

DENOMINATOR IDENTITIES FOR FINITE-DIMENSIONAL LIE SUPERALGEBRAS AND HOWE DUALITY FOR COMPACT DUAL PAIRS

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ABSTRACT. We provide formulas for the denominator and superdenominator of a basic classical type Lie superalgebra for any set of positive roots. We establish a connection between certain sets of positive roots and the theory of reductive dual pairs of real Lie groups, and, as an application of these formulas, we recover the Theta correspondence for compact dual pairs. Along the way we give an explicit description of the real forms of basic classical type Lie superalgebras.

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1. INTRODUCTION

The Weyl denominator identity

$$(1.1) \quad \prod_{\alpha \in \Delta^+} (1 - e^{-\alpha}) = \sum_{w \in W} \text{sgn}(w) e^{w(\rho) - \rho}$$

is one of the most intriguing combinatorial identities in the character ring of a complex finite-dimensional simple Lie algebra. It admits far reaching generalizations to the Kac-Moody setting, where it provides a proof for the Macdonald's identities (see [11]). Its role in representation theory is well-understood, since the inverse of the l.h.s. of (1.1) is the character of the Verma module $M(0)$ with the highest weight 0.

In the first part of this paper we provide a generalization of formula (1.1) to the setting of basic classical type Lie superalgebras. Deepening this problem, we came across an interesting connection with representation theory of Lie groups which is the theme of the second part of the paper.

By a basic classical type Lie superalgebra we mean an almost simple finite-dimensional Lie superalgebra $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ with a non-degenerate invariant supersymmetric bilinear form (\cdot, \cdot) and \mathfrak{g}_0 reductive. Choosing a Cartan subalgebra \mathfrak{h} of \mathfrak{g}_0 , we get the set of roots $\Delta = \Delta_0 \cup \Delta_1$, where Δ_i is the set of roots of \mathfrak{h} in \mathfrak{g}_i , $i = 0, 1$. Choosing a set of positive roots Δ^+ in Δ , we let $\Delta_i^+ = \Delta^+ \cap \Delta_i$. In trying to extend (1.1) to a Lie superalgebra \mathfrak{g} , it is natural to replace the l.h.s of (1.1) with the character of the Verma module $M(0)$ over \mathfrak{g} , which is the inverse of

$$(1.2) \quad R = \frac{\prod_{\alpha \in \Delta_0^+} (1 - e^{-\alpha})}{\prod_{\alpha \in \Delta_1^+} (1 + e^{-\alpha})},$$

called the *denominator*. Beyond the denominator R , very important for us will be the *superdenominator*, defined as

$$(1.3) \quad \check{R} = \frac{\prod_{\alpha \in \Delta_0^+} (1 - e^{-\alpha})}{\prod_{\alpha \in \Delta_1^+} (1 - e^{-\alpha})}.$$

Generalizations of formulas for R and \check{R} to affine superalgebras and their connection with number theory and the theory of special functions are thoroughly discussed in [14]. The striking differences which make the super case very different from the purely even one are the following. First, it is no more true that subsets of positive roots are conjugate under the Weyl group: to get transitivity on the sets of positive roots one has to consider Serganova's odd reflections, see (2.5). Moreover the restriction of the nondegenerate invariant bilinear form (\cdot, \cdot) to the real span V_Δ of roots may be indefinite, hence isotropic sets of roots appear naturally. One defines the defect $d = \text{def } \mathfrak{g}$ as the dimension of a maximal isotropic subspace of V_Δ . A subset of Δ , consisting of linearly independent pairwise orthogonal isotropic roots is called *isotropic*. It is known that any maximal isotropic subset of Δ consists of d roots ([14]).

In this paper we settle completely the problem of finding an analogue of (1.1) for basic classical type Lie superalgebras, by providing an expression for the r.h.s. which incorporates the dependence on the set of positive roots. The following result is known.

Theorem KWG. *Let \mathfrak{g} be a basic classical type Lie superalgebra and let Δ^+ be a set of positive roots such that there exists a maximal isotropic subset S of Δ^+ , contained in the set of simple roots corresponding to Δ^+ . Then*

$$(1.4) \quad e^\rho R = \sum_{w \in W^\#} \operatorname{sgn}(w) w \left(\frac{e^\rho}{\prod_{\beta \in S} (1 + e^{-\beta})} \right),$$

$$(1.5) \quad e^\rho \check{R} = \sum_{w \in W^\#} \operatorname{sgn}'(w) w \left(\frac{e^\rho}{\prod_{\beta \in S} (1 - e^{-\beta})} \right).$$

In the above statement $\rho = \rho_0 - \rho_1$, where $\rho_i = \frac{1}{2} \sum_{\alpha \in \Delta_i^+} \alpha$, $W^\#$ is a subgroup of the Weyl group $W_{\mathfrak{g}}$ of \mathfrak{g} defined in (2.1) and sgn' is defined in (2.3). This result has been stated by Kac and Wakimoto in [14], and fully proved for $d = 1$ by using representation theoretical methods. A combinatorial proof for arbitrary d has been recently obtained by Gorelik [4].

Theorem KWG will be of fundamental importance for our generalization, which is as follows: given any subset of positive roots Δ^+ , we want to express $e^\rho R$ by a formula like the r.h.s. of (1.4), in which at the numerator appears the ρ corresponding to Δ^+ and at the denominator a suitable maximal isotropic subset S of Δ^+ , so that the exponents at the denominator are linear combinations with non-positive integer coefficients of simple roots: see (1.10), (1.11). We also provide a formula in which the denominator is exactly as in the r.h.s. of (1.4), but we have to perform a correction on ρ , depending on S : see (1.15).

Let us discuss more in detail these formulas. First we construct a certain class \mathcal{S} of maximal isotropic subsets by the following procedure.

Definition 1.1. *Denote by $\mathcal{S}(\Delta^+)$ the collection of maximal isotropic subsets of Δ^+ of the form $S = S_1 \cup \dots \cup S_t$ of Δ^+ , where S_1 is a non empty isotropic subset of the set of simple roots, and, inductively, S_i is such a subset in the set of indecomposable roots of $S_{i-1}^\perp \setminus S_{i-1}$, where $S_{i-1}^\perp = \{\alpha \in \Delta^+ \mid (\alpha, \beta) = 0 \forall \beta \in S_{i-1}\}$.*

Denote by Q, Q_0 the lattices spanned over \mathbb{Z} by all roots and even roots, respectively, and let, for $S \in \mathcal{S}(\Delta^+)$ and $\gamma \in S$

$$(1.6) \quad \varepsilon(\eta) = \begin{cases} 1 & \text{if } \eta \in Q_0, \\ -1 & \text{if } \eta \in Q \setminus Q_0, \end{cases}$$

$$(1.7) \quad \gamma^\leq = \{\beta \in S \mid \beta \leq \gamma\}, \quad \gamma^< = \{\beta \in S \mid \beta < \gamma\},$$

$$(1.8) \quad \llbracket \gamma \rrbracket = \sum_{\beta \in \gamma^\leq} \varepsilon(\gamma - \beta) \beta, \quad \llbracket \gamma \rrbracket = \sum_{\beta \in \gamma^<} \varepsilon(\gamma - \beta) \beta,$$

$$(1.9) \quad \operatorname{sgn}(\gamma) = (-1)^{|\gamma^\leq|+1},$$

where, as usual, $\beta \leq \gamma$ if $\gamma - \beta$ is a sum of positive roots or zero and $|X|$ stands for the cardinality of the set X .

Theorem 1.1. *Let $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be a basic classical type Lie superalgebra having defect d , where $\mathfrak{g} = A(d-1, d-1)$ is replaced by $gl(d, d)$. Let Δ^+ be any set of*

positive roots. For any $S \in \mathcal{S}(\Delta^+)$ we have

$$(1.10) \quad C \cdot e^\rho R = \sum_{w \in W_{\mathfrak{g}}} \operatorname{sgn}(w) w \left(\frac{e^\rho}{\prod_{\gamma \in S} (1 + \operatorname{sgn}(\gamma) e^{-\llbracket \gamma \rrbracket})} \right),$$

$$(1.11) \quad C \cdot e^\rho \check{R} = \sum_{w \in W_{\mathfrak{g}}} \operatorname{sgn}'(w) w \left(\frac{e^\rho}{\prod_{\gamma \in S} (1 - e^{-\llbracket \gamma \rrbracket})} \right),$$

where

$$(1.12) \quad C = \frac{C_{\mathfrak{g}}}{\prod_{\gamma \in S} \frac{ht(\gamma)+1}{2}},$$

and $C_{\mathfrak{g}} = |W_{\mathfrak{g}}/W^{\sharp}|$. Moreover, there exists $S^{\text{nice}} \in \mathcal{S}(\Delta^+)$ such that $\llbracket \gamma \rrbracket \in Q^+ = \mathbb{Z}_+ \Delta^+$ for any $\gamma \in S^{\text{nice}}$.

The explicit values of $C_{\mathfrak{g}}$ are the following.

$$(1.13) \quad C_{\mathfrak{g}} = \begin{cases} d! & \text{if } \mathfrak{g} = A(m-1, n-1), \\ 2^d d! & \text{if } \mathfrak{g} = B(m, n), \\ 2^d d! & \text{if } \mathfrak{g} = D(m, n), m > n, \\ 2^{d-1} d! & \text{if } \mathfrak{g} = D(m, n), n \geq m, \\ 1 & \text{if } \mathfrak{g} = C(n), \\ 2 & \text{if } \mathfrak{g} = D(2, 1, \alpha), F(4), G(3). \end{cases}$$

If $\operatorname{def}(\mathfrak{g}) = 1$, and $S \in \mathcal{S}(\Delta^+)$, then S consists of a single simple root. Hence we are in the hypothesis of Theorem KWG, in which case (1.10), (1.11) hold with $C = C_{\mathfrak{g}}$, and these formulas coincide with (1.4), (1.5). Therefore we have to deal only with Lie superalgebras of type $gl(m, n)$, $B(m, n)$, $D(m, n)$, which have defect $d = \min\{m, n\}$. We also treat the case $A(d-1, d-1)$, see Subsection 3.2.2.

In these cases we introduce a combinatorial encoding of the elements of $\mathcal{S}(\Delta^+)$ using the notion of an arc diagram (see Definition 3.1). To any arc diagram X we associate a maximal isotropic set $S(X)$, and in Proposition 3.2 we show that this is a bijection between all arc diagrams and $\mathcal{S}(\Delta^+)$.

We introduce two types of operations on arc diagrams: odd reflections r_v and interval reflections $r_{[v,w]}$. They have the following features:

- (1) for any arc diagram X there exists a finite sequence of odd and interval reflections which change X into an arc diagram X' such that $S(X')$ consists of simple roots (cf. Lemma 3.4);
- (2) we are able to relate $\sum_{w \in W} \operatorname{sgn}(w) w \left(\frac{e^\rho}{\prod_{\gamma \in S(X)} (1 - e^{-\llbracket \gamma \rrbracket})} \right)$ to the similar sums where the product in the denominator of the l.h.s ranges over $r_v X$ and over $r_{[v,w]} X$ (cf. (3.7)).

The above properties allow to prove formula (1.11) starting from Theorem KWG applied to $S(X')$. In Lemma 3.7 we show that formula (1.10) can be derived from (1.11). We also single out a special set of arc diagrams (see Definition 3.2) which have the property described in the last sentence of Theorem 1.1.

Further, we provide a different generalization of (1.5), in which only positive roots appear in the denominator, namely we have

Proposition 1.2. *Let $S \in \mathcal{S}(\Delta^+)$. Then*

$$(1.14) \quad e^\rho R = \sum_{w \in W^\#} \operatorname{sgn}(w) w \left(\frac{(-1)^{\sum_{\gamma \in S} |\gamma^<|} e^{\rho + \sum_{\gamma \in S} \mathbb{I}\gamma\mathbb{I}}}{\prod_{\beta \in S} (1 + e^{-\beta})} \right),$$

$$(1.15) \quad e^\rho \check{R} = \sum_{w \in W^\#} \operatorname{sgn}'(w) w \left(\frac{e^{\rho + \sum_{\gamma \in S} \mathbb{I}\gamma\mathbb{I}}}{\prod_{\beta \in S} (1 - e^{-\beta})} \right).$$

The main application of our formulas is a conceptually uniform derivation of the Theta correspondence for compact dual pairs. Basic classical type Lie superalgebras are linked to Howe theory of dual pairs through the notion of a *distinguished* set of positive roots, which we introduce in Definition 4.1: it requires that the depth of the grading, which assigns value 1 (resp. 0) to each generator corresponding to a positive odd (resp. even) simple root, does not exceed 2. The distinguished sets of positive roots for the basic classical type Lie superalgebras (which are not Lie algebras) are classified in Subsection 4.4.

To understand the relationship with Howe theory, consider the complex symplectic space $(\mathfrak{g}_1, \langle \cdot, \cdot \rangle)$, where $\langle \cdot, \cdot \rangle = (\cdot, \cdot)|_{\mathfrak{g}_1}$. A choice of a set of positive roots Δ^+ determines a polarization $\mathfrak{g}_1 = \mathfrak{g}_1^+ \oplus \mathfrak{g}_1^-$, where $\mathfrak{g}_1^\pm = \bigoplus_{\alpha \in \Delta_1^\pm} \mathfrak{g}_\alpha$. Hence we can consider the Weyl algebra $W(\mathfrak{g}_1)$ of $(\mathfrak{g}_1, \langle \cdot, \cdot \rangle)$ and construct the $W(\mathfrak{g}_1)$ -module

$$(1.16) \quad M^{\Delta^+}(\mathfrak{g}_1) = W(\mathfrak{g}_1)/W(\mathfrak{g}_1)\mathfrak{g}_1^+,$$

with action by left multiplication. The module $M^{\Delta^+}(\mathfrak{g}_1)$ is also a $sp(\mathfrak{g}_1, \langle \cdot, \cdot \rangle)$ -module with $T \in sp(\mathfrak{g}_1, \langle \cdot, \cdot \rangle)$ acting by left multiplication by

$$(1.17) \quad \theta(T) = -\frac{1}{2} \sum_{i=1}^{\dim \mathfrak{g}_1} T(x_i) x^i,$$

where $\{x_i\}$ is any basis of \mathfrak{g}_1 and $\{x^i\}$ is its dual basis w.r.t. $\langle \cdot, \cdot \rangle$, i.e. $\langle x_i, x^j \rangle = \delta_{ij}$. It is easy to check that, in $W(\mathfrak{g}_1)$, relation

$$(1.18) \quad [\theta(T), x] = T(x)$$

holds for any $x \in \mathfrak{g}_1$. This implies that we have an \mathfrak{h} -module isomorphism

$$(1.19) \quad M^{\Delta^+}(\mathfrak{g}_1) \cong S(\mathfrak{g}_1^-) \otimes \mathbb{C}_{-\rho_1}$$

where $S(\mathfrak{g}_1^-)$ is the symmetric algebra of \mathfrak{g}_1^- , and $\mathbb{C}_{-\rho_1}$ is the 1-dimensional \mathfrak{h} -module with highest weight $-\rho_1$. Hence the \mathfrak{h} -character of $M^{\Delta^+}(\mathfrak{g}_1)$ is given by

$$(1.20) \quad \operatorname{ch} M^{\Delta^+}(\mathfrak{g}_1) = \frac{e^{-\rho_1}}{\prod_{\alpha \in \Delta_1^+} (1 - e^{-\alpha})}.$$

Since $\operatorname{ad}_{\mathfrak{g}_1}(\mathfrak{g}_0) \subset sp(\mathfrak{g}_1, \langle \cdot, \cdot \rangle)$, we obtain an action of \mathfrak{g}_0 on $M^{\Delta^+}(\mathfrak{g}_1)$. Upon multiplication by $e^{\rho_0} \prod_{\alpha \in \Delta_0^+} (1 - e^{-\alpha})$ the r.h.s. of (1.20) becomes $e^\rho \check{R}$ and equating it to our formula we obtain the \mathfrak{g}_0 -character of $M^{\Delta^+}(\mathfrak{g}_1)$.

