coefficients is \mathbb{Z} , but we now wish to define the group using a non-trivial local system on the space \mathcal{B} of orbifold connections on S^3 , as in [10].

We review the construction of the local system. Let \mathcal{S} denote the ring

$$\begin{aligned} \mathcal{S} &= \mathbb{Q}[u^{-1}, u] \\ &= \mathbb{Q}[\mathbb{Z}]. \end{aligned}$$

We can regard \mathcal{S} as contained in the ring $\mathbb{Q}[\mathbb{R}]$ and use the notation u^x ($x \in \mathbb{R}$) for generators of the larger ring. Given a continuous function

$$\mu : \mathcal{B} \rightarrow \mathbb{R}/\mathbb{Z},$$

we define a local system of free rank-1 \mathcal{S} -modules by defining

$$\Gamma_a = u^{\mu(a)} \mathcal{S}$$

for $a \in \mathcal{B}$. The particular μ we wish to use follows the scheme of equation (89) of [10]. That is, we frame the link K with the Seifert framing and for each orbifold connection A we define

$$\mu([A]) \in U(1) = \mathbb{R}/\mathbb{Z}$$

by taking the product of the holonomies of $[A]$ along the components of K . The framing is used to resolve the ambiguity in this definition in the orbifold context: see [10]. We write

$$I^\sharp(K; \Gamma)$$

for the corresponding Floer homology group. Note that we do not make use of the extra two components belonging to the Hopf link in K^\sharp in the definition of Γ . (The definition in this paper is essentially identical to that in [10], except that we were previously not adding the Hopf link to K : instead, we considered $I^w(T^3 \# (S^3, K); \Gamma)$, where w was a standard circle in the T^3 . The variable u was previously called t .)

The homology group $I^\sharp(K; \Gamma)$ is naturally $\mathbb{Z}/4$ graded, and with one choice of standard conventions the homology of the unknot U is given by

$$I^\sharp(U; \Gamma) = \mathcal{S}\mathbf{u}_+ \oplus \mathcal{S}\mathbf{u}_-$$

where \mathbf{u}_+ and \mathbf{u}_- have mod-4 gradings 1 and -1 respectively.

The following points are proved in [10]. First, an oriented cobordism $\Sigma \subset [0, 1] \times S^3$ between links induces a map $\psi^\sharp(\Sigma)$ of their instanton homology groups. In the case of a cobordism between knots, the map $\psi^\sharp(\Sigma)$ has degree 0 or 2 mod 4, according as the genus of Σ is even or odd. Using a blow-up construction, this definition can be extended to “immersed cobordisms” $\Sigma \subset [0, 1] \times S^3$ with normal crossings. We can then consider how the map $\psi^\sharp(\Sigma)$ corresponding to an immersed cobordism Σ changes when we change Σ by one of the local moves indicated schematically in Figure 2. (The figure represents analogous moves for an immersed arc in \mathbb{R}^2 .)

Proposition 3.1 ([10, Proposition 5.2]). *Let Σ^* be obtained from Σ by either a positive twist move (introducing a positive double-point), a negative twist move (introducing a negative double-point), or a finger move (introducing a cancelling pair of double-points). Then we have, respectively:*

- (a) $\psi^\sharp(\Sigma^*) = (1 - u^2)\psi^\sharp(\Sigma)$ for the positive twist move;
- (b) $\psi^\sharp(\Sigma^*) = \psi^\sharp(\Sigma)$ for the negative twist move;
- (c) $\psi^\sharp(\Sigma^*) = (1 - u^2)\psi^\sharp(\Sigma)$ for the finger move.

□

From these properties one can deduce the following. For any knot K , the torsion submodule of the \mathcal{S} -module $I^\sharp(K; \Gamma)$ is annihilated by some power of

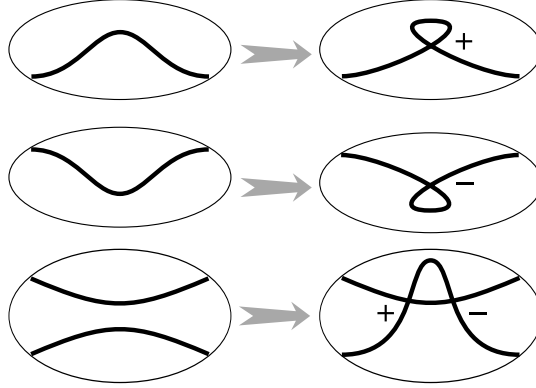


Figure 2: Schematic pictures in lower dimension, representing the positive and negative twist moves and the finger move.

