Cyclic elements in semisimple Lie algebras

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0 Introduction

Let $\mathfrak{g}$ be a semisimple finite-dimensional Lie algebra over an algebraically closed field $\mathbb{F}$ of characteristic 0 and let e be a non-zero nilpotent element of $\mathfrak{g}$. By the Morozov–Jacobson theorem, the element e can be included in an $s\ell_2$-triple $\mathfrak{s} = \{e, h, f\}$, so that $[e, f] = h$, $[h, e] = 2e$, $[h, f] = -2f$. Then the eigenspace decomposition of $\mathfrak{g}$ with respect to ad h is a $\mathbb{Z}$-grading of $\mathfrak{g}$:

$$(0.1) \qquad \mathfrak{g} = \bigoplus_{j=-d}^{d} \mathfrak{g}_j \,,$$

where $\mathfrak{g}_{\pm d} \neq 0$. The positive integer d is called the *depth* of this $\mathbb{Z}$-grading, and of the nilpotent element e. This notion was previously studied e.g. in [P1].

An element of $\mathfrak{g}$ of the form $e + F$, where F is a non-zero element of $\mathfrak{g}_{-d}$, is called a *cyclic element*, associated with e. In [K1] Kostant proved that any cyclic element, associated with a principal (= regular) nilpotent element e, is regular semisimple, and in [S] Springer proved that any cyclic element, associated with a subregular nilpotent element of a simple exceptional Lie algebra, is regular semisimple as well, and, moreover, found two more distinguished nilpotent conjugacy classes in E_8 with the same property. Both Kostant and Springer use this property in order to exhibit an explicit connection between these nilpotent conjugacy classes and conjugacy classes of certain regular elements of the Weyl group of $\mathfrak{g}$.

A completely different use of cyclic elements was discovered by Drinfeld and Sokolov [DS]. They used a cyclic element, associated with a principal nilpotent element of a simple Lie algebra $\mathfrak{g}$, to construct a bi-Hamiltonian hierarchy of integrable evolution PDE of KdV type (the case $\mathfrak{g} = s\ell_2$ produces the KdV hierarchy). In a number of subsequent papers, [W], [GHM], [BGHM], [FHM], [DF], [F],... the method of Drinfeld and Sokolov was extended to some other nilpotent elements. Namely, it was established that one gets a bi-Hamiltonian integrable hierarchy for any nilpotent element e of a simple Lie algebra, provided that there exists a semisimple cyclic element, associated with e. One of the results of the present paper is a description of all nilpotent elements with this property in all semisimple Lie algebras.

We say that a non-zero nilpotent element e (and its conjugacy class) is of *nilpotent* (resp. *semisimple* or *regular semisimple*) *type* if any cyclic element, associated with e, is nilpotent (resp. any generic cyclic element, associated with e, is semisimple or regular semisimple). If neither of the above cases occurs, we say that e is of *mixed type*.

If $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ and $e = e_1 + e_2$, where $e_i \in \mathfrak{g}_i$ are non-zero nilpotent elements, then either $d_1 = d_2$, in which case e is of nilpotent (resp. semisimple) type iff e_i is such in $\mathfrak{g}_i$, $i = 1, 2$, or else we may assume that $d_1 > d_2$, in which case e is of nilpotent type iff e_1 is, and e is of mixed type otherwise. This remark reduces the problem of determination of types of nilpotent elements in semisimple Lie algebras to the case when $\mathfrak{g}$ is simple.

By a general simple argument we prove that a nilpotent e of a simple Lie algebra $\mathfrak{g}$ is of nilpotent type if and only if its depth d is odd (Theorem 1.1).

For nilpotent elements of even depth the problem can be studied, using the theory of theta groups [Ka1], [V], as follows.

Let e be a nilpotent element of $\mathfrak{g}$ and (0.1) the corresponding $\mathbb{Z}$-grading. One can pick a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}_0$ of $\mathfrak{g}$ and a set of simple roots $\{\alpha_1, \ldots, \alpha_r\}$, such that the root subspace, attached to a simple root α_i, lies in $\mathfrak{g}_{s_i}$ with $s_i \geq 0$, for all $1 \leq i \leq r$. It is a well-known result of Dynkin [D] that the only possible values of s_i are 0, 1, or 2. This associates to e a labeling by 0, 1, 2 of the Dynkin diagram of $\mathfrak{g}$, called the characteristic of e, which uniquely determines e. The element e is even iff all the labels are 0 or 2. If $-\alpha_0 = \sum_{i=1}^{r} a_i \alpha_i$ is the highest root of $\mathfrak{g}$, then the depth d of e is determined by the formula

$$(0.2) \qquad d = \sum_{i=1}^{r} a_i s_i .$$

We associate to e a labeled by 0, 1 and 2 extended Dynkin diagram by putting $s_0 = 2$ at the extra node and the labels s_i of the characteristic of e at all other nodes. Let $m = d + 2$, and let ε be a primitive m^{th} root of 1. Define an automorphism σ_e of $\mathfrak{g}$ by letting

$$(0.3) \qquad \sigma_e(e_{\alpha_i}) = \varepsilon^{s_i} e_{\alpha_i} , \ \sigma_e(e_{-\alpha_i}) = \varepsilon^{-s_i} e_{-\alpha_i} , \quad i = 1, \ldots, r, ,$$

where $e_{\pm \alpha_i}$ are some root vectors, attached to $\pm \alpha_i$. The order of σ_e is m if e is odd, and $m/2$ if e is even.

The automorphism σ_e defines a $\mathbb{Z}/m\mathbb{Z}$-grading

$$(0.4) \qquad \mathfrak{g} = \bigoplus_{j \in \mathbb{Z}/m\mathbb{Z}} \mathfrak{g}^j .$$

Here $\mathfrak{g}^0 = \mathfrak{g}_0$ is a reductive subalgebra of $\mathfrak{g}$, whose semisimple part has the Dynkin diagram, obtained from the extended Dynkin diagram of $\mathfrak{g}$ by removing the nodes with non-zero labels. Note also that the lowest weight vectors of the $\mathfrak{g}^0$-module $\mathfrak{g}^1 = \mathfrak{g}_1$ are the roots α_i with $s_i = 1$ [Ka2], [OV]. If e is even, then the lowest weight vectors of the $\mathfrak{g}^0$-module $\mathfrak{g}^2 = \mathfrak{g}_2 + \mathfrak{g}_{-d}$ are the roots α_i with $s_i = 2$ (including the lowest root α_0).

In the case when e is even, all the labels s_i are even, and it is convenient to divide them by 2 (which is being done in Section 6).

The connected linear algebraic group $G^0|\mathfrak{g}^j$ for each $j \in \mathbb{Z}/m\mathbb{Z}$ is called a theta group.

One of the basic facts of the theory of theta groups is that the G^0-orbit of an element $v \in \mathfrak{g}^j$ is closed (resp. contains 0 in its closure) if and only if v is a semisimple (resp. nilpotent) element of $\mathfrak{g}$ [V]. Since a cyclic element $e + F$ lies in $\mathfrak{g}^2$, this reduces the study of its semisimplicity or nilpotence to the corresponding properties of its G^0-orbit.

In Section 2 we study the *rank* rk e of a nilpotent element e, defined as $\dim\left(\mathfrak{g}^2/G^0\right)$. It follows from Theorem 1.1 that rk e > 0 if and only if the depth of e is even. Several other equivalent definitions of the rank of e are given by Proposition 2.1. Furthermore, we prove that if e is a nilpotent element of mixed type, then a cyclic element $e + F$ is never semisimple (Proposition 2.2(b)). Thus, the element e is of semisimple type if and only if there exists a semisimple cyclic element associated with it.

In Section 3 we introduce the notion of a *reducing subalgebra* for a nilpotent element e of even depth in a simple Lie algebra $\mathfrak{g}$. It is a semisimple subalgebra $\mathfrak{q}$, normalized by $\mathfrak{s}$, such that $Z(\mathfrak{s})(\mathfrak{q} \cap \mathfrak{g}_{-d})$ is Zariski dense in $\mathfrak{g}_{-d}$. We have: $\mathfrak{s} \subset \mathfrak{n}(\mathfrak{q}) = \mathfrak{q} \oplus \mathfrak{z}(\mathfrak{q})$.

By a case-wise consideration, we find a reducing subalgebra $\mathfrak{q}$ for each nilpotent element e, such that the projection of e to $\mathfrak{q}$ along $\mathfrak{z}(\mathfrak{q})$ has regular semisimple type in $\mathfrak{q}$. This leads to a characterization of nilpotent elements of semisimple type and to a kind of a Jordan decomposition of nilpotent elements. The latter decomposition allows us to combine all conjugacy classes of nilpotent elements of $\mathfrak{g}$ in *bushes*, which consist of nilpotent classes with the same reducing subalgebra $\mathfrak{q}$ and the same projection to $\mathfrak{q}$. Each bush contains a unique nilpotent class of semisimple type, and all other are of mixed type, but have the same depth d, rank r, and the conjugacy class of the semisimple part of a generic cyclic element; we denote by $\mathfrak{a}$ the derived subalgebra of the centralizer of this semisimple part.

In Section 4 we describe explicitly some minimal reducing subalgebras of nilpotent elements of even depth in all simple classical Lie algebras. This leads to a description of types of all nilpotent elements and their bushes in these algebras (Theorem 4.3).

We also find some minimal reducing subalgebras for all nilpotent elements in the exceptional simple Lie algebras. This allows us to combine the nilpotent classes in bushes and find their types. All bushes of nilpotent classes in all exceptional simple Lie algebras, along with their invariants d, r and $\mathfrak{a}$, and a minimal reducing subalgebra, are given in Tables 5.1-5.4 in Section 5, and in Table 1.1 (the bush of 0).

In Table 0.1 we list the numbers of all conjugacy classes of non-zero nilpotent elements and those of semisimple, regular semisimple and nilpotent type in the exceptional simple Lie algebras.

Table 0.1

$\mathfrak{g}$ \ #	non-zero nilpotent classes	nilpotent type classes	semisimple type classes	regular semisimple type classes
E_6	20	2	13	5
E_7	44	3	21	5
E_8	69	7	27	7
F_4	15	2	11	4
G_2	4	1	3	2

In particular, among the distinguished nilpotent conjugacy classes we find 7 regular semisimple and 4 mixed type classes in E_8, 3 regular semisimple and 3 mixed type classes in E_7, only regular semisimple type classes in E_6, F_4 and G_2 (3, 4 and 2 classes respectively).

In the case when a nilpotent element e is of regular semisimple type, i.e. the corresponding generic cyclic element $e + F$ is regular (by a result of Springer [S] this holds for all distinguished nilpotents of semisimple type), the centralizer of $e + F$ is a Cartan subalgebra $\mathfrak{h}'$ of $\mathfrak{g}$, and σ_e

induces an element w_e of the Weyl group $W = W(\mathfrak{h}')$ of $\mathfrak{g}$. We thus get a map from the set of conjugacy classes of nilpotent elements of $\mathfrak{g}$ of regular semisimple type to the set of conjugacy classes of regular elements of W, extending that of Kostant and Springer [K1], [S].

As an application, we find in Section 6 the diagrams of all regular elements in exceptional Weyl groups. Another application is a classification of all theta groups which have a closed orbit with a finite stabiliser (Proposition 6.10). In the cases $\mathfrak{g} = E_{6,7,8}$ this list was found in [G] as a part of a long classification of theta groups of positive rank.

In a similar fashion we also construct a map from the set of conjugacy classes of all nilpotent elements of semisimple type (and even of "quasi-semisimple type", see Remark 6.2) to the set of conjugacy classes of W. It would be interesting to compare this map with the one, constructed in [KL] and in [L].

Throughout the paper the base field $\mathbb{F}$ is algebraically closed of characteristic zero. Also, we use throughout the following notation: $(.,.)$ is the Killing form on $\mathfrak{g}$; G is a simply connected connected algebraic group over $\mathbb{F}$ with the Lie algebra $\mathfrak{g}$; $A|V$ denotes an algebraic group A acting linearly on a vector space V, V/A denotes the categorical quotient (which exists if A is reductive), and $\mathrm{rk}\,(A|V) = \dim\,(V/A)$ is called the rank of this linear algebraic group; $Z(\mathfrak{a})$ (resp. $\mathfrak{z}(\mathfrak{a})$) denotes the centralizer of a subset $\mathfrak{a}$ of $\mathfrak{g}$ in G (resp. $\mathfrak{g}$), and $N(\mathfrak{a})$ (resp. $\mathfrak{n}(\mathfrak{a})$) denotes the normaliser of a subalgebra $\mathfrak{a}$. We denote by $\mathfrak{p}'$ the derived subalgebra of a Lie algebra $\mathfrak{p}$.

1 The case of odd depth

In this section we prove the following theorem.

Theorem 1.1. *A nilpotent element e of a simple Lie algebra $\mathfrak{g}$ is of nilpotent type if and only if its depth d is odd.*

Consider the $\mathbb{Z}/m\mathbb{Z}$-grading (0.4) of $\mathfrak{g}$, corresponding to e, so that any cyclic element $e + F$ (where $F \in \mathfrak{g}_{-d}$) lies in $\mathfrak{g}^2$.

Recall that, for any $\mathbb{Z}/m\mathbb{Z}$-grading of $\mathfrak{g}$, the dimension of a maximal abelian subspace of $\mathfrak{g}^j$, consisting of semisimple elements, is called the rank of the G^0-module $\mathfrak{g}^j$. We denote it by $\mathrm{rk}\,(G^0|\mathfrak{g}^{\mathrm{j}})$. It is equal to the dimension of the categorical quotient $\mathfrak{g}^j/G^0$. If $\gcd(j_1, m) = \gcd(j_2, m)$, then $\mathrm{rk}\,(G^0|\mathfrak{g}^{\mathrm{j_1}}) = \mathrm{rk}\,(G^0|\mathfrak{g}^{\mathrm{j_2}})$ [V].