So far we have not used the special features of distinguished sets of positive roots Δ^+ . Restricting to distinguished sets of positive roots Δ^+ for Lie superalgebras

of type $gl(m, n)$, $B(m, n)$, $D(m, n)$, we are able to build up a real form V of \mathfrak{g}_1 endowed with a standard symplectic basis $\{e_\alpha, f_\alpha\}_{\alpha \in \Delta_1^+}$ such that

$$\bigoplus_{\alpha \in \Delta_1^+} \mathbb{C}(e_\alpha \pm \sqrt{-1}f_\alpha) = \mathfrak{g}_1^\pm.$$

It turns out that $sp(V) \cap ad_{|\mathfrak{g}_1}(\mathfrak{g}_0) = \mathfrak{s}_1 \times \mathfrak{s}_2$, \mathfrak{s}_i , $i = 1, 2$, being the Lie algebras of a compact dual pair in $Sp(V)$. As stated in Proposition 4.8, distinguished sets of positive roots turn out to correspond in this way to all compact dual pairs. Howe theory and our denominator formula allow to recover the explicit computation of the Theta correspondence done by Kashiwara-Vergne [15] and Li-Paul-Tan-Zhu [17] and related results by Enright [2]. Before discovering formula (1.10), we used this argument the other way around, to deduce the denominator identities from the knowledge of the Theta correspondence: the relevant details are given in [13].

In Section 4.8 we show that relaxing the condition on the depth of the grading which defines the distinguished sets of positive roots it is possible to generalize the previous approach to all but one noncompact type I dual pairs (cf. Proposition 4.2). We plan to investigate in a subsequent paper if our methods can be pushed further to obtain information on the Theta correspondence in these cases too.

In conclusion of the paper we use our superdenominator formula to confirm the validity of the Kac-Wakimoto conjecture [14] for the natural representations of the classical Lie superalgebras. We intend to extend this method to a large class of representations in a subsequent publication.

The paper is organized as follows. In Section 2 we collect some basic notation and definitions. In Section 3 we develop the combinatorial machinery needed to prove our superdenominator formulas: we introduce arc diagrams and study the effect of odd and interval reflection on arc diagrams. We finally prove the superdenominator formula for the non-exceptional basic classical type Lie superalgebras and also prove the denominator formula. The final subsection is devoted to the proof of formula (1.15). In Section 4 we relate the distinguished sets of positive roots to Howe theory: after a preliminary discussion on Cartan involutions we introduce and analyze the definition of distinguished set of positive roots. Then we relate distinguished sets of positive roots to real forms: as a byproduct of our analysis we obtain a conceptual proof of the classification of real forms of basic classical type Lie superalgebras which is in the same spirit as the classification of simple Lie algebra involutions, given in [11]. This classification has been previously obtained by Kac, Parker and Serganova, (cf. [9], [18], [21]). Finally we provide a concise discussion about certain sets of positive roots related to noncompact dual pairs. In the following four Sections we deduce from our denominator formula the explicit form of the Theta correspondence for all compact dual pairs. It is worthwhile to note that the pair $(O(2m), Sp(2n, \mathbb{R}))$ has additional complications which require to use a version of the character formula for disconnected compact groups due to Kostant [16]. In Section 9 we prove the Kac-Wakimoto conjecture for the natural representation.

The ground field is \mathbb{C} throughout the paper, unless otherwise specified.

2. SETUP

In this section we collect some notation and definitions which will be constantly used throughout the paper. Let \mathfrak{g} be a basic classical type Lie superalgebra. This

means that $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ is an almost simple finite-dimensional Lie superalgebra with reductive \mathfrak{g}_0 and that \mathfrak{g} admits a nondegenerate invariant supersymmetric bilinear form (\cdot, \cdot) . Almost simple means that $[\mathfrak{g}, \mathfrak{g}]$ modulo its center is simple. Supersymmetric means that $(\mathfrak{g}_0, \mathfrak{g}_1) = 0$ and the restriction of (\cdot, \cdot) to \mathfrak{g}_0 (resp. \mathfrak{g}_1) is symmetric (resp. skewsymmetric). A complete list of the simple ones consists of four series $A(m, n)$, $B(m, n)$, $C(n)$, $D(m, n)$ and three exceptional Lie superalgebras $D(2, 1, \alpha)$, $F(4)$, $G(3)$, and the non-simple ones are obtained by adding to a simple one or to $gl(n+1, n+1)$ a central ideal. These Lie superalgebras can be constructed, starting with a Cartan matrix, like simple finite-dimensional Lie algebras, though inequivalent Cartan matrices may correspond to the same algebra (see [9]).

Choose a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}_0$, and let $\Delta, \Delta_0, \Delta_1 \subset \mathfrak{h}^*$ be the set of roots, even roots, odd roots, respectively. Let $W_{\mathfrak{g}} \subset GL(\mathfrak{h}^*)$ be the group generated by the reflections s_{α} w.r.t. *even* roots $\alpha \in \Delta_0$. Choose a set of positive roots $\Delta^+ \subset \Delta$ and set $\Delta_i^+ = \Delta_i \cap \Delta^+$, $i = 0, 1$. Let Π be the set of simple roots corresponding to the choice of Δ^+ , i.e. the set of indecomposable roots in Δ^+ . Set also, as usual, for $i = 0, 1$, $\rho_i = \frac{1}{2} \sum_{\alpha \in \Delta_i^+} \alpha$, $\rho = \rho_0 - \rho_1$. From time to time we will write $\rho(\Delta^+)$ or $\rho(\Pi)$ to emphasize the dependence of ρ from the choice of the set of positive roots.

Next, recall the notation skipped in the Introduction. Let h^{\vee} be the dual Coxeter number of \mathfrak{g} , i.e. $2h^{\vee}$ is the eigenvalue of the Casimir operator of $(\mathfrak{g}, (\cdot, \cdot))$ in the adjoint representation. If $h^{\vee} \neq 0$ (which holds unless \mathfrak{g} is of type $A(n, n)$, $D(n+1, n)$ or $D(2, 1, \alpha)$), we let [14]

$$(2.1) \quad \begin{aligned} \Delta_0^{\sharp} &= \{\alpha \in \Delta_0 \mid h^{\vee}(\alpha, \alpha) > 0\}, \\ W^{\sharp} &= \langle s_{\beta} \in W_{\mathfrak{g}} \mid \beta \in \Delta_0^{\sharp} \rangle, \end{aligned}$$

We refer to [14, Remark 1.1, b)] for the definition of W^{\sharp} when $h^{\vee} = 0$. Set

$$(2.2) \quad \overline{\Delta}_0 = \{\alpha \in \Delta_0 \mid \frac{1}{2}\alpha \notin \Delta\}, \quad \overline{\Delta}_1 = \{\alpha \in \Delta_1 \mid (\alpha, \alpha) = 0\}.$$

Finally, for $w \in W_{\mathfrak{g}}$, set

$$(2.3) \quad \text{sgn}(w) = (-1)^{\ell(w)}, \quad \text{sgn}'(w) = (-1)^m,$$

where ℓ is the usual length function on $W_{\mathfrak{g}}$ and m is the number of reflections from $\overline{\Delta}_0^+$ occurring in an expression of w . Note that

$$w(e^{\rho} \check{R}) = \text{sgn}'(w) e^{\rho} \check{R}.$$

In particular, sgn' is well-defined. Note also that $\text{sgn} = \text{sgn}'$ in types A and D .

Let now recall the definition of odd reflections [19]: for an isotropic root $\alpha \in \Pi$ we define

$$(2.4) \quad r_{\alpha}(\Delta^+) = (\Delta^+ \setminus \{\alpha\}) \cup \{-\alpha\}.$$

It is easy to prove that $r_{\alpha}(\Delta^+)$ is a set of positive roots for \mathfrak{g} and that if we set

$$(2.5) \quad r_{\alpha}(\beta) = \begin{cases} \alpha + \beta & \text{if } (\alpha, \beta) \neq 0 \\ \beta & \text{if } (\alpha, \beta) = 0, \alpha \neq \beta \\ -\alpha & \text{if } \beta = \alpha \end{cases}$$

for $\beta \in \Pi$, then $r_{\alpha}(\Pi) = \{r_{\alpha}(\beta) \mid \beta \in \Pi\}$ is the corresponding set of simple roots.

The importance of odd reflections lies in the fact that, up to $W_{\mathfrak{g}}$ -action, any two sets of positive roots can be obtained one from the other by applying a finite sequence of odd reflections. Note that, for an isotropic α ,

$$(2.6) \quad \rho(r_{\alpha}(\Delta^+)) = \rho(\Delta^+) + \alpha,$$

from which one deduces that

$$(2.7) \quad e^{\rho(\Delta^+)} \check{R}(\Delta^+) = -e^{\rho(r_{\alpha}(\Delta^+))} \check{R}(r_{\alpha}(\Delta^+)) \quad \forall \alpha \in \Pi.$$

3. DENOMINATOR FORMULAS

3.1. Arc diagrams. Let \mathfrak{g} be a Lie superalgebra of type $gl(m, n), B(m, n), C(m), D(m, n)$. We first need an encoding of the sets of positive roots. The explicit realizations of these Lie superalgebras given in [9], [10, Section 4] leads to a description of their roots in terms of functionals $\epsilon_i, \delta_i \in \mathfrak{h}^*$.

We let

$$\mathcal{E} = \{\epsilon_1, \dots, \epsilon_m\}, \quad \mathcal{D} = \{\delta_1, \dots, \delta_n\}$$

if \mathfrak{g} is of type $gl(m, n), B(m, n)$ and let

$$\mathcal{E} = \{\epsilon_1, \dots, \epsilon_{m-1}, \epsilon_m\} \text{ or } \mathcal{E} = \{\epsilon_1, \dots, \epsilon_{m-1}, -\epsilon_m\}, \quad \mathcal{D} = \{\delta_1, \dots, \delta_n\}$$

if \mathfrak{g} is of type $C(m)$ or $D(m, n)$. Let

$$\mathcal{B} = \mathcal{E} \cup \mathcal{D}.$$

Then \mathcal{B} is a basis of \mathfrak{h}^* (for $C(m)$ one has $n = 1$). We call two elements $v_1, v_2 \in \mathcal{B}$ *elements of the same type* if $\{v_1, v_2\} \subset \mathcal{E}$ or $\{v_1, v_2\} \subset \mathcal{D}$ and *elements of different types* otherwise.

Recall the structure of Δ in terms of \mathcal{B} : in type $gl(m, n)$ we have

$$\begin{aligned} \Delta_0 &= \{\epsilon_i - \epsilon_j \mid 1 \leq i \neq j \leq m\} \cup \{\delta_i - \delta_j \mid 1 \leq i \neq j \leq n\}, \\ \Delta_1 &= \{\pm(\epsilon_i - \delta_j) \mid 1 \leq i \leq m, 1 \leq j \leq n\}. \end{aligned}$$

As an invariant bilinear form on $gl(m, n)$ we choose the supertrace form $(a, b) = str(ab)$, str being the supertrace of a matrix in $gl(m, n)$, so that

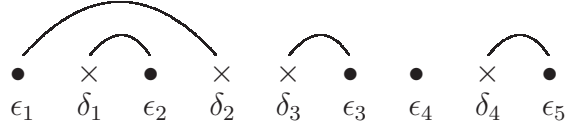
$$(\epsilon_i, \epsilon_j) = \delta_{ij} = -(\delta_i, \delta_j).$$

In the other classical types see (5.1), (6.1) for the description of roots; the invariant bilinear form is the restriction of the supertrace form.

Given a total order $>$ on $\mathcal{B} = \{\xi_1 > \dots > \xi_{m+n}\}$, we define a set of simple roots $\Pi(\mathcal{B}, >)$ for \mathfrak{g} as follows

\mathfrak{g}	$\Pi(\mathcal{B}, >)$
$gl(m, n)$	$\{\xi_i - \xi_{i+1}\}_{i=1}^{m+n-1}$
$B(m, n)$	$\{\xi_i - \xi_{i+1}\}_{i=1}^{m+n-1} \cup \{\xi_{m+n}\}$
$C(m)$	$\{\xi_i - \xi_{i+1}\}_{i=1}^m \cup \{2\xi_{m+1}\}$ if $\xi_{m+1} \in \mathcal{E}$
	$\{\xi_i - \xi_{i+1}\}_{i=1}^m \cup \{\xi_m + \xi_{m+1}\}$ if $\xi_{m+1} \in \mathcal{D} = \{\delta_1\}$
$D(m, n)$	$\{\xi_i - \xi_{i+1}\}_{i=1}^{m+n-1} \cup \{2\xi_{m+n}\}$ if $\xi_{m+n} \in \mathcal{D}$
	$\{\xi_i - \xi_{i+1}\}_{i=1}^{m+n-1} \cup \{\xi_{m+n-1} + \xi_{m+n}\}$ if $\xi_{m+n} \in \mathcal{E}$.

Using Kac's description of Borel subalgebras (see [9]), it is not difficult to prove the following result.


 FIGURE 1. An arc diagram X for $gl(5, 4)$.

Lemma 3.1. *Up to $W_{\mathfrak{g}}$ -equivalence, any set of positive roots for \mathfrak{g} has $\Pi(\mathcal{B}, >)$ as a set of simple roots for some total order $>$ on \mathcal{B} .*

It may happen in type $D(m, n)$ that two different choices of \mathcal{B} give rise to the same set of simple roots Π , so we make the following choice. Let $h \in \mathfrak{h}$ be such that $\alpha(h) = 1$ for each $\alpha \in \Pi$. Choose $\mathcal{B} = \{\epsilon_1, \dots, \epsilon_m\} \cup \{\delta_1, \dots, \delta_n\}$ if $\epsilon_m(h) \geq 0$, and $\mathcal{B} = \{\epsilon_1, \dots, -\epsilon_m\} \cup \{\delta_1, \dots, \delta_n\}$ if $\epsilon_m(h) < 0$.

Instead of the Dynkin diagram, we encode $\Pi(\mathcal{B}, >)$ by the ordered sequence \mathcal{B} , which is pictorially represented as an array of dots and crosses, the former corresponding to vertices in \mathcal{E} and the latter to vertices in \mathcal{D} .

Examples for $gl(5, 4)$, $D(4, 3)$ are given in Figures 1, 2, 3 disregarding the arcs. In the first case, we start from the total order $\{\epsilon_1 > \delta_1 > \epsilon_2 > \delta_2 > \delta_3 > \epsilon_3 > \epsilon_4 > \delta_4 > \epsilon_5\}$, which corresponds to the set of simple roots $\{\epsilon_1 - \delta_1, \delta_1 - \epsilon_2, \epsilon_2 - \delta_2, \delta_2 - \delta_3, \delta_3 - \epsilon_3, \epsilon_3 - \epsilon_4, \epsilon_4 - \delta_5\}$.

In Figure 2, we start from the total order $\epsilon_1 > \delta_1 > \epsilon_2 > \delta_2 > \epsilon_3 > \delta_3 > \epsilon_4$, which gives rise to the set of simple roots $\{\epsilon_1 - \delta_1, \delta_1 - \epsilon_2, \epsilon_2 - \delta_2, \delta_2 - \epsilon_3, \epsilon_3 - \delta_3, \delta_3 \pm \epsilon_4\}$, while in Figure 3 the total order is $\epsilon_1 > \delta_1 > \epsilon_2 > \delta_2 > \epsilon_3 > -\epsilon_4 > \delta_4$, which corresponds to the set of simple roots $\{\epsilon_1 - \delta_1, \delta_1 - \epsilon_2, \epsilon_2 - \delta_2, \delta_2 - \epsilon_3, \epsilon_3 + \epsilon_4, -\epsilon_4 \pm \delta_3\}$.