$(1 - u^2)$. Second, just as for the unknot, the quotient $I^\sharp(K; \Gamma)/\text{torsion}$ is a free module of rank 2, with generators in mod-4 gradings 1 and -1

$$I^\sharp(K; \Gamma)/\text{torsion} = \mathcal{S}z_+ \oplus \mathcal{S}z_-.$$

So if Σ is a cobordism from the unknot U to K , then the induced map on $I^\sharp/\text{torsion}$ is determined by a pair of elements $\sigma_+(\Sigma)$, $\sigma_-(\Sigma)$ in \mathcal{S} , in that we have either

$$\begin{aligned} \psi^\sharp(\Sigma)(\mathbf{u}_+) &= \sigma_+(\Sigma)z_+ \\ \psi^\sharp(\Sigma)(\mathbf{u}_-) &= \sigma_-(\Sigma)z_- \end{aligned} \quad (2a)$$

or

$$\begin{aligned} \psi^\sharp(\Sigma)(\mathbf{u}_+) &= \sigma_+(\Sigma)z_- \\ \psi^\sharp(\Sigma)(\mathbf{u}_-) &= \sigma_-(\Sigma)z_+ \end{aligned} \quad (2b)$$

modulo torsion elements, according as the genus of Σ is even or odd. Furthermore, these elements depend only on the genus of Σ . The dependence is as follows. If Σ_1 denotes a surface of genus 1 larger, then

$$\begin{aligned} \sigma_+(\Sigma_1) &= q(u)\sigma_-(\Sigma) \\ \sigma_-(\Sigma_1) &= p(u)\sigma_+(\Sigma) \end{aligned} \quad (3)$$

where $p(u)$ and $q(u)$ are two elements of \mathcal{S} , independent of K and Σ . They are determined by the map ψ^\sharp arising from a genus-1 cobordism from U to U . See equation (105) in [10]. From that paper we record the fact that

$$p(u)q(u) = 4u^{-1}(u-1)^2. \quad (4)$$

Later in this paper we will compute p and q , but for now the above relation is all that we require.

Let us now pass to the completion of the local ring of \mathcal{S} at $u = 1$: the ring of formal power series in the variable $u - 1$. In this ring of formal power series, let λ be the solution to the equation

$$\lambda^2 = u^{-1}(u - 1)^2$$

given by

$$\lambda = (u - 1) - \frac{1}{2}(u - 1)^2 + \dots,$$

so that the completed local ring can be identified with the ring $\mathcal{R} = \mathbb{Q}[[\lambda]]$. We can then write

$$\begin{aligned} \sigma_+(\Sigma) &\sim \lambda^{m_+^\sharp} \\ \sigma_-(\Sigma) &\sim \lambda^{m_-^\sharp} \end{aligned} \tag{5}$$

in \mathcal{R} , for unique natural numbers m_+^\sharp and m_-^\sharp depending on Σ . We define

$$m^\sharp(\Sigma) = \frac{1}{2}(m_+^\sharp + m_-^\sharp).$$

From the relation (4), it follows that $m^\sharp(\Sigma_1) = m^\sharp(\Sigma) + 1$. Therefore, if we define

$$s^\sharp = 2(g(\Sigma) - m^\sharp(\Sigma)),$$

then s^\sharp is independent of Σ and is an invariant of K alone. This is our definition of the instanton variant $s^\sharp(K)$. The factor of 2 is included only to match the definition of $s(K)$ from [12]. From the definition, it is clear that

$$s^\sharp(K) \leq 2g_*(K)$$

for any knot K , just as with Rasmussen's invariant.

If we write $\hat{\Gamma}$ for the local system $\Gamma \otimes_{\mathcal{S}} \mathcal{R}$, then we can phrase the whole definition in terms of the \mathcal{R} -module $I^\sharp(K; \hat{\Gamma})$. Let us abbreviate our notation and write

$$I(K) = I^\sharp(K; \hat{\Gamma})$$

and

$$I'(K) = I(K)/\text{torsion}.$$

Then the integers m_+^\sharp and m_-^\sharp corresponding to a cobordism Σ from U to K completely describe the map $I'(U) \rightarrow I'(K)$ arising from Σ , just as in the Khovanov case.

3.2 Mirror images

Our invariant changes sign when we replace K by its mirror:

Lemma 3.2. *If K^\dagger is the mirror image of K , then $s^\sharp(K^\dagger) = -s^\sharp(K)$.*

Proof. The complex C that computes $I(K^\dagger)$ can be identified with $\text{Hom}(C, \mathcal{R})$, so that $I(K^\dagger)$ is related to $I(K)$ as cohomology is to homology. The universal coefficient theorem allows us to identify

$$I'(K^\dagger) = \text{Hom}(I'(K), \mathcal{R}).$$

Under this identification, the canonical mod-4 grading changes sign. Furthermore, if Σ is a cobordism from U to K , then we can regard it (with the same orientation) also as a cobordism Σ^\dagger from K^\dagger to U and the resulting map $\psi^\sharp(\Sigma^\dagger) : I'(K^\dagger) \rightarrow I'(U)$ is the dual of the map $\psi^\sharp(\Sigma) : I'(U) \rightarrow I'(K)$. So if \mathbf{z}_+ and \mathbf{z}_- are standard generators for $I'(K^\dagger)$ in mod-4 gradings 1 and -1 , we have