Lemma 1.2. *For the $\mathbb{Z}/m\mathbb{Z}$-grading of $\mathfrak{g}$ defined above, all elements of $\mathfrak{g}^1$ are nilpotent.*

Proof. Under our definition of the $\mathbb{Z}/m\mathbb{Z}$-grading of $\mathfrak{g}$, one has $\mathfrak{g}^0 = \mathfrak{g}_0$ and $\mathfrak{g}^1 = \mathfrak{g}_1$, and hence all elements of $\mathfrak{g}^1$ are nilpotent. □

The "if" part of Theorem 1.1 follows immediately from this lemma and the preceding remark.

Lemma 1.3. *Let $e \in \mathfrak{g}$ be a nilpotent element of even depth d. Then*

(a) *$((\mathrm{ad}\,e)^d x, x)$ is a $Z(\mathfrak{s})$-invariant non-degenerate quadratic form on $\mathfrak{g}_{-d}$.*

(b) *The G^0-invariant polynomial $f(u + x) = ((\mathrm{ad}\,u)^d x, x)$ on $\mathfrak{g}^2 = \mathfrak{g}_2 \oplus \mathfrak{g}_{-d}$ is non-zero.*

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Proof. The subspaces $\mathfrak{g}_{-d}$ and $\mathfrak{g}_d$ are dual with respect to the Killing form, and the operator $(ade)^d$ induces an isomorphism from $\mathfrak{g}_{-d}$ to $\mathfrak{g}_d$. If d is even, this operator is symmetric, whence (a) follows. Clearly, this implies (b). $\square$

The "only if" part of Theorem 1.1 follows Lemma 1.3(b), since $e + F$ is not nilpotent if $f(e + F) \neq 0$.

Nilpotent elements of nilpotent type in classical Lie algebras are described in Section 4. We list in Table 1.1 below all conjugacy classes of nilpotent elements e of nilpotent type in exceptional Lie algebras $\mathfrak{g}$. In the last column we give the type of a generic (nilpotent) cyclic element $e + F$, associated with e.

Table 1.1. Nilpotent orbits of nilpotent type in exceptional Lie algebras

$\mathfrak{g}$	e	d	$e + F$
E_6	$3A_1$	3	D_4
E_6	$2A_2 + A_1$	5	E_6
E_7	$[3A_1]'$	3	D_4
E_7	$4A_1$	3	$D_4 + A_1$
E_7	$2A_2 + A_1$	5	E_6
E_8	$3A_1$	3	D_4
E_8	$4A_1$	3	$D_4 + A_1$
E_8	$2A_2 + A_1$	5	E_6
E_8	$2A_2 + 2A_1$	5	$E_6 + A_1$
E_8	$2A_3$	7	D_7
E_8	$A_4 + A_3$	9	E_8
E_8	A_7	15	E_8
F_4	$A_1 + \widetilde{A}_1$	3	C_3
F_4	$\widetilde{A}_2 + A_1$	5	F_4
G_2	$\widetilde{A}_1$	3	G_2

Remark 1.4. Let $\mathfrak{g} = \bigoplus_{j=-d'}^{d'} \mathfrak{g}'_j$ be a $\mathbb{Z}$-grading of $\mathfrak{g}$ of depth d', such that $e \in \mathfrak{g}'_2$. The same proof as that of Theorem 1.1 shows that e is of nilpotent type for this grading if d' is odd, and that e is not of nilpotent type for this grading if $d' = d$ and d is even. (The only difference is that the quadratic form from Lemma 1.3(a) may be degenerate in this case but it is still non-zero, since $\mathfrak{g}_d \subset \mathfrak{g}'_d$ if $d' = d$.) Note that from the classification of good $\mathbb{Z}$-gradings, given in [EK], it is easy to see that the depth of all good gradings for e is the same as for the Dynkin grading (0.1) for all $\mathfrak{g}$, except for $\mathfrak{g} = C_n$, when the depth may increase by 1 or 2 for e, corresponding to the partitions with all parts even of multiplicity 2, and by 1 for e, corresponding to all other partitions with maximal part even of multiplicity 2.

2   The rank of a nilpotent element of even depth

In this section e is a non-zero nilpotent element of even depth d in a semisimple Lie algebra $\mathfrak{g}$ and $\mathfrak{s} = \{e, h, f\}$ is an $\mathfrak{sl}_2$-triple containing e. Let (0.1) be the corresponding $\mathbb{Z}$-grading of $\mathfrak{g}$, and let $\mathfrak{g} = \oplus_{j \in \mathbb{Z}/m\mathbb{Z}} \mathfrak{g}^j$ be the $\mathbb{Z}/m\mathbb{Z}$-grading (0.4), defined by the characteristic of e as in the Introduction. In particular, $\mathfrak{g}^0 = \mathfrak{g}_0$ and $\mathfrak{g}^2 = \mathfrak{g}_2 + \mathfrak{g}_{-d}$.

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Recall that an action of a reductive algebraic group is called stable if its generic orbits are closed. In this case, the codimension of a generic orbit is equal to the dimension of the categorical quotient. Clearly, e is of semisimple type if and only if the action $G^0|\mathfrak{g}^2$ is stable.

It is known that any orthogonal representation of a reductive algebraic group is stable [Lun2]. By Lemma 1.3(a) the representation $Z(\mathfrak{s})|\mathfrak{g}_{-d}$ is orthogonal and hence stable.

Proposition 2.1. *The following numbers are equal:*

1) $\dim(\mathfrak{g}^2/G^0)(= \mathrm{rk}\,(G^0|\mathfrak{g}^2))$;

2) $\dim(\mathfrak{g}_{-d}/Z(\mathfrak{s}))$;

3) the codimension of a generic orbit of the action $G^0|\mathfrak{g}^2$;

4) the codimension of a generic orbit of the action $Z(\mathfrak{s})|\mathfrak{g}_{-d}$.

We shall call each of these equal positive numbers the *rank* of e in $\mathfrak{g}$ and denote it by $\mathrm{rk}\,e$ or, more precisely, $\mathrm{rk}_{\mathfrak{g}}\,e$.

Proof. Since the orbit $G^0 e$ is open in $\mathfrak{g}^2$, the codimensions of generic orbits for the actions $G^0|\mathfrak{g}^2$ and $Z(\mathfrak{s})|\mathfrak{g}_{-d}$ coincide. But the former is equal to $\dim\,(\mathfrak{g}^2/G^0)$, since the fibers of the factorization morphism $\mathfrak{g}^2 \to \mathfrak{g}^2/G^0$ consist of finitely many orbits [V], while the latter is equal to $\dim\,(\mathfrak{g}_{-d}/Z(\mathfrak{s}))$, since the action $Z(\mathfrak{s})|\mathfrak{g}_{-d}$ is stable. $\qquad\square$

The rank of a principal nilpotent element of a simple Lie algebra is equal to 1. If $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ and $e_1 \in \mathfrak{g}_1$, $e_2 \in \mathfrak{g}_2$ are nilpotent elements of the same even depth, then $\mathrm{rk}\,e = \mathrm{rk}\,e_1 + \mathrm{rk}\,e_2$.

Recall that for a reductive algebraic linear group $H|V$ and a semisimple element $v \in V$ (i.e. such that its orbit is closed) a slice S_v at v is a plane of the form $v + N_v$, where $N_v \subset V$ is an H_v-invariant complementary subspace of the tangent space of the orbit Hv at v. Recall that the linear group $H|V$ is stable if and only if the group $H_v|N_v$ is stable [Lun].

Proposition 2.2. (a) *If a cyclic element $e + F$ is semisimple, then F is semisimple for the action $Z(\mathfrak{s})|\mathfrak{g}_{-d}$.*

(b) *If there exist semisimple cyclic elements associated with e, then a generic cyclic element is semisimple.*

Proof. (a) If $e + F$ is semisimple, then the orbit $G^0(e + F)$ is closed. But

$$e + Z(\mathfrak{s})F = G^0(e + F) \cap (e + \mathfrak{g}_{-d}),$$

whence $Z(\mathfrak{s})F$ is closed.

(b) Let a cyclic element $e + F$ be semisimple. Then, by (a), the orbit $Z(\mathfrak{s})F$ is closed. Let R be the stabilizer of F in $Z(\mathfrak{s})$, and let S be a slice at F for the action $Z(\mathfrak{s})|\mathfrak{g}_{-d}$. (One may assume that S is a subspace of $\mathfrak{g}_{-d}$ containing F.) Then R is the stabilizer of $e + F$ in G^0, and $e + S$ is a slice at $e + F$ for the action $G^0|\mathfrak{g}^2$. Since the action $Z(\mathfrak{s})|\mathfrak{g}_{-d}$ is stable, the action $R|S$ is also stable, and hence the action $G^0|\mathfrak{g}^2$ is stable as well, so a generic cyclic element is semisimple.

$\qquad\square$

Corollary 2.3. *If e is a nilpotent element, for which there exist a semisimple cyclic element $e + F$, then e is of semisimple type.*

3 Reducing subalgebras

We retain the assumptions and notation of the previous section.

Definition 3.1. *A semisimple subalgebra $\mathfrak{q} \subset \mathfrak{g}$ is called a reducing subalgebra for e, if it is normalized by $\mathfrak{s}$ and if $\overline{Z(\mathfrak{s})\mathfrak{q}_{-d}} = \mathfrak{g}_{-d}$ (i.e. a generic orbit of the action $Z(\mathfrak{s})|\mathfrak{g}_{-d}$ intersects $\mathfrak{q}_{-d}$). We will denote by Q the connected simply connected algebraic group with Lie algebra $\mathfrak{q}$.*

Remark 3.2. If $\mathfrak{q}$ is a reductive subalgebra normalized by $\mathfrak{s}$ satisfying the condition $\overline{Z(\mathfrak{s})\mathfrak{q}_{-d}} = \mathfrak{g}_{-d}$, then its semisimple part $\mathfrak{q}' = [\mathfrak{q}, \mathfrak{q}]$ is a reducing subalgebra for e. Indeed, it is obviously normalized by $\mathfrak{s}$, while $\mathfrak{q}'_{-d} = \mathfrak{q}_{-d}$.

If $\mathfrak{q} \subset \mathfrak{g}$ is a semisimple subalgebra normalized by $\mathfrak{s}$, then

$$\mathfrak{s} \subset \mathfrak{n}(\mathfrak{q}) = \mathfrak{q} \oplus \mathfrak{z}(\mathfrak{q}).$$

We denote by $e_{\mathfrak{q}}$ (resp. $\mathfrak{s}_{\mathfrak{q}}$) the projection of e (resp. $\mathfrak{s}$) to $\mathfrak{q}$ along $\mathfrak{z}(\mathfrak{q})$. Clearly, $\mathfrak{s}_{\mathfrak{q}}$ is an $\mathfrak{sl}_2$-triple containing $e_{\mathfrak{q}}$. The following theorem is a convenient criterion for $\mathfrak{q}$ to be reducing for e.

Theorem 3.3. *A semisimple subalgebra $\mathfrak{q} \subset \mathfrak{g}$ normalized by $\mathfrak{s}$ is reducing for e if and only if $e_{\mathfrak{q}}$ has the same depth and rank in $\mathfrak{q}$ as e in $\mathfrak{g}$.*

Proof. Let $\mathfrak{m}$ be an $(\mathfrak{q} + \mathfrak{s})$-invariant complementary subspace of $\mathfrak{q}$ in $\mathfrak{g}$. Then

$$\mathfrak{z}(\mathfrak{s}) = \mathfrak{z}_{\mathfrak{q}}(\mathfrak{s}_{\mathfrak{q}}) + \mathfrak{z}_{\mathfrak{m}}(\mathfrak{s}),$$

and, for any $x \in \mathfrak{q}$, we have

$$(3.1) \qquad\qquad [\mathfrak{z}(\mathfrak{s}), x] = [\mathfrak{z}_{\mathfrak{q}}(\mathfrak{s}_{\mathfrak{q}}), x] + [\mathfrak{z}_{\mathfrak{m}}(\mathfrak{s}), x],$$

the second summand lying in $\mathfrak{m}$. The condition $\overline{Z(\mathfrak{s})\mathfrak{q}_{-d}} = \mathfrak{g}_{-d}$ means that, for a generic $x \in \mathfrak{q}_{-d}$, one has

$$(3.2) \qquad\qquad \mathfrak{q}_{-d} + [\mathfrak{z}(\mathfrak{s}), x] = \mathfrak{g}_{-d}.$$

Due to (3.1), this is equivalent to the equality

$$[\mathfrak{z}_{\mathfrak{m}}(\mathfrak{s}), x] = \mathfrak{m}_{-d}.$$

It also follows from (3.1) that the codimension of $Z(\mathfrak{s})x$ in $\mathfrak{g}_{-d}$ is equal to the codimension of $Z_H(\mathfrak{s}_{\mathfrak{q}})x$ in $\mathfrak{g}_{-d}$ plus the codimension of $[\mathfrak{z}_{\mathfrak{m}}(\mathfrak{s}), x]$ in $\mathfrak{m}_{-d}$. Hence, (3.1) holds if and only if these codimensions are equal. $\qquad\square$

Remark 3.4. Let $\mathfrak{l}$ be the semisimple part of the centralizer of a Cartan subalgebra of the centraliser of $\mathfrak{s}$. This is the well-known minimal semisimple Levi subalgebra, containing $\mathfrak{s}$. It is interesting that the depth of e in $\mathfrak{l}$ is always equal to d, provided that d is even. We can check this by a case-wise verification but we have no conceptual proof of this fact. Hence $\mathfrak{l}$ is a good candidate for a reducing subalgebra for e. For example, by Theorems 1.1 and 3.3, $\mathfrak{l}$ is a reducing subalgebra for e, provided that $\operatorname{rk}_{\mathfrak{g}} e = 1$.

Remark 3.5. The minimal semisimple Levi subalgebra $\mathfrak{l}$, contaning e, is a reducing subalgebra if and only if the stabiliser in $Z(\mathfrak{s})$ of a generic element of $\mathfrak{g}_{-d}$ is a maximal torus of $Z(\mathfrak{s})$. Indeed, $\mathfrak{l}$ is the semisimple part of the centralizer of a maximal torus of $Z(\mathfrak{s})$, hence $\mathfrak{l} \cap \mathfrak{g}$ is the fixed point set of this torus on $\mathfrak{g}$.