For $v, w \in \mathcal{B}$, if $v \geq w$, let $[v, w] = \{u \in \mathcal{B} \mid v \geq u \geq w\}$. If $\mathcal{B}' \subset \mathcal{B}$, we denote by $W_{\mathcal{B}'}$ the subgroup of $W_{\mathfrak{g}}$ consisting of (non-signed) permutations of $\mathcal{B}' \cap \mathcal{E}$ and of $\mathcal{B}' \cap \mathcal{D}$, so that we have $W_{\mathcal{B}'} \cong S_k \times S_l$, where $|\mathcal{B}' \cap \mathcal{E}| = k$, $|\mathcal{B}' \cap \mathcal{D}| = l$.

We now introduce arc diagrams. These are combinatorial data that encode some maximal isotropic subsets of the set of positive roots.

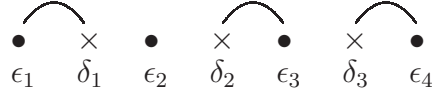
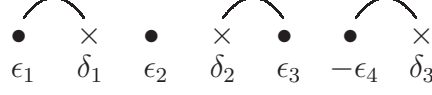
Definition 3.1. *An arc diagram is the datum consisting of the ordered sequence of vertices representing \mathcal{B} , and of arcs between some of the vertices, satisfying the following properties:*

- (i) *the vertices at the ends of each arc are of different type;*
- (ii) *the arcs do not intersect (including the end points);*
- (iii) *for each arc \widehat{vw} the interval $[v, w]$ contains the same number of elements of \mathcal{E} and of \mathcal{D} : $|[v, w] \cap \mathcal{E}| = |[v, w] \cap \mathcal{D}|$;*
- (iv) *the number of arcs is $\min(m, n)$.*

We also define the *support* of an arc diagram X as

$$\text{Supp}(X) = \{v \in \mathcal{B} \mid v \text{ is an end of an arc in } X\}.$$

If Δ^+ is the set of positive roots corresponding to $\Pi(\mathcal{B}, >)$, we denote by $\mathcal{A}(\Delta^+)$ the set of arc diagrams whose underlying set of vertices is \mathcal{B} and the underlying total order is $>$.

FIGURE 2. An arc diagram for $D(4, 3)$.FIGURE 3. Another arc diagram for $D(4, 3)$.

3.1.1. *The maximal isotropic set $S(X)$.* Consider an arc diagram X . It encodes the following isotropic set of positive roots:

$$S(X) = \{v - w \mid v, w \in \mathcal{B} \text{ are connected by an arc and } v > w\}.$$

For example, the arc diagram in Figure 1 encodes the isotropic subset $S = \{\delta_1 - \epsilon_2, \epsilon_1 - \delta_2, \delta_3 - \epsilon_3, \delta_4 - \epsilon_5\}$. The arc diagram in Figure 2 encodes the isotropic subset $S = \{\epsilon_1 - \delta_1, \delta_2 - \epsilon_3, \delta_3 - \epsilon_4\}$, whereas that in Figure 3 encodes the isotropic subset $S' = \{\epsilon_1 - \delta_1, \delta_2 - \epsilon_3, -\delta_3 - \epsilon_4\}$.

We will often denote by $\Delta^+(X)$, $\Delta_0^+(X)$, $\Delta_1^+(X)$, $\Pi(X)$ the sets of positive, positive even, positive odd, and simple roots associated to the ordering of vertices of \mathcal{B} underlying X (the arc structure of X is irrelevant for that). One readily sees that $S(X) \subset \Delta^+(X)$ and $S(X)$ is a basis of a maximal isotropic subspace in \mathfrak{h}^* (since the cardinality of $S(X)$ equals the defect of \mathfrak{g}). This shows that $S(X)$ is indeed a maximal set of isotropic roots.

For each $\beta \in \Delta_1$ let

$$\text{sn } \beta = (\beta, \sum_{i=1}^m \epsilon_i)$$

(that is $\text{sn}(\epsilon_i \pm \delta_j) = 1$, $\text{sn}(-\epsilon_i \pm \delta_j) = -1$). For $\beta \in S(X)$ one has $\text{sn } \beta = 1$ (resp., -1) if the left end of the corresponding arc is in \mathcal{E} (resp., in \mathcal{D}). For each $\alpha \in S(X)$ we have (cf. (1.8))

$$(3.1) \quad \llbracket \alpha \rrbracket = \sum_{\beta \in S(X), \beta \leq \alpha} \text{sn } \alpha \cdot \text{sn } \beta \cdot \beta,$$

namely

$$\llbracket \epsilon_i - \delta_j \rrbracket = \sum_{\xi \in [\epsilon_i, \delta_j] \cap \mathcal{E}} \xi - \sum_{\xi \in [\epsilon_i, \delta_j] \cap \mathcal{D}} \xi, \quad \llbracket \delta_i - \epsilon_j \rrbracket = \sum_{\xi \in [\delta_i, \epsilon_j] \cap \mathcal{D}} \xi - \sum_{\xi \in [\delta_i, \epsilon_j] \cap \mathcal{E}} \xi.$$

E.g., with reference to Figure 1, we have

$$\llbracket \epsilon_1 - \delta_2 \rrbracket = \epsilon_1 + \epsilon_2 - \delta_1 - \delta_2, \quad \llbracket \delta_1 - \epsilon_2 \rrbracket = \delta_1 - \epsilon_2.$$

Notice that for an arc \widehat{vw} the element $\llbracket v - w \rrbracket$ is $W_{[v,w]}$ -invariant.

3.1.2. *Special arc diagrams.*

Definition 3.2. *We say that an arc is simple if it connects consecutive vertices. We call an arc diagram simple if each arc is simple.*

We call an arc diagram X nice if for each $\alpha, \beta \in S(X)$ such that $\alpha < \beta$ one has $\text{sn } \alpha = \text{sn } \beta$ (equivalently, if for any arc, its left end is of the same type as the left ends of all arcs which are below this arc).

One readily sees that simple diagrams correspond to the case when $S(X)$ consists of simple roots, and nice arc diagrams correspond to the case when $[\gamma] = \sum_{\beta \in S(X), \beta \leq \gamma} \beta$. In particular, for a nice arc diagram X , $[\gamma] \in Q^+$ for each $\gamma \in S(X)$. We denote by $\mathcal{A}^{nice}(\Delta^+)$ the set of nice arc diagrams X such that $\Delta^+ = \Delta^+(X)$.

3.1.3. Existence of arc diagrams and of nice arc diagrams. Fix an order $>$ on \mathcal{B} , set $\Pi = \Pi(\mathcal{B}, >)$ and let Δ^+ be the corresponding set of positive roots. Arc diagrams can be easily constructed inductively. We start by drawing an arc between any two consecutive vertices v, w of different types. Then we throw away these two vertices and obtain an ordered set $\mathcal{E}' \cup \mathcal{D}'$, where the cardinality of \mathcal{E}' (resp., of \mathcal{D}') is less by one than the cardinality of \mathcal{E} (resp. of \mathcal{D}). We take any arc diagram for $\mathcal{E}' \cup \mathcal{D}'$, and we let $\Sigma(X')$ be the set of its arcs. Then $\Sigma(X) = \Sigma(X') \cup \{\widehat{vw}\}$ is the set of arcs of an arc diagram X for \mathcal{B} .

More generally, we can start by drawing any possible arc (i.e., the arc between two vertices v, w of different types such that $[v, w]$ has the same numbers of vertices lying in \mathcal{E} and in \mathcal{D}). Then we can construct an arc diagram for the sequence which is below the arc (i.e., for $[v, w] \setminus \{v, w\}$) and for the ordered set $\mathcal{B} \setminus [v, w]$. The set of arcs in the resulting arc diagram is the union of the sets of arcs in these two diagrams and the arc \widehat{vw} . As a result, there is an arc diagram X such that $\beta \in \Delta_1^+$ belongs to $S(X)$ iff the interval defined by β (i.e., $[v, w]$ for $\beta = v - w$) contains the same numbers of vertices lying in \mathcal{E} and in \mathcal{D} .

Look now at the above procedure from an algebraic point of view: removing a pair $\{u, v\}$ of vertices of different type (or more generally a set $Z = \{u_i, v_i\}_{i=1}^k$ of pairs of vertices of different type) from the ordered set \mathcal{B} gives the ordered set corresponding to $S^\perp \setminus S$ where $S = \{u - v\}$ (resp, $S = \{u_i - v_i\}_{i=1}^k$). This is clearly related to the procedure explained in Definition 1.1. More precisely, we have

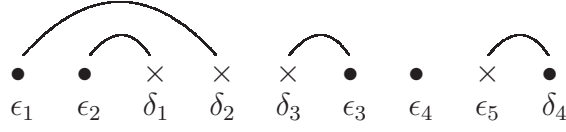
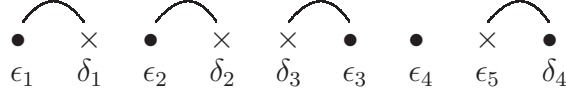
Proposition 3.2. *If X is an arc diagram, then $S(X) \in \mathcal{S}(\Delta^+)$. Viceversa, any $S \in \mathcal{S}(\Delta^+)$ is of the form $S(X)$ for some arc diagram X .*

Proof. Both assertions follow easily from the remark in the previous paragraph after it is proved that an arc diagram has always a simple arc. To prove this, remark that axioms in Definition 3.1 imply that all vertices below a given arc are ends of some arcs. Thus the “lowest” arcs (the ones which do not have arcs below them) are necessarily simple. \square

Let us now explain how to construct nice diagrams.

Proposition 3.3. $\mathcal{A}^{nice}(\Delta^+) \neq \emptyset$.

Proof. Let $\epsilon_1 > \epsilon_2 > \dots > \epsilon_m$ and $\delta_1 > \delta_2 > \dots > \delta_n$. Assume that $\epsilon_1 > \delta_1$, so that our sequence $A_0 = \mathcal{B}$ starts with $\epsilon_1, \dots, \epsilon_k, \delta_1$ ($k \geq 1$). We draw the arc $a_1 = \widehat{\epsilon_k \delta_1}$ and consider the sequence $A_1 = \mathcal{B} \setminus \{\epsilon_k, \delta_1\}$. We repeat the procedure until the first element of the sequence A_{s+1} is not in \mathcal{E} . Note that the right end of the arc a_j obtained in the j th step is δ_j . Let U be the union of the ends in all arcs a_j , $j = 1, \dots, s$. Let us show that $U = [\epsilon_1, \delta_s]$. Indeed, by the above, for $j = 1, \dots, s$ the right end of a_j is δ_j so both ends lie in $[\epsilon_1, \delta_j] \subset [\epsilon_1, \delta_s]$. Therefore $U \subset [\epsilon_1, \delta_s]$. Moreover, $\delta_1, \dots, \delta_s \in U$ so $([\epsilon_1, \delta_s] \setminus U) \subset \mathcal{E}$. Since the first element in $A_{s+1} = A_0 \setminus U$ lies in \mathcal{D} , we conclude that $[\epsilon_1, \delta_s] = U$ as required. Thus the sequence $[\epsilon_1, \delta_s]$ with the ends of the arcs in U is a nice diagram X and each element

FIGURE 4. $r_{\delta_1 - \epsilon_2}(X)$.FIGURE 5. $r_{[\epsilon_1, \delta_2]}(X)$.

of this sequence is an end of an arc. Now we construct a nice diagram Y for the sequence A_{s+1} (which is shorter than A_0). The union of X and Y is a nice arc diagram for the sequence A_0 . \square

3.1.4. *Operations with the arc diagrams.* Consider all arc diagrams on the set \mathcal{B} endowed by a total order and introduce the following operations.

Odd reflections r_{v-w} . Let v, w be two consecutive vertices of an arc diagram X connected by an arc \widehat{vw} . We define $r_{v-w}(X)$ to be the arc diagram with v, w switched (i.e., $r_{v-w}(X) = (X \setminus \{\widehat{vw}\}) \cup \{\widehat{wv}\}$; the total order on \mathcal{B} for $r_{v-w}(X)$ is obtained from the total order for X by interchanging v and w). Figure 4 displays $r_{\delta_1 - \epsilon_2}(X)$ where X is the arc diagram of Figure 1. The odd reflection on an arc diagram corresponds to an odd reflection with respect to a simple root lying in $S(X)$ (cf. (2.4)):

$$\begin{aligned} \Pi(r_{v-w}(X)) &= r_{v-w}(\Pi(X)), \\ S(r_{v-w}(X)) &= S(X) \setminus \{v - w\} \cup \{w - v\}. \end{aligned}$$

Notice that $\text{sn } \alpha \cdot \alpha = \text{sn}(-\alpha) \cdot (-\alpha)$ and thus for $\gamma \in S(X) \cap S(r_{v-w}(X))$ the element $\llbracket \gamma \rrbracket$ defined for X and for $r_{v-w}(X)$ is the same.

Interval reflections $r_{[e_1, d_k]}$. Suppose X has a subsequence $e_1, d_1, e_2, d_2, \dots, e_k, d_k$ ($k > 1$), where e_1, \dots, e_k are of the same type and d_1, \dots, d_k are of another type, with the arcs $\widehat{e_1 d_k}, \widehat{d_1 e_2}, \widehat{e_2 d_3}, \dots, \widehat{d_{k-1} e_k}$. We define $r_{[e_1, d_k]}(X)$ as the arc diagram with the same total order on \mathcal{B} , where the above arcs are substituted by the arcs $\widehat{e_1 d_1}, \widehat{e_2 d_2}, \widehat{e_3 d_3}, \dots, \widehat{e_k d_k}$. Hence

$$\begin{aligned} S(r_{[e_1, d_k]}(X)) &= \\ &= (S(X) \setminus \{e_1 - d_k, d_1 - e_2, \dots, e_k - d_k\}) \cup \{e_1 - d_1, e_2 - d_2, \dots, e_k - d_k\}. \end{aligned}$$

Figure 5 displays $r_{[\epsilon_1, \delta_2]}(X)$ where X is the arc diagram of Figure 1. Figure 6 provides further examples: the arc diagram X in the left display is not nice, since $\llbracket \epsilon_1 - \delta_2 \rrbracket = (\epsilon_1 - \delta_2) - (\delta_1 - \epsilon_2)$. The middle display represents $r_{\delta_1 - \epsilon_2}(X)$, which is nice, and the right display represents $r_{[\epsilon_1, \delta_2]}(X)$, which is simple (hence, in particular, nice). This example also shows that both odd and interval reflections are necessary moves to change an arc diagram into a simple one. They are also sufficient, as we show next.

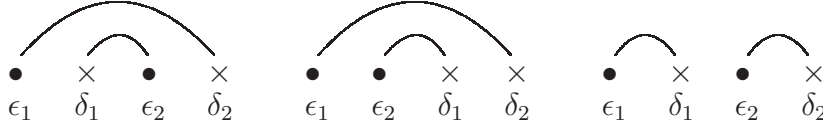


FIGURE 6.