$$\begin{aligned} \psi^\sharp(\Sigma^\dagger) : \mathbf{z}_+ &\mapsto \sigma_-(\Sigma)\mathbf{u}_+ \\ \psi^\sharp(\Sigma^\dagger) : \mathbf{z}_- &\mapsto \sigma_+(\Sigma)\mathbf{u}_- \end{aligned}$$

if the genus is even or

$$\begin{aligned} \psi^\sharp(\Sigma^\dagger) : \mathbf{z}_+ &\mapsto \sigma_-(\Sigma)\mathbf{u}_- \\ \psi^\sharp(\Sigma^\dagger) : \mathbf{z}_- &\mapsto \sigma_+(\Sigma)\mathbf{u}_+ \end{aligned}$$

if the genus is odd. Here $\sigma_\pm(\Sigma)$ are the elements of \mathcal{S} associated earlier to the cobordism Σ from U to K .

Now take any oriented cobordism T from U to K^\dagger . For convenience arrange that Σ and T both have even genus. Consider the composite cobordism from U to U . It is given by

$$\begin{aligned} \mathbf{u}_+ &\mapsto \sigma_-(\Sigma)\sigma_+(T)\mathbf{u}_+ \\ \mathbf{u}_- &\mapsto \sigma_+(\Sigma)\sigma_-(T)\mathbf{u}_-. \end{aligned}$$

For the composite cobordism, we therefore have

$$m^\sharp(\Sigma \circ T) = m^\sharp(\Sigma) + m^\sharp(T),$$

so

$$\begin{aligned} 0 &= \frac{1}{2}s^\sharp(U) \\ &= g(\Sigma \circ T) - m^\sharp(\Sigma \circ T) \\ &= g(\Sigma) - m^\sharp(\Sigma) + g(T) - m^\sharp(T) \\ &= \frac{1}{2}s^\sharp(K^\dagger) - \frac{1}{2}s^\sharp(K). \end{aligned}$$

□

4 Equality of the two invariants

To show that $s^\sharp(K) = s(K)$ for all knots K , it will suffice to prove the inequality

$$s^\sharp(K) \geq s(K) \tag{6}$$

because both invariants change sign when K is replaced by its mirror image.

4.1 The cube and its filtration

Let D be a planar diagram for an oriented link K , and let the resolutions K_v and the Khovanov cube $\mathbf{C}(D)$ be defined as in section 2.1. According to [9], there is a spectral sequence whose E_1 term is isomorphic to the Khovanov complex corresponding to this diagram and which abuts to the instanton homology I^\sharp . The spectral sequence was defined in [9] using \mathbb{Z} coefficients, and we need to adapt it now to the case of the local coefficient system Γ or $\hat{\Gamma}$.

We continue to write $I(K)$ as an abbreviation for $I^\sharp(K; \hat{\Gamma})$ and we write $CI(K)$ for the complex that computes it. The complex depends on choices of metrics and perturbations. We form a cube $\mathbf{C}^\sharp(D)$ from the instanton chains complexes of all the resolutions:

$$\mathbf{C}^\sharp(D) = \bigoplus_v CI(K_v).$$

Although we previously used \mathbb{Z} coefficients, we can carry over verbatim from [9] the definition of a differential d^\sharp on $\mathbf{C}^\sharp(D)$ with components

$$d_{vu}^\sharp : CI(K_v) \rightarrow CI(K_u)$$

for all vertices $v \geq u$ on the cube. The homology of the cube $(\mathbf{C}^\sharp(D), d^\sharp)$ is isomorphic to the instanton homology $I(K)$: the proof is easily adapted from [9].

As in [9], the differential d^\sharp respects the decreasing filtration defined by the grading h on cube, so we may consider the associated spectral sequence. We have the following:

Proposition 4.1. *The page (E_1, d_1) of the spectral sequence associated to the filtered complex $(\mathbf{C}^\sharp(D), d^\sharp)$ is isomorphic to the Khovanov cube $(\mathbf{C}(D), d)$ in the version defined in section 2.1.*

Proof. The definition of the filtered complex tells us that the E_1 page is the sum of the instanton homologies of the unlinks K_v :

$$E_1 = \bigoplus_{v \in \{0,1\}^N} I(K_v).$$

If K_v is an unlink of $p(v)$ components, then the excision argument of [9] provides a natural isomorphism

$$\gamma : I(K_v) \rightarrow I(U)^{\otimes p(v)}$$

where U is the unknot. To compute $I(U)$, we return to the definitions: the critical points of the unperturbed Chern-Simons functional comprise an S^2 , so $I(U)$ is a free \mathcal{R} -module of rank 2, with generators differing by 2 in the $\mathbb{Z}/4$ grading. Our conventions put these generators in degrees 1 and -1 . In order to nail down particular generators, we use the disk cobordisms D_+ from the empty link U_0 to the unknot $U = U_1$, and D_0 from U_1 to U_0 . We define $\mathbf{v}_+ \in I(U_1)$ to be the image of the canonical generator $1 \in \mathcal{R} \cong I(U_0)$ under the map $\psi^\sharp(D_+)$. We define $\mathbf{v}_- \in I(U_1)$ to be the unique element with $\psi^\sharp(D_-)(\mathbf{v}_-) = 1$.