Remark 3.6. By Lemma 1.3(b), the representation of the reductive group $Z(\mathfrak{s})$ on $\mathfrak{g}_{-d}$ is orthogonal. Hence, if $\dim \mathfrak{g}_{-d} = 1$, then the representation of $\mathfrak{z}(\mathfrak{s})$ on $\mathfrak{g}_{-d}$ is trivial. Furthermore, if $\dim \mathfrak{g}_{-d} = 2$, this representation is trivial as well. Indeed, this representation cannot contain SO_2, since in this case $r = 1$, hence, by Remark 3.4, $\mathfrak{l}$ is a reducing subalgebra, and, by Remark 3.5, the generic stabilizer must contain a maximal torus of SO_2, a contradiction. Finally, if $\dim \mathfrak{g}_{-d} = 3$, the only possibilities for the linear group $Z^0(\mathfrak{s})|\mathfrak{g}_{-d}$ are: trivial, SO_3, and $T_1 \subset SO_3$, and in the last case $\mathfrak{l}$ is not a reducing subalgebra.

The meaning of the notion of a reducing subalgebra is explained by the following theorem.

Theorem 3.7. *Let $\mathfrak{q} \subset \mathfrak{g}$ be a reducing subalgebra for e. Then e is of semisimple type if and only if $e \in \mathfrak{q}$ and e is of semisimple type in $\mathfrak{q}$.*

Proof. By the definition of a reducing subalgebra, a generic cyclic element $e + F$ (where $F \in \mathfrak{g}_{-d}$) is $Z(\mathfrak{s})$-conjugate to an element of $e + \mathfrak{q}_{-d} \subset \mathfrak{q} \oplus \mathfrak{z}(\mathfrak{q})$. The latter is semisimple if and only if its projections to both $\mathfrak{q}$ and $\mathfrak{z}(\mathfrak{q})$ are semisimple, which is only possible if $e \in \mathfrak{q}$. $\square$

Theorem 3.8. *Under the assumptions of Theorem 3.7, the projection $e_{\mathfrak{z}}$ of e to $\mathfrak{z}(\mathfrak{q})$ is a nilpotent element of depth $< d$ in the semisimple part $\mathfrak{z}(\mathfrak{q})'$ of $\mathfrak{z}(\mathfrak{q})$.*

Proof. Suppose that the depth of $e_{\mathfrak{z}}$ in $\mathfrak{z}(\mathfrak{q})'$ is equal to d. Then there is a non-zero semisimple element in $\mathfrak{z}(\mathfrak{q})^2$, whence $\mathrm{rk}_{\mathfrak{g}} e > \mathrm{rk}_{\mathfrak{q}} e_{\mathfrak{q}}$, which contradicts Theorem 3.3. $\square$

Proposition 3.9. *Let $\mathfrak{q} \subset \mathfrak{g}$ be a reducing subalgebra for e. Then*

> *(a) Any semisimple subalgebra normalized by $\mathfrak{s}$ and containing $\mathfrak{q}$ is also a reducing subalgebra for e.*

> *(b) Any reducing subalgebra for $e_{\mathfrak{q}}$ in $\mathfrak{q}$ is a reducing subalgebra for e in $\mathfrak{g}$.*

Proof. The assertion (a) is obvious. The assertion (b) follows from the inclusion $Z_H(\mathfrak{s}_{\mathfrak{q}}) \subset Z(\mathfrak{s})$. $\square$

Due to Proposition 3.9(b), there is a minimal reducing subalgebra for e. An open question is whether it is unique up to conjugation.

The simplest case, when the condition $\overline{Z(\mathfrak{s})\mathfrak{q}_{-d}} = \mathfrak{g}_{-d}$ is satisfied, is when $\mathfrak{q}_{-d} = \mathfrak{g}_{-d}$. As the following proposition shows, there is a unique minimal reducing subalgebra with the last property.

Proposition 3.10. *Let $\mathfrak{m} \subset \mathfrak{g}$ be the $\mathfrak{s}$-submodule generated by $\mathfrak{g}_{-d}$ (the isotypic component corresponding to the highest weight d). Then the subalgebra $\mathfrak{q}$ generated by $\mathfrak{m}$ is semisimple.*

Proof. Let $\mathfrak{n}$ be the unipotent radical of $\mathfrak{q}$, and $\mathfrak{q} = \mathfrak{l} + \mathfrak{n}$ be a Levi decomposition. One may assume that the reductive subalgebra $\mathfrak{l}$ is normalized by $\mathfrak{s}$. Since the scalar product is non-degenerate on $\mathfrak{m}$, we have $\mathfrak{m} \cap \mathfrak{n} = 0$, so $\mathfrak{m} \subset \mathfrak{l}$; but then $\mathfrak{q} = \mathfrak{l}$ (and $\mathfrak{n} = 0$). Thus, $\mathfrak{q}$ is reductive. Clearly, $\mathfrak{g}_{-d}$ lies in the semisimple part $\mathfrak{q}' = [\mathfrak{q}, \mathfrak{q}]$ of $\mathfrak{q}$. Since $\mathfrak{q}'$ is normalized by $\mathfrak{s}$, we obtain $\mathfrak{m} \subset \mathfrak{q}'$, and hence $\mathfrak{q} = \mathfrak{q}'$ is semisimple. $\square$

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Another way to get reducing subalgebras is given by the following proposition.

Proposition 3.11. *Let K be any reductive subgroup of a generic stabilizer of the action $Z(\mathfrak{s})|\mathfrak{g}_{-d}$. Then the semisimple part $(\mathfrak{g}^K)'$ of the subalgebra $\mathfrak{g}^K$ of K-fixed elements of $\mathfrak{g}$ is a reducing subalgebra for e.*

Proof. Clearly, the subalgebra $(\mathfrak{g}^K)'$ is normalized by $Z(\mathfrak{s})$. By the Luna– Richardson theorem [LR], any closed orbit of the action $Z(\mathfrak{s})|\mathfrak{g}_{-d}$ intersects $\mathfrak{g}_{-d}^K$, and hence $(\mathfrak{g}^K)'$ is a reducing subalgebra for e. $\qquad\square$

Corollary 3.12. *There is a reducing subalgebra $\mathfrak{q} \subset \mathfrak{g}$ for e such that a generic stabilizer for the action $Z_Q(\mathfrak{s_q})|\mathfrak{q}_{-d}$ lies in the center of Q.*

Proof. It suffices to take for K the whole of a generic stabilizer of the action $Z(\mathfrak{s})|\mathfrak{g}_{-d}$. $\qquad\square$

Corollary 3.13. *There is a reducing subalgebra $\mathfrak{q} \subset \mathfrak{g}$ for e such that $e_\mathfrak{q}$ is an even nilpotent element of $\mathfrak{q}$.*

Proof. If e is odd, it suffices to take for K the center of the connected subgroup S with $\operatorname{Lie} S = \mathfrak{s}$, which acts trivially on $\mathfrak{g}_{-d}$. In other words, $\mathfrak{q} = \sum_j \mathfrak{g}_{2j}$. $\qquad\square$

The main general result of the present paper is that, for any nilpotent element e of even depth, there is a reducing subalgebra $\mathfrak{q} \subset \mathfrak{g}$ such that $e_\mathfrak{q}$ is of semisimple type in $\mathfrak{q}$. Unfortunately, we can do it only by presenting such a subalgebra in each case. This is done in the next sections. But if it is known that such a subalgebra exists, then the following theorem together with Proposition 3.9(b) shows that there also exists a reducing subalgebra $\mathfrak{q}$ such that $e_\mathfrak{q}$ is of regular semisimple type in $\mathfrak{q}$.

Theorem 3.14. *Let $e \in \mathfrak{g}$ be a nilpotent element of semisimple type. Then there exists a reducing subalgebra $\mathfrak{q}$ for e such that e is of regular semisimple type in $\mathfrak{q}$.*

Proof. Let $e+F \in \mathfrak{g}^2$ be a generic cyclic element associated with e. Note that $K = Z(e+F)\cap G^0$ is the stabilizer of F for the action $Z(\mathfrak{s})|\mathfrak{g}_{-d}$.

If e is not of regular semisimple type, then $\mathfrak{z}(e + F)$ is a non-abelian reductive $\mathbb{Z}/m\mathbb{Z}$-graded subalgebra of $\mathfrak{g}$. Hence, $\mathfrak{z}(e + F)^0 \neq 0$ (see, e.g., [GOV], Theorem 3.3.7). By Proposition 3.11 $\mathfrak{q} = (\mathfrak{g}^K)'$ is a reducing subalgebra for e (containing e and F). Since a generic stabilizer for the action $Z_Q(\mathfrak{s})|\mathfrak{q}_{-d}$ coincides with the center of Q (see Corollary 3.12), we have $\mathfrak{z}(e + F)\cap\mathfrak{q}^0 = 0$, which implies that e is of regular semisimple type in $\mathfrak{q}$. $\qquad\square$

4 Cyclic elements in classical Lie algebras

Let V be a vector space of dimension n. Let $G \subset GL(V)$ be one of the classical linear groups $SL(V)$, $SO(V)$, $Sp(V)$ so that $\mathfrak{g} := \operatorname{Lie}G = \mathfrak{sl}(V), \mathfrak{so}(V)$, $\mathfrak{sp}(V)$, respectively.

In the last two cases the space V is endowed with a non-degenerate symmetric or skew-symmetric bilinear form, called the scalar product. We will refer to these two types of vector spaces with a scalar product as quadratic and symplectic spaces. The tensor product of two quadratic or two symplectic spaces (resp. of a quadratic space and a symplectic space) is naturally a quadratic (resp. symplectic) space.

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We use the notation of $\mathfrak{so}^+(V)$ (resp. $\mathfrak{sp}^+(V)$) for the space of symmetric linear operators in a quadratic (resp. symplectic) vector space V.

It is well-known (see e.g. [C]) that, up to an automorphism of $\mathfrak{g}$, a nilpotent element $e \in \mathfrak{g}$ is defined by the orders $n_1, \ldots, n_p$ of its Jordan blocks. We shall assume that $n_1 \geq n_2 \geq \cdots \geq n_p(> 0)$.

In the case $G = SO(V)$ (resp. $G = Sp(V)$) the sequence $(n_1, \ldots, n_p)$ contains each even (resp. odd) number with even multiplicity.

Let, as before, $\mathfrak{s} = \{e, h, f\} \subset \mathfrak{g}$ be an $\mathfrak{sl}_2$-triple containing e. The space V decomposes into a direct sum

$$(4.1) \qquad\qquad V = V_1 \oplus \cdots \oplus V_p$$

of $\mathfrak{s}$-invariant subspaces of dimensions $n_1, \ldots, n_p$. In the case $G = SO(V)$ (resp. $G = Sp(V)$) one may assume that the summands of even (resp. odd) dimension are grouped in pairs of dual isotropic subspaces so that their sums and the remaining single summands are mutually orthogonal.

The eigenvalues of h in V_i are $n_i - 1, n_i - 3, \ldots - (n_i - 3), -(n_i - 1))$. Looking at the roots of $\mathfrak{g}$, one can observe that e is even if and only if $n_1, \ldots, n_p$ are of the same parity. The depth d of e is equal to $2n_1 - 2$, except for the cases $G = SO(V)$, n_1 odd, $n_1 - n_2 = 1$, when $d = 2n_1 - 3$, and $G = SO(V)$, n_1 odd, $n_1 - n_2 = 2$, when $d = 2n_1 - 4$. In particular, by Theorem 1.1, a nilpotent element e of a classical Lie algebra $\mathfrak{g}$ is of nilpotent type (equivalently, $\mathrm{rk}\, e = 0$) if and only if $\mathfrak{g} = \mathfrak{so}(V)$, $dim\, V \geq 7$, n_1 is odd and $n_1 - n_2 = 1$; it is easy to see that the associated with this e generic cyclic element is a nilpotent element, corresponding to the partition with the first part $3n_1 - 2$, the part n_2 having multiplicity two less than for e, and all other parts unchanged. So, we will not consider the latter case in the rest of this section.

We are now going to describe the behavior of cyclic elements associated with nilpotent elements of the classical Lie algebras $\mathfrak{g} = \mathfrak{sl}(V), \mathfrak{so}(V), \mathfrak{sp}(V)$. In particular, we shall obtain the Jordan decomposition for them.

As before, an important role is played by the centralizer $Z(\mathfrak{s})$ of the $\mathfrak{sl}_2$-triple $\mathfrak{s} = \{e, h, f\}$. It is a reductive subgroup of G, leaving invariant each component of the grading (0.1) of $\mathfrak{g}$ and each isotypic component of the representation of $\mathfrak{s}$ in V.

Theorem 4.1. *Let e be a non-zero nilpotent element of even depth d in a classical Lie algebra $\mathfrak{g}$. Then there exists a reducing subalgebra $\mathfrak{g}^s \subset \mathfrak{g}$ such that the projection e^s of e to $\mathfrak{g}^s$ is a nilpotent element of regular semisimple type in $\mathfrak{g}^s$.*

Proof. In what follows the theorem will be proved in seven possible cases: one for $G = SL(V)$, two for $G = Sp(V)$, and four for $G = SO(V)$. The subalgebra $\mathfrak{g}^s$ will be explicitly described in terms of the decomposition (4.1). The property 1) of the definition of a reducing subalgebra will be clear from this description. Moreover, in all the cases but one e^s will be a principal nilpotent element of $\mathfrak{g}^s$ and thereby a nilpotent element of regular semisimple type in $\mathfrak{g}^s$.

Case $G = SL(V)$.

Let $n_1 = \cdots = n_r > n_{r+1}$ (assuming $n_{p+1} = 0$). Set

$$\widetilde{V} = V_1 \oplus \cdots \oplus V_r, \quad \widetilde{\mathfrak{g}} = \mathfrak{sl}(\widetilde{V}).$$

Clearly, $\mathfrak{g}_{-d} = \widetilde{\mathfrak{g}}_{-d}$. Replacing $\mathfrak{g}$ by $\widetilde{\mathfrak{g}}$ and the linear operators e, h, f by their restrictions to $\widetilde{V}$, one can reduce the proof to the case, when $r = p$, i.e. all Jordan blocks of e are of the same size.

