THE ROUND SPHERE MINIMIZES ENTROPY AMONG CLOSED SELF-SHRINKERS

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ABSTRACT. The entropy of a hypersurface is a geometric invariant that measures complexity and is invariant under rigid motions and dilations. It is given by the supremum over all Gaussian integrals with varying centers and scales. It is monotone under mean curvature flow, thus giving a Lyapunov functional. Therefore, the entropy of the initial hypersurface bounds the entropy at all future singularities. We show here that not only does the round sphere have the lowest entropy of any closed singularity, but there is a gap to the second lowest.

0. INTRODUCTION

The F-functional of a hypersurface Σ of Euclidean space $\mathbf{R}^{n+1}$ is the Gaussian surface area[1]

$$(0.1) \qquad F(\Sigma) = (4\pi)^{-\frac{n}{2}} \int_\Sigma e^{-\frac{|x|^2}{4}},$$

whereas the Gaussian entropy is the supremum over all Gaussian surface areas given by

$$(0.2) \qquad \lambda(\Sigma) = \sup \ (4\pi t_0)^{-\frac{n}{2}} \int_\Sigma e^{-\frac{|x-x_0|^2}{4t_0}}.$$

Here the supremum is taking over all $t_0 > 0$ and $x_0 \in \mathbf{R}^{n+1}$. Entropy is invariant under rigid motions and dilations.

Mean curvature flow ("MCF") is an evolution equation where a one-parameter family of hypersurfaces $M_t \subset \mathbf{R}^{n+1}$ evolves over time to minimize volume, satisfying the equation

$$(0.3) \qquad (\partial_t x)^\perp = -H\,\mathbf{n}.$$

Here $H = \text{div}_\Sigma\,\mathbf{n}$ is the mean curvature, $\mathbf{n}$ is the outward pointing unit normal and x is the position vector. As a consequence of Huisken's monotonicity, entropy is monotone nonincreasing under MCF.

A hypersurface $\Sigma \subset \mathbf{R}^{n+1}$ is a *self-shrinker* if it is the $t = -1$ time-slice of a MCF moving by rescalings, i.e., where $\Sigma_t \equiv \sqrt{-t}\,\Sigma$ is a MCF; see [A], [Ch], [KKM], [M] and [N] for examples. This is easily seen to be equivalent to the equation

$$(0.4) \qquad H = \frac{\langle x, \mathbf{n}\rangle}{2}.$$

[1]Gaussian surface area has also been studied in convex geometry (see [B] and [Na]) and in theoretical computer science (see [K] and [KDS]).

Since (0.4) is the Euler-Lagrange equation for F, self-shrinkers are critical points for F.

Under MCF, every closed hypersurface becomes singular and the main problem is to understand the singularities. By Huisken's monotonicity and an argument of [I1], [W2], blow-ups of singularities of a MCF are self-shrinkers.

By section 7 in [CM1], the entropy of a self-shrinker is equal to the value of F and, thus, no supremum is needed. In [St], Stone computed the F functional, and therefore the entropy, for generalized cylinders $\mathbf{S}^k \times \mathbf{R}^{n-k}$. He showed that $\lambda(\mathbf{S}^n)$ is decreasing in n and

$$(0.5) \qquad \lambda(\mathbf{S}^1) = \sqrt{\frac{2\pi}{e}} \approx 1.5203 > \lambda(\mathbf{S}^2) = \frac{4}{e} \approx 1.4715 > \lambda(\mathbf{S}^3) > \cdots > 1 = \lambda(\mathbf{R}^n) \,.$$

Moreover, a simple computation shows that $\lambda(\Sigma \times \mathbf{R}) = \lambda(\Sigma)$.

It follows from Brakke's theorem, [Br], that $\mathbf{R}^n$ has the least entropy of any self-shrinker and, in fact, there is a gap to the next lowest. Our main result is that the round sphere has the least entropy of any *closed* self-shrinker.

Theorem 0.6. Given n, there exists $\epsilon = \epsilon(n) > 0$ so that if $\Sigma \subset \mathbf{R}^{n+1}$ is a closed self-shrinker not equal to the round sphere, then $\lambda(\Sigma) \geq \lambda(\mathbf{S}^n) + \epsilon$. Moreover, if $\lambda(\Sigma) \leq \min\{\lambda(\mathbf{S}^{n-1}), \frac{3}{2}\}$, then Σ is diffeomorphic to $\mathbf{S}^n$.[2]

Theorem 0.6 is suggested by the dynamical approach to MCF of [CM1] and [CM2]. The idea is that MCF starting at a closed M becomes singular, the corresponding self-shrinker has lower entropy and, by [CM1], the only self-shrinkers that cannot be perturbed away are $\mathbf{S}^{n-k} \times \mathbf{R}^k$ and $\lambda(\mathbf{S}^{n-k} \times \mathbf{R}^k) \geq \lambda(\mathbf{S}^n)$. However, we are not able to make this approach rigorous. One of the difficulties is that if the self-shrinker is non-compact and we perturb it, then *a priori* it may flow smoothly without ever becoming singular.[3] To overcome this, we combine results from [CM1] and [IW].

The dynamical picture also suggests two closely related conjectures; the first is for any closed hypersurface and the second is for self-shrinkers:

Conjecture 0.7. Theorem 0.6 holds with $\epsilon = 0$ for any closed hypersurface M^n with $n \leq 6$.

Conjecture 0.8. Theorem 0.6 holds for any *non-flat* self-shrinker $\Sigma^n \subset \mathbf{R}^{n+1}$ with $n \leq 6$.

Both conjectures are true for curves, i.e., when $n = 1$. The first conjecture follows for curves by combining Grayson's theorem, [G] (cf. [GaHa]), and the monotonicity of λ under curve shortening flow. The second conjecture follows for curves from the classification of self-shrinkers by Abresch and Langer, [AbL].