Lemma 3.4. *There exists a finite sequence of odd and interval reflections which change any arc diagram into a simple arc diagram.*

Proof. We proceed by induction on the number k of non-simple arcs in the arc diagram X . If $k = 0$, then X is a simple arc diagram and there is nothing to prove. Assume $k \geq 1$ and let \widehat{ed} be a non-simple arc such that each arc \widehat{vw} , $e > v > w > d$ is simple. Then, necessarily, we have $[e, d] = ea_1a'_1a_2a'_2 \dots a_ha'_hd$, with arcs \widehat{ed} , $\widehat{a_i a'_i}$, $i = 1, \dots, h$. Use the odd reflections $r_{a_i - a'_i}$, if necessary, to modify $[e, d]$ in such a way that vertices of different type alternate from e to d : call the resulting diagram X' . Then $r_{[e,d]}(X')$ has $k - 1$ non-simple arcs and we are done by induction. \square

3.2. Proof of Theorem 1.1. In this Section we prove formulas (1.10), (1.11). For future applications we prove a slightly more general result (see Proposition 3.8).

For any subset U of $W_{\mathfrak{g}}$ and any rational function Y with $\{e^b \mid b \in \mathcal{B}\}$ as set of variables, introduce the sum

$$(3.2) \quad \mathcal{F}_U(Y) = \sum_{w \in U} \text{sgn}(w) w(Y).$$

We will need also the sum

$$(3.3) \quad \check{\mathcal{F}}_U(Y) = \sum_{w \in U} \text{sgn}'(w) w(Y).$$

Notation 3.1. *If U is a subgroup of $W_{\mathfrak{g}}$, and $W \subset W_{\mathfrak{g}}$ is stable under the right action of U , then W is a union of right cosets in $W_{\mathfrak{g}}/U$. We denote by W/U a set of representatives.*

Note that

$$\mathcal{F}_W(Y) = \mathcal{F}_{W/U}(\mathcal{F}_U(Y)), \quad \check{\mathcal{F}}_W(Y) = \check{\mathcal{F}}_{W/U}(\check{\mathcal{F}}_U(Y)).$$

The formula displayed on the right follows since $\text{sgn}' : W_{\mathfrak{g}} \rightarrow \{\pm 1\}$ is a homomorphism.

Lemma 3.5. *Let Π be a set of simple roots for $\mathfrak{g} = \mathfrak{gl}(k|k)$, $k \geq 2$, all of which are isotropic. Let $\{\beta_1, \dots, \beta_k\} \subset \Pi$ be the maximal isotropic subset. Then*

$$(3.4) \quad \check{\mathcal{F}}_{W_{\mathfrak{g}}}\left(\frac{1}{\prod_{i=2}^k (1 - e^{-\beta_i})}\right) = \frac{1 - e^{-\sum_{i=1}^k \beta_i}}{k} \check{\mathcal{F}}_{W_{\mathfrak{g}}}\left(\frac{1}{\prod_{i=1}^k (1 - e^{-\beta_i})}\right).$$

Proof. Let $A = \frac{1}{\prod_{i=1}^k (1 - e^{-\beta_i})}$, $x_0 = \check{\mathcal{F}}_{W_{\mathfrak{g}}}(A)$ and for $j = 1, \dots, k$ let

$$x_j = \check{\mathcal{F}}_{W_{\mathfrak{g}}}(Ae^{-\beta_1 - \beta_2 - \dots - \beta_j}).$$

Then the left-hand side of formula (3.4) is equal to $x_0 - x_1$: $LHS = x_0 - x_1$.

Write $\Pi = \{\epsilon_1 - \delta_1, \delta_1 - \epsilon_2, \dots, \epsilon_k - \delta_k\}$ and $\beta_i = \epsilon_i - \delta_i$. Observe that $W_{\mathfrak{g}}$ contains a subgroup permuting the β_i 's, and all elements of this subgroup are even

(the permutations $(\epsilon_i \epsilon_j)(\delta_i \delta_j)$ switches β_i and β_j). Since $\check{\mathcal{F}}_{W_{\mathfrak{g}}}(A')$ is $W_{\mathfrak{g}}$ -skew-invariant for any rational function A' , we conclude that

$$(3.5) \quad x_j = \check{\mathcal{F}}_{W_{\mathfrak{g}}}(Ae^{-\sum_{i \in J} \beta_i})$$

for any subset $J \subset \{1, 2, \dots, k\}$ of cardinality j . Since the permutation $(\epsilon_1 \epsilon_2)$ stabilizes A , one has

$$\check{\mathcal{F}}_{W_{\mathfrak{g}}}(A \prod_{i=1}^s (1 - e^{-\beta_i})) = 0$$

for $s = 2, \dots, k$. Hence, using (3.5), we obtain that $\sum_{j=0}^s (-1)^j \binom{s}{j} x_j = 0$ for $s = 2, \dots, k$. We deduce by induction on $j \geq 2$ that $x_j = j(x_1 - x_0) + x_0$ and thus

$$LHS = x_0 - x_1 = \frac{x_0 - x_k}{k} = \frac{1}{k} \check{\mathcal{F}}_{W_{\mathfrak{g}}}((1 - e^{-\sum_{i=1}^k \beta_i})Y) = \frac{1 - e^{-\sum_{i=1}^k \beta_i}}{k} \check{\mathcal{F}}_{W_{\mathfrak{g}}}(A)$$

which is (3.4). \square

For an arc diagram X we denote by ρ_X the element $\rho_{\Pi(X)}$ and by R_X, \check{R}_X the fractions R, \check{R} constructed for $\Pi(X)$, that is

$$R_X = \frac{\prod_{\alpha \in \Delta_0^+(X)} (1 - e^{-\alpha})}{\prod_{\alpha \in \Delta_1^+(X)} (1 + e^{-\alpha})}, \quad \check{R}_X = \frac{\prod_{\alpha \in \Delta_0^+(X)} (1 - e^{-\alpha})}{\prod_{\alpha \in \Delta_1^+(X)} (1 - e^{-\alpha})}.$$

Let

$$(3.6) \quad \mathcal{P}(X) = \prod_{\gamma \in S(X)} \frac{\text{ht } \gamma + 1}{2} \cdot \frac{e^{\rho_X}}{\prod_{\gamma \in S(X)} (1 - e^{-\llbracket \gamma \rrbracket})}.$$

Corollary 3.6. *Let X be an arc diagram and $r_{[v,w]}$ be an interval reflection. For any subset W of the Weyl group which is stable under the right action of $W_{\text{Supp}(X)}$ one has*

$$(3.7) \quad \check{\mathcal{F}}_W(\mathcal{P}(X)) = \check{\mathcal{F}}_W(\mathcal{P}(r_{[v,w]}(X))).$$

Proof. Denote by Y the arc subdiagram corresponding to the interval $[v, w]$ and let $Y' = r_{[v,w]}(Y)$. Then $X' = r_{[v,w]}(X)$ is obtained from X by substituting Y by Y' . View Y, Y' as arc diagrams of $gl(k, k)$ -type. Then $G = W_{[v,w]}$ is the Weyl group of Y and of Y' .

Notice that $(\rho_X - \rho_Y, \alpha) = 0$ for each $\alpha \in \Pi(Y)$ and thus $\rho_X - \rho_Y$ is G -invariant. Since $[v, w] \subseteq \text{Supp}(X)$, we have that $WG = W$. Therefore

$$\check{\mathcal{F}}_W(\mathcal{P}(X)) = \check{\mathcal{F}}_{W/G} \left(\frac{e^{\rho_X - \rho_Y}}{\prod_{\beta \in S(X) \setminus S(Y)} (1 - e^{-\llbracket \beta \rrbracket})} \cdot \check{\mathcal{F}}_G(\mathcal{P}(Y)) \right)$$

and a similar formula holds for X', Y' respectively. One has $S(X) \setminus S(Y) = S(X') \setminus S(Y')$. Moreover, since $\Pi(Y) = \Pi(Y')$ and $\Pi(X) = \Pi(X')$, one has $\rho_Y = \rho_{Y'}$ and $\rho_X = \rho_{X'}$. Thus formula (3.7) follows from the following equality

$$(3.8) \quad \check{\mathcal{F}}_G(\mathcal{P}(Y)) = \check{\mathcal{F}}_G(\mathcal{P}(Y')),$$

which we now prove.

Recall that, by definition of interval reflection, the interval $[v, w]$ is of the form $v = e_1 > d_1 > e_2 > d_2 > \dots > e_s > d_s = w$, where the e_i 's are of the same type and the d_i 's are of another type. Then $S(Y) = \{\alpha\} \cup S'$, where $\alpha = v - w = e_1 - d_s$ and $S' = \{d_i - e_{i+1}\}_{i=1}^{s-1}$. Since $S' \subset \Pi(Y)$ one has $\llbracket \beta \rrbracket = \beta$ for $\beta \in S'$. Recall that

$[\alpha]$ is G -invariant and note that ρ_Y is also G -invariant (since $\Pi(Y)$ consists of odd roots). Therefore

$$\check{\mathcal{F}}_G(\mathcal{P}(Y)) = \frac{e^{\rho_Y}}{1 - e^{-[\alpha]}} \cdot \check{\mathcal{F}}_G\left(\frac{1}{\prod_{\beta \in S'}(1 - e^{-\beta})}\right).$$

Consider the simple arc diagram Z with the order $d_1 > e_2 > d_2 > \dots > d_s > e_1$; then $S(Z) = S' \cup \{-\alpha\}$. Using Lemma 3.5 we obtain

$$\check{\mathcal{F}}_G\left(\frac{1}{\prod_{\beta \in S'}(1 - e^{-\beta})}\right) = \frac{1 - e^{-\sum_{\beta \in S'} \beta + \alpha}}{s} \check{\mathcal{F}}_G\left(\frac{1}{\prod_{\beta \in S(Z)}(1 - e^{-\beta})}\right).$$

Observe that $s = \frac{\text{ht } \alpha + 1}{2} = \frac{\text{ht}(v-w)+1}{2}$ and $[\alpha] = \alpha - \sum_{\beta \in S'} \beta = \rho_Z - \rho_Y$. Summarizing, we obtain

$$\begin{aligned} \check{\mathcal{F}}_G(\mathcal{P}(Y)) &= e^{\rho_Y} \frac{1 - e^{[\alpha]}}{1 - e^{-[\alpha]}} \check{\mathcal{F}}_G\left(\frac{1}{\prod_{\beta \in S(Z)}(1 - e^{-\beta})}\right) \\ &= -e^{\rho_Z} \check{\mathcal{F}}_G\left(\frac{1}{\prod_{\beta \in S(Z)}(1 - e^{-\beta})}\right) = -C \check{R}_Z e^{\rho_Z}, \end{aligned}$$

where the last equality follows from (1.5) (because Z is a simple arc diagram). Since $S(Y') \subset \Pi(Y')$, Theorem KWG gives also $\check{\mathcal{F}}_G(\mathcal{P}(Y')) = C \check{R}_{Y'} e^{\rho_{Y'}}$.

Since $\Pi(Z)$ corresponds to the total order $d_1 > e_2 > d_2 > \dots > d_s > e_1$ and $\Pi(Y')$ corresponds to the total order $e_1 > d_1 > e_2 > d_2 > \dots > d_s$, one has $\Pi(Z) = r_{e_1-d_s} \dots r_{e_1-e_2} r_{e_1-d_1}(\Pi(Y'))$. Using (3.10) we get

$$\check{R}_{Y'} e^{\rho_{Y'}} = (-1)^{2s-1} \check{R}_Z e^{\rho_Z} = -\check{R}_Z e^{\rho_Z}.$$

This establishes (3.8) and completes the proof. \square

To complete the proof of Theorem 1.1, we will need the following observation.

Lemma 3.7. *Formula (1.10) follows from (1.11).*

Proof. Fix any set of positive roots Δ^+ and let Π be the associated set of simple roots. Take $h \in \mathfrak{h}$ such that $\alpha(h) = 0$ if $\alpha \in \Pi \cap \Delta_0$ and $\alpha(h) = 1$ if $\alpha \in \Pi \cap \Delta_1$. Then $\alpha(h) \equiv 0 \pmod{2}$ if $\alpha \in \Delta_0$ and $\alpha(h) \equiv 1 \pmod{2}$ if $\alpha \in \Delta_1$. We claim that $F(e^\alpha) = e^{\pi\sqrt{-1}\alpha(h)} e^\alpha$ changes (1.11) to (1.10). Let $k = e^{\pi\sqrt{-1}\rho(h)}$. Then, obviously, $F(e^\rho) = k e^\rho$, so we have $F(e^\rho \check{R}) = k e^\rho R$. We need only to check that $F(e^{w(\rho)}) = k \frac{\text{sgn}(w)}{\text{sgn}'(w)} e^{w(\rho)}$. First of all observe that $F(w(e^\rho \check{R})) = \text{sgn}'(w) F(e^\rho \check{R}) = \text{sgn}'(w) k e^\rho R$. On the other hand this equals $F(e^{w(\rho)} w \check{R}) = F(e^{w(\rho)}) F(w \check{R})$. Since w permutes the roots of the same parity, we have that $F(w \check{R}) = w(F(\check{R}))$ and in turn $F(e^{w(\rho)}) F(w \check{R}) = F(e^{w(\rho)}) w(F(\check{R}))$. It follows that $F(e^{w(\rho)} w \check{R}) = e^{\pi\sqrt{-1}w(\rho)(h)} e^{w(\rho)} w(R) = e^{\pi\sqrt{-1}w(\rho)(h)} \text{sgn}(w) e^\rho R$. Hence

$$e^{\pi\sqrt{-1}w(\rho)(h)} \text{sgn}(w) = k \text{sgn}'(w),$$

and this relation implies our claim. \square

We are finally ready to give the proof of Theorem 1.1: in the current setting, formula (1.11) becomes

$$(3.9) \quad \check{\mathcal{F}}_{W_{\mathfrak{g}}}(\mathcal{P}(X)) = C_{\mathfrak{g}} e^{\rho_X} \check{R}_X.$$

By Lemma 3.4, any arc diagram can be transformed to a simple one by a sequence of odd and interval reflections. By Theorem KWG, $\check{\mathcal{F}}_{W_{\mathfrak{g}}}(\mathcal{P}(X)) = C_{\mathfrak{g}} \check{R}_X e^{\rho_X}$ if X

is a simple diagram. For any odd reflection r_α of Π one has $\rho_{r_\alpha(\Pi)} = \rho_\Pi + \alpha$ (cf. (2.6)), and so

$$(3.10) \quad e^{\rho_{r_\alpha \Pi}} \check{R}_{r_\alpha \Pi} = -e^{\rho_\Pi} \check{R}_\Pi.$$

Since for $\alpha \in S(X)$ one has $\mathcal{P}(r_\alpha(X)) = -\mathcal{P}(X)$, the fraction $\check{\mathcal{F}}_{W_{\mathfrak{g}}}(\mathcal{P}(X))/(\check{R}_X e^{\rho_X})$ is not changed by the action of r_α . The interval reflections do not change $\check{R}_X e^{\rho_X}$, since they do not change $\Pi(X)$; moreover, applying Corollary 3.6 with $W = W_{\mathfrak{g}}$, they do not change $\check{\mathcal{F}}_{W_{\mathfrak{g}}}(\mathcal{P}(X))$. This proves formula (3.9), hence (1.11). Lemma 3.7 implies that (1.10) also holds. By choosing X to be a nice arc diagram, we have that $[\gamma] \in Q^+$ for any $\gamma \in S(X)$. This concludes the proof of Theorem 1.1.