Under this condition, the $\mathfrak{s}$-module V can be represented as

$$V = V_0 \otimes R \,,$$

where V_0 is a simple $\mathfrak{s}$-module of dimension n_1, while R is a trivial $\mathfrak{s}$-module of dimension p. In these terms we have

(a) $e = e_0 \otimes 1$, where e_0 is a principal nilpotent element of $\mathfrak{sl}(V_0)$;

(b) $h = h_0 \otimes 1$, where h_0 is a characteristic of e_0 in $\mathfrak{sl}(V_0)$;

(c) $\mathfrak{g} = \mathfrak{sl}(V_0) \otimes g\ell(R)$;

(d) $\mathfrak{g}_{-d} = \mathfrak{sl}(V_0)_{-d} \otimes g\ell(R)$, where the grading of $\mathfrak{sl}(V_0)$ is defined by h_0.

The group $Z(\mathfrak{s})$ is a finite extension of $1 \otimes SL(R)$ (the intersection of $1 \otimes GL(R)$ with $SL(V)$). It acts on $\mathfrak{g}_{-d}$ by conjugations of the second tensor factor. In this way, a generic element of $g\ell(R)$ can be put in a diagonal form, which means that the subalgebra

$$\mathfrak{g}^s = \mathfrak{sl}(V_1) \oplus \cdots \oplus \mathfrak{sl}(V_p) \subset \mathfrak{g}$$

satisfies property 2) of a reducing subalgebra.

Remark 4.2. A cyclic element $e + F_0 \otimes A$ is semisimple if and only if A is semisimple and non-degenerate.

Case $G = Sp(V)$, n_1 even.

As in the previous case, the proof reduces to the case, when all Jordan blocks of e have the same size. Under this condition, the symplectic $\mathfrak{s}$-module V is represented as

$$V = V_0 \otimes R \,,$$

where V_0 is a simple symplectic $\mathfrak{s}$-module of dimension n_1, while R is a quadratic vector space of dimension p, on which $\mathfrak{s}$ acts trivially. We also have

(a) $e = e_0 \otimes 1$, where e_0 is a principal nilpotent element of $\mathfrak{sp}(V_0)$;

(b) $h = h_0 \otimes 1$ where h_0 is a characteristic of e_0;

(c) $\mathfrak{g} = \mathfrak{sp}(V_0) \otimes \mathfrak{so}^+(R) + \mathfrak{sp}^+(V_0) \otimes \mathfrak{so}(R)$;

(d) $\mathfrak{g}_{-d} = \mathfrak{sp}(V_0)_{-d} \otimes \mathfrak{so}^+(R)$.

The group $Z(\mathfrak{s}) \simeq O(R)$ acts on $\mathfrak{g}_{-d}$ by conjugations of the second tensor factor. In this way, a generic element of $\mathfrak{so}^+(R)$ can be put in a diagonal form in an orthogonal basis. Assuming that $V_1, \ldots, Vp$ are mutually orthogonal, we thus see that

$$\mathfrak{g}^s = \mathfrak{sp}(V_1) \oplus \cdots \oplus \mathfrak{sp}(V_p) \subset \mathfrak{g}$$

satisfies property 2) of a reducing subalgebra.

Case $G = Sp(V)$, n_1 odd.

The proof reduces to the case, when $n_1 = \cdots = n_{2r}$ with $2r = p$. Under this assumption,

$$V = V_0 \otimes R,$$

where V_0 is a simple quadratic $\mathfrak{s}$-module of dimension n_1, while R is a symplectic vector space of dimension $2r$, on which $\mathfrak{s}$ acts trivially. We have

(a) $e = e_0 \otimes 1$, where e_0 is a principal nilpotent element of $\mathfrak{so}(V_0)$;

(b) $h = h_0 \otimes 1$, where h_0 is a characteristic of e_0;

(c) $\mathfrak{g} = \mathfrak{so}^+(V_0) \otimes \mathfrak{sp}(R) + \mathfrak{so}(V_0) \otimes \mathfrak{sp}^+(R)$;

(d) $\mathfrak{g}_{-d} = \mathfrak{so}^+(V)_{-d} \otimes \mathfrak{sp}(R)$.

The group $Z(\mathfrak{s}) \simeq Sp(R)$ acts on $\mathfrak{g}_{-d}$ by conjugations of the second tensor factor. In this way, a generic element of $\mathfrak{sp}(R)$ can be put in a diagonal form in a symplectic basis. Assuming that the summands of (4.1) are grouped in pairs of dual isotropic subspaces $V_1, V_1^*, \ldots, V_r, V_r^*$ so that their sums are mutually orthogonal, we come to the conclusion that the algebra

$$\mathfrak{g}^s = \mathfrak{sl}(V_1) \oplus \cdots \oplus \mathfrak{sl}(V_r)$$

naturally embedded in $\mathfrak{sp}(V)$, satisfies property 2) of a reducing subalgebra.

Case $G = SO(V)$, n_1 even.

The proof goes as in the previous case, but this time both V_0 and R are symplectic vector spaces, and $\mathfrak{g}_{-d} = \mathfrak{sp}(V_0) \otimes \mathfrak{sp}^+(R)$. Assuming that the summands of (4.1) are grouped in pairs as in the previous case (but with respect to the symmetric scalar product in V), the subalgebra $\mathfrak{g}^s$ is defined in the same way.

Case $G = SO(V)$, n_1 odd, $n_1 = n_2$.

As in the previous cases, one may assume that $n_1 = n_2 = \cdots = n_p$. Under this assumption,

$$V = V_0 \otimes R,$$

where V_0 is a simple $\mathfrak{s}$-module of dimension n_1, while R is a quadratic vector space of dimension p, on which $\mathfrak{s}$ acts trivially. We have

(a) $e = e_0 \otimes 1$, where e_0 is a principal nilpotent element of $\mathfrak{so}(V_0)$;

(b) $h = h_0 \otimes 1$, where h_0 is a characteristic of e_0;

(c) $\mathfrak{g} = \mathfrak{so}^+(V_0) \otimes \mathfrak{so}(R) + \mathfrak{so}(V_0) \otimes \mathfrak{so}^+(R)$;

(d) $\mathfrak{g}_{-d} = \mathfrak{so}^+(V)_{-d} \otimes \mathfrak{so}(R)$.

The group $Z(\mathfrak{s}) \simeq SO(R)$ acts on $\mathfrak{g}_{-d}$ by conjugations of the second tensor factor.

If $p = 2r$, then a generic element of $\mathfrak{so}(R)$ can be put in a diagonal form in a basis, $(e_1, e_1', e_2, e_2', \ldots, e_z, e_z')$ such that $(e_i, e_i') = 1$, while all the other pairs of basis vectors are orthogonal. Assuming that the summands of (4.1) are grouped in pairs of dual isotropic subspaces $V_1, V_1^*, \ldots, V_r, V_r^*$, so that their sums are mutually orthogonal, we see that the algebra

$$\mathfrak{g}^s = \mathfrak{sl}(V_1) \oplus \cdots \oplus \mathfrak{sl}(V_r)$$

naturally embedded in $\mathfrak{g}$, satisfies property 2) of the definition of a reducing subalgebra.

The case $p = 2r + 1$ is a bit different. Here we have one extra basis vector e_p of R with $(e_p, e_p) = 1$, orthogonal to all the other basis vectors, and, when putting an element of $\mathfrak{so}(R)$ in a diagonal form, the pth diagonal entry is always zero. So if we assume that the first $2r$ summands of (4.1) are grouped in pairs as above, while V_p is orthogonal to all of them, then the subalgebra $\mathfrak{g}^s$ defined as above still satisfies property 2) of the definition of a reducing subalgebra.

In both cases the projection of e to $\mathfrak{g}^s$ is a principal nilpotent element of $\mathfrak{g}^s$.

Case $G = SO(V)$, n_1 odd, $n_1 - n_2 = 2$.

Let $n_2 = \cdots = n_r > n_{r+1}$. Set

$$\widetilde{V} = V_1 \oplus V_2 \oplus \cdots \oplus V_r, \quad \widetilde{\mathfrak{g}} = \mathfrak{so}(\widetilde{V}).$$

Then $\mathfrak{g}_{-d} = \widetilde{\mathfrak{g}}_{-d}$, so the proof reduces to the case, when $r = p$.

Under this condition, the quadratic $\mathfrak{s}$-module V is represented as

$$V = V_1 \oplus (V_0 \otimes R),$$

where V_0 is a simple quadratic $\mathfrak{s}$-module of dimension n_2, while R is a quadratic vector space of dimension $p - 1$, on which $\mathfrak{s}$ acts trivially. We have

(a) $e = e_1 \oplus (e_0 \otimes 1)$, where e_1 (resp. e_0) is a principal nilpotent element of $\mathfrak{so}(V_1)$ (resp. $\mathfrak{so}(V_0)$);

(b) $h = h_1 \oplus (h_0 \otimes 1)$, where h_1 (resp. h_0) is a characteristic of e_1 (resp. e_0);

(c) $\mathfrak{g} = \mathfrak{so}(V_1) \oplus (V_1 \otimes V_0 \otimes R) \oplus \mathfrak{so}(V_0 \otimes R)$, where the second summand is understood as the space of skew-symmetric linear operators taking V_1 to $V_0 \otimes R$ and $V_0 \otimes R$ to V_1;

(d) $d = 2n_1 - 4$ and

$$\mathfrak{g}_{-d} = \mathfrak{so}(V_1)_{-d} \oplus (V_1(-n_1 + 1) \otimes V_0(-n_1 + 3) \otimes R),$$

where $V_1(\lambda)$ (resp. $V_0(\lambda)$) denotes the eigenspace of h_1 (resp. h_0) of the eigenvalue λ.

Note that the spaces $\mathfrak{so}(V_1)_{-d}$, $V_1(-n_1 + 1)$, $V_0(-n_1 + 3)$ are one-dimensional. The unity component of $Z(\mathfrak{s})$ acts trivially on the first summand of $\mathfrak{g}_{-d}$ and by unimodular orthogonal

transformations of R on the second summand. Any non-isotropic vector of R can be taken to a vector of a fixed non-isotropic line by such a transformation. This means that the subalgebra

$$\mathfrak{g}^s = \mathfrak{so}(V_1 \oplus V_2)$$

(assuming that V_2 is non-degenerate) satisfies property 2) of the definition of a reducing sub-algebra. The projection of e to $\mathfrak{g}^s$ is not a principal nilpotent element of $\mathfrak{g}^s$, but we shall show that it is of regular semisimple type in $\mathfrak{g}^s$.

Let $n_1 = 2m + 1$, $n_2 = 2m - 1$. Changing the notation, assume that V is a $4m$-dimensional quadratic vector space with a basis $\{e_i | i \in \mathbb{Z}/4m\mathbb{Z}\}$ such that $(e_i, e_{-i}) = 1$, while all the other pairs of basis vectors are orthogonal. Let $A_i = -A_{-i-1}$ be the linear operator, taking e_i to e_{i+1}, e_{-i-1} to $-e_{-i}$, and all the other basis vectors to 0. It is easy to see that it is skew-symmetric. The operator

$$A = A_0 + A_1 + ... + A_{2m-1}$$

cyclically permutes the pairs $\pm e_0$, $\pm e_1$, ..., $\pm e_{4k-1}$ and, hence, is a regular semisimple element of $\mathfrak{so}(V)$.

The space V decomposes into the orthogonal sum $V = V_1 \oplus V_2$, where V_1 (resp. V_2) is spanned by e_{3m}, e_{3m-1}, ..., e_{m+1}, e_m (resp. by e_{m-1}, e_{m-2}, ..., $e_{-(m-2)}$, $e_{-(m-1)}$). Correspondingly, the operator A decomposes as $A = e + F$, where $F = A_{m-1}$ and

$$e = (A_m + A_{m+1} + ... + A_{2m-1}) + (A_0 + A_1 + ... + A_{m-2}) \in \mathfrak{so}(V_1) \oplus \mathfrak{so}(V_2) \subset \mathfrak{so}(V)$$

is a nilpotent element of type $(2m + 1, 2m - 1)$ and of depth $d = 4m - 2$.

A characteristic of e is the operator h multiplying e_i by $2i$ for $i = m - 1$, $m - 2$, ..., $-(m - 2)$, $-(m - 1)$ and by $2i - 4m$ for $i = 3m$, $3m - 1$, ..., $m + 1$, m. It follows that

$$F = A_{m-1} \in \mathfrak{so}(V)_{-d}.$$

Thus, e is a nilpotent element of regular semisimple type in $\mathfrak{so}(V)$.

Case $G = SO(V)$, n_1 odd, $n_1 - n_2 > 2$ or $p = 1$.

In this case $d = 2n_1 - 4$ and $\mathfrak{g}_{-d} = \mathfrak{so}(V_1)_{-d}$, so one can take $\mathfrak{g}^s = \mathfrak{so}(V_1)$. The projection of e to $\mathfrak{g}^s$ is a principal nilpotent element of $\mathfrak{g}^s$.

$\square$

The description of $\mathfrak{g}^s$ given in the previous proof implies that in all the cases but one the semisimple part $\mathfrak{g}^n$ of the centralizer of $\mathfrak{g}^s$ is a classical linear Lie algebra of the same type as $\mathfrak{g}$. The exceptional case is $G = SO(V)$, $n_1 = 2$, when, if the multiplicity of n_1 is $2r$, $\mathfrak{g}^n$ is $\mathfrak{so}_{n-4r}$ plus the sum of r copies of sl_2; but the projection of e to $\mathfrak{g}^n$ still lies in $\mathfrak{so}_{n-4r}$.

In all the cases but two the type of the projection e^n of e to $\mathfrak{g}^n$ is obtained from that of e by deleting all the maximal parts. The exceptional cases are $G = SO(V)$, n_1 odd of multiplicity $2r + 1$, when one should only delete $2r$ maximal parts, and $G = SO(V)$, n_1 odd, $n_1 - n_2 = 2$, when one should delete n_1 and n_2.