At this point we have an isomorphism of \mathcal{R} -modules,

$$E_1 = \bigoplus_{v \in \{0,1\}^N} V^{\otimes p(v)},$$

and so $E_1 = \mathbf{C}(D)$. The differential d_1 on the E_1 page arises from the maps

$$\psi_{vu}^\sharp : I(K_v) \rightarrow I(K_u)$$

obtained from the elementary cobordisms $K_v \rightarrow K_u$ corresponding to the edges of the cube. As in [9], and adopting the notation from there, we need only examine the pair-of-pants cobordisms

$$\begin{aligned} \Pi &: U_2 \rightarrow U_1 \\ \Pi &: U_1 \rightarrow U_2 \end{aligned}$$

and show that the corresponding maps

$$\begin{aligned} \psi^\sharp(\Pi) &: V \otimes V \rightarrow V \\ \psi^\sharp(\Pi) &: V \rightarrow V \otimes V \end{aligned}$$

are equal respectively to the maps ∇ and Δ from the definition of $Kh(K)$ in section 2.1.

Because of the mod 4 grading, we know that we can write

$$\begin{aligned}\psi^\sharp(\Pi)(\mathbf{v}_+) &= a(\mathbf{v}_+ \otimes \mathbf{v}_-) + b(\mathbf{v}_+ \otimes \mathbf{v}_-) \\ \psi^\sharp(\Pi)(\mathbf{v}_-) &= c(\mathbf{v}_+ \otimes \mathbf{v}_+) + d(\mathbf{v}_- \otimes \mathbf{v}_-)\end{aligned}$$

for some coefficients a, b, c, d in \mathcal{R} . Simple topological arguments, composing with the cobordisms D_+ and D_- , show that $a = b = d = 1$. From Poincaré duality, then see that

$$\begin{aligned}\psi^\sharp(\Pi)(\mathbf{v}_+ \otimes \mathbf{v}_+) &= \mathbf{v}_+ \\ \psi^\sharp(\Pi)(\mathbf{v}_+ \otimes \mathbf{v}_-) &= \mathbf{v}_- \\ \psi^\sharp(\Pi)(\mathbf{v}_- \otimes \mathbf{v}_+) &= \mathbf{v}_- \\ \psi^\sharp(\Pi)(\mathbf{v}_- \otimes \mathbf{v}_-) &= c\mathbf{v}_+.\end{aligned}$$

To compute the unknown element $c \in \mathcal{R}$ we compose Π and Π to form a genus-1 cobordism from U_1 to U_1 . In the basis $\mathbf{v}_+, \mathbf{v}_-$, the composite cobordism gives rise to the map

$$\phi = \begin{pmatrix} 0 & 2c \\ 2 & 0 \end{pmatrix}. \quad (7)$$

Comparing this with (3), we see that $q(u) = 2$ and $p(u) = 2c$. We also have the relation (4): in terms of the variable λ , this says $p(u)q(u) = 4\lambda^2$. We deduce that $c = \lambda^2$. Thus $\psi^\sharp(\Pi)$ and $\psi^\sharp(\Pi)$ are precisely the maps ∇ and Δ from the definition of $Kh(K)$. This completes the proof. \square

There is a further simplification we can make, following [6, section 2]. The critical point set for the unperturbed Chern-Simons functional for the each unlink K_v is a product spheres, $(S^2)^{p(v)}$. We can therefore choose a perturbation so that the complex $CI(K_v)$ has generators whose indices are all equal mod 2. The differential on $CI(K_v)$ is the zero, and the page E_1 is canonically isomorphic to $\mathbf{C}^\sharp(D)$ (i.e. to the E_0 page). At this point we can reinterpret the above proposition in slightly different language:

Proposition 4.2. *With perturbations chosen as above, there is a canonical isomorphism of cubes $\mathbf{C}^\sharp(D) \rightarrow \mathbf{C}(D)$. Under this isomorphism, the differential d^\sharp can be written as $d + x$, where d is the Khovanov differential on $\mathbf{C}(D)$ (which increases the h grading by 1) and x is a sum of terms all of which increase the h grading by more than 1.* \square

Remark. The term x has odd degree with respect to the mod 2 grading defined by h .