3.2.1. For future applications we will need a slightly stronger version of Theorem 1.1. Given an arc diagram X , let \mathcal{B}' be a subset of \mathcal{B} containing $\text{Supp}(X)$. Let $\Delta(\mathcal{B}')$ be the set of roots that are linear combinations of the simple roots that are in the span of \mathcal{B}' . Assume for simplicity that $\Delta(\mathcal{B}')$ is irreducible and let $\Delta^\sharp(\mathcal{B}')$ be the irreducible component of $\Delta_0(\mathcal{B}')$ which is not the smallest one in the sense of [4, Section 1.2]. Let $W(\mathcal{B}')$ and $W^\sharp(\mathcal{B}')$ be the corresponding Weyl groups. Clearly $W_{\mathcal{B}'} W^\sharp(\mathcal{B}')$ is a subgroup of $W(\mathcal{B}')$ and let $T = (W_{\mathcal{B}'} W^\sharp(\mathcal{B}')) \setminus W(\mathcal{B}')$. Set $Z = W_{\mathfrak{g}}/W(\mathcal{B}')$ and $W_0 = ZW_{\mathcal{B}'} W^\sharp(\mathcal{B}')$ so that $W_{\mathfrak{g}} = W_0 T$.

Proposition 3.8.

$$(3.11) \quad \check{\mathcal{F}}_{W_0}(\mathcal{P}(X)) = \frac{C_{\mathfrak{g}}}{|T|} e^{\rho_X} \check{R}_X.$$

Proof. Recall that any arc diagram can be transformed into a simple arc diagram by a sequence of odd and interval reflections. Note that these reflections permute the ends of arcs and do not change the positions of other vertices (the interval reflections do not change the order of vertices and the odd reflections permute two vertices connected by an arc). Thus these reflections do not change \mathcal{B}' and $\text{Supp}(X)$. Since $W_0 W_{\text{Supp}(X)} = W_0$, we can argue as in the proof of Theorem 1.1 and assume that X is simple. Let Y be the same arc diagram viewed as a diagram for $\Delta(\mathcal{B}')$. Then, since $(\rho_X - \rho_Y, \alpha) = 0$ for any simple root of $\Delta(\mathcal{B}')$, we have that $\rho_X - \rho_Y$ is $W(\mathcal{B}')$ -invariant. Since X is simple, Y is also simple and (1.5) gives

$$\mathcal{F}_{W^\sharp(\mathcal{B}')}(\mathcal{P}(Y)) = \frac{1}{|T|} \mathcal{F}_{W^\sharp(\mathcal{B}')T}(\mathcal{P}(Y)).$$

It follows that

$$\begin{aligned} \check{\mathcal{F}}_{W_0}(\mathcal{P}(X)) &= \check{\mathcal{F}}_{W_0/W^\sharp(\mathcal{B}')} (e^{\rho_X - \rho_Y} \cdot \mathcal{F}_{W^\sharp(\mathcal{B}')}(\mathcal{P}(Y))) \\ &= \frac{1}{|T|} \check{\mathcal{F}}_{W_0/W^\sharp(\mathcal{B}')} (e^{\rho_X - \rho_Y} \cdot \mathcal{F}_{W^\sharp(\mathcal{B}')T}(\mathcal{P}(Y))) \\ &= \frac{1}{|T|} \check{\mathcal{F}}_{W_{\mathfrak{g}}}(\mathcal{P}(X)) = \frac{C_{\mathfrak{g}}}{|T|} e^{\rho_X} \check{R}_X. \end{aligned}$$

This completes the proof. \square

3.2.2. *Comments on type A.* Note that if $m \neq n$, formula (1.11) restricts plainly to $A(m, n) = sl(m+1, n+1)$. If instead $m = n$, the formula does not restrict to \mathfrak{h} when $[\gamma] = \sum_{i=1}^{n+1} (\delta_i - \epsilon_i)$ for $\gamma \in S$. Note that the factor $\frac{1}{1-e^{-[\gamma]}}$ is $W_{\mathfrak{g}}$ -invariant,

hence it can be taken out of the sum. Since the left hand side of (1.11) restricts to \mathfrak{h} , the sum

$$\check{\mathcal{F}}_{W_{\mathfrak{g}}}\left(\frac{e^{\rho}}{\prod_{\beta \in S \setminus \{\gamma\}} (1 - e^{-\llbracket \beta \rrbracket})}\right)$$

is divisible by $1 - e^{-\llbracket \gamma \rrbracket}$. After simplifying, we may restrict to the Cartan subalgebra of $A(n, n)$ getting a superdenominator formula in this type too.

3.3. Proof of Proposition 1.2. Recall from (1.8) the definition of $\llbracket \gamma \rrbracket$; note that it may combinatorially rewritten as $\llbracket \gamma \rrbracket = \sum_{\beta \in \gamma^<} \text{sn } \beta \cdot \text{sn } \gamma \cdot \beta$ and that $\llbracket \gamma \rrbracket = 0$ if $\gamma \in \Pi$. Let

$$\mathcal{Q}(X) = \frac{e^{\rho + \sum_{\gamma \in S} \llbracket \gamma \rrbracket}}{\prod_{\beta \in S(X)} (1 - e^{-\beta})}.$$

By Theorem KWG, we have $\check{\mathcal{F}}_{W^{\#}}(\mathcal{Q}(X)) = \check{R}_X e^{\rho_X}$ if X is a simple arc diagram. Notice that an odd reflection changes the sign of $\check{R}_X e^{\rho_X}$ and does the same on $\mathcal{Q}(X)$. The interval reflections do not change $R_X e^{\rho_X}$, since they do not change $\Pi(X)$. Thus, due to Lemma 3.4, in order to prove (1.15), it is enough to verify that the interval reflections do not change $\check{\mathcal{F}}_W(\mathcal{Q}(X))$. This is done in Lemma 3.9 below.

Lemma 3.9. *Let X be an arc diagram and $r_{[v,w]}$ be an interval reflection. Then*

$$\check{\mathcal{F}}_{W^{\#}}(\mathcal{Q}(X)) = \check{\mathcal{F}}_{W^{\#}}(\mathcal{Q}(r_{[v,w]}(X))).$$

Proof. Denote by Y the arc subdiagram corresponding to the interval $[v, w]$ and let $Y' = r_{[v,w]}(Y)$. Then $X' = r_{[v,w]}(X)$ is obtained from X by substituting Y by Y' . View Y, Y' as arc diagrams of $gl(k, k)$ -type. Then $G = W_{[v,w]} \cap W^{\#}$ is $W^{\#}$ constructed for Y and for Y' .

Let e, d be vertices such that $e - d \in S(X)$. Denote by $]e, d[$ the interval $[e, d] \setminus \{e, d\}$. Then $\llbracket e - d \rrbracket = \pm(\sum_{\xi \in]e, d[\cap \mathcal{E}} \xi - \sum_{\xi \in]e, d[\cap \mathcal{D}} \xi)$ with the sign ”+” if $e \in \mathcal{E}$ and the sign ”-” if $e \in \mathcal{D}$. We see that $\llbracket e - d \rrbracket$ is $W_{]e, d[}$ -invariant. In particular, $\llbracket \gamma \rrbracket$ is G -invariant for each $\gamma \in S(X) \setminus S(Y)$, since if v', w' are vertices such that $v' - w' \in S(X) \setminus S(Y)$, then $[v, w] \cap [v', w'] = \emptyset$ or $[v, w] \subset [v', w']$ (because the arcs do not intersect). Notice that $(\rho_X - \rho_Y, \alpha) = 0$ for each $\alpha \in \Pi(Y)$, hence $\rho_X - \rho_Y$ is G -invariant. Therefore

$$\check{\mathcal{F}}_{W^{\#}}(\mathcal{Q}(X)) = \check{\mathcal{F}}_{W^{\#}/G}\left(\frac{e^{\rho_X - \rho_Y + \sum_{\gamma \in S(X) \setminus S(Y)} \llbracket \gamma \rrbracket}}{\prod_{\beta \in S(X) \setminus S(Y)} (1 - e^{-\beta})} \cdot \check{\mathcal{F}}_G(\mathcal{Q}(Y))\right)$$

and the similar formula holds for X', Y' respectively. Notice that $S(X) \setminus S(Y) = S(X') \setminus S(Y')$. Since $\Pi(Y) = \Pi(Y')$ and $\Pi(X) = \Pi(X')$, one has $\rho_Y = \rho_{Y'}$, $\rho_X = \rho_{X'}$. We see that the required equality $\check{\mathcal{F}}_{W^{\#}}(\mathcal{Q}(X)) = \check{\mathcal{F}}_{W^{\#}}(\mathcal{Q}(X'))$ follows from the equality

$$(3.12) \quad \check{\mathcal{F}}_G(\mathcal{Q}(Y)) = \check{\mathcal{F}}_G(\mathcal{Q}(Y')),$$

which we verify below.

Recall that, by the definition of the interval reflection, the interval $[v, w]$ is of the form $v = e_1 > d_1 > e_2 > d_2 > \dots > e_s > d_s = w$, where the e_i 's are of the same type

and the d_i 's are of another type. Then $S(Y) = \{\alpha\} \cup S'$, where $\alpha = v - w = e_1 - d_s$ and $S' = \{d_i - e_{i+1}\}_{i=1}^{s-1}$. Since $S' \subset \Pi(Y)$ one has

$$\sum_{\gamma \in S(Y)} \llbracket \gamma \rrbracket \alpha \llbracket = \sum_{i=1}^s (e_i - d_i) - \alpha.$$

Consider the simple arc diagram Z with the order $d_1 > e_2 > d_2 > \dots > d_s > e_1$; then $S(Z) = S' \cup \{-\alpha\}$ and

$$\rho_Z = \frac{1}{2} \sum_{i=1}^s (e_i - d_i) = -\rho_Y, \quad \text{that is } \rho_Z = \rho_Y + \alpha \llbracket \alpha \llbracket.$$

We have

$$\check{F}_G(\mathcal{Q}(Y)) = \check{F}_G\left(\frac{e^{\rho_Y \llbracket \alpha \llbracket}}{\prod_{\beta \in S(Y)} (1 - e^{-\beta})}\right) = -\check{F}_G\left(\frac{e^{\rho_Y \llbracket \alpha \llbracket + \alpha}}{\prod_{\beta \in S(Z)} (1 - e^{-\beta})}\right) = -\check{F}_G(\mathcal{Q}(Z)).$$

Since Z and Y' are simple, one has $\check{F}_G(\mathcal{Q}(Z)) = e^{\rho_Z} \check{R}_Z$ and $\check{F}_G(\mathcal{Q}(Y')) = e^{\rho_{Y'}} \check{R}_{Y'}$. Arguing as in the proof of Corollary 3.6, one shows that $e^{\rho_Z} \check{R}_Z = -e^{\rho_{Y'}} \check{R}_{Y'}$, and this completes the proof of (3.12). \square

Remark 3.1. Arguing as in Lemma 3.7, one deduces (1.14) from (1.15).

4. DISTINGUISHED SETS OF POSITIVE ROOTS AND COMPACT DUAL PAIRS

4.1. Dual pairs and Theta correspondence. Let us now recall what dual pairs and the Theta correspondence are: this involves some basic and well-known facts on the oscillator representations of symplectic groups (see e.g. [1] for more details and a rich list of references).

Let $(V, \langle \cdot, \cdot \rangle)$ be a $2n$ -dimensional real symplectic vector space. Fix a polarization $V = A^+ \oplus A^-$ (A^\pm are isotropic subspaces) and a standard symplectic basis w.r.t. $\langle \cdot, \cdot \rangle$: $A^+ = \bigoplus_{i=1}^n \mathbb{R}e_i$, $A^- = \bigoplus_{i=1}^n \mathbb{R}f_i$, so that $\langle e_i, f_j \rangle = \delta_{ij}$.

Starting from this polarization of V we can construct a complex polarization of $V_{\mathbb{C}} = V \otimes_{\mathbb{R}} \mathbb{C}$ by setting

$$V_{\mathbb{C}}^+ = \bigoplus_{i=1}^n \mathbb{C}(e_i + \sqrt{-1}f_i), \quad V_{\mathbb{C}}^- = \bigoplus_{i=1}^n \mathbb{C}(e_i - \sqrt{-1}f_i).$$

This polarization is “totally complex”, i.e. $V_{\mathbb{C}}^+ \cap V = \{0\}$. As in (1.16) we can consider the representation $M = W(V_{\mathbb{C}})/W(V_{\mathbb{C}})V_{\mathbb{C}}^+$ of the Weyl algebra $W(V_{\mathbb{C}})$ of $(V_{\mathbb{C}}, \langle \cdot, \cdot \rangle_{\mathbb{C}})$, and, by means of (1.17), we define an action of $sp(V_{\mathbb{C}}, \langle \cdot, \cdot \rangle_{\mathbb{C}})$, on M . This representation is usually called the oscillator representation. The choice of a totally complex polarization is equivalent to assigning a compatible complex structure J on V (i.e., $J \in Sp(V)$ such that $J^2 = -1$). Explicitly J is defined by setting $J(e_i) = -f_i$ and $J(f_i) = e_i$. Let W be the space V seen as a complex space via the complex structure J . The elements $g \in Sp(V)$ commuting with J form a maximal compact subgroup K of $Sp(V)$ and we let \mathfrak{k} be its complexified Lie algebra viewed as a subalgebra of $sp(V_{\mathbb{C}}, \langle \cdot, \cdot \rangle_{\mathbb{C}})$. Since K commutes with J , we may let it act \mathbb{C} -linearly on W . We let $\det(k)$ be the determinant of the action of $k \in K$ on W . If \tilde{K} is the $\sqrt{\det}$ cover of K , then M has an action of \tilde{K} whose differential coincides with the action of \mathfrak{k} as a subalgebra of $sp(V_{\mathbb{C}}, \langle \cdot, \cdot \rangle_{\mathbb{C}})$.

For future reference we describe explicitly this action. Recall that

$$\tilde{K} = \{(g, z) \in K \times \mathbb{C}^\times \mid z^2 = \det g\}.$$

The covering map is the projection π on the first factor. The Lie algebra of \tilde{K} is the subalgebra of $\mathfrak{k} \times \mathbb{C}$ given by

$$\tilde{\mathfrak{k}} = \left\{ \left(A, \frac{\text{tr}(A)}{2} \right) \mid A \in \mathfrak{k} \right\}.$$

Note that $d\pi$ is the projection on the first factor and provides an isomorphism between $\tilde{\mathfrak{k}}$ and \mathfrak{k} . We want to define an action of \tilde{K} on M in such a way that

$$(4.1) \quad \exp(X) \cdot v = e^{d\pi(X)}(v)$$

for any $X \in \tilde{\mathfrak{k}}$. Identify $V_{\mathbb{C}}^+$ with W by mapping $v \otimes (a + ib)$ to $av + J(bv)$. Then M is linearly isomorphic to the polynomial algebra $P(W)$ on W by identifying $v \in V_{\mathbb{C}}^-$ with the linear function on $V_{\mathbb{C}}^+$ given by $v(u) = \langle u, v \rangle$. Recall that K acts on W , hence also on $P(W)$. With this identifications we can define an action of \tilde{K} on $M \simeq P(W)$ by

$$(4.2) \quad (g, z) \cdot p = z^{-1}g \cdot p.$$

We now check that (4.1) holds. According to our definitions, if $X = \left(A, \frac{\text{tr}(A)}{2} \right) \in \tilde{\mathfrak{k}}$ then $\exp(X) \cdot p = e^{-\frac{\text{tr}(A)}{2}} e^A \cdot p$. On the other hand, according to (1.17), A acts on M by left multiplication by $\theta(A)$. Now, applying (1.18), we see that

$$\theta(A)p = [\theta(A), p] + p\theta(A) \cdot 1 = A \cdot p + p\theta(A) \cdot 1.$$

Choose a basis $\{x_i\}$ of $V_{\mathbb{C}}^-$ and let $\{y_i\}$ be the basis of $V_{\mathbb{C}}^+$ such that $\langle x_i, y_j \rangle = \delta_{ij}$. Then $\{x_i, y_i\}$ is a basis of $V_{\mathbb{C}}$ and $\{y_i, -x_i\}$ is its dual basis. Hence

$$\begin{aligned} \theta(A) \cdot 1 &= \frac{1}{2} \sum_i A(y_i)x_i \cdot 1 = \frac{1}{2} \sum_{i,j} \langle x_j, A(y_i) \rangle y_j x_i \cdot 1 \\ &= -\frac{1}{2} \sum_i \langle x_i, A(y_i) \rangle = -\frac{\text{tr}(A)}{2}. \end{aligned}$$

Therefore

$$\theta(A)p = A \cdot p - \frac{\text{tr}(A)}{2}p.$$

Exponentiating, we find (4.1).