In the following theorem, we retain the preceding notation and the assumptions of Theorem 4.1. In particular, $(n_1, \ldots, n_p)$ is the partition corresponding to the nilpotent element $e \in \mathfrak{g}$.

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Theorem 4.3. *1) If $G = SL(V)$ or $Sp(V)$, then e is of semisimple type if and only if the corresponding partition is of the form $(n_1, \ldots, n_1, 1, \ldots, 1)$, and of regular semisimple type if and only if 1 occurs at most once.*

2) If $G = SO(V)$, then e is of semisimple type if and only if one of the following three possibilities holds:

(a) n_1 has even multiplicity and all the other parts are 1;

(b) $n_1 = 2m + 1, n_2 = 2m - 1 \, (m \geq 1)$ and all the other parts are 1;

(c) $n_1 \geq 5$ and all the other parts are 1.

Moreover, e is of regular semisimple type if and only if n_1 is odd and 1 occurs at most twice in case (a), $p \leq 4$ in case (b), and $p \leq 2$ in case (c).

Proof. Let $\mathfrak{g}^s$ be the reducing subalgebra constructed in the proof of Theorem 4.1. It follows from Proposition 3.8 that the element e is of semisimple type if and only if it lies in $\mathfrak{g}^s$. The explicit description of $\mathfrak{g}^s$ given in the proof of Theorem 4.1 in the seven considered cases, permits to determine whether e is of semisimple type, in terms of the corresponding partition.

Assume now that $e \in \mathfrak{g}^s$. Then automatically $\mathfrak{s} \subset \mathfrak{g}^s$. With one exception described below, denote by V' the sum of all one-dimensional components of the decomposition (4.1) and by V^s the sum of all the other components. The exceptional case is the nilpotent element of type $(3, 1, \ldots, 1)$ in $\mathfrak{so}(V)$, when we set $V^s = V_1 + V_2$ and $V' = V_3 + \ldots + V_p$, assuming V_2 to be orthogonal to all the other one-dimensional components.

In all the cases, denote by $\mathfrak{g}(V^s)$ (resp. $\mathfrak{g}(V')$) the classical linear Lie algebra of the same type as $\mathfrak{g}$ acting on V^s (resp. on V'). Clearly, $\mathfrak{g}^s \subset \mathfrak{g}(V^s)$. Since a generic cyclic element $e + F$ is $Z(\mathfrak{s})$-conjugate to an element of $\mathfrak{g}^s$, one may assume that it lies in $\mathfrak{g}^s$. Then its centralizer contains $\mathfrak{g}(V')$. This implies that e may be of regular semisimple type only if the algebra $\mathfrak{g}(V')$ is commutative, which means that $\dim V' \leq 1$ in the cases $G = SL(V)$ or $Sp(V)$ and $\dim V' \leq 2$ in the case $G = SO(V)$.

Moreover, in subcase (c), if $\dim V' = 2$, then $\mathfrak{g}(V^s) = \mathfrak{so}(V^s)$ does not contain regular semisimple elements of $\mathfrak{g} = \mathfrak{so}(V)$, so e cannot be of regular semisimple type.

Note that the eigenvalues of any semisimple cyclic element associated with a principal nilpotent element of $\mathfrak{sl}_m$, are the m-th roots of some non-zero number. In particular, if m is even, they decompose into pairs of opposite numbers. It follows that such an element is not regular in $\mathfrak{so}_{2m}$, where $\mathfrak{sl}_m$ is embedded in the natural way. This implies that in subcase (a), if n_1 is even, a semisimple cyclic element associated with e cannot be regular in $\mathfrak{g} = \mathfrak{so}(V)$.

It remains to check that in all the other cases the element e is of regular semisimple type. For $G = SL(V)$, for $G = Sp(V)$ and n_1 odd, and for $G = SO(V)$ in subcase (a), this follows from the above description of the eigenvalues of semisimple cyclic elements associated with principal nilpotent elements of $\mathfrak{sl}_m$. For $G = Sp(V)$ and n_1 even, this follows from the analogous description of the eigenvalues of semisimple cyclic elements associated with principal nilpotent elements of $\mathfrak{sp}_m$, which are also the m-th roots of some non-zero number.

For a nilpotent element of type $(2m + 1, 2m - 1)$ in so_{4m}, the eigenvalues of the associated semisimple cyclic element constructed in the proof of Theorem 4.1 are $4m$-th roots of some number. This implies that in subcase (b), the element e is of regular semisimple type, provided $\dim V' \leq 2$.

Finally, for a principal nilpotent element of $\mathfrak{so}_{2m+1}$, the eigenvalues of any associated semisimple cyclic element are $2m$-th roots of some number and 0. This implies that in subcase (c), the element e is of regular semisimple type, provided $\dim V' \leq 1$. $\qquad \square$

The rank of a nilpotent element e of even depth in a classical Lie algebra $\mathfrak{g}$ is determined as follows. In all the cases when the projection of e to $\mathfrak{g}^s$ is a principal nilpotent element of $\mathfrak{g}^s$, the rank $\operatorname{rk} e$ is equal to the number of simple factors of $\mathfrak{g}^s$. In the exceptional case, when $G = SO(V)$, n_1 odd, $n_1 - n_2 = 2$, one has $\operatorname{rk} e = 2$.

Note also that a bush of a semisimple type element e in a classical Lie algebra consists of all nilpotents, corresponding to (admissible) partitions, obtained from the partition, associated with e, by replacing the set of its 1's by arbitrary parts strictly smaller than the largest part, and also, in case $G = SO(V)$ and n_1 of odd multiplicity > 1, one of the replacing parts can be n_1.

Finally, the minimal semisimple Levi subalgebra, containing a nilpotent element e of a classical Lie algebra $\mathfrak{g}$ is a reducing subalgebra, except for the following cases:

(a) $\mathfrak{g} = \mathfrak{sp}_n$, n_1 even, $n_1 = n_2$;

(b) $\mathfrak{g} = \mathfrak{so}_n$, n_1 odd, $n_1 - n_2 = 2$, and the multiplicity of n_2 is even.

5 Nilpotents of semisimple and mixed type in exceptional Lie algebras

For all exceptional simple Lie algebras $\mathfrak{g}$ we list in Tables 5.1-5.4 below all bushes of conjugacy classes of nilpotent elements of even depth. In the first row of a bush we give the type of the conjugacy class of the nilpotent of semisimple type, and in the rest of the rows of the bush that of mixed type. All nilpotent conjugacy classes of a bush have the same depth d and rank r, the same derived subalgebra $\mathfrak{a}$ of the centralizer of the semisimple part of a generic cyclic element $e + F$, the same minimal reducing subalgebra $\mathfrak{g}^s$ and the semisimple part of the centralizer $\mathfrak{g}^n$ of $\mathfrak{g}^s$ in $\mathfrak{g}$, and the same projection e^s of e on $\mathfrak{g}^s$ (for some e in the conjugacy class).

Nilpotent conjugacy classes in a bush differ only by their projection e^n on $\mathfrak{g}^n$, listed in the next to last column. All subalgebras $\mathfrak{a}$, $\mathfrak{g}^s$ and $\mathfrak{g}^n$ are semisimple subalgebras of $\mathfrak{g}$. In almost all cases their conjugacy class is determined by their types, listed in the tables. The only exceptions are $\mathfrak{g} = E_7$, when $[A_5]'$, $[A_3 + A_1]'$, and $[3A_1]'$ (resp. $[A_5]''$, $[A_3 + A_1]''$, and $[3A_1]''$) denote the semisimple Levi subalgebras, contained (resp. not contained) in A_7, and $\mathfrak{g} = G_2$ and F_4, when A_1 and A_2 (resp. $\widetilde{A}_1$ and $\widetilde{A}_2$) denote the semisimple Levi subalgebras, whose roots are long (resp. short).

Since $\mathfrak{a}$ is a semisimple Levi subalgebra, for a nilpotent e of semisimple type we have:

$$\dim \mathfrak{z}_{\mathfrak{g}} e + F = \dim \mathfrak{a} + \operatorname{rank} \mathfrak{g} - \operatorname{rank} \mathfrak{a}.$$

Also, such e is of regular semisimple type iff $\mathfrak{a} = 0$.

In the last column we list the unity component of the linear algebraic group, which is the image of the action of $Z(\mathfrak{s})$ on $\mathfrak{g}_{-d}$. Here n denotes the trivial linear group acting on the n-dimensional vector space, G_2 and F_4 denote the 7- and 26-dimensional representations respectively of these exceptional algebraic groups, and $\oplus$ denotes the direct sum of linear algebraic groups.

The procedure of computing the entries of Tables 5.1–5.4 is as follows.

Table 5.1. Nilpotent orbits of semisimple and mixed type in E_6

e	d	r	$\mathfrak{a}$	$\mathfrak{g}^s$	e^s	$\mathfrak{g}^n$	e^n	$Z(\mathfrak{s})^0|\mathfrak{g}_{-d}$
A_1	2	1	A_5	A_1	2	A_5	0	1
$2A_1$	2	2	A_3	$2A_1$	$(2;2)$	A_3	0	$SO_7 \oplus 1$
A_2	4	1	$2A_2$	A_2	(3)	$2A_2$	0	1
A_2+A_1							$(2,1;1^3)$	1
A_2+2A_1							$(2,1;2,1)$	SO_3
$2A_2$	4	2	A_2	$2A_2$	$(3;3)$	A_2	0	$G_2 \oplus 1$
A_3	6	1	A_3	A_3	(4)	$2A_1$	0	1
A_3+A_1							$(2;1^2)$	1
$D_4(a_1)$	6	2	0	D_4	$(5,3))$	0	0	2
A_4	8	1	A_1	A_4	(5)	A_1	0	1
A_4+A_1							(2)	1
D_4	10	1	$2A_2$	B_3	(7)	A_1	0	1
$D_5(a_1)$							(2)	1
A_5	10	1	A_2	A_5	(6)	A_1	0	1
$E_6(a_3)$	10	2	0	F_4	$F_4(a_2)$	0	0	2
D_5	14	1	0	B_4	(9)	0	0	1
$E_6(a_1)$	16	1	0	E_6	$E_6(a_1)$	0	0	1
E_6	22	1	0	F_4	F_4	0	0	1

First, we find the depth d by Remark 1.4, using the list of characteristics from [D] (see also [E]).

Second, we compute the last column and deduce from it the value of the rank r. By Remark 3.6, the representation of $\mathfrak{z}(\mathfrak{s})$ on $\mathfrak{g}_{-d}$ is trivial if $\dim \mathfrak{g}_{-d} = 1$ or 2, hence in these cases $r = \dim \mathfrak{g}_{-d}$. Also, if e is distinguished, then $\mathfrak{z}(\mathfrak{s}) = 0$, and again $r = \dim \mathfrak{g}_{-d}$. The remaining cases are few (three in E_6 and F_4, nine in E_7, and eleven in E_8). The reductive Lie algebras $\mathfrak{z}(\mathfrak{s})$ were computed in [E], and it is straightforward to compute their representations on $\mathfrak{g}_{-d}$; the number r of generating invariants in all these cases is well known.

Third, by Remark 3.5, we now know from the last column all nilpotent elements e, for which the minimal semisimple Levi subalgebra $\mathfrak{l}$, containing e, is a reducing subalgebra. Then, of course, e is distingushed in $\mathfrak{l}$. We see that there are only five nilpotent conjugacy classes in all the exceptional Lie algebras when this is not the case: nilpotent classes of type $A_3 + A_2$ and $A_3 + A_2 + A_1$ in E_7 and E_8, and nilpotent class of type $\widetilde{A}_1$ in F_4. For E_7 there is only one minimal regular subalgebra, different from $\mathfrak{l}$, containing these e, namely $D_4 + 2A_1$ [D], and we check that the summand D_4 is a reducing subalgebra. The same D_4 is also a reducing subalgebra for nilpotent class of the same type in E_8. Hence all these four nilpotent classes are of mixed type. In the case of the nilpotent class $\widetilde{A}_1$ in F_4 the only other minimal regular subalgebra, containing e, is $A_1 \oplus A_1$ [D], and it is easy check that it is a reducing subalgebra, in which e is of semisimple type. Hence, by Theorem 3.7, this nilpotent class in F_4 is of semisimple type.