4.2 Comparing maps from cobordisms

Let Σ be a cobordism from the unknot U_1 to a knot K . This cobordism provides maps

$$\begin{aligned}\psi(\Sigma) &: V \rightarrow Kh(K) \\ \psi^\sharp(\Sigma) &: V \rightarrow I(K).\end{aligned}$$

Given a diagram D for K , we can represent both $Kh(K)$ and $I(K)$ as the homologies of a cube $\mathbf{C}(D) = \mathbf{C}^\sharp(D)$, with two different differentials, as above. At the chain level, the maps $\psi(\Sigma)$ and $\psi^\sharp(\Sigma)$ can be represented by maps to the cube:

$$\begin{aligned}\Psi(\Sigma) &: V \rightarrow \mathbf{C}(D) \\ \Psi^\sharp(\Sigma) &: V \rightarrow \mathbf{C}(D).\end{aligned}$$

Since Khovanov homology is a graded theory with respect to h , the map $\Psi(\Sigma)$ respects the h -grading, so the image of $\Psi(\Sigma)$ lies in grading $h = 0$. This does not hold of $\Psi^\sharp(\Sigma)$. We shall show:

Proposition 4.3. *For a suitable choice of Σ , the maps $\Psi(\Sigma)$ and $\Psi^\sharp(\Sigma)$ can be chosen at the chain level so that*

$$\Psi^\sharp(\Sigma) = a\Psi(\Sigma) + \mathbf{X},$$

where $a \in \mathcal{R}$ is a unit and \mathbf{X} strictly increases the h -grading.

Remark. The authors presume that a can be shown to be ± 1 , but the argument presented below will leave that open.

Proof of the Proposition. The construction of a chain map $\Psi^\sharp(\Sigma)$ at the level of the cubes $\mathbf{C}(D)$ is presented in [6]. The construction depends on decomposing Σ as a composite of elementary cobordisms, each of which corresponds either to a Reidemeister move or to the addition of handle of index 0, 1 or 2. The map $\Psi(\Sigma)$ on the Khovanov complex is constructed in a similar manner. So it is sufficient to consider such elementary cobordisms. Thus we consider two knots or links K_1 and K_0 , with diagrams D_1 and D_0 , differing either by a Reidemeister move or by a handle-addition, and a standard oriented cobordism Σ between them. (The cobordism is topologically a cylinder in the case of a Reidemeister move.) There is a chain map

$$\Psi^\sharp(\Sigma) : \mathbf{C}(D_1) \rightarrow \mathbf{C}(D_0)$$

representing the map $\psi^\sharp(\Sigma)$.

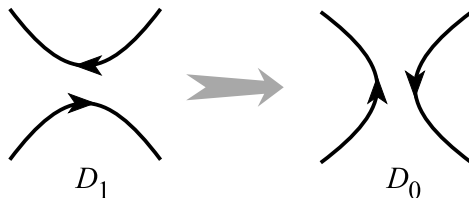


Figure 3: Changing an oriented link diagram by the addition of an oriented 1-handle.

In [6] it is shown how to define $\Psi^\sharp(\Sigma)$ so that it respects the decreasing filtration defined by h . (See Proposition 1.5 of [6]. The term $S \cdot S$ is absent in our case because our cobordism is orientable.) So we can write

$$\Psi^\sharp(\Sigma) = \Psi_0 + \mathbf{X}$$

where Ψ_0 respects the h -grading, and \mathbf{X} strictly increases it. From Proposition 4.2 we see that Ψ_0 is a chain map for the Khovanov differential d . If we check that Ψ_0 induces the map $a\psi(\Sigma)$ on Khovanov homology, for some unit a , then we will be done.

Since this verification seems a little awkward for the Reidemeister III move in particular, we simplify the work needed here by exploiting the fact that we are free to choose Σ as we wish. We start with a crossing-less diagram for the unknot U_1 and then perform 1-handle additions to this diagram to obtain an unlink with many components. Next we perform Reidemeister I moves to components of this unlink, at most one per component, to introduce some crossings. We then perform more 1-handle additions on the diagram. In this way we can arrive at a cobordism from U_1 to any knot K , using only 1-handles together with Reidemeister I moves applied to circles without crossings in the diagram.

In fact, because we need to compare the leading term Ψ_0 with the map $\psi(\Sigma)$ defined in [12], we should treat only the *negative* Reidemeister I move directly: the positive version, Reidemeister I_+ , can be obtained as the composite of a Reidemeister II and the inverse of a Reidemeister I_- .

With this simplification, we see that we need only check two cases:

- (a) introducing a single negative crossing in a circle without crossings, by a Reidemeister I_- move;
- (b) introducing two crossings in a circle without crossings, by a Reidemeister II move;
- (c) altering the diagram by the addition of an oriented 1-handle, as shown in Figure 3.