Conjecture 0.7 would allow us to carry out the outline above to show that any closed hypersurface has entropy at least that of the sphere, proving Conjecture 0.8.

Furthermore, one could ask which self-shrinker has the third least entropy, etc. It is easy to see that the entropy of the "Simons cone" over $\mathbf{S}^k \times \mathbf{S}^k$ in $\mathbf{R}^{2k+2}$ is asymptotic to $\sqrt{2}$ as $k \to \infty$, which is also the limit of $\lambda(\mathbf{S}^{2k+1})$. Thus, as the dimension increases, the Simons cones have lower entropy than some of the generalized cylinders. For example, the cone over $\mathbf{S}^2 \times \mathbf{S}^2$ has entropy $\frac{3}{2} < \lambda(\mathbf{S}^1 \times \mathbf{R}^4)$. In other words, already for $n = 5$, $\mathbf{S}^k \times \mathbf{R}^{n-k}$ is not a complete list of the lowest entropy self-shrinkers.

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[2] If $n > 2$, then $\lambda(\mathbf{S}^{n-1}) < \frac{3}{2}$ and the minimum is unnecessary.

[3] Theorem 0.10 in [CM1] is proven using an argument along these lines.

It is easy to see that if an immersed hypersurface has entropy strictly less than two, then it is embedded, hence, we will always assume embeddness below.

1. Perturbing a self-shrinker

Throughout this paper, unless otherwise mentioned, L will be the second variation operator for the F functional from section 4 of [CM1], that is,

$$(1.1) \qquad L = \mathcal{L} + |A|^2 + \frac{1}{2} = \Delta - \langle \frac{x}{2}, \nabla \cdot \rangle + |A|^2 + \frac{1}{2} \,,$$

where A is the second fundamental form and the second equality defines the operator $\mathcal{L}$.

The first step is to perturb a closed self-shrinker inside itself, while reducing the entropy and making the self-shrinker version of mean curvature positive. This uses the classification of stable self-shrinkers from [CM1].

Lemma 1.2. If $\Sigma^n \subset \mathbf{R}^{n+1}$ is a closed self-shrinker (for any n) and Σ is not round, then there is a nearby hypersurface Γ with the following properties:

(1) $\lambda(\Gamma) < \lambda(\Sigma)$.
(2) Γ is inside of Σ, i.e., the compact region of $\mathbf{R}^{n+1}$ bounded by Σ contains Γ.
(3) $\left(H - \frac{1}{2}\langle x, \mathbf{n}\rangle\right) > 0$ on Γ.

Proof. Let Σ_s be a one-parameter family of normal graphs given by

$$(1.3) \qquad \Sigma_s = \left\{ x + s\, u(x)\, \mathbf{n}(x) \,|\, x \in \Sigma \right\},$$

where $\mathbf{n}(x)$ is the outward unit normal to Σ. Equation (4.10) in [CM1] computes that

$$(1.4) \qquad \frac{d}{ds}\Big|_{s=0} \left(H - \frac{\langle x, \mathbf{n}\rangle}{2} \right) = -L\, u \,.$$

The following three facts are proved in [CM1][4]:

- The lowest eigenfunction u for L is positive and has $L\, u = \mu\, u$ for some $\mu > 1$.
- There exists $\epsilon > 0$, so that for every s with $0 < |s| < \epsilon$ we have $\lambda(\Sigma_s) < \lambda(\Sigma)$. Here Σ_s is the graph of $s\, u$.
- With s as above, Σ_s has

$$(1.5) \qquad \left(H - \frac{1}{2}\langle x, \mathbf{n}\rangle \right) < 0 \text{ if } s > 0 \,,$$

$$(1.6) \qquad \left(H - \frac{1}{2}\langle x, \mathbf{n}\rangle \right) > 0 \text{ if } s < 0 \,.$$

Thus, we see that for $s \in (-\epsilon, 0)$, $\Gamma = \Sigma_s$ has the three properties. $\qquad\square$

[4]See [CM1] corollary 5.15 and theorem 4.30, theorem 0.15, and equation (4.10), respectively.

2. Rescaled MCF

A one-parameter family of hypersurfaces M_t evolves by rescaled MCF if it satisfies

$$(2.1) \qquad \partial_t x = - \left(H - \frac{1}{2} \langle x, \mathbf{n} \rangle \right) \mathbf{n} \,.$$

The rescaled MCF is equivalent to MCF, up to rescalings in space and a reparameterization of time. Namely, if Σ_t is a MCF, then $t \to \Sigma_{-\mathrm{e}^{-t}}/\sqrt{\mathrm{e}^{-t}}$ is a solution to the rescaled MCF equation and vice versa. Moreover, rescaled MCF is the negative gradient flow for the F functional and self-shrinkers are the fixed points for this flow.

2.1. Mean convex rescaled MCF.
The next ingredient is a result from [IW] about "mean convex rescaled MCF" where

$$(2.2) \qquad \left(H - \frac{1}{2} \langle x, \mathbf{n} \rangle \right) \geq 0 \,.$$

We will often refer to this quantity as the rescaled mean curvature.

This result is the rescaled analog of that mean convexity is preserved for MCF.

Lemma 2.3. Let $M_t \subset \mathbf{R}^{n+1}$ be a smooth rescaled MCF of closed hypersurfaces for $t \in [0, T]$.

(1) If (2.2) holds on M_0, then it also holds on M_t for every $t \in [0, T]$.
(2) If in addition $\left(H - \frac{1}{2} \langle x, \mathbf{n} \rangle \right) > 0$ at least at one point of M_0, then $\left(H - \frac{1}{2} \langle x, \mathbf{n} \rangle \right) > 0$ on M_t for all $t > 0$.

In particular, the flow is monotone in that M_t is inside M_s for $t > s$.