Let H be the element of $sp(V_{\mathbb{C}}, \langle \cdot, \cdot \rangle_{\mathbb{C}})$ such that $H|_{V_{\mathbb{C}}^\pm} = \pm I$. Then bracketing with H defines a \mathbb{Z} -gradation

$$sp(V_{\mathbb{C}}, \langle \cdot, \cdot \rangle_{\mathbb{C}}) = \bigoplus_{n \in \mathbb{Z}} sp(V_{\mathbb{C}}, \langle \cdot, \cdot \rangle_{\mathbb{C}})_n.$$

We set $\mathfrak{p} = \bigoplus_{n \geq 0} sp(V_{\mathbb{C}}, \langle \cdot, \cdot \rangle_{\mathbb{C}})_n$. Clearly \mathfrak{p} is a parabolic subalgebra of $sp(V_{\mathbb{C}}, \langle \cdot, \cdot \rangle_{\mathbb{C}})$.

A reductive dual pair is a pair of real Lie subgroups G_1, G_2 of $Sp(V)$ which act reductively on V and such that each is the centralizer of the other in $Sp(V)$. We say that the dual pair is *compact* if one of the two subgroup is compact. In the following we deal always with compact dual pairs, assuming G_1 compact. We also assume that $G_1 \subset K$ and let \mathfrak{s}_i be the Lie algebras of G_i ($i = 1, 2$). Denote by $\mathfrak{s}_i^{\mathbb{C}}, i = 1, 2$ their complexifications. Let \tilde{G}_1 be the lift of G_1 to \tilde{K} . Since $G_1 \subset K$ it follows that

the center of K is contained in G_2 . The center of K is $\{exp(\sqrt{-1}tH) \mid t \in \mathbb{R}\}$, thus we can conclude that $H \in \mathfrak{s}_2^{\mathbb{C}}$. It follows that $\mathfrak{p}_2 = \mathfrak{p} \cap \mathfrak{s}_2^{\mathbb{C}}$ is a parabolic subalgebra of $\mathfrak{s}_2^{\mathbb{C}}$. Howe duality in this setting gives the following result (see also [15]):

Theorem 4.1. [6] *There is a set Σ of irreducible finite-dimensional representations of \widetilde{G}_1 such that, as $\widetilde{G}_1 \times \mathfrak{s}_2^{\mathbb{C}}$ -module,*

$$M = \bigoplus_{\eta \in \Sigma} \eta \otimes \tau(\eta),$$

where $\tau(\eta)$ are irreducible quotients of \mathfrak{p}_2 -parabolic Verma modules.

The map $\eta \mapsto \tau(\eta)$ is called the Theta correspondence.

We need also to recall the following structure theorem (cf. [7]).

Proposition 4.2.

- (1) *A compact dual pair (G_1, G_2) is of type I, i.e., $G_1 G_2$ acts irreducibly on V .*
- (2) *A reductive dual pair (G_1, G_2) is of type I if and only if there exists a division algebra D over \mathbb{R} with involution e , an Hermitian right D -vector space $(W_1, (\cdot, \cdot)_1)$, a skew-Hermitian left D -vector space $(W_2, (\cdot, \cdot)_2)$ and an isomorphism $V \cong W_1 \otimes_D W_2$ of \mathbb{R} -vector spaces such the symplectic form $\langle \cdot, \cdot \rangle$ corresponds to $Tr_{D/\mathbb{R}}((\cdot, \cdot)_1 \otimes e \circ (\cdot, \cdot)_2)$ and under which G_1 and G_2 map to the isometry groups $U(G_1, (\cdot, \cdot)_1)$, $U(G_2, (\cdot, \cdot)_2)$, respectively.*

4.2. Cartan involutions. Suppose that \mathfrak{g} is a Lie superalgebra of basic classical type. If \mathfrak{g} is simple of type $A(1, 1)$ let Δ be the set of roots of $gl(2, 2)$. In all other cases we let Δ be, as usual, the set of roots of \mathfrak{g} . Choose a set Δ^+ of positive roots and let $\Pi = \{\alpha_1, \dots, \alpha_n\}$ be the corresponding set of simple roots. If \mathfrak{g} is not of type $A(1, 1)$ then the root spaces have dimension 1, so we can choose for each $\alpha \in \Delta^+$ root vectors $X_\alpha \in \mathfrak{g}_\alpha$ and $X_{-\alpha} \in \mathfrak{g}_{-\alpha}$ with the property that $(X_\alpha, X_{-\alpha}) = 1$, and let $h_\alpha = [X_\alpha, X_{-\alpha}]$. We set $e_i = X_{\alpha_i}$ and $f_i = X_{-\alpha_i}$. If \mathfrak{g} is simple of type $A(1, 1)$, then, given $\alpha \in \Delta$, we let X_α be the projection on \mathfrak{g} of the corresponding root vector in $gl(2, 2)$.

Recall from [9] that \mathfrak{g} is the minimal \mathbb{Z} -graded Lie superalgebra with local part

$$\bigoplus_{i=1}^n \mathbb{C}f_i \oplus \mathfrak{h} \oplus \bigoplus_{i=1}^n \mathbb{C}e_i$$

and relations

$$(4.3) \quad [e_i, f_j] = \delta_{ij} h_{\alpha_i}, \quad [h_{\alpha_i}, e_j] = (\alpha_i, \alpha_j) e_j, \quad [h_{\alpha_i}, f_j] = -(\alpha_i, \alpha_j) f_j,$$

on the local part. From now on we assume that $(\alpha_i, \alpha_j) \in \mathbb{R}$ for any i, j . In particular, if \mathfrak{g} is of type $D(2, 1, \alpha)$, we assume that $\alpha \in \mathbb{R}$.

We let $N_{\alpha, \beta}$ be the structure constants for the chosen basis of root vectors:

$$[X_\alpha, X_\beta] = N_{\alpha, \beta} X_{\alpha+\beta}, \quad \text{if } \alpha, \beta, \alpha + \beta \in \Delta.$$

Set $\sigma_\alpha = -1$ if α is an odd negative root and $\sigma_\alpha = 1$ otherwise, so that $(X_\alpha, X_{-\alpha}) = \sigma_\alpha$. We also let $p(\alpha)$ be the parity of α : $p(\alpha) = 1$ if α is odd and $p(\alpha) = 0$ if α is even. The following statement is a reformulation of Lemma 3.2 of [8].

Lemma 4.3. *Given $\alpha, \beta \in \Delta$ such that $\alpha + \beta \in \Delta$, let p, q be non-negative integers such that $\beta + i\alpha \in \Delta \cup \{0\}$, $i \in \mathbb{Z}$ if and only if $-p \leq i \leq q$.*

- (1) If α is even, then $N_{\alpha,\beta}N_{-\alpha,\alpha+\beta} = \frac{1}{2}q(p+1)(\alpha, \alpha)$
 (2) If α is odd and $(\alpha, \alpha) \neq 0$, then

$$N_{\alpha,\beta}N_{-\alpha,\alpha+\beta} = \begin{cases} -\sigma_\alpha \frac{1}{2}q(\alpha, \alpha) & \text{if } p \text{ is even} \\ \sigma_\alpha \frac{1}{2}(p+1)(\alpha, \alpha) & \text{if } p \text{ is odd} \end{cases}.$$

- (3) If α is odd and $(\alpha, \alpha) = 0$, then $N_{\alpha,\beta}N_{-\alpha,\alpha+\beta} = \begin{cases} \sigma_\alpha(\alpha, \beta) & \text{if } p \text{ is even} \\ 0 & \text{if } p \text{ is odd} \end{cases}.$

Proof. The first statement follows as in the case when \mathfrak{g} is a Lie algebra.

For the second statement, let us first assume that α and β are not proportional. Then, as in [8, Lemma 3.2], we obtain

$$N_{\alpha,\beta}N_{-\alpha,\alpha+\beta} = \sigma_\alpha \sum_{i=0}^p (-1)^i (\beta - i\alpha)(h_\alpha).$$

Since 2α is an even root, we see that the subspaces

$$\sum_{-p \leq 2i \leq q} \mathbb{C}\mathfrak{g}_{\beta+2i\alpha}, \quad \sum_{-p \leq 2i+1 \leq q} \mathbb{C}\mathfrak{g}_{\beta+(2i+1)\alpha}$$

are the irreducible components of $\sum_{-p \leq i \leq q} \mathbb{C}\mathfrak{g}_{\beta+i\alpha}$ viewed as a $\langle X_{2\alpha}, h_{2\alpha}, X_{-2\alpha} \rangle$ -module. It follows that $\beta(h_\alpha) + q(\alpha, \alpha) = -\beta(h_\alpha) + p(\alpha, \alpha)$ if $p - q$ is even while, if $q - p$ is odd, then $\beta(h_\alpha) + q(\alpha, \alpha) = -\beta(h_\alpha) + (p-1)(\alpha, \alpha)$ and $\beta(h_\alpha) + (q-1)(\alpha, \alpha) = -\beta(h_\alpha) + p(\alpha, \alpha)$. This implies that $p - q$ is even and $\beta(h_\alpha) = \frac{p-q}{2}(\alpha, \alpha)$. Substituting we find the statement. Suppose now that α and β are proportional. There are only two possibilities: $\beta = \alpha$ or $\beta = -2\alpha$. Both cases follow directly from the Jacobi identity

$$[[X_\alpha, X_{-\alpha}], X_\beta] = [X_\alpha, [X_{-\alpha}, X_\beta]] + (-1)^{p(\alpha)p(\beta)} [X_\alpha, X_\beta], X_{-\alpha}].$$

Finally, if $(\alpha, \alpha) = 0$, then, as in Lemma 3.2 of [8], we obtain

$$N_{\alpha,\beta}N_{-\alpha,\alpha+\beta} = \sigma_\alpha \sum_{i=0}^p (-1)^i (\beta - i\alpha)(h_\alpha),$$

hence the statement follows readily. \square

As shown in the proof Lemma 3.3 of [8], we have that

$$N_{-\alpha,\alpha+\beta} = (-1)^{p(\alpha)} \frac{\sigma_{\alpha+\beta}}{\sigma_\beta} N_{-\beta,-\alpha}.$$

Substituting in Lemma 4.3, and using the fact that if $\alpha, \beta, \alpha + \beta \in \Delta$ then $N_{\alpha,\beta} \neq 0$, we find

$$(4.4) \quad N_{\alpha,\beta}N_{-\beta,-\alpha} = \frac{\sigma_\beta}{2\sigma_{\alpha+\beta}} q(p+1)(\alpha, \alpha)$$

if α is even,

$$(4.5) \quad N_{\alpha,\beta}N_{-\beta,-\alpha} = \begin{cases} \frac{\sigma_\alpha \sigma_\beta}{2\sigma_{\alpha+\beta}} q(\alpha, \alpha) & \text{if } p \text{ is even} \\ \frac{-\sigma_\alpha \sigma_\beta}{2\sigma_{\alpha+\beta}} (p+1)(\alpha, \alpha) & \text{if } p \text{ is odd} \end{cases}$$

if α is odd and $(\alpha, \alpha) \neq 0$, and

$$(4.6) \quad N_{\alpha,\beta}N_{-\beta,-\alpha} = \frac{-\sigma_\alpha \sigma_\beta}{\sigma_{\alpha+\beta}} (\alpha, \beta)$$

if α is odd and $(\alpha, \alpha) = 0$.

Introduce the anti-involution T of \mathfrak{g} defined on the local part by

$$\begin{cases} T(h) = h & \text{if } h \in \mathfrak{h}, \\ T(e_i) = f_i & \text{if } \alpha_i \text{ is even,} \\ T(e_i) = \sqrt{-1}f_i & \text{if } \alpha_i \text{ is odd.} \end{cases}$$

Since it preserves relations (4.3), it can be extended to \mathfrak{g} , and $T(\mathfrak{g}_\alpha) = \mathfrak{g}_{-\alpha}$. As shown in Lemma 3.8 of [8], we can choose $X_\alpha, X_{-\alpha}$ in such a way that $(X_\alpha, X_{-\alpha}) = 1$ and $T(X_\alpha) = \sqrt{-1}^{p(\alpha)}\sigma_\alpha X_{-\alpha}$, and we can still assume that $X_{\alpha_i} = e_i$ and $X_{-\alpha_i} = f_i$.

It follows that, if $\alpha, \beta, \alpha + \beta \in \Delta$, then

$$\begin{aligned} N_{\alpha, \beta} \sqrt{-1}^{p(\alpha+\beta)} \sigma_{\alpha+\beta} X_{-\alpha-\beta} &= T([X_\alpha, X_\beta]) \\ &= [T(X_\beta), T(X_\alpha)] = N_{-\beta, -\alpha} \sqrt{-1}^{p(\alpha)+p(\beta)} \sigma_\alpha \sigma_\beta X_{-\alpha-\beta}, \end{aligned}$$

hence

$$(4.7) \quad N_{\alpha, \beta} = (-1)^{p(\alpha)p(\beta)} \frac{\sigma_\alpha \sigma_\beta}{\sigma_{\alpha+\beta}} N_{-\beta, -\alpha} = -\frac{\sigma_\alpha \sigma_\beta}{\sigma_{\alpha+\beta}} N_{-\alpha, -\beta}.$$

Combining (4.7) with (4.4), (4.5), (4.6), it follows that

$$(4.8) \quad N_{\alpha, \beta}^2 = \frac{1}{2}q(p+1)(\alpha, \alpha)$$

if α is even,

$$(4.9) \quad N_{\alpha, \beta}^2 = \begin{cases} (-1)^{p(\beta)} \frac{q(\alpha, \alpha)}{2} & \text{if } p \text{ is even} \\ -(-1)^{p(\beta)} \frac{(p+1)(\alpha, \alpha)}{2} & \text{if } p \text{ is odd} \end{cases}$$

if α is odd and $(\alpha, \alpha) \neq 0$, and

$$(4.10) \quad N_{\alpha, \beta}^2 = -(-1)^{p(\beta)}(\alpha, \beta).$$

if α is odd and $(\alpha, \alpha) = 0$. Thus $N_{\alpha, \beta}$ is either real or pure imaginary.

If $(\alpha, \alpha) \neq 0$, let $\epsilon_\alpha = \text{sgn}(\alpha, \alpha)$. Let

$$\xi_\alpha = \begin{cases} \epsilon_\alpha & \text{if } p(\alpha) = 0, \\ 1 & \text{if } p(\alpha) = 1. \end{cases}$$

Lemma 4.4.

$$(4.11) \quad \overline{N_{\alpha, \beta}} = \frac{\xi_\alpha \xi_\beta}{\xi_{\alpha+\beta}} N_{\alpha, \beta}.$$

Proof. Since $N_{\alpha, \beta} = \pm N_{\beta, \alpha}$ it is enough consider the following three cases:

- (1) α is even;
- (2) α is odd non-isotropic and β is odd;
- (3) α and β are isotropic.