Case (c) is the most straightforward of these three. If D_1 and D_0 are diagrams as shown in the figure, then we can draw a third diagram D_2 representing a link with an extra crossing, so that D_1 and D_0 are the two diagrams obtained by smoothing the one extra crossing in the two standard ways. The cube $\mathbf{C}(D_2)$ can then be regarded as $\mathbf{C}(D_1) \oplus \mathbf{C}(D_0)$, and the map $\Psi^\sharp(\Sigma) : \mathbf{C}(D_1) \rightarrow \mathbf{C}(D_0)$ corresponding to the addition of a 1-handle is equal to the corresponding component of the differential on $\mathbf{C}(D_2)$. (See [6, section 10].) The same is true for the corresponding map in the Khovanov theory, as described in [12]; so the fact that the leading term of $\Psi^\sharp(\Sigma)$ is the Khovanov map is just an instance of Proposition 4.2, applied to the diagram D_2 .

Cases (a) and (b) can be treated together, as follows. Since the Reidemeister move only involves a single circle without crossings, we can pass to the simple case in which D_1 itself is a single circle. This is because both the instanton theory and the Khovanov theory have tensor product rule for split diagrams, and it is easy to check that the two product rules agree to leading order. In the case of a single circle, we have then two chain maps

$$\begin{aligned}\Psi_0 &: V \rightarrow \mathbf{C}(D_0, d) \\ \Psi &: V \rightarrow \mathbf{C}(D_0, d)\end{aligned}$$

where the first map is the leading term of Ψ^\sharp and the second map is the one defined for the Khovanov theory in [12]. The maps ψ_0 and ψ which these give rise to on homology are both isomorphisms, from one free module of rank 2 to another. Since both preserve the same grading mod 4, it follows that ψ_0 and ψ differ at most by an automorphism δ of V of the form

$$\begin{aligned}\mathbf{v}_+ &\mapsto a\mathbf{v}_+ \\ \mathbf{v}_- &\mapsto b\mathbf{v}_-, \end{aligned}$$

where a and b are units in \mathcal{R} . We must only check that $a = b$. Because we have already dealt with 1-handle additions, we can see that $a = b$ as a consequence of the fact that δ must commute with the map ϕ from (7). This map corresponds a cobordism from U_1 to itself genus 1, and can be realized as the addition of two 1-handles. Both ψ_0 and ψ intertwine the map $\phi : V \rightarrow V$ by a map on the homology of $\mathbf{C}(D_2, d)$ arising from a chain map

$$\Phi : \mathbf{C}(D_2, d) \rightarrow \mathbf{C}(D_2, d)$$

given (in both cases) by applying ϕ or $\phi \otimes 1$ as appropriate at each vertex of the cube. In the case of ψ_0 , this follows from the naturality of instanton homology, while in the case of ψ it can be verified directly. \square

The inequality (6) – and hence the equality (1) – can be deduced from Proposition 4.3 as follows. From the definitions of s and s^\sharp , it is apparent that it will suffice to show

$$\begin{aligned} m_+^\sharp &\leq m_+ \\ m_-^\sharp &\leq m_-, \end{aligned}$$

where m_+^\sharp etc. are the numbers associated to the cobordism Σ . Consider the first of these two inequalities. (The second is not essentially different.) Although the definition of m_+^\sharp was in terms of $I'(K) = I(K)/\text{torsion}$, we can use the fact that all the torsion is annihilated by some power of λ to rephrase the definition: m_+^\sharp is the largest integer m such that, for all sufficiently large k , the element

$$a_k^\sharp := \psi^\sharp(\Sigma)(\lambda^k \mathbf{v}_+)$$

is divisible by λ^{k+m} . This means that the chain representative

$$\mathbf{a}_k^\sharp = \Psi^\sharp(\Sigma)(\lambda^k \mathbf{v}_+)$$

is d^\sharp -homologous to an element of $\mathbf{C}(D)$ that is divisible by λ^{k+m} :

$$\mathbf{a}_k^\sharp = \lambda^{k+m} \mathbf{b}^\sharp + d^\sharp \mathbf{c}^\sharp \tag{8}$$

for $m = m_+^\sharp$.

Let us write \mathcal{F}^i for the subcomplex of $\mathbf{C}(D)$ generated by the elements in h -grading $\geq i$. In (8), the element \mathbf{a}_k^\sharp on the left belongs to \mathcal{F}^0 and its leading term in degree $h = 0$ is equal to

$$\mathbf{a}_k = \Psi(\Sigma)(\lambda^k \mathbf{v}_+)$$

by Proposition 4.3. A priori, we do not know which level of the filtration the elements \mathbf{b}^\sharp and \mathbf{c}^\sharp belong to. But we have the following lemma.

Lemma 4.4. *After perhaps increasing k , we can arrange that*

$$\mathbf{a}_k^\sharp = \lambda^{k+m} \mathbf{b}^\sharp + d^\sharp \mathbf{c}^\sharp \tag{9}$$

where \mathbf{b}^\sharp lies in \mathcal{F}^0 and \mathbf{c}^\sharp lies in \mathcal{F}^{-1} .