The key to proving this is a Simons type equation for rescaled MCF

$$(2.4) \qquad (\partial_t - L) \left(H - \frac{\langle x, \mathbf{n} \rangle}{2} \right) = 0 \,,$$

which follows from (1.4) with $u = - \left(H - \frac{\langle x, \mathbf{n} \rangle}{2} \right)$.

Proof of Lemma 2.3. Set $\phi = H - \frac{1}{2} \langle x, \mathbf{n} \rangle$. Equation (2.4) is equivalent to that

$$(2.5) \qquad (\partial_t - \Delta)\, \phi = \left(|A|^2 + \frac{1}{2} \right) \phi - \langle \frac{x}{2}, \nabla \phi \rangle \,.$$

Since $\phi(x, 0) \geq 0$ by assumption, the parabolic maximum principle gives $\phi(x, t) \geq 0$ for all $t \geq 0$, giving (1). The parabolic strong maximum principle then gives (2). The last claim follows immediately since $\mathbf{n}$ is the outward pointing normal. $\qquad \square$

3. Simons type identity for the rescaled A

Using covariant derivatives, the operators ∂_t and L can be extended to tensors (see Hamilton, [Ha]; cf. lemma 10.8 in [CM1]). We will next show that the Simons type identity (2.4) is in reality the trace of a tensor equation:

Lemma 3.1. If M_t is flowing by rescaled MCF and $L = \Delta - \langle \frac{x}{2}, \nabla \cdot \rangle + |A|^2 + \frac{1}{2}$, then

$$(3.2) \qquad (\partial_t - L)\, A = -A\,,$$

$$(3.3) \qquad (\partial_t - L)\, H = -H\,,$$

$$(3.4) \qquad (\partial_t - L)\, \langle x, \mathbf{n} \rangle = -2\, H\,,$$

$$(3.5) \qquad (\partial_t - L)\, e^t \left(A - \frac{\langle x, \mathbf{n} \rangle}{2n}\, g \right) = \frac{e^t}{n} \left(H - \frac{\langle x, \mathbf{n} \rangle}{2} \right) g\,.$$

We will later use that the first equation is equivalent to that $(\partial_t - L)\, e^t A = 0$.

Proof. Given a hypersurface and an orthonormal frame $\{e_i\}$, we follow the convention in [CM1] by setting

$$(3.6) \qquad a_{ij} = A(e_i, e_j) = \langle \nabla_{e_i} e_j, \mathbf{n} \rangle\,,$$

so that $H = -\sum_{i=1}^n a_{ii}$. For a general hypersurface, the Laplacian of A is (see, e.g., lemma B.8 in [Si] where the sign convention for the a_{ij}'s is reversed)

$$(3.7) \qquad (\Delta A)_{ij} = -|A|^2\, a_{ij} - H\, a_{ik}\, a_{jk} - H_{ij}\,.$$

Here H_{ij} is the ij component of the Hessian of H.

If a hypersurface evolves by $\partial_t x = f\, \mathbf{n}$ for a function f, then we have the following standard formulas for the variations of the components of the metric g_{ij}, the components of the second fundamental form a_{ij}, and the unit normal $\mathbf{n}$ (see, e.g., lemmas 7.4 and 7.6 in [HP])

$$(3.8) \qquad \partial_t\, g_{ij} = -2\, f\, a_{ij}\,,$$

$$(3.9) \qquad \partial_t \mathbf{n} = -\nabla f\,,$$

$$(3.10) \qquad \partial_t\, a_{ij} = f_{ij} - f\, a_{ik}\, a_{kj}\,.$$

In the rescaled MCF, we have $f = \langle x, \mathbf{n} \rangle / 2 - H$.

We will extend ∂_t to tensors by using the covariant derivative ∇_{∂_t} in the metric $g_{ij} \times dt^2$ (cf. [Ha] or the appendix in [W4]). If we let e_i be the evolving frame on M_t coming from pushing forward the frame on M_0, then the formula for the Christoffel symbols gives

$$(3.11) \qquad \partial_t e_i = \frac{1}{2}\, g^{jk}(\partial_t\, g_{ij})\, e_k = -f\, g^{jk}\, a_{ij}\, e_k\,.$$

Using the Leibniz rule, we have

$$(3.12) \qquad \partial_t g_{ij} = \partial_t\, (g(e_i, e_j)) = (\partial_t g)\, (e_i, e_j) + g(\partial_t e_i, e_j) + g(\partial_t e_j, e_i)\,,$$

so we see that this choice makes $\partial_t g = 0$.

Similarly, working at a point where the e_i's are orthonormal, the Leibniz rule gives that

$$(3.13) \qquad \begin{aligned} (\partial_t\, A)\, (e_i, e_j) &= \partial_t a_{ij} - A(\partial_t e_i, e_j) - A(e_i, \partial_t e_j) \\ &= (f_{ij} - f\, a_{ik} a_{kj}) - 2\, (-f\, a_{ik}\, a_{kj}) = f_{ij} + f\, a_{ik}\, a_{kj}\,. \end{aligned}$$

Combining this with (3.7), we compute the heat operator on A

$$(3.14) \qquad \begin{aligned} (\partial_t\, A - \Delta A)\, (e_i, e_j) &= f_{ij} + f\, a_{ik}\, a_{kj} + |A|^2\, a_{ij} + H\, a_{ik} a_{kj} + H_{ij} \\ &= (f + H)_{ij} + (f + H)\, a_{ik}\, a_{kj} + |A|^2\, a_{ij} \\ &= \frac{1}{2}\, \langle x, \mathbf{n} \rangle_{ij} + \frac{1}{2}\, \langle x, \mathbf{n} \rangle\, a_{ik}\, a_{kj} + |A|^2\, a_{ij}\,. \end{aligned}$$