Table 5.2. Nilpotent orbits of semisimple and mixed type in E_7

e	d	r	$\mathfrak{a}$	$\mathfrak{g}^s$	e^s	$\mathfrak{g}^n$	e^n	$Z(\mathfrak{s})^0\vert\mathfrak{g}_{-d}$
A_1	2	1	D_6	A_1	(2)	D_6	0	1
$2A_1$	2	2	$D_4 \oplus A_1$	$2A_1$	(2;2)	$D_4 \oplus A_1$	0	$SO_9 \oplus 1$
$[3A_1]''$	2	3	D_4	$[3A_1]''$	(2;2;2)	D_4	0	$F_4 \oplus 1$
A_2	4	1	$[A_5]''$	A_2	(3)	$[A_5]''$	0	1
$A_2 + A_1$							$(2,1^4)$	1
$A_2 + 2A_1$							$(2^2,1^2)$	SO_3
$A_2 + 3A_1$							(2^3)	G_2
$2A_2$	4	2	A_2	$2A_2$	(3;3)	A_2	0	$G_2 \oplus SO_3$
A_3	6	1	$D_4 \oplus A_1$	A_3	(4)	$[A_3 \oplus A_1]''$	0	1
$[A_3 + A_1]'$							$(2,1^2;1^2)$	1
$[A_3 + A_1]''$							$(1^4;2)$	1
$A_3 + 2A_1$							$(2,1^2;2)$	1
$D_4(a_1)$	6	2	$[3A_1]''$	D_4	(5,3)	$[3A_1]''$	0	2
$D_4(a_1) + A_1$							$(2;1^2;1^2)$	2
$A_3 + A_2$							$(2;2;1^2)$	$T_1 \subset SO_3$
$A_3 + A_2 + A_1$							$(2;2;2)$	$S^2SO_3/1$
A_4	8	1	A_2	A_4	(5)	A_2	0	1
$A_4 + A_1$							$(2,1)$	1
$A_4 + A_2$							(3)	SO_3
D_4	10	1	$[A_5]''$	D_4	(7,1)	$[3A_1]''$	0	1
$D_4 + A_1$							$(2;1^2;1^2)$	1
$D_5(a_1)$							$(2;2;1^2)$	1
$D_5(a_1) + A_1$							$(2;2;2)$	1
$[A_5]'$	10	1	$A_2 \oplus 3A_1$	$[A_5]'$	(6)	A_1	0	1
$[A_5]''$	10	1	D_4	$[A_5]''$	(6)	A_2	0	1
$A_5 + A_1$							$(2,1)$	1
$D_6(a_2)$	10	2	A_2	D_6	(7,5)	A_1	0	2
$E_6(a_3)$	10	2	$[3A_1]''$	F_4	$F_4(a_2)$	A_1	0	2
$E_7(a_5)$	10	3	0	E_7	$E_7(a_5)$	0	0	3
A_6	12	1	0	A_6	(7)	0	0	SO_3
D_5	14	1	$[3A_1]''$	B_4	(9)	$2A_1$	0	1
$D_5 + A_1$							$(2;1^2)$	1
$D_6(a_1)$							$(1^2;2)$	1
$E_7(a_4)$							$(2;2)$	1
$E_6(a_1)$	16	1	0	E_6	$E_6(a_1)$	0	0	1
D_6	18	1	A_2	D_6	(11,1)	A_1	0	1
$E_7(a_3)$							(2)	1
E_6	22	1	$[3A_1]''$	F_4	F_4	A_1	0	1
$E_7(a_2)$							(2)	1
$E_7(a_1)$	26	1	0	E_7	$E_7(a_1)$	0	0	1
E_7	34	1	0	E_7	E_7	0	0	1

Table 5.3. Nilpotent orbits of semisimple and mixed type in E_8

e	d	r	$\mathfrak{a}$	$\mathfrak{g}^s$	e^s	$\mathfrak{g}^n$	e^n	$Z(\mathfrak{s})^0|\mathfrak{g}_{-d}$
A_1	2	1	E_7	A_1	2	E_7	0	1
$2A_1$	2	2	D_6	$2A_1$	(2;2)	D_6	0	$SO_{13} \oplus 1$
A_2	4	1	E_6	A_2	(3)	E_6	0	1
$A_2 + A_1$							A_1	1
$A_2 + 2A_1$							$2A_1$	SO_3
$A_2 + 3A_1$							$3A_1$	G_2
$2A_2$	4	2	$2A_2$	$2A_2$	(3;3)	$2A_2$	0	$G_2 \oplus G_2$
A_3	6	1	D_6	A_3	(4)	D_5	0	1
$A_3 + A_1$							$(2^2, 1^6)$	1
$A_3 + 2A_1$							$(2^4, 1^2)$	1
$D_4(a_1)$	6	2	D_4	D_4	(5,3)	D_4	0	2
$D_4(a_1) + A_1$							$(2^2, 1^4)$	2
$A_3 + A_2$							(2^4)	$T_1 \subset SO_3$
$A_3 + A_2 + A_1$							$(3, 2^2, 1)$	$S^2 SO_3/1$
$D_4(a_1) + A_2$							$(3^2, 1^2)$	$Ad(SL_3)$
A_4	8	1	A_4	A_4	(5)	A_4	0	1
$A_4 + A_1$							$(2, 1^3)$	1
$A_4 + 2A_1$							$(2^2, 1)$	1
$A_4 + A_2$							$(3, 1^2)$	SO_3
$A_4 + A_2 + A_1$							$(3,2)$	SO_3
D_4	10	1	E_6	B_3	(7)	B_4	0	1
$D_4 + A_1$							$(2, 2, 1^5)$	1
$D_5(a_1)$							$(3, 1^6)$	1
$D_5(a_1) + A_1$							$(3, 2^2, 1^2)$	1
$D_4 + A_2$							$(3^2, 1^3)$	1
$D_5(a_1) + A_2$							(3^3)	1
A_5	10	1	$D_4 \oplus A_2$	A_5	(6)	$A_2 \oplus A_1$	0	1
$A_5 + A_1$							$(1^3; 2)$	1
$E_6(a_3)$	10	2	D_4	F_4	$F_4(a_2)$	G_2	0	2
$E_6(a_3) + A_1$							A_1	2
$D_6(a_2)$	10	2	$2A_2$	D_6	(7,5)	$2A_1$	0	2
$E_7(a_5)$	10	3	A_2	E_7	$E_7(a_5)$	A_1	0	3
$E_8(a_7)$	10	4	0	E_8	$E_8(a_7)$	0	0	4

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e	d	r	$\mathfrak{a}$	$\mathfrak{g}^s$	e^s	$\mathfrak{g}^n$	e^n	$Z(\mathfrak{s})^0\vert\mathfrak{g}_{-d}$
A_6	12	1	A_1	A_6	(7)	A_1	0	SO_3
$A_6 + A_1$							(2)	SO_3
D_5	14	1	D_4	B_4	(9)	B_3	0	1
$D_5 + A_1$							$(2^2, 1^3)$	1
$D_6(a_1)$							$(3, 1^4)$	1
$D_5 + A_2$							$(3^2, 1)$	1
$D_7(a_2)$							$(5, 1^2)$	1
$E_7(a_4)$							$(3, 2^2)$	1
$E_6(a_1)$	16	1	A_2	E_6	$E_6(a_1)$	A_2	0	1
$E_6(a_1) + A_1$							$(2,1)$	1
$E_8(b_6)$							(3)	1
D_6	18	1	A_4	B_5	(11)	B_2	0	1
$E_7(a_3)$							$(2^2, 1)$	1
$D_7(a_1)$							$(3, 1^2)$	1
$E_8(a_6)$	18	2	0	E_8	$E_8(a_6)$	0	0	2
E_6	22	1	D_4	F_4	F_4	G_2	0	1
$E_6 + A_1$							A_1	1
$E_7(a_2)$							$\widetilde{A}_1$	1
$E_8(b_5)$							$G_2(a_1)$	1
D_7	22	1	$2A_2$	B_6	(13)	A_1	0	1
$E_8(a_5)$	22	2	0	E_8	$E_8(a_5)$	0	0	2
$E_7(a_1)$	26	1	A_1	E_7	$E_7(a_1)$	A_1	0	1
$E_8(b_4)$							(2)	1
$E_8(a_4)$	28	1	0	E_8	$E_8(a_4)$	0	0	1
E_7	34	1	A_2	E_7	E_7	A_1	0	1
$E_8(a_3)$							(2)	1
$E_8(a_2)$	38	1	0	E_8	$E_8(a_2)$	0	0	1
$E_8(a_1)$	46	1	0	E_8	$E_8(a_1)$	0	0	1
E_8	58	1	0	E_8	E_8	0	0	1

Table 5.4. Nilpotent orbits of semisimple and mixed type in F_4 and G_2

$\mathfrak{g}$	e	d	r	$\mathfrak{a}$	$\mathfrak{g}^s$	e^s	$\mathfrak{g}^n$	e^n	$Z(\mathfrak{s})^0|\mathfrak{g}_{-d}$
G_2	A_1	2	1	$\widetilde{A}_1$	A_1	(2)	$\widetilde{A}_1$	0	1
	$G_2(a_1)$	4	1	0	A_2	(3)	0	0	1
	G_2	10	1	0	G_2	G_2	0	0	1
F_4	A_1	2	1	C_3	A_1	(2)	C_3	0	1
	$\widetilde{A}_1$	2	2	B_2	$2A_1$	$(2;2)$	$2A_1$	0	$SO_6 \oplus 1$
	A_2	4	1	$\widetilde{A}_2$	A_2	(3)	$\widetilde{A}_2$	0	1
	$A_2 + \widetilde{A}_1$							$(2,1)$	SO_3
	$\widetilde{A}_2$	4	1	A_2	$\widetilde{A}_2$	(3)	A_2	0	G_2
	B_2	6	1	B_2	B_2	(5)	$2A_1$	0	1
	$C_3(a_1)$							$(2;1^2)$	1
	$F_4(a_3)$	6	2	0	D_4	$(5,3)$	0	0	2
	B_3	10	1	$\widetilde{A}_2$	B_3	(7)	0	0	1
	C_3	10	1	A_2	C_3	(6)	A_1	0	1
	$F_4(a_2)$	10	2	0	F_4	$F_4(a_2)$	0	0	2
	$F_4(a_1)$	14	1	0	B_4	(9)	0	0	1
	F_4	22	1	0	F_4	F_4	0	0	1

Now we turn to the nilpotent classes e, for which the subalgebra $\mathfrak{l}$ is a reducing subalgebra. Since e is distinguished in $\mathfrak{l}$, and, by Theorem 3.7, e is of semisimple type in $\mathfrak{g}$ if and only if it is in $\mathfrak{l}$, we need to figure out which of the distingushed nilpotent classes are of semisimple type. For this we use the following lemma.

Lemma 5.1. *(a) Let e be a distingushed (hence even) nilpotent element of semisimple type in a simple Lie algebra $\mathfrak{g}$. As described in the introduction, we associate with e an automorphism σ_e of $\mathfrak{g}$ of order $m = \frac{1}{2}d + 1$, where d is the depth of e (recall that for even e we divide the RHS of (0.2) by 2). Then m is a regular number of $\mathfrak{g}$, defined in Section 6, and we have: $|\Delta| = m \dim \mathfrak{g}^\sigma$, where Δ is the set of roots of $\mathfrak{g}$.*

(b) The following is a complete list of distinguished nilpotent conjugacy classes of semisimple (hence of regular semisimple [S]) type in all simple Lie algebras:

(i) regular nilpotent classes;

(ii) subregular nilpotent classes $X(a_1)$ in all exceptional Lie algebras X;

(iii) nilpotent classes, corresponding to the partition $(2k+1, 2k-1, 1)$ in B_{2k}, and to the partition $(2k+1, 2k-1)$ in D_{2k} (denoted by $D_{2k}(a_{k-1})$);

(iv) nilpotent classes $F_4(a_2)$, $E_7(a_5)$, $E_8(a_i)$ for $i = 2, 4, 5, 6, 7$.

Proof. Let e be an even nilpotent element of $\mathfrak{g}$ of regular semisimple type, and let σ_e be the corresponding automorphism of order m of $\mathfrak{g}$, desribed in the introduction. By the Kostant-Springer construction (see Section 6), the centralizer of a generic cyclic element $e+F$, associated with e, is a Cartan subalgebra $\mathfrak{h}'$ of $\mathfrak{g}$, on which σ_e induces a regular element w_e of the Weyl group. It has only zero fixed points in $\mathfrak{h}'$, and $e + F$ is its regular eigenvector whose eigenvalue is m-th primitive root ε of 1. Hence w_e is a regular element of the Weyl group, and its order m is a regular number of $\mathfrak{g}$, equal to $d/2 + 1$, proving (a).

The classification of all distinguished elements of semisimple type in classical Lie algebras follows from Theorem 4.3.

All regular numbers are listed in [S] (see also Section 6), and the condition that $d/2 + 1$ is a regular number rules out all distingushed nilpotents in exceptional Lie algebras, which are not listed in (b), with the exception of the nilpotent $E_8(b_5)$. The latter is ruled out by the last equation in (a) since in this case $\dim \mathfrak{g}^{\sigma_e} = 22$.

By the results of Kostant [K1] and Springer [S], the regular and subregular nilpotents of exceptional Lie algebras are of regular semisimple type, as well as the subsubregular nilpotent $E_8(a_2)$.

In the remaining cases, listed in (b)iv, the elements σ_e are powers of the above, and they still have only zero fixed points in $\mathfrak{h}'$ and a regular eigenvector with eigenvalue the corresponding power of ε. Hence all the corresponding theta groups are stable, i.e. all these distinguished nilpotent elements are of regular semisimple type.

Here is a simple proof that the regular nilpotent element e has semisimple type, using the theory of theta groups (which works over any field). Indeed, in this case G^0 is an r-dimensional torus and a generic cyclic element is of the form $\sum_{i=0}^{r} e_{\alpha_i}$ in the notaton of the introduction. Since 0 lies inside the convex hull of its weights α_i, $i = 0, ..., r$, it follows that the orbit of such element is closed. A similar proof works, for example, for the subprinciplal nilpotent classes in exceptional Lie algebras. $\square$

We thus get the list of all nilpotent classes of semisimple type in all exceptional Lie algebras. For each of them we choose as a reducing subalgebra $\mathfrak{g}^s$ the minimal semisimple Levi subalgebra $\mathfrak{l}$, containing e, with the following exceptions, when we take for $\mathfrak{g}^s$ a smaller subalgebra: if e is of type D_n, E_6, $E_6(a_3)$, $F_4(a_1)$, $F_4(a_3)$, $G_2(a_1)$ in $\mathfrak{l}$, then we take $\mathfrak{g}^s = B_{n-1}$, F_4, F_4, B_4, D_4, A_2 respectively.

Next, for each nilpotent element e^s of semisimple type we compute the centarizer of a generic cyclic element; its derived subalgebra is $\mathfrak{a}$. This computation is done on the computer, using the programm by W. de Graaf [G].

After that, for each e^s we combine all nilpotent classes, having the same d, r, and the conjugacy class of $\mathfrak{a}$, as e^s, in one bush. We compute $\mathfrak{g}^n = \mathfrak{z}(\mathfrak{g}^s)'$, and we check that all nilpotents of mixed type in one bush lie in $\mathfrak{g}^s \oplus \mathfrak{g}^n$ and have projection e^s on $\mathfrak{g}^s$. For computation of $\mathfrak{g}^n$ the following remark is useful.

Remark 5.2. Since the semisimple part of a cyclic element lies in $\mathfrak{g}^s$, it follows that $\mathfrak{g}^n \subset \mathfrak{a}$. This inclusion is an equality in many cases.

The type of a nilpotent element $e^s + e^n$ with $e^n \in \mathfrak{g}^n$ is established by computing the dimension and the reductive part of its centralizer, using the programm by W. de Graaf [G].