Proof. We deal with \mathbf{b}^\sharp first. Suppose that \mathbf{b}^\sharp does not lie in \mathcal{F}^0 but lies in \mathcal{F}^{-n} for some $n > 0$, with a non-zero leading term b_{-n} :

$$\mathbf{b}^\sharp = b_{-n} + z.$$

Writing $d^\sharp = d + x$ as before, the degree-1 term in the equality $d^\sharp \mathbf{b}^\sharp = 0$ tells us that $db_{-n} = 0$. Since the torsion-free part of $H(\mathbf{C}(D), d)$ is supported in h -grading 0, the cycle $\lambda^l b_{-n}$ must be d -exact once l is large enough:

$$\lambda^l b_{-n} = d\beta$$

for some β in h -grading $-n - 1$. If we define

$$\tilde{\mathbf{b}}^\sharp = \lambda^l \mathbf{b}^\sharp - d^\sharp \beta,$$

then $\tilde{\mathbf{b}}^\sharp$ belongs to \mathcal{F}^{-n+1} and is d^\sharp -homologous to \mathbf{a}_{k+l}^\sharp . We replace k by $k + l$, and replace \mathbf{b}^\sharp with $\tilde{\mathbf{b}}^\sharp$. The effect is to decrease n by at least 1. In this way we eventually arrive at a situation with $\mathbf{b}^\sharp \in \mathcal{F}^0$.

Once we have \mathbf{b}^\sharp in \mathcal{F}^0 , a similar argument applies to \mathbf{c}^\sharp . Suppose that (9) holds, with \mathbf{b}^\sharp in \mathcal{F}^0 and \mathbf{c}^\sharp in \mathcal{F}^{-n} with a non-zero leading term c_{-n} in h -grading $-n$, for some $n > 1$. In (9), the only term in degree $-n + 1$ is dc_{-n} , so dc_{-n} is zero. Using again the fact that the d -homology in negative h -gradings is torsion, we conclude that $\lambda^l c_{-n}$ is d -exact, for some l :

$$\lambda^l c_{-n} = d\gamma.$$

If we set

$$\tilde{\mathbf{c}}^\sharp = \lambda^l \mathbf{c}^\sharp - d^\sharp \gamma,$$

then we have $d^\sharp \tilde{\mathbf{c}}^\sharp = \lambda^l d^\sharp \mathbf{c}^\sharp$, so

$$\mathbf{a}_{k+l}^\sharp = \lambda^{l+k+m} \mathbf{b}^\sharp + d^\sharp \tilde{\mathbf{c}}^\sharp.$$

On the other hand, $\tilde{\mathbf{c}}^\sharp$ lies in \mathcal{F}^{-n+1} , so we are better off by at least 1. Eventually, we arrive in \mathcal{F}^{-1} . \square

Given \mathbf{a}_k^\sharp , \mathbf{b}^\sharp and \mathbf{c}^\sharp as in the lemma, let us write \mathbf{a}_k , \mathbf{b} and \mathbf{c} for the components of these in h -gradings 0, 0 and -1 respectively. We have $\mathbf{a}_k = \Psi(\Sigma)(\lambda^k \mathbf{v}_+)$ as just noted, and the leading term of the identity in the lemma is

$$\mathbf{a}_k = \lambda^{k+m} \mathbf{b} + d\mathbf{c}.$$

So in the homology group $Kh(K) = H(\mathbf{C}(D), d)$, the element $\psi(\Sigma)(\lambda^k \mathbf{v}_+)$ is represented at the chain level by a chain that is divisible by λ^{k+m} , where $m = m_+^\sharp$. The largest possible power of λ by which a chain-representative of this class can be divisible is m_+ , by definition, so $m_+ \geq m_+^\sharp$. This completes the proof that $s^\sharp(K) \geq s(K)$.

5 Negative-definite manifolds and connected sums

We give the proof of Corollary 1.1.

5.1 Three lemmas

As mentioned earlier and proved in [9] and [10], the invariant $I^\sharp(K)$ is functorial not just for cobordisms in $[0, 1] \times S^3$, but more generally for oriented cobordisms of pairs (W, Σ) , where W is a 4-dimensional cobordism from S^3 to S^3 and Σ is an embedded 2-dimensional cobordism. We can also extend the definition to immersed cobordisms $\Sigma \looparrowright W$ with normal crossings, just as we did before, by blowing up and taking proper transforms.