Rewriting this in terms of the L operator (using x^T for the tangential part of x) gives

$$(\partial_t A - LA)(e_i, e_j) = (\partial_t A - \Delta A)(e_i, e_j) + \left(\frac{1}{2}\nabla_{x^T} A - \frac{1}{2} A - |A|^2 A\right)(e_i, e_j)$$

$$(3.15) \qquad = \frac{1}{2}\left\{\langle x, \mathbf{n}\rangle_{ij} + \langle x, \mathbf{n}\rangle\, a_{ik}\, a_{kj} + (\nabla_{x^T} A)(e_i, e_j) - a_{ij}\right\}.$$

The next step is to compute the operators on $\langle x, \mathbf{n}\rangle$. To simplify the calculation, let e_i be an orthonormal frame and work at a point where its tangential covariant derivatives vanish. In particular, at this point we have $\nabla_{e_i} e_j = a_{ij}\, \mathbf{n}$ and $\nabla_{e_i}\mathbf{n} = -a_{ij}\, e_j$. Using this and the general fact that $\nabla_{e_i} x = e_i$ (cf. the proof of lemma 5.5 in [CM1]), we get at this point

$$(3.16) \qquad \langle x, \mathbf{n}\rangle_i = \nabla_{e_i}\langle x, \mathbf{n}\rangle = \langle e_i, \mathbf{n}\rangle + \langle x, \nabla_{e_i}\mathbf{n}\rangle = -A(x^T, e_i).$$

To compute the Hessian, we will differentiate this. Using first the Leibniz rule, then the fact that ∇A is fully symmetric in all three inputs by the Codazzi equations and the fact that

$$(3.17) \qquad \langle \nabla_{e_j} x^T, e_k\rangle = \langle \nabla_{e_j} x, e_k\rangle - \langle \nabla_{e_j} x^\perp, e_k\rangle = \delta_{jk} + a_{jk}\langle x, \mathbf{n}\rangle,$$

we get that

$$\langle x, \mathbf{n}\rangle_{ij} = -\left(\nabla_{e_j} A\right)(x^T, e_i) - A\left(\left(\nabla_{e_j} x^T\right)^T, e_i\right) - A(x^T, \nabla_{e_j} e_i)$$

$$(3.18) \qquad = -(\nabla_{x^T} A)(e_i, e_j) - A(e_i, e_j) - a_{ij}\, a_{kj}\langle x, \mathbf{n}\rangle,$$

Using (3.18) in (3.15) gives the first claim. The second claim follows from the first since traces commute with covariant derivatives (g is parallel, even with respect to time). The third claim follows from the second claim and the Simons equation (2.4) for $H - \frac{1}{2}\langle x, \mathbf{n}\rangle$.

For the last claim, we use that $(\partial_t - L)\, e^t A = 0$ by the first claim and, since the metric is time-parallel, the third claim gives that

$$(3.19) \qquad e^{-t}(\partial_t - L)\left(e^t\langle x, \mathbf{n}\rangle g\right) = \langle x, \mathbf{n}\rangle g + ((\partial_t - L)\langle x, \mathbf{n}\rangle) g = (\langle x, \mathbf{n}\rangle - 2 H)\, g.$$

$$\square$$

3.1. Bounding A.

In this subsection, we use the matrix equation in combination with the parabolic maximum principle to bound A for the rescaled MCF by the rescaled mean curvature when the latter is nonnegative.

To achieve this bound, we will need some simple computations. First we will need the quotient rule. Let f be a symmetric 2-tensor and h a positive function

$$(3.20) \qquad (\partial_t - \Delta)\frac{f}{h} = \frac{(\partial_t - \Delta) f}{h} - \frac{f(\partial_t - \Delta) h}{h^2} + 2\frac{\nabla_{\nabla h} f}{h^2} - 2\frac{f\, |\nabla h|^2}{h^3}$$

$$= \frac{(\partial_t - \Delta) f}{h} - \frac{f(\partial_t - \Delta) h}{h^2} + \frac{2}{h}\nabla_{\nabla h}\left(\frac{f}{h}\right).$$

Next let V be a vector field, and set $\mathcal{L} f = \Delta f + \nabla_V f$; we will also use that

$$(3.21) \qquad (\partial_t - \mathcal{L})\,|f|^2 = 2\langle f, (\partial_t - \mathcal{L}) f\rangle - 2\,|\nabla f|^2 \le 2\langle f, (\partial_t - \mathcal{L}) f\rangle,$$

$$(3.22) \qquad (\partial_t - \mathcal{L})\left|\frac{f}{h}\right|^2 \le 2\langle\frac{f}{h}, (\partial_t - \mathcal{L})\frac{f}{h}\rangle,$$

$$(3.23) \qquad (\partial_t - \mathcal{L})\frac{f}{h} = \frac{(\partial_t - \mathcal{L}) f}{h} - \frac{f(\partial_t - \mathcal{L}) h}{h^2} + \frac{2}{h}\nabla_{\nabla h}\left(\frac{f}{h}\right).$$

In particular, if $L f = \mathcal{L} f + K f$ for some function K, $(\partial_t - L) f = 0$, and $(\partial_t - L) h = 0$, then the two previous equations give

$$(\partial_t - \mathcal{L}) \left| \frac{f}{h} \right|^2 \leq 2 \left\langle \frac{f}{h}, (\partial_t - \mathcal{L}) \frac{f}{h} \right\rangle = \frac{4}{h} \left\langle \nabla_{\nabla h} \left(\frac{f}{h} \right), \frac{f}{h} \right\rangle$$

$$\text{(3.24)} \qquad = \frac{2}{h} \left\langle \nabla \left| \frac{f}{h} \right|^2, \nabla h \right\rangle = 2 \left\langle \nabla \left| \frac{f}{h} \right|^2, \nabla \log h \right\rangle.$$