In case (1), by (4.8), we have $\overline{N_{\alpha, \beta}} = \xi_\alpha N_{\alpha, \beta}$. Thus the result follows obviously if β is odd. If β is even, then observe that (4.8) implies that $N_{\alpha, \beta} \neq 0$ if and only if $\epsilon_\alpha = \epsilon_\beta = \epsilon_{\alpha+\beta}$. This observation implies the result.

In case (2), by Lemma 4.3 (2), the product $N_{\alpha, \beta} N_{-\alpha, \alpha+\beta}$ is real; by (1), $N_{-\alpha, \alpha+\beta}$ is real iff $\xi_{\alpha+\beta} = 1$, so $N_{\alpha, \beta}$ is real iff $\xi_{\alpha+\beta} = 1$ as required.

In case (3), we have $2(\alpha, \beta) = (\alpha + \beta, \alpha + \beta)$, so, by (4.10), $\overline{N_{\alpha, \beta}} = \xi_{\alpha+\beta} N_{\alpha, \beta}$ as desired.

□

In the above setting, given complex numbers $\lambda_1, \dots, \lambda_n$ such that $\lambda_i \in \sqrt{-1}\mathbb{R}$ if $p(\alpha_i) = 1$ and $\lambda_i \in \mathbb{R}$ if $p(\alpha_i) = 0$, one can define an antilinear involution $\omega : \mathfrak{g} \rightarrow \mathfrak{g}$ setting

$$(4.12) \quad \omega(e_i) = \lambda_i f_i, \quad \omega(f_i) = \bar{\lambda}_i^{-1} e_i, \quad \omega(h_{\alpha_i}) = -h_{\alpha_i}, \quad 1 \leq i \leq n.$$

Since ω preserves relations (4.3), it follows that it extends to \mathfrak{g} .

Clearly $\omega(X_\alpha)$ is a multiple of $X_{-\alpha}$. We define λ_α for each $\alpha \in \Delta$ by

$$(4.13) \quad \omega(X_\alpha) = -\sigma_\alpha \xi_\alpha \lambda_\alpha X_{-\alpha}.$$

Hence, if $\alpha, \beta, \alpha + \beta \in \Delta$, since ω is an antilinear automorphism, we have

$$-\sigma_{\alpha+\beta} \xi_{\alpha+\beta} \lambda_{\alpha+\beta} \bar{N}_{\alpha,\beta} = \sigma_\alpha \sigma_\beta \xi_\alpha \xi_\beta \lambda_\alpha \lambda_\beta N_{-\alpha,-\beta}.$$

Using (4.7) and (4.11), we deduce that

$$(4.14) \quad \lambda_{\alpha+\beta} = \lambda_\alpha \lambda_\beta, \quad \lambda_\alpha \lambda_{-\alpha} = 1.$$

It follows that, if $\alpha = \sum_{i=1}^n n_i \alpha_i$, then

$$(4.15) \quad \lambda_\alpha = \prod_i (-\xi_{\alpha_i} \lambda_i)^{n_i}.$$

Endow \mathfrak{g} with the \mathbb{Z} -grading

$$(4.16) \quad \mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{q}_i$$

which assigns degree 0 to $h \in \mathfrak{h}$ and to e_i and f_i if α_i is even, and degree 1 to e_i and degree -1 to f_i , if α_i is odd.

Let $\pi = \{i \mid 1 \leq i \leq n, p(\alpha_i) = 1\}$, $\pi^c = \{1, \dots, n\} \setminus \pi$.

Proposition 4.5.

- (1) The set \mathfrak{g}_0^ω of ω -fixed points in \mathfrak{g}_0 is a real form of \mathfrak{g}_0 .
- (2) \mathfrak{q}_0^ω is a compact form of \mathfrak{q}_0 if and only if $\lambda_i(\alpha_i, \alpha_i) < 0$ for all $i \in \pi^c$.
- (3) If $\lambda_i(\alpha_i, \alpha_i) < 0$ for all $i \in \pi^c$ and $\sqrt{-1}\lambda_i$ have the same sign for all $i \in \pi$, then, for any positive integer r , $(X_\alpha, \omega(X_\alpha))(\alpha, \alpha) < 0$ if $X_\alpha \in \mathfrak{q}_{4r} \oplus \mathfrak{q}_{-4r}$ and $(X_\alpha, \omega(X_\alpha))(\alpha, \alpha) > 0$ if $X_\alpha \in \mathfrak{q}_{4r-2} \oplus \mathfrak{q}_{-4r+2}$.

Proof. The proof of the first assertion is standard. For the latter two claims observe that, by (4.13),

$$(4.17) \quad (X_\alpha, \omega(X_\alpha)) = -\sigma_\alpha \xi_\alpha \lambda_\alpha.$$

If α is an even root, the r.h.s. of (4.17) is a real number whose sign does not depend on the choice of the basis of root vectors X_α . In fact, if $\{X'_\alpha\}$ is another basis with $[X'_\alpha, X'_{-\alpha}] = \sigma_\alpha h_\alpha$, then $X'_\alpha = c_\alpha X_\alpha$ for suitable complex numbers c_α and $(X'_\alpha, \omega(X'_\alpha)) = -|c_\alpha|^2 \sigma_\alpha \xi_\alpha \lambda_\alpha$. Therefore we can use formula (4.15) to prove the statements in a straightforward way. As for (2), recall that a real Lie algebra is compact if the form $(\cdot, \omega(\cdot))$ has signature opposite to that of the invariant form (\cdot, \cdot) . If $X_\alpha \in \mathfrak{q}_0$ then $\lambda_\alpha = \prod (-\xi_{\alpha_i} \lambda_i)^{n_i}$. Therefore $-\xi_\alpha \sigma_\alpha \lambda_\alpha = -\xi_\alpha \prod (-\xi_{\alpha_i} \lambda_i)^{n_i} > 0$ for any α if and only if $\lambda_i(\alpha_i, \alpha_i) < 0$ for all $i \in \pi^c$. Claim (3) is proved in a similar way. □

Let

$$V = \text{fixed point set of } \omega|_{\mathfrak{g}_1}.$$

Clearly V is a real form of \mathfrak{g}_1 . Since a basis of V is given by

$$(4.18) \quad \{X_\alpha + \omega(X_\alpha) \mid X_\alpha \in \mathfrak{q}_i, i \in 2\mathbb{Z} + 1\},$$

it is checked easily that $\langle \cdot, \cdot \rangle = (\cdot, \cdot)|_V$ is a real nondegenerate symplectic form. Since $ad_{|\mathfrak{g}_1}((\mathfrak{g}_0)_{\mathbb{R}}) \subset ad_{|\mathfrak{g}_1}(\mathfrak{g}_0) \cap sp(V)$, we have that

$$ad_{|\mathfrak{g}_1}((\mathfrak{g}_0)_{\mathbb{R}}) = ad_{|\mathfrak{g}_1}(\mathfrak{g}_0) \cap sp(V)$$

and that $ad_{|\mathfrak{g}_1}(\mathfrak{g}_0) \cap sp(V)$ is a real form of $ad_{|\mathfrak{g}_1}(\mathfrak{g}_0)$.

4.3. Dynkin diagrams. We will briefly recall the usual encoding of sets of positive roots by means of Dynkin diagrams. In the following $N = n + m + 1$ and \circ, \otimes, \square correspond respectively to even, isotropic and nonisotropic (both even and odd) roots. A \cdot is a placeholder for even or isotropic roots. The possible diagrams in each type are listed in [9]. We reproduce this list below.

4.3.1. *Type $gl(m + 1, n + 1)$.*

$$(4.19) \quad \begin{array}{ccccccc} \cdot & \text{---} & \cdot & \text{---} & \dots & \text{---} & \cdot & \text{---} & \cdot \\ \alpha_1 & & \alpha_2 & & & & \alpha_N & & \alpha_{N+1} \end{array}$$

4.3.2. *Types $B(m + 1, n + 1)$ and $D(m + 1, n + 1)$.* The possible diagrams in these types are

$$(4.20) \quad \begin{array}{ccccccc} \cdot & \text{---} & \cdot & \text{---} & \dots & \text{---} & \cdot & \rightleftharpoons & \square \\ \alpha_1 & & \alpha_2 & & & & \alpha_N & & \alpha_{N+1} \end{array}$$

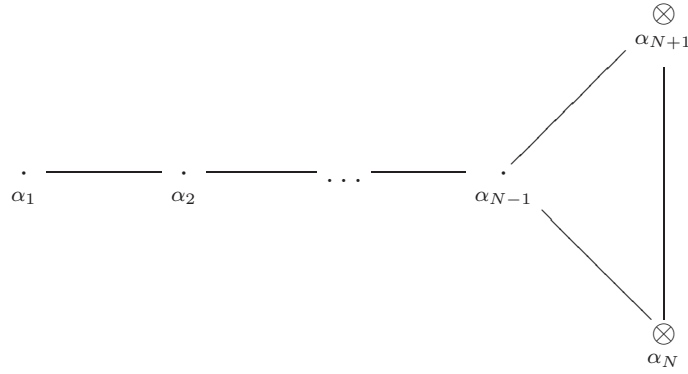
for type $B(m + 1, n + 1)$, and

$$(4.21) \quad \begin{array}{ccccccc} \cdot & \text{---} & \cdot & \text{---} & \dots & \text{---} & \circ & \rightleftharpoons & \circ \\ \alpha_1 & & \alpha_2 & & & & \alpha_N & & \alpha_{N+1} \end{array}$$

$$(4.22) \quad \begin{array}{ccccccc} \cdot & \text{---} & \cdot & \text{---} & \dots & \text{---} & \cdot & \leftarrow & \otimes & \rightarrow & \circ \\ \alpha_1 & & \alpha_2 & & & & \alpha_{N-1} & & \alpha_N & & \alpha_{N+1} \end{array}$$

$$(4.23) \quad \begin{array}{ccccccc} & & & & & & \circ & & & & \\ & & & & & & \alpha_{N+1} & & & & \\ & & & & & & \nearrow & & & & \\ \cdot & \text{---} & \cdot & \text{---} & \dots & \text{---} & \cdot & & & & \\ \alpha_1 & & \alpha_2 & & & & \alpha_{N-1} & & & & \\ & & & & & & \searrow & & & & \\ & & & & & & \alpha_N & & & & \end{array}$$

(4.24)


 for $D(m+1, n+1)$.

4.4. Distinguished sets of positive roots.

Definition 4.1. *We say that a set of positive roots Δ^+ is distinguished if $\mathfrak{q}_i = \{0\}$ for $|i| > 2$ in grading (4.16). We say that a set of positive roots Δ^+ is very distinguished if the corresponding set of simple roots contains a unique odd root.*

Since $\max\{a_j \mid \sum_{i=1}^n a_i \alpha_i \in \Delta^+, p(\alpha_j) = 1\} = 2$ for any choice of Δ^+ , we have that if Δ^+ is very distinguished, then it is distinguished. The sets of positive roots which in [10] and in most of the subsequent literature are called distinguished are all “very distinguished”. We have preferred to change the terminology because of the relationships with the Theta correspondence.

We now discuss the possible distinguished subsets of positive roots up to $W_{\mathfrak{g}}$ -equivalence. The classification is basically a case by case inspection, looking at all possible diagrams (cf. [9] and [22] for the exceptional types) and calculating the coefficients which express the positive roots as a linear combination of the simple roots.

4.4.1. $gl(m, n)$. Given non negative integers p, q with $p + q = m$, we let $\Delta_{gl}^{(p,q)}$ be the set of positive roots for $gl(m, n)$, corresponding to the following set of simple roots

$$\Pi_{gl}^{(p,q)} = \{\epsilon_1 - \epsilon_2, \dots, \epsilon_p - \delta_1, \delta_1 - \delta_2, \dots, \delta_n - \epsilon_{p+1}, \epsilon_{p+1} - \epsilon_{p+2}, \dots, \epsilon_{m-1} - \epsilon_m\}.$$

This is essentially the only possibility for a distinguished set of positive roots (including the possibility $p = 0$ or $q = 0$, in which case $\mathfrak{q}_i = \{0\}$ for $|i| > 1$). The other one is to take non negative integers r, s such that $r + s = n$ and exchanging ϵ 's with δ 's. But this case is equivalent to the above by exchanging m and n , so we will not distinguish these two possibilities.

4.4.2. $B(m, n)$. There is a unique, up to $W_{\mathfrak{g}}$ -action, distinguished set of positive roots Δ_B^+ . The corresponding set of simple roots, with notation as in [9], is

$$(4.25) \quad \Pi_B = \{\delta_1 - \delta_2, \dots, \delta_n - \epsilon_1, \epsilon_1 - \epsilon_2, \dots, \epsilon_{m-1} - \epsilon_m, \epsilon_m\}.$$

We also allow the case $m = 0$, in which the set of simple roots is $\{\delta_1 - \delta_2, \dots, \delta_{n-1} - \delta_n, \delta_n\}$.

4.4.3. $D(m, n)$. There are three distinguished sets of positive roots, up to $W_{\mathfrak{g}}$ -action: $\Delta_{D1}^+, \Delta_{D2}^+, \Delta_{D2'}^+$. The corresponding sets of simple roots are

$$\begin{aligned}\Pi_{D1} &= \{\delta_1 - \delta_2, \dots, \delta_n - \epsilon_1, \epsilon_1 - \epsilon_2, \dots, \epsilon_{m-1} - \epsilon_m, \epsilon_{m-1} + \epsilon_m\}, \\ \Pi_{D2} &= \{\epsilon_1 - \epsilon_2, \dots, \epsilon_{m-1} - \epsilon_m, \epsilon_m - \delta_1, \delta_1 - \delta_2, \dots, \delta_{n-1} - \delta_n, 2\delta_n\}, \\ \Pi_{D2'} &= \{\epsilon_1 - \epsilon_2, \dots, \epsilon_{m-1} + \epsilon_m, -\epsilon_m - \delta_1, \delta_1 - \delta_2, \dots, \delta_{n-1} - \delta_n, 2\delta_n\}.\end{aligned}$$

4.4.4. $C(n+1)$. There are three distinguished sets of positive roots, up to $W_{\mathfrak{g}}$ -action: $\Delta_{C1}^+, \Delta_{C2}^+, \Delta_{C2'}^+$. The corresponding sets of simple roots are

$$\begin{aligned}\Pi_{C1} &= \{\delta_1 - \delta_2, \dots, \delta_{n-1} - \delta_n, \delta_n - \epsilon, \delta_n + \epsilon\}, \\ \Pi_{C2} &= \{\epsilon - \delta_1, \delta_1 - \delta_2, \dots, \delta_{n-1} - \delta_n, 2\delta_n\}, \\ \Pi_{C2'} &= \{-\epsilon - \delta_1, \delta_1 - \delta_2, \dots, \delta_{n-1} - \delta_n, 2\delta_n\}.\end{aligned}$$

4.4.5. *Exceptional types*. A direct inspection shows that the distinguished sets of positive roots correspond to the following diagrams

$$(4.26) \quad \begin{array}{c} \circ \\ \diagup \\ \otimes \\ \diagdown \\ \circ \end{array} \quad \begin{array}{c} \otimes \text{---} \circ \text{---} \circ \text{---} \circ \\ \leftarrow \leftarrow \leftarrow \leftarrow \end{array}$$

for $D(2, 1, \alpha)$ (left display; it will be denoted by $\Delta_{D(2,1,\alpha)}^+$), $G(3)$ (right display; it will be denoted by Δ_G^+), and to

$$(4.27) \quad \begin{array}{c} \otimes \text{---} \circ \text{---} \circ \text{---} \circ \\ \leftarrow \leftarrow \leftarrow \leftarrow \end{array} \quad \begin{array}{c} \circ \text{---} \circ \text{---} \otimes \text{---} \circ \text{---} \circ \\ \leftarrow \leftarrow \leftarrow \leftarrow \end{array}$$

for $F(4)$ (they will be denoted by $\Delta_{F1}^+, \Delta_{F2}^+$, respectively).