Let K be a knot in S^3 , and let (W, Σ) be such an oriented cobordism of pairs (with Σ embedded, not immersed), from (S^3, U) to (S^3, Σ) . Using the induced map $\psi(W, \Sigma)$ to $I^\sharp(K; \Gamma)$, we define elements

$$\sigma_+(W, \Sigma), \sigma_-(W, \Sigma) \in \mathcal{S},$$

well-defined up to multiplication by units, just as we did at (2). Passing to the completion \mathcal{R} , we obtain non-negative numbers $m_-^\sharp(W, \Sigma)$ and $m_+^\sharp(W, \Sigma)$. We take these to be $+\infty$ if $\sigma_\pm(W, \Sigma)$ is zero. We then define

$$m^\sharp(W, \Sigma) = \frac{1}{2}(m_+^\sharp(W, \Sigma) + m_-^\sharp(W, \Sigma))$$

and

$$s^\sharp(W, \Sigma) = 2(g(\Sigma) - m^\sharp(W, \Sigma)),$$

as we did in the case that W was a cylinder. The proofs from the previous case carry over, to establish the following to lemmas.

Lemma 5.1. *If Σ and Σ' are two embedded cobordisms in the same W and if Σ is homotopic to Σ' relative to their common boundary, then $s^\sharp(W, \Sigma) = s^\sharp(W, \Sigma')$. \square*

Lemma 5.2. *If Σ_1 is obtained from Σ by forming the connected sum with an embedded torus inside a standard 4-ball in W , then $s^\sharp(W, \Sigma) = s^\sharp(W, \Sigma_1)$. \square*

The third lemma that we shall need is a connected sum theorem for W :

Lemma 5.3. *Suppose that W' is obtained from W as a connected sum $W' = W \# N$, with the sum being made in a ball in W disjoint from Σ . Suppose that N has $b_1(N) = 0$ and $b_2^+(N) = 0$. Then $s^\sharp(W', \Sigma) = s^\sharp(W, \Sigma)$.*

Proof. The essential parts of this connect-sum theorem are proved in [1]. We are not assuming that N is simply connected, so the instanton moduli space $M_0(N)$ for instanton number 0 (i.e. the space of flat $SU(2)$ connections) is potentially non-trivial. After choosing suitable holonomy perturbations on N as in [1], we may arrange that the perturbed moduli space consists of the following:

- (a) one reducible flat connection for each homomorphism $\pi_1(N) \rightarrow \{\pm 1\} \subset SU(2)$;
- (b) one S^1 -reducible connection for each of the finitely many S^1 -reducible flat connections on N .

Furthermore, for the perturbed-flat connections A of the second kind, we can be assured that the kernel of the linearized perturbed equations with gauge-fixing is zero.

We now follow the standard argument. We consider a 1-parameter family of metrics on W' in which the neck of the connected sum is stretched. The usual dimension-counting shows that the only contributions come from flat (or perturbed-flat) connections on N . Following the analysis of the orientations from [1], we see that

$$\psi(W', \Sigma) = (a + 2b)\psi(W, \Sigma)$$

where a is the number of points in $M_0(N)$ of the first kind and b is the number of points of the second kind. The quantity $a + 2b$ is the order of $H_1(N; \mathbb{Z})$. Since in any event this number is non-zero, the result follows. \square

5.2 Conclusion of the proof

As a corollary of these three lemmas, we deduce:

Corollary 5.4. *If W has $b_1(W) = 0$ and $b_2^+(W) = 0$, and if the inclusion $\Sigma \hookrightarrow W$ is homotopic to map whose image is contained in the union of $\partial W = S^3 \cup S^3$ together with an arc joining them, then $s^\sharp(W, \Sigma)$ coincides with the standard invariant $s^\sharp(K)$.*

Proof. After adding additional handles to Σ (and applying the second of the three lemmas above), we may assume that Σ is homotopic to an embedded surface Σ' entirely contained in a neighborhood of $S^3 \cup S^3$ and the arc joining them. We may replace Σ with Σ' , because of the first lemma. At this point, we can regard W as a connected sum of a standard cylinder $[0, 1] \times S^3$ with

a negative definite manifold N . Our surface is disjoint from N , so the third lemma applies. That is, $s^\sharp(W, \Sigma)$ is equal to $s^\sharp([0, 1] \times S^3, \Sigma)$. The latter is, by definition, the invariant $s^\sharp(K)$. \square

Corollary 5.5. *Let X be an oriented 4-manifold with boundary S^3 that has $b_1(X) = b_2^+(X) = 0$, and let Σ be a properly embedded, connected oriented surface with boundary a knot K in S^3 . Suppose that the inclusion of Σ in X is homotopic to the boundary S^3 relative to K . Then $2g(\Sigma) \geq s^\sharp(K)$.*

Proof. We may regard (X, Σ) as providing a cobordism (W, Σ') by removing a standard ball-pair. From the definition of $s^\sharp(W, \Sigma')$, we have $s^\sharp(W, \Sigma') \leq 2g(\Sigma)$. But $s^\sharp(W, \Sigma') = s^\sharp(K)$ by the previous corollary. \square

Corollary 1.1 in the introduction is simply a restatement of the last corollary above, in light of the equality $s^\sharp = s$.

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