Applying the above to $f = e^t A$ and $h = H - \frac{\langle x, \mathbf{n} \rangle}{2}$ and using Lemma 3.1 yields the following differential inequality for the ratio

$$\text{(3.25)} \qquad B = \frac{f}{h} = \frac{e^t A}{H - \frac{\langle x, \mathbf{n} \rangle}{2}} :$$

Lemma 3.26. If $M_t \subset \mathbf{R}^{n+1}$ are hypersurfaces flowing by the rescaled MCF, $H - \frac{\langle x, \mathbf{n} \rangle}{2} > 0$ on the initial hypersurface, and B is as above, then

$$\text{(3.27)} \qquad (\partial_t - \mathcal{L}) |B|^2 \leq 2 \left\langle \nabla |B|^2, \nabla \log \left(H - \frac{\langle x, \mathbf{n} \rangle}{2} \right) \right\rangle.$$

The parabolic maximum principle now implies the following:

Corollary 3.28. If $M_t \subset \mathbf{R}^{n+1}$ are closed hypersurfaces flowing by the rescaled MCF with $H - \frac{\langle x, \mathbf{n} \rangle}{2} > 0$ on the initial hypersurface, then

$$\text{(3.29)} \qquad |A|^2 \leq C \, e^{-2t} \left| H - \frac{\langle x, \mathbf{n} \rangle}{2} \right|^2,$$

for some constant C depending on the initial hypersurface.

Proof. This follows from the parabolic maximum principle and Lemma 3.26. $\qquad \square$

3.2. **Curvature bounds for blow ups.** The next lemma establishes that H bounds $|A|$ on the regular part of any multiplicity one tangent flow when the initial hypersurface is closed and has positive rescaled mean curvature. In particular, any such tangent flow has $H \geq 0$.

Lemma 3.30. Let $M_t \subset \mathbf{R}^{n+1}$ be closed hypersurfaces flowing by the rescaled MCF with $H - \frac{\langle x, \mathbf{n} \rangle}{2} \geq 0$ on the initial hypersurface M_0. There exists $C > 0$ so that if T_t is a tangent flow at the first singular time and T_{-1} is multiplicity one, then:

- $|A| \leq C H$ on the regular set $\text{Reg}(T_{-1})$.

Proof. Let τ be the first singular time and $y \in M_\tau$ the singular point.

By Lemma 2.3, every M_t has $H - \frac{1}{2} \langle x, \mathbf{n} \rangle \geq 0$, so the flow is nested and

$$\text{(3.31)} \qquad \frac{1}{2} |\langle x, \mathbf{n} \rangle| \leq \frac{|x|}{2} \leq C_1 \equiv \frac{1}{2} \max_{M_0} |x|.$$

Corollary 3.28 gives a constant $C_0 > 0$ depending on the initial hypersurface M_0 so that

$$\text{(3.32)} \qquad |A| \leq C_0 \, e^{-t} \left(H - \frac{\langle x, \mathbf{n} \rangle}{2} \right) \leq C_0 \left(H - \frac{\langle x, \mathbf{n} \rangle}{2} \right) \leq C_0 H + C_0 C_1.$$

Fix a compact subset $\Omega \subset \text{Reg}(T_{-1})$. Since Ω is smooth and multiplicity one, White's version of Brakke's regularity theorem, [W2] (cf. Allard's theorem; see [Al] or [Si]), gives a sequence $h_i \to 0$ so that (a subset of)

$$(3.33) \qquad \Sigma_i \equiv \frac{1}{h_i}\left(M_{\tau - h_i^2} - y\right)$$

smoothly converges to Ω. Let A_i and H_i be the second fundamental form and mean curvature, respectively, of Σ_i. It follows from (3.32) that

$$(3.34) \qquad |A_i| = h_i\,|A| \le h_i\,(C_0\,H + C_0\,C_1) = C_0\,H_i + h_i\,C_0\,C_1\,.$$

Since we have smooth convergence to Ω and $h_i \to 0$, we can pass to limits to get that

$$(3.35) \qquad |A_\Omega| \le C_0\,H_\Omega\,.$$

The lemma follows since Ω is arbitrary. $\qquad \square$

Remark 3.36. A similar statement holds for blow ups at the first singular time.

4. FINITE TIME BLOW UP FOR RESCALED MCF

The next result shows that there is finite time blow up for solutions of the rescaled MCF when the initial closed hypersurface has a strict positive lower bound for $H - \frac{1}{2}\langle x, \mathbf{n}\rangle$.

Lemma 4.1. Given $c > 0$, there exists T_c so that if M_0 is a closed hypersurface in $\mathbf{R}^{n+1}$ with $H - \frac{1}{2}\langle x, \mathbf{n}\rangle \ge c$, then the rescaled MCF M_t hits a singularity for $t \le T_c$.

Proof. Set $\phi = H - \frac{1}{2}\langle x, \mathbf{n}\rangle$. Let $m(t)$ be the minimum of ϕ at time t. By (2.5), we have

$$(4.2) \qquad (\partial_t - \Delta)\,\mathrm{e}^{-\frac{t}{2}}\,\phi \ge -\langle\frac{x}{2}, \nabla\mathrm{e}^{-\frac{t}{2}}\,\phi\rangle\,,$$

so the parabolic maximum principle gives that the minimum of $\mathrm{e}^{-\frac{t}{2}}\,\phi$ is non-decreasing in time. This gives exponential growth of $m(t)$, i.e.,

$$(4.3) \qquad m(t) \ge \mathrm{e}^{\frac{t}{2}}\,m(0) \ge \mathrm{e}^{\frac{t}{2}}\,c > 0\,.$$