Remark 5.3. In most of the cases bushes are compatible with inclusions of semisimple Lie algebras $\mathfrak{k} \subset \mathfrak{g}$ of the same rank. Namely, if e^s is a nilpotent element of semisimple type in $\mathfrak{k}$, such that its minimal semisimple Levi subalgebra in $\mathfrak{k}$ is a reducing subalgebra, then e^s is of semisimple type in $\mathfrak{g}$ as well, and the bush of e^s in $\mathfrak{k}$ is contained in the bush of e^s in $\mathfrak{g}$ (some of the nilpotent orbits of the former combine in one nilpotent orbit in the latter).

For example, the bush of the nilpotent class D_4 of semisimple type in E_8 comes from this nipotent class in the subalgebra D_8, where it corresonds to the partition $(7, 1^9)$. The bush of it in D_8 consists of 7 nilpotent orbits (as described in Section 4), two of which get joined in a single orbit in E_8. As a result, we get all 6 nilpotent orbits of the bush of the nilpotent class D_4 in E_8.

Note that for the nilpotent class of type D_5 in E_7 we have: $\mathfrak{g}^n = A_1 \oplus A_1$. One of these A_1's corresponds to a root of E_7, and its centralizer in E_7 is D_6. The latter contains a direct sum of B_4 and the other A_1, which does not correspond to a root of E_7 (it is of type $2A_1$).

In all these considerations paper [La] was very useful.

Remark 5.4. Let us call a nilpotent element e (and its conjugacy class) *irreducible* if there is no proper reducing subalgebra for it. From the discussion in Section 4 and Tables 5.1–5.4 we obtain the following complete list of conjugacy classes of irreducible nilpotent elements in all simple Lie algebras: A_{2n}, $B_n(n \neq 3)$, C_n, $D_{2n}(a_{n-1})$, G_2, F_4, $F_4(a_2)$, $E_6(a_1)$, E_7, $E_7(a_1)$, $E_7(a_5)$, E_8, $E_8(a_i)$ for $i = 1, 2, 4, 5, 6, 7$.

In Tables 5.5 and 5.6 below, we list all nilpotent classes e of mixed type (combined in bushes) in the exceptional Lie algebras. For each e, we give the dimension of the centralizer $\mathfrak{z}(e + F)$ of a generic cyclic element $e + F$, and the nilpotent part of $e + F$ in $\mathfrak{a}$ (the derived subalgebra of the centralizer of the semisimple part of $e + F$). For this we again used the programm by W. de Graaf [G].

Table 5.5. Nilpotent orbits of mixed type in E_6, E_7, F_4, and G_2

$\mathfrak{g}$	e	$\dim \mathfrak{z}(e + F)$	$\mathfrak{a}$	nilpotent part of $e + F$ in $\mathfrak{a}$
E_6	$A_2 + A_1$	14	$2A_2$	$(1^3; 2, 1)$
	$A_2 + 2A_1$	10		$(2,1 ; 2,1)$
	$A_3 + A_1$	12	A_3	$(2, 1^2)$
	$A_4 + A_1$	6	A_1	(2)
	$D_5(a_1)$	10	$2A_2$	$(2,1;2,1)$
E_7	$A_2 + A_1$	27	$[A_5]''$	$(2, 1^4)$
	$A_2 + 2A_1$	21		$(2^2, 1^2)$
	$A_2 + 3A_1$	19		(2^3)
	$[A_3 + A_1]''$	31	$D_4 \oplus A_1$	$(1^8; 2)$
	$[A_3 + A_1]'$	23		$(2^2, 1^4; 1^2)$
	$A_3 + 2A_1$	21		$(2^2, 1^4; 2)$
	$D_4(a_1) + A_1$	11	$[3A_1]''$	$(1^2; 1^2; 2)$
	$A_3 + A_2$	9		$(1^2; 2; 2)$
	$A_3 + A_2 + A_1$	7		$(2; 2; 2)$
	$D_4 + A_1$	27	$[A_5]''$	$(2, 1^4)$
	$D_5(a_1)$	21		$(2^2, 1^2)$
	$D_5(a_1) + A_1$	19		(2^3)
	$A_4 + A_1$	9	A_2	$(2,1)$
	$A_4 + A_2$	7		(3)
	$A_5 + A_1$	21	D_4	$(2^2, 1^4)$
	$D_5 + A_1$	11	$[3A_1]''$	$(1^2; 1^2; 2)$
	$D_6(a_1)$	9		$(1^2; 2; 2)$
	$E_7(a_4)$	7		$(2;2;2)$
	$E_7(a_3)$	9	A_2	$(2,1)$
	$E_7(a_2)$	7	$[3A_1]''$	$(2;2;2)$
F_4	$A_2 + \widetilde{A}_1$	6	$\widetilde{A}_2$	$(2,1)$
	$C_3(a_1)$	8	B_2	$(2^2, 1)$

Table 5.6. Nilpotent orbits of mixed type in E_8

e	$\dim_{\mathfrak{z}}(e+F)$	$\mathfrak{a}$	nilpotent part of $e+F$ in $\mathfrak{a}$
$A_2 + A_1$	58	E_6	A_1
$A_2 + 2A_1$	48		$2A_1$
$A_2 + 3A_1$	40		$3A_1$
$A_3 + A_1$	50	D_6	$(2^2, 1^8)$
$A_3 + 2A_1$	40		$(2^4, 1^4)$
$D_4(a_1) + A_1$	22	D_4	$(2^2, 1^4)$
$A_3 + A_2$	20		(2^4)
$A_3 + A_2 + A_1$	16		$(3, 2^2, 1)$
$D_4(a_1) + A_2$	14		$(3^2, 1^2)$
$A_4 + A_1$	20	A_4	$(2, 1^3)$
$A_4 + 2A_1$	16		$(2^2, 1)$
$A_4 + A_2$	14		$(3, 1^2)$
$A_4 + A_2 + A_1$	12		$(3,2)$
$D_4 + A_1$	58	E_6	A_1
$D_5(a_1)$	48		$2A_1$
$D_5(a_1) + A_1$	40		$3A_1$
$D_4 + A_2$	38		A_2
$D_5(a_1) + A_2$	30		$2A_1 + A_2$
$A_5 + A_1$	28	$D_4 + A_2$	$(2^2, 1^4; 1^3)$
$E_6(a_3) + A_1$	22	D_4	$(2^2, 1^4)$
$A_6 + A_1$	8	A_1	(2)
$D_5 + A_1$	22	D_4	$(2^2, 1^4)$
$D_6(a_1)$	20		(2^4)
$E_7(a_4)$	16		$(3, 2^2, 1)$
$D_5 + A_2$	14		$(3^2, 1^2)$
$D_7(a_2)$	12		(4^2)
$E_6(a_1) + A_1$	10	A_2	$(2,1)$
$E_8(b_6)$	8		(3)
$E_7(a_3)$	20	A_4	$(2, 1^3)$
$D_7(a_1)$	16		$(2^2, 1)$
$E_6 + A_1$	22	D_4	$(2^2, 1^4)$
$E_7(a_2)$	16		$(3, 2^2, 1)$
$E_8(b_5)$	14		$(3^2, 1^2)$
$E_8(b_4)$	8	A_1	(2)
$E_8(a_3)$	10	A_2	$(2,1)$

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6 Diagrams of regular elements of exceptional Weyl groups

Let $\mathfrak{h}$ be a Cartan subalgebra of a simple Lie algebra $\mathfrak{g}$ and let $W \subset \mathrm{Aut}_\mathbb{F}\mathfrak{h}$ be the Weyl group of $\mathfrak{g}$. An element $w \in W$ is called *regular* [S] if it has an eigenvector $a \in \mathfrak{h}$, which is a regular element (i.e., the centralizer of a in $\mathfrak{g}$ is $\mathfrak{h}$), and $w \neq 1$. The order of a regular element of W is called a *regular number.* Springer proved in [S] that, up to conjugacy, there exists at most one, up to conjugacy, regular element in W of a given order and listed all possible regular numbers, along with the characteristic polynomials of the corresponding regular elements.

By definition, if e is a nilpotent element of regular semisimple type, then the corresponding generic cyclic element $e + F$ is regular semisimple, hence its centralizer is a Cartan subalgebra, which we denote by $\mathfrak{h}'$. As explained in the introduction, we associate with e a $\mathbb{Z}/m\mathbb{Z}$ grading (0.4), hence an inner automorphism of $\mathfrak{g}$ of order m, denoted by σ_e, such that $\sigma_e(e) = \varepsilon e$, where ε is m-th primitive root of 1. The order m is given by the RHS of (0.2) if e is odd, and by its half if e is even.

The automorphism σ_e leaves $\mathfrak{h}'$ invariant and induces on it a regular element of the Weyl group, which we denote by w_e. Note that σ_e and w_e have the same order if e is even. It follows from the classification of nilpotent elements of regular semisimple type, given in Sections 4 and 5, that, with the exception of the nilpotent of $\mathfrak{sl}_n$, corresponding to the partition $(p^m, 1)$ (hence $n = mp + 1$), where p is even, all nipotent elements of $\mathfrak{g}$ of regular semisimple type are even.

In the following Tables 6.1 - 6.5 we list the diagrams of all inner automorphisms σ of all exceptional Lie algebras $\mathfrak{g}$, which induce the regular elements w of the Weyl group W and have the same order m as w. In the third column we list $\mathfrak{g}^\sigma$ and in the last one the dimension of $\mathfrak{h}^w$, if it is non-zero. By Kostant's theorem [K2], there is a unique lift of $w \in W$ if $\mathfrak{h}^w = 0$. However there can be several conjugacy classes of such σ, even of the same order as w, if $\mathfrak{h}^w \neq 0$. We list in the lower parts of the tables the σ of the form $\sigma = \sigma_e$, where e are distinguished nilpotent elements of (regular) semisimple type (for all of them $\mathfrak{h}^w$ must be 0). In the middle part the tables we list σ of the form σ_e, corresponding to remaining nilpotents of regular semisimple type (among exceptional Lie algebras they exist only for E_6 and E_7); for each regular number they appear first in the tables. After them we list those σ of the same order, which induce the same w. Finally, it turns out that in all exceptional Lie algebras (in fact, in all simple Lie algebras, except for $\mathfrak{sl}_n$ with n odd) all regular numbers are divisors of the orders of the elements w_e, corresponding to nilpotents e of regular semisimple type. Hence, taking the appropriate powers of σ_e, we get the diagrams of finite order automorphisms of $\mathfrak{g}$ which induce all the remaining regular elements of W. They are given in the upper parts of the tables. It turns out that in all cases, except for $E_6^{(1)}$ and $E_7^{(1)}$, we have $\mathfrak{h}^w = 0$ for all regular w. In the last column of Tables 6.1 and 6.2 we list $\dim\mathfrak{h}^w$ when it is not 0 (in all Tables 6.3-6.5 it is 0).

It is interesting to note that all elements of the form $e^{\frac{\pi i \rho^\vee}{m}}$, where $\rho^\vee$ is the half of the sum of positive coroots and m is a regular number, are conjugate to an element from the tables, which is the first one for given m. It turns out that the conjugacy classes of the σ represented in the tables contain all m-th order inner automorphisms of $\mathfrak{g}$, whose fixed point set has minimal possible dimension among all m-th order inner automorphisms, cf. [EKV]. Note also that the second element for given m in the tables is a conjugate of the element of the form $e^{\frac{\pi i(\rho^\vee - \omega)}{m}}$

where ω is the fundamental coweight, attached to the branching node of the Dynkin diagram. Of course in the case when the fixed point subalgebra $\mathfrak{g}^\sigma$ is $\mathfrak{h}$, we get the well-known conjugacy class of Coxeter–Kostant elements of order equal to the Coxeter number h.

Table 6.1. Diagrams of regular elements of W_{E_6}

order m	$E_6^{(1)}$	fixed point set	dim $\mathfrak{h}^w$
2	0 1 00000 0	$A_1 \oplus A_5$	2
3	0 00100	$A_2 \oplus A_2 \oplus A_2$	
4	1 0 00100 1	$A_1 \oplus A_2 \oplus A_2 \oplus T_1$	2
4	0 1 0 0 1 0 1	$A_3 \oplus A_1 \oplus T_2$	2
8	1 10101 0	$A_1 \oplus A_1 \oplus T_4$	1
8	1 1 1 0 11	$A_1 \oplus A_1 \oplus T_4$	1
6	1 0 10101 1	$A_1 \oplus A_1 \oplus A_1 \oplus T_3$	
9	1 11011 1	$A_1 \oplus T_5$	
12	1 11111	T_6	

Remark 6.1. The elements $\sigma(m) = e^{\frac{\pi i \rho^\vee}{m}}$, where m is a regular number, are liftings of all regular elements of W for all classical simple Lie algebras $\mathfrak{g}$ as well. In the case $m = h$ it is the Coxeter-Kostant element, whose diagram has all labels equal 1. In addition we have $\sigma(n-1)$ for $\mathfrak{g} = \mathfrak{sl}_n$, whose diagram has one 0 label and all other labels equal 1, and we have $\sigma(n)$ for $\mathfrak{g} = \mathfrak{so}_{2n}$, whose diagram is as follows for n even and odd respectively:

$$
\begin{array}{ccccccc}
 & 1 & & & 1 & & \\
1 & 0 & 1 & 0 & \ldots & 1 & 0 & 1
\end{array}
\qquad
\begin{array}{ccccccc}
 & 1 & & & 1 & & \\
1 & 0 & 1 & 0 & \ldots & 1 & 0 & 1 & 0
\end{array}
$$

The d-th powers of the above elements, where d divide their orders, are liftings of all regular elements of W, up to conjugacy. Removing 1 at the extra node, we obtain the characteristic of

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a nilpotent element of $\mathfrak{so}_{2n}$ of semisimple type, corresponding to the partition $(n+1, n-1)$ if n is even, and to the partition (n, n) if n is odd.