Since M_t lies in a bounded set by Lemma 2.3 and $|\mathbf{n}| = 1$, we have a constant C so that $\left|\frac{1}{2}\langle x, \mathbf{n}\rangle\right| \le |x|/2 \le C$ for all $t \ge 0$. Combining this with (4.3), we can choose $T_1 > 0$ so that if $t \ge T_1$, then $H \ge \frac{\phi}{2}$. In particular, when $t \ge T_1$, this means that

$$(4.4) \qquad n\,|A|^2 \ge H^2 \ge \frac{\phi^2}{4}\,.$$

Using this in (2.5), we get for $t \ge T_1$ that

$$(4.5) \qquad (\partial_t - \Delta)\,\phi = \left(|A|^2 + \frac{1}{2}\right)\phi - \frac{1}{2}\langle x, \nabla\phi\rangle \ge \frac{1}{4n}\,\phi^3 - \frac{1}{2}\langle x, \nabla\phi\rangle\,.$$

Using this, the parabolic maximum principle gives the differential inequality (for $t \ge T_1$)

$$(4.6) \qquad m'(t) \ge \frac{1}{4n}\,m^3(t)\,.$$

Comparing this with the ODE $f'(t) \ge \frac{1}{4n}f^3(t)$, this gives finite time blow up for ϕ which implies the finite time blow up for $|A|$, giving the desired singularity. $\qquad \square$

5. Tangent cones to self-shrinkers

In this section, we will study two types of n-dimensional rectifiable integral varifolds. The first are weak solutions of the self-shrinker equation (0.4); these are critical points for the F functional and are called F-stationary. The second are stationary with respect to the Euclidean volume and will simply be called stationary; these arise as blow ups of the first.

The singular set and regular set of a rectifiable varifold Σ are denoted by $\mathrm{Sing}(\Sigma)$ and $\mathrm{Reg}(\Sigma)$, respectively.

The main result of this section is the following:

Proposition 5.1. If $\Sigma \subset \mathbf{R}^{n+1}$ is a F-stationary rectifiable varifold, $\lambda(\Sigma) < \frac{3}{2}$, and there is a constant $C > 0$ so that

$$(5.2) \qquad\qquad |A| \leq C\,H \text{ on the regular set } \mathrm{Reg}(\Sigma) \,,$$

then Σ is smooth.

We will use a blow up analysis, where we analyze the tangent cones, to prove the proposition. It follows from [Al] (cf. section 42 in [Si]) that an integral n-rectifiable varifold has stationary integral rectifiable tangent cones as long as the generalized mean curvature H is locally in L^p for some $p > n$. This is trivially satisfied for stationary varifolds where $H = 0$ and for F-stationary varifolds where H is locally in L_∞.

Lemma 5.3. Suppose that $\Sigma \subset \mathbf{R}^{n+1}$ is a F-stationary rectifiable varifold satisfying (5.2). If Γ is any multiplicity one blow up of Σ, then Γ is stationary and $|A| \equiv 0$ on $\mathrm{Reg}(\Gamma)$.

Proof. By definition, there are sequences $s_i \to 0$ and $y_i \to y \in \mathbf{R}^{n+1}$ so that

$$(5.4) \qquad\qquad \Sigma_i \equiv \frac{1}{s_i}\,(\Sigma - y_i)$$

converges with multiplicity one to Γ. Hence, the regular part of each Σ_i satisfies $|A_i| \leq C\,H_i$ with the constant C from (5.2) since this inequality is scale invariant. Since $H = \frac{1}{2}\,\langle x, \mathbf{n} \rangle$ is locally bounded on Σ, the limit Γ is stationary.

Since Γ is multiplicity one, Allard's theorem[5] (see [Al] or [Si]) implies that the Σ_i's converge smoothly on the regular set $\mathrm{Reg}(\Gamma)$ and, thus,

$$(5.5) \qquad\qquad |A_\Gamma| \leq C\,H_\Gamma \text{ on } \mathrm{Reg}(\Gamma) \,.$$

Since Γ is stationary, we have $H = 0$ on $\mathrm{Reg}(\Gamma)$ and the lemma follows.   □

Tangent cones are limits of rescalings about a fixed point. An iterated tangent cone is a tangent cone to a tangent cone to a tangent cone... (with finitely many iterations).

Corollary 5.6. Let $\Sigma \subset \mathbf{R}^{n+1}$ be a F-stationary rectifiable varifold satisfying (5.2). If $\lambda(\Sigma) < 2$, then every iterated tangent cone Γ is stationary, has $\lambda(\Gamma) \leq \lambda(\Sigma)$, has multiplicity one, and satisfies $|A| \equiv 0$ on $\mathrm{Reg}(\Gamma)$.

Proof. This follows from Lemma 5.3 since iterated tangent cones can be realized as blow ups by taking a diagonal sequence and any limit has multiplicity one since $\lambda(\Sigma) < 2$.   □

We will need two elementary lemmas from blow up analysis:

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[5]Since the sequence of rescalings has locally bounded H, Allard gives uniform local $C^{1,\alpha}$ estimate graphical estimates. Elliptic theory then gives uniform estimates on higher derivatives and, thus, smooth convergence.

Lemma 5.7. Suppose that $\Gamma \subset \mathbf{R}^{n+1}$ is an n-dimensional stationary rectifiable varifold with $|A| \equiv 0$ on $\mathrm{Reg}(\Gamma)$. If Γ is a cone, $\mathrm{Sing}(\Gamma) \subset \{0\}$ and $n > 1$, then $\mathrm{Sing}(\Gamma) = \emptyset$.

Proof. Since $|A| \equiv 0$ on the regular set, the closure of each connected component of $\mathrm{Reg}(\Gamma)$ is contained in a hyperplane. Since 0 is the only singular point and Γ is a cone, the closure of each connected component of $\mathrm{Reg}(\Gamma)$ must in fact be a hyperplane through 0. However, since two distinct hyperplanes in $\mathbf{R}^{n+1}$ with 0 in their closure always intersect away from 0 (since $n > 1$), Γ consists of a single hyperplane through the origin and thus is, in particular, smooth, though of course it could have multiplicity. □

The above lemma does not extend to $n = 1$; in particular, three half-lines meeting at 0 with angles $\frac{2\pi}{3}$ is a stationary configuration. Thus something else, like an entropy bound, is needed to rule out such a singularity. Also, as noted, it could have multiplicity.

The second elementary lemma that we will need is often implicitly used in Federer type dimension reduction arguments. It is the following:

Lemma 5.8. Suppose that $\Gamma \subset \mathbf{R}^{n+1}$ is a stationary rectifiable varifold and Γ is a cone. If $y \in \mathrm{Sing}(\Gamma) \setminus \{0\}$, then every tangent cone to Γ at y is of the form $\Gamma' \times \mathbf{R}_y$ where $\mathbf{R}_y$ is the line in the direction y and Γ' is a conical stationary varifold in $\mathbf{R}^n$.

Proof of Proposition 5.1. By Allard's regularity theorem, it is enough to show that every tangent cone to Σ is a multiplicity one hyperplane. We will prove this by contradiction. Suppose, therefore, that $x \in \mathrm{Sing}(\Sigma)$, Γ_1 is a tangent cone to Σ at x, and Γ_1 is not a multiplicity one hyperplane.

By assumption, $\lambda(\Sigma) < 3/2$ and Σ satisfies (5.2); therefore, Corollary 5.6 implies that Γ_1 is stationary, satisfies $|A| \equiv 0$ on $\mathrm{Reg}(\Gamma_1)$, and

$$\lambda(\Gamma_1) \leq \lambda(\Sigma) < 3/2 \,. \tag{5.9}$$

Since Γ_1 is not a hyperplane (x is singular), Lemma 5.7 gives a singular point $y_1 \in \mathrm{Sing}(\Gamma_1) \setminus \{0\}$. Thus, Lemma 5.8 gives a multiplicity one tangent cone Γ_2 to Γ_1 at y_1 with

$$\Gamma_2 = \Gamma_2' \times \mathbf{R}_{y_1} \,, \tag{5.10}$$

where Γ_2' is a stationary cone in $\mathbf{R}^n$. Since y was a singular point, Γ_2' is not a hyperplane. Furthermore, since Corollary 5.6 applies to all iterated tangent cones, we know that Γ_2, and thus also Γ_2' have $|A| \equiv 0$ on their regular parts.

We can now repeat the argument to get a stationary cone $\Gamma_3' \subset \mathbf{R}^{n-1}$ that is multiplicity one, is not a hyperplane, and also has $|A| \equiv 0$ on its regular part.

Repeating this $n-1$ times eventually gives a stationary cone $\Gamma_n' \subset \mathbf{R}^2$ that is not a line and where $\Gamma_n' \times \mathbf{R}^{n-1}$ is an iterated tangent cone for Γ. In particular, since entropy is preserved under products with $\mathbf{R}$, we must have that

$$\lambda(\Gamma_n') < 3/2 \,. \tag{5.11}$$

This implies that Γ_n' consists of at most two rays through the origin. However, the only such configuration that is stationary is when there are exactly two rays meeting at angle π to form a line. This contradiction completes the proof. □

6. Low entropy singularities for mean convex rescaled MCF

An important tool in this paper is a classification of singularities for rescaled MCF starting from a closed rescaled mean convex hypersurface with low entropy. Here, "rescaled mean convex" means that (2.2) holds. This classification will be given in Corollary 6.4 below. The first step in this direction is the following regularity theorem and partial classification:

Proposition 6.1. Let M_t be a rescaled MCF of closed hypersurfaces in $\mathbf{R}^{n+1}$, satisfying (2.2) on $[0, T)$ and $\lambda(M_0) < 3/2$. If there is a singularity at (y, T), then there is a tangent flow T_t with:

- T_{-1} is a smooth, embedded, self-shrinker with $\lambda(T_{-1}) < 3/2$ and multiplicity one.
- T_{-1} has $H \geq 0$ and is not flat.

Proof. The blow up argument given in lemma 8 of [I1] (cf. [W1], [W3]) gives a tangent flow T_t so that T_{-1} is an F-stationary rectifiable varifold. Huisken's monotonicity and the properties of the entropy in [CM1] (see lemma 1.11 and section 7 there) give that

$$(6.2) \qquad\qquad \lambda(T_{-1}) \leq \lambda(M_0) < 3/2 \,,$$

so that T_{-1} has multiplicity one and is embedded. Hence, by Lemma 3.30,

- $|A| \leq C H$ on the regular set $\mathrm{Reg}(T_{-1})$.

Finally, Proposition 5.1 gives that M_{-1} is smooth. $\qquad\qquad\qquad\square$

We will combine Proposition 6.1 with the following classification of smooth, embedded mean convex self-shrinkers in arbitrary dimension from theorem 0.17 in [CM1]:

Theorem 6.3 ([CM1]). $\mathbf{S}^k \times \mathbf{R}^{n-k}$ are the only smooth complete embedded self-shrinkers without boundary, with polynomial volume growth, and $H \geq 0$ in $\mathbf{R}^{n+1}$.

The $\mathbf{S}^k$ factor in Theorem 6.3 is round and has radius $\sqrt{2k}$; we allow the possibilities of a sphere ($n - k = 0$) or a hyperplane (i.e., $k = 0$), although Brakke's theorem rules out the multiplicity one hyperplane as a tangent flow at a singular point.

The classification of smooth embedded self-shrinkers with $H \geq 0$ began with [H1], where Huisken showed that round spheres are the only closed ones. In [H2], Huisken showed that generalized cylinders $\mathbf{S}^k \times \mathbf{R}^{n-k} \subset \mathbf{R}^{n+1}$ are the only open ones with polynomial volume growth and $|A|$ bounded. Theorem 0.17 in [CM1] completed the classification by removing the $|A|$ bound.