Table 6.2. Diagrams of regular elements of W_{E_7}

order m	$E_7^{(1)}$	fixed point set	$\dim \mathfrak{h}^w$
2	1 0000000 0	A_7	
3	0000100 0	$A_2 \oplus A_5$	1
7	0101001 0	$A_1 \oplus A_1 \oplus A_1 \oplus A_2 \oplus T_2$	1
9	0101011 1	$A_1 \oplus A_1 \oplus A_1 \oplus A_1 \oplus T_3$	1
9	1 100101 0	$A_2 \oplus A_1 \oplus T_4$	1
6	1001001 1	$A_1 \oplus A_2 \oplus A_2 \oplus T_2$	
14	1110111 1	$A_1 \oplus T_6$	
18	1111111	T_7	

Table 6.3. Diagrams of regular elements of W_{E_8}

order m	$E_8^{(1)}$	fixed point set
2	0 00000001	D_8
3	1 00000000	A_8
4	0 00010000	$A_3 \oplus D_5$
5	0 00001000	$A_4 \oplus A_4$
8	0 01000100	$A_1 \oplus A_1 \oplus A_2 \oplus A_3 \oplus T_1$
6	0 10001000	$A_3 \oplus A_4 \oplus T_1$
10	0 10100100	$A_1 \oplus A_1 \oplus A_2 \oplus A_2 \oplus T_2$
12	0 10100101	$A_1 \oplus A_1 \oplus A_1 \oplus A_2 \oplus T_3$
15	0 11010101	$A_1 \oplus A_1 \oplus A_1 \oplus A_1 \oplus T_4$
20	1 11101011	$A_1 \oplus A_1 \oplus T_6$
24	1 11111011	$A_1 \oplus T_7$
30	1 11111111	T_8

Table 6.4. Diagrams of regular elements of W_{F_4}

order m	$F_4^{(1)}$ $\circ - \circ - \circ \Rightarrow \circ - \circ$	fixed point set
2	01000	$A_1 \oplus C_3$
3	00100	$A_2 \oplus A_2$
4	10100	$A_1 \oplus A_2 \oplus T_1$
6	10101	$A_1 \oplus A_1 \oplus T_2$
8	11101	$A_1 \oplus T_3$
12	11111	T_4

Table 6.5. Diagrams of regular elements of W_{G_2}

order m	$G_2^{(1)}$ $\circ - \circ \Rightarrow \circ$	fixed point set
2	0 1 0	$A_1 \oplus A_1$
3	1 1 0	$A_1 \oplus T_1$
6	1 1 1	T_2

Remark 6.2. Let e be a nilpotent element of $\mathfrak{g}$ of semisimple type. Then we can canonically associate with e (rather its conjugacy class) a conjugacy class in W as follows. Let $e + F$ be a generic cyclic element, associated with e, which, by definition, is semisimple. As has been explained in the Introduction, we associate with e a finite order inner automorphism σ_e of $\mathfrak{g}$, with the diagram being the extended Dynkin diagram whose labels s_i on the Dynkin subdiagram are the labels of the characteristic of e (resp. the halves of these labels) if e is an odd (resp. even) nilpotent, and the label s_0 of the extra node is 2 (resp. 1). Let $\mathfrak{a}$ be the semisimple part of the centralizer $\mathfrak{z}(e + F)$ of $e + F$ in $\mathfrak{g}$. Since $e + F$ is an eigenvector of σ_e, the reductive subalgebra $\mathfrak{z}(e + F)$, and hence $\mathfrak{a}$, is σ_e-invariant. Let $\mathfrak{h}_0$ be a Cartan subalgebra of the fixed point set of σ_e in $\mathfrak{a}$ and let $\mathfrak{h}'$ be the centralizer of $\mathfrak{h}_0$ in $\mathfrak{z}(e + F)$. Then by [Ka2], Lemma 8.1b, $\mathfrak{h}'$ is a Cartan subalgebra of $\mathfrak{z}(e+F)$, hence of $\mathfrak{g}$. Therefore σ_e induces an element w_e of $W(\mathfrak{h}')$, which fixes $\mathfrak{h}_0$ pointwise. Hence w_e is independent of the choice of $\mathfrak{h}_0$. We thus obtain a well defined map from the set of conjugacy classes of nilpotent elements of semisimple type in $\mathfrak{g}$ to the set of conjugacy classes in W. This map can be extended to the conjugacy classes of nilpotent elements of "quasi-semisimple" type as follows. By results of Sections 4 and 5, for any nilpotent element e, which is not of regular semisimple type, there exists a proper regular reducing subalgebra $\mathfrak{q}$. If e is of mixed type, it has decomposition $e = e^s + e^n$, where $e^s \in \mathfrak{q}$ is of semisimple type, and $e^n \in \mathfrak{z}(\mathfrak{q})$ is a nilpotent element of lower depth that e. Let $w_1 \in W$ be an element, corresponding to e, as constructed above. If e^n is of mixed type, let $w_2 \in W$ be the element, corresponding to $(e^n)^s$. If e^n is of semisimple type in $\mathfrak{g}$, we let $w = w_1 w_2 \in W$ correspond to e. Otherwise, we repeat this procedure for the element $(e^n)^n$, etc. If this sequence of steps brings us to a nilpotent element of semisimple type, we call e a nilpotent element of quasi-semisimple type. Thus, to any nilpotent conjugacy class of quasi-semisimple type in $\mathfrak{g}$ we have associated a conjugacy class in W. This map is probably closely related to that, constructed in [KL], [L], on the set of all nilpotent classes. Note that in $\mathfrak{sl}_n$ and $\mathfrak{sp}_n$ all nilpotent classes are of quasi-semisimple type, that nilpotent classes of nilpotent type are not of quasi-semisimple type, and only they are not of quasi-semisimple type in all exceptional $\mathfrak{g}$, with the exception of E_8, where there are four more such classes.

Example 6.3. Let $e = e_{-\alpha_0}$ be the root vector attached to the highest root $-\alpha_0$ of a simple Lie algebra $\mathfrak{g}$. Its orbit has minimal dimension among all non-zero nilpotent orbits. The depth of the corresponding $\mathbb{Z}$-grading (0.1) is 2, $\mathfrak{g}_{\pm 2}$ being $\mathbb{F}e_{\mp\alpha_0}$. In all cases e is a nilpotent element of semisimple type. If $\mathfrak{g} \neq \mathfrak{sp}_{2n}$, then e is odd and all the non-zero labels of the diagram of σ_e are: 2 at the extra node and 1 at the adjacent to it nodes; we have: $|\sigma_e| = 4$ and w_e is the reflection with respect to α_0. If $\mathfrak{g} = \mathfrak{sp}_n$, then e is even and all the non-zero labels of the diagram of σ_e are 1 at the two end nodes; we have: $|\sigma_e| = 2$ and w_e is again the reflection with respect to α_0.

Definition 6.4. *We call an element $w \in W$ of order m quasiregular if for any proper divisor s of m the element w^s does not fix a root.*

Lemma 6.5. *If $w \in W$ is regular, then it is quasiregular.*

Proof. This follows from the fact, proved in [S], that the eigenvalue of a regular eigenvector of a regular element w of order m is an m-th primitive root of unity. □

Theorem 6.6. *If w is a quasiregular element of order m in W and it has a lift σ of the same order in the normalizer of $\mathfrak{h}$ in the adjoint group, then*

$$(6.1) \qquad \frac{|\Delta|}{m} = \dim \mathfrak{g}^\sigma - \dim \mathfrak{h}^w \,,$$

where $|\Delta|$ is the number of roots of $\mathfrak{g}$ and $\mathfrak{g}^\sigma$ (resp. $\mathfrak{h}^w$) denotes the fixed point set of σ in $\mathfrak{g}$ (resp. of w in $\mathfrak{h}$).

Proof. Let C (resp. $\widetilde{C}$) be the cyclic group, generated by w (resp. σ). Then, by Lemma 6.5, $|C(\beta)| = m = |\widetilde{C}(e_\beta)|$ for any root β and a root vector e_β, attached to β. Hence $\dim \mathfrak{g}^\sigma = \frac{|\Delta|}{m} + \dim \mathfrak{h}^w$. □

Remark 6.7. By the construction of w_e, we have:

$$\dim \mathfrak{h}^{w_e} \leq \operatorname{rank} \mathfrak{r} \,,$$

where $\mathfrak{r}$ is the reductive part of the centralizer of e in $\mathfrak{g}$. It follows that $\mathfrak{h}^{w_e} = 0$ if e is a distinguished nilpotent of $\mathfrak{g}$. Using formula (6.1) one can compute $\dim \mathfrak{h}^{w_e}$ for any regular $w \in W$.

Remark 6.8. In the cases when a regular $w \in W$ has two liftings, the second one in Tables 6.1 and 6.2 is obtained using good gradings for e, different from Dynkin gradings, found in [EK]. One of the E_6 examples was found in [P2].

The group S of symmetries of the Dynkin diagram of $\mathfrak{g}$ acts on $\mathfrak{h}$ by permuting simple roots, and we may consider the twisted Weyl group $W^{tw} = S \ltimes W$. In [S] Springer also studied the regular elements of $W^{tw} \backslash W$, called twisted regular, and showed that again there is at most one such twisted regular element of given order and found these orders. Lifts of twisted regular elements are outer automorphisms of finite order of $\mathfrak{g}$, hence they are described by labeled twisted extended Dynkin diagrams [Ka2], [OV].

Given a symmetry ν of the Dynkin diagram of a simple Lie algebra, let $\mathfrak{p}$ be its fixed point set. Then the corresponding twisted extended Dynkin diagram is obtained by adding to the Dynkin diagram of $\mathfrak{p}$ one extra node. If now e is an even nilpotent elements of $\mathfrak{p}$, we associate to it a labeling of the twisted extended Dynkin diagram by putting halves of the labels of the characteristic of e in $\mathfrak{p}$ at the nodes of the Dynkin diagram of $\mathfrak{p}$ and 1 at the extra node. If e is of regular semisimple type in $\mathfrak{p}$, we obtain a finite order automorphism σ_e of $\mathfrak{g}$ and the corresponding element w_e of the same order of $W^{tw} \backslash W$ acting on the centralizer of a generic cyclic element $e + F$.

In the following Tables 6.6 and 6.7 we list in the lower part the diagrams of outer finite order automorphisms of E_6 and D_4 from the connected component of ν of order 2 and 3 respectively, which induce the regular elements of $W^{tw} \backslash W$, corresponding to distinguished nilpotent elements of $\mathfrak{p}$, and in the upper part the lifts of the remaining twisted regular elements.

Table 6.6 Diagrams of regular elements of $W_{E_6}^{tw}$

order m	$E_6^{(2)}$ ∘ − ∘ − ∘ ⇐ ∘ − ∘	fixed point set	$\dim \mathfrak{h}^w$
2	0 0 0 0 1	C_4	
4	0 0 0 1 0	$A_3 \oplus A_1$	
8	1 0 0 1 1	$A_2 + T_2$	1
8	0 1 0 1 0	$A_1 \oplus A_1 \oplus A_1 \oplus T_1$	1
6	1 0 0 1 0	$A_2 \oplus A_1 \oplus T_1$	
12	1 1 0 1 1	$A_1 \oplus T_3$	
18	1 1 1 1 1	T_4	

Table 6.7. Diagrams of regular elements of $W_{D_4}^{tw}$

order m	$D_4^{(3)}$ ∘ − ∘ ⇚ ∘	fixed point set
3	0 0 1	A_2
6	1 0 1	$A_1 \oplus T_1$
12	1 1 1	T_2

Remark 6.9. Theorem 6.6 holds also for twisted regular elements. It follows that for all cases listed in Tables 6.6 and 6.7 we have $\mathfrak{h}^w = 0$, except for the element w of order 8 in $W_{E_6}^{tw}$, when $\dim \mathfrak{h}^w = 1$. The latter element is the only twisted regular element which is not a power of an element of the form σ_e, but the diagrams of its liftings can be easily obtained using formula (6.1) (one can see that $\dim \mathfrak{h}^w = 1$ from Table 8 of [S]). It turns out that again, all automorphisms are of the form $\nu e^{\frac{\pi i \rho^\vee}{m}}$, where m is from the first column, occur in these tables (in the case $m = 8$ it is the first one).

Proposition 6.10. *Let $G^0|\mathfrak{g}^1$ be a theta group, attached to an inner finite order automorphism σ of $\mathfrak{g}$, such that there exists $v \in \mathfrak{g}^1$ whose G^0-orbit is closed and the stabilizer G_v^0 is finite. Then the centralizer $\mathfrak{h} := \mathfrak{g}_v$ of v in $\mathfrak{g}$ is a Cartan subalgebra of $\mathfrak{g}$, hence σ induces an element w of the Weyl group W, attached to $\mathfrak{h}$. Moreover, w is a regular element of W and $\mathfrak{h}^w = 0$. Conversely, all regular elements of W with zero fixed point set on $\mathfrak{h}$ are obtained in this way.*

Proof. Since the G^0-orbit of v is closed, v is a semisimple element of $\mathfrak{g}$ [V], hence its centralizer $\mathfrak{g}_v$ is reductive. The automorphism σ induces a finite order automorphism of $\mathfrak{g}_v$ with zero fixed point set. Hence $\mathfrak{h} := \mathfrak{g}_v$ is a Cartan subalgebra [Ka1], Chapter 8, and v is a regular element of $\mathfrak{g}$, so σ induces a regular Weyl group element w on $\mathfrak{h}$. We have $\mathfrak{h}^w = 0$ since the fixed point set of σ on $\mathfrak{g}_v$ is zero.

Conversely, if w is a regular element of W, with a regular eigenvector $v \in \mathfrak{h}$ and such that $\mathfrak{h}^w = 0$, then its (unique, up to conjugacy) lift σ in G obviously defines a theta group $G^0|\mathfrak{g}^1$ for which v has a closed G^0-orbit with a finite stabilizer. $\square$

Remark 6.11. All theta groups with properties, described in Proposition 6.10, have been independently found in a recent paper [G] on classification of theta groups of positive rank. Their tables are in complete agreement with our Tables 5.1–5.4 and 6.6, 6.7, as predicted by Proposition 6.10.

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