Combining Proposition 6.1 with Theorem 6.3 gives:

Corollary 6.4. Let M_t be a rescaled MCF of closed hypersurfaces in $\mathbf{R}^{n+1}$, satisfying (2.2) on $[0, T)$ and $\lambda(M_0) < 3/2$. If there is a singularity at (y, T), then there is a multiplicity one tangent flow T_t of the form $\mathbf{S}^k \times \mathbf{R}^{n-k}$ for some $k > 0$.

Combining this with the monotonicity of the entropy will give the following lower bound for entropy and topological rigidity for rescaled mean convex hypersurfaces:

Proposition 6.5. Let M^n be a closed hypersurface satisfying (2.2). If $\lambda(M) < \min\{\lambda(\mathbf{S}^{n-1}), \frac{3}{2}\}$, then M is diffeomorphic to $\mathbf{S}^n$ and $\lambda(M) \geq \lambda(\mathbf{S}^n)$.

Proof. We can assume that $H - \frac{1}{2}\langle x, \mathbf{n}\rangle > 0$ at least at one point. To see this, suppose instead that $H - \frac{1}{2}\langle x, \mathbf{n}\rangle \equiv 0$, so that M is a self-shrinker. If $M = \mathbf{S}^n$, then we are done. Otherwise, Lemma 1.2 gives a nearby graph $\tilde{M}$ over M with $H - \frac{1}{2}\langle x, \mathbf{n}\rangle > 0$ and $\lambda(\tilde{M}) < \lambda(M)$.

Let M_t be the rescaled MCF with $M_0 = M$, so that Lemma 2.3 gives

$$(6.6) \qquad\qquad H - \frac{1}{2}\langle x, \mathbf{n}\rangle > 0 \text{ for all } t > 0.$$

Lemmas 4.1 and 2.3 give a singularity in finite time inside of M. Thus, Corollary 6.4 gives a multiplicity one tangent flow T_t at this point of the form $\mathbf{S}^k \times \mathbf{R}^{n-k}$ for some $k > 0$. By the monotonicity of the entropy under MCF (lemma 1.11 in [CM1]), its invariance under dilation, and its lower semi-continuity under limits, we have

$$(6.7) \qquad\qquad \lambda(T_{-1}) \le \lambda(M) < \min\{\lambda(\mathbf{S}^{n-1}), 3/2\}.$$

By [St], we have $\lambda(\mathbf{S}^n) < \lambda(\mathbf{S}^{n-1} \times \mathbf{R}) < \ldots$, so we conclude that $k = n$ and $\lambda(\mathbf{S}^n) \le \lambda(M)$. Finally, White's version, [W2], of Brakke's theorem implies that T_{-1} is the smooth limit of rescalings of the M_t's. In particular, M itself is diffeomorphic to $T_{-1} = \mathbf{S}^n$. $\qquad\square$

7. Proof of the main theorem

Proof of Theorem 0.6. Let Σ be a closed self-shrinker in $\mathbf{R}^{n+1}$ with $\lambda(\Sigma) < \min\{\lambda(\mathbf{S}^{n-1}), 3/2\}$. By Proposition 6.5, $\tilde{\Sigma}$ is diffeomorphic to $\mathbf{S}^n$ and $\lambda(\tilde{\Sigma}) \ge \lambda(\mathbf{S}^n)$.

We will show that there is a gap, i.e., some $\epsilon > 0$ so that $\lambda(\Sigma) \ge \lambda(\mathbf{S}^n) + \epsilon$ if Σ is not round. Suppose instead that there is a sequence of closed self-shrinkers $\Sigma_i \ne \mathbf{S}^n$ with

$$(7.1) \qquad\qquad \lambda(\mathbf{S}^n) \le \lambda(\Sigma_i) < \lambda(\mathbf{S}^n) + 2^{-i}.$$

By Huisken, [H1], none of the Σ_i's is strictly convex since they are not round.

Perturbing the Σ_i's with Lemma 1.2 and then applying rescaled MCF to the perturbations gives a sequence of rescaled MCF's $\tilde{M}_{i,t}$ so that

- Each initial hypersurface $\tilde{M}_{i,0}$ is not strictly convex and has $\lambda(\tilde{M}_{i,0}) < \lambda(\Sigma_i)$.
- Each $\tilde{M}_{i,t}$ has a multiplicity one spherical singularity in finite time.

The second claim follows from Lemma 4.1.

We can now create a new sequence of rescaled MCF's $M_{i,t}$ by rescaling the $\tilde{M}_{i,t}$'s about the spherical singularity so that the new flows satisfy:

(1) Each $M_{i,t}$ converges smoothly to the round sphere as $t \to \infty$.
(2) Each initial hypersurface $M_{i,0}$ is a C^2 graph over $\mathbf{S}^n$ with C^2 norm exactly $\epsilon > 0$.
(3) Every hypersurface $M_{i,t}$ for $t \ge 0$ is a C^2 graph over $\mathbf{S}^n$ with C^2 norm at most ϵ.

By (3), we can choose a subsequence that converges smoothly to a limiting rescaled MCF M_t. Now consider the F functional along M_t. By (1), we have

$$(7.2) \qquad\qquad \lim_{t\to\infty} F(M_t) = \lambda(\mathbf{S}^n).$$

On the other hand, we know that

$$(7.3) \qquad\qquad F(M_0) \le \lambda(M_0) \le \liminf \lambda(\Sigma_i) = \lambda(\mathbf{S}^n).$$

Since the rescaled MCF is the gradient flow for F, we see that the flow must be constant and, thus, every M_t is round. This contradicts (2) which says that the initial hypersurfaces are strictly away from the round sphere, completing the proof.

$\square$

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