4.5. **Distinguished sets of positive roots and real forms.** If Δ^+ is distinguished we choose ω corresponding to $\lambda_i = -\epsilon_{\alpha_i}$ if α_i is an even simple root and $\lambda_i = \sqrt{-1}$ if α_i is an odd simple root. Set, for any $\alpha \in \Delta_1^+$,

$$e_\alpha = \frac{1}{\sqrt{2}}(X_\alpha - \sqrt{-1}X_{-\alpha}), \quad f_\alpha = \frac{1}{\sqrt{2}}(X_{-\alpha} - \sqrt{-1}X_\alpha).$$

Since $\mathfrak{g}_1 = \mathfrak{q}_1 + \mathfrak{q}_{-1}$, applying formula (4.18) with our choice of ω , we get that these vectors form a standard symplectic basis of V . Since

$$\bigoplus_{\alpha \in \Delta_1^+} \mathbb{C}(e_\alpha \pm \sqrt{-1}f_\alpha) = \mathfrak{g}_1^\pm,$$

the oscillator representation is exactly $M^{\Delta^+}(\mathfrak{g}_1)$ as a $sp(\mathfrak{g}_1)$ -module.

Observe now that for any Lie superalgebra of basic classical type we have

$$(4.28) \quad \mathfrak{g}_0 = \mathfrak{g}_0^1 \times \mathfrak{g}_0^2$$

with $\mathfrak{g}_0^1, \mathfrak{g}_0^2$ reductive Lie algebras. Set

$$(4.29) \quad \mathfrak{s}_i = ad_{|\mathfrak{g}_1}(\mathfrak{g}_0^i) \cap sp(V), \quad i = 1, 2.$$

A case by case check in the distinguished sets of positive roots shows that we can always choose $\mathfrak{g}_0^1, \mathfrak{g}_0^2$ so that:

$$\mathfrak{g}_0^1 \subset \mathfrak{q}_0, \quad \mathfrak{g}_0^2 = (\mathfrak{q}_0 \cap \mathfrak{g}_0^2) \oplus (\mathfrak{q}_{-2} + \mathfrak{q}_2).$$

In the case of $\mathfrak{g} = gl(m, n)$, we choose $\mathfrak{g}_0^1 = gl(m), \mathfrak{g}_0^2 = gl(n)$ or the other way around. Combining these observations with Lemma 4.5 we readily obtain the following result.

Proposition 4.6. *If Δ^+ is any distinguished set of positive roots of a Lie superalgebra of basic classical type, then \mathfrak{s}_i is a real form of $ad_{|\mathfrak{g}_1}(\mathfrak{g}_0^i)$. Moreover \mathfrak{s}_1 is a compact form of $ad_{|\mathfrak{g}_1}(\mathfrak{g}_0^1)$ and there is a Cartan decomposition $\mathfrak{k} \oplus \mathfrak{r}$ of \mathfrak{s}_2 such that $\mathfrak{k}^{\mathbb{C}} = ad_{|\mathfrak{g}_1}(\mathfrak{g}_0^2 \cap \mathfrak{q}_0)$ and $\mathfrak{r}^{\mathbb{C}} = ad_{|\mathfrak{g}_1}(\mathfrak{q}_2 \oplus \mathfrak{q}_{-2})$.*

In particular, if $\mathfrak{q}_2 = \mathfrak{q}_{-2} = \{0\}$ then $\mathfrak{s}_1 \times \mathfrak{s}_2$ is a compact form of $ad_{|\mathfrak{g}_1}(\mathfrak{q}_0) = ad_{|\mathfrak{g}_1}(\mathfrak{g}_0)$.

4.6. Detour: classification of real forms. We now deepen some aspects of the theory developed in the last two Subsections, in order to obtain an explicit construction of all real forms of Lie superalgebras of basic classical type, which have been classified in [9], [18] and [21]. In this section when referring to type $A(m, n)$ we mean either $\mathfrak{g} = gl(m, n)$ or the simple superalgebra of type $A(m, n)$.

Let $\mathfrak{g}_{\mathbb{R}}$ be a real form of \mathfrak{g} and let ω be the corresponding complex conjugation. Then $\omega|_{\mathfrak{g}_0}$ is an antilinear involution of the Lie algebra \mathfrak{g}_0 , hence there is a corresponding Cartan decomposition $\mathfrak{g}_0 = \mathfrak{k} \oplus \mathfrak{p}$, with Cartan involution ζ_{ω} . We will describe explicitly the involution ζ_{ω} .

Let K_0 be a compact Lie group having \mathfrak{k}^{ω} as Lie algebra and let $\mathfrak{t}_{\mathbb{R}}$ be the Lie algebra of a torus in K_0 . Then $\mathfrak{t} = \mathfrak{t}_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$ is a Cartan subalgebra of \mathfrak{k} . Let \mathfrak{h} be the centralizer in \mathfrak{g}_0 of \mathfrak{t} . It is clear that both \mathfrak{t} and \mathfrak{h} are ω -stable. Since \mathfrak{h} is ζ_{ω} -stable, we have $\mathfrak{h} = (\mathfrak{h} \cap \mathfrak{k}) \oplus (\mathfrak{h} \cap \mathfrak{p})$; since $\omega\zeta_{\omega} = \zeta_{\omega}\omega$, we have $\omega(\mathfrak{h} \cap \mathfrak{p}) \subseteq \mathfrak{h} \cap \mathfrak{p}$. So $\mathfrak{h} \cap \mathfrak{p} = (\mathfrak{h} \cap \mathfrak{p})^{\omega} + \sqrt{-1}(\mathfrak{h} \cap \mathfrak{p})^{\omega}$. It is known that $\mathfrak{t}_{\mathbb{R}} \oplus \sqrt{-1}(\mathfrak{h} \cap \mathfrak{p})^{\omega}$ is a Cartan subalgebra of a compact real form of \mathfrak{g}_0 ; therefore roots are purely imaginary-valued on it. So the real span of roots is contained in $\sqrt{-1}\mathfrak{t}_{\mathbb{R}} \oplus (\mathfrak{h} \cap \mathfrak{p})^{\omega}$. Now fix a set of positive roots Δ^+ and let ω_0 be the involution given by (4.12) with the following choice of parameters:

$$(4.30) \quad \lambda_i = -\epsilon_{\alpha_i} \text{ for } i \notin \pi, \quad \lambda_i = \sqrt{-1} \text{ for } i \in \pi.$$

Then

$$(\omega \circ \omega_0)|_{\mathfrak{t}} = I, \quad (\omega \circ \omega_0)|_{\mathfrak{h} \cap \mathfrak{p}} = -I.$$

In particular, \mathfrak{t} is a Cartan subalgebra of $\mathfrak{g}_0^{\omega \circ \omega_0}$. It follows from [11, Chapter 8] that $(\omega \circ \omega_0)|_{\mathfrak{g}_0} = \eta_0 \circ e^{ad(h)}$ with η_0 a diagram automorphism of \mathfrak{g}_0 and $h \in \mathfrak{t}$. Clearly $\eta = \omega \circ \omega_0 \circ e^{-ad(h)}$ is an extension of η_0 to \mathfrak{g} hence we can write that

$$\omega = \eta \circ e^{ad(h)} \circ \omega_0,$$

with η an automorphism of \mathfrak{g} such that $\eta|_{\mathfrak{g}_0}$ is a diagram automorphism of \mathfrak{g}_0 . We now list the possible choices for automorphisms η of \mathfrak{g}_0 , such that $\eta|_{\mathfrak{g}_0}$ is a diagram automorphism of \mathfrak{g}_0 .

Lemma 4.7. *There exists $\eta \in Aut(\mathfrak{g})$ such that $\eta|_{\mathfrak{g}_0}$ is a nontrivial diagram automorphism if and only if \mathfrak{g} is of type $A(m, n), D(m, n), D(2, 1, \alpha), \alpha \in \{1, (-2)^{\pm 1}\}$, and $\eta|_{\mathfrak{g}_0}$ is as follows.*

- (1) If \mathfrak{g} is of type $A(n, m)$ with $n \neq m$, then $\eta|_{\mathfrak{g}_0}$ restricts to the nontrivial diagram automorphism of both A_n and A_m .
- (2) If \mathfrak{g} is of type $A(n, n)$, then $\eta|_{\mathfrak{g}_0}$ is either the nontrivial diagram automorphism of both A_n components, or it is the flip automorphism between the two A_n components, or the composition of these two automorphisms.
- (3) If \mathfrak{g} is of type $D(m, n)$, $m > 2$, then $\eta|_{\mathfrak{g}_0}$ is the unique diagram automorphism of \mathfrak{g}_0 .
- (4) If \mathfrak{g} is of type $D(2, 1) \cong D(2, 1, \alpha)$, $\alpha \in \{1, (-2)^{\pm 1}\}$, then η is the unique diagram automorphism of the diagram in the left display of (4.26).

Proof. No η as in the statement can exist if \mathfrak{g}_0 has no nontrivial diagram automorphisms. This rules out the cases $F(4), G(3), B(m, n)$ with $(m, n) \neq (2, 2)$. The case when \mathfrak{g} is of type $B(2, 2)$ is also excluded because the flip automorphism of $\mathfrak{g}_0 \cong B_2 \times B_2$ sends the unique odd simple root to a weight which is not a root. Let us now deal with type $A(m, n)$. An automorphism of \mathfrak{g} of type $A(m, n)$ which restricts to the nontrivial diagram automorphisms of both A_n and A_m is given by $\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-st} = \begin{pmatrix} -A^t & C^t \\ -B^t & -D^t \end{pmatrix}$. The flip automorphism F is given by $F \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} D & C \\ B & A \end{pmatrix}$ and the composition $F \circ (-st)$ gives the remaining automorphism. We are left with proving that there is no automorphism of \mathfrak{g} of type $A(m, n)$ which restricts trivially to, say, A_n and nontrivially to A_m or it is obtained from this one by composing it with the flip. This is easily checked by observing that these latter automorphisms map the simple odd root to a weight that is not a root.

If \mathfrak{g} is of type $D(m, n)$, an automorphism that restricts to the unique diagram automorphism of \mathfrak{g}_0 is $Ad(J)$ with $J = \begin{pmatrix} I_{m-1,1} & 0 \\ 0 & I_n \end{pmatrix}$, where $I_{m-1,1} = \begin{pmatrix} I_{m-1} & 0 \\ 0 & -1 \end{pmatrix} \in gl(m)$.

We now deal with the case when \mathfrak{g} is of type $D(2, 1, \alpha)$. If $\alpha \in \{1, (-2)^{\pm 1}\}$, then \mathfrak{g} is isomorphic to $D(2, 1)$ and claim $Ad(J)$ gives as above the desired automorphism. We need to check that, if $\alpha \notin \{1, (-2)^{\pm 1}\}$, then there is no automorphism η of \mathfrak{g} which restricts to a non trivial diagram automorphism of \mathfrak{g}_0 and that, if $\alpha = 1$, then $Ad(J)|_{\mathfrak{g}_0}$ is the only one. We check this as follows. Recall that $\mathfrak{g}_0 \cong (sl(2, \mathbb{C}))^{\times 3}$, and let $\beta_1, \beta_2, \beta_3$ be the simple roots of these copies of $sl(2, \mathbb{C})$. Since the form $(\eta(\cdot), \eta(\cdot))$ is invariant, then there exists a constant c such that $(\eta(\cdot), \eta(\cdot)) = c(\cdot, \cdot)$. If η restricts to a diagram automorphism of \mathfrak{g}_0 , then $\eta(\beta_i) = \beta_{\sigma(i)}$, $i = 1, 2, 3$, for a suitable permutation σ . Hence $(\beta_{\sigma(i)}, \beta_{\sigma(i)}) = c^{-1}(\beta_i, \beta_i)$. We check directly that this is not possible for a real value of α different from $1, (-2)^{\pm 1}$ and also that, if $\alpha = 1$, then the only possibility is given by $Ad(J)$. \square

The case of \mathfrak{g} being the simple Lie superalgebra of type $A(1, 1)$ needs special care, so we postpone the discussion of this case to the end of this section. Until then, we are excluding this case from our discussion.

In order to complete the classification we choose a very distinguished set of positive roots. If \mathfrak{g} is of type $D(m, n)$, we choose Δ_{D1}^+ . We wish to compute the action of ω on the generators e_i, f_i . We will often use the following result of Serganova.

Let δ_λ be the map on \mathfrak{g} defined by setting

$$(4.31) \quad \delta_{\lambda|_{\mathfrak{g}_0}} = Id, \quad \delta_{\lambda|_{\mathfrak{g}_1^+}} = \lambda Id, \quad \delta_{\lambda|_{\mathfrak{g}_1^-}} = \lambda^{-1} Id.$$

If \mathfrak{g} is of type I , then δ_λ is an automorphism of \mathfrak{g} for any $\lambda \in \mathbb{C}$. If \mathfrak{g} is of type II , then δ_λ is an automorphism of \mathfrak{g} if and only if $\lambda = \pm 1$. By [20, Lemmas 1 and 2], an automorphism of \mathfrak{g} that restricts to $Id_{\mathfrak{g}_0}$ is necessarily δ_λ .

Since the form $(\eta(\cdot), \eta(\cdot))$ is invariant, we have that there is $s_\eta \in \mathbb{C}$ such that $(\eta(\cdot), \eta(\cdot)) = s_\eta(\cdot, \cdot)$. Since $\eta^2 = Id$ on \mathfrak{g}_0 , we see that $s_\eta = \pm 1$. Let now i_0 be the index of the unique odd simple root. Set $\gamma = \eta(\alpha_{i_0})$. Then there is $\lambda \in \mathbb{C}$ such that $\eta(e_{i_0}) = \lambda X_\gamma$ so that $\eta(f_{i_0}) = s_\eta \sigma_\gamma \lambda^{-1} X_{-\gamma}$. If $\eta|_{\mathfrak{g}_0} = Id_{\mathfrak{g}_0}$, then we set $\eta_1 = Id_{\mathfrak{g}}$. If $\eta|_{\mathfrak{g}_0} \neq Id_{\mathfrak{g}_0}$ and \mathfrak{g} is of type $A(m, n)$, then we set $\eta_1 = \eta \circ \delta_{\lambda^{-1}}$, while, if \mathfrak{g} is of type $D(m, n)$ we set $\eta_1 = Ad(J)$ with J as in the proof of Lemma 4.7. Then, in all cases $\eta_1(e_{i_0}) = X_\gamma$ and we can write

$$\omega = \eta_1 \delta_\lambda e^{ad(h)} \omega_0.$$

We want to find conditions on h and λ for which $\omega = \eta_1 \delta_\lambda e^{ad(h)} \omega_0$ has order 2.

First assume that $\eta|_{\mathfrak{g}_0} = Id$. In this case $\mathfrak{h} = \mathfrak{t}$ and $\eta = \delta_\lambda$ for some $\lambda \in \mathbb{C}$. We see that $\omega(e_{i_0}) = \sqrt{-1} \lambda^{-1} e^{-\alpha_{i_0}(h)} f_{i_0}$, $\omega(e_i) = -\epsilon_{\alpha_i} e^{-\alpha_i(h)} f_i$ if $i \neq i_0$, $\omega(f_{i_0}) = \sqrt{-1} \lambda e^{\alpha_{i_0}(h)} e_{i_0}$, $\omega(e_i) = -\epsilon_{\alpha_i} e^{-\alpha_i(h)} f_i$ if $i \neq i_0$. Since ω is an involution we obtain that $\lambda e^{\alpha_{i_0}(h)} \in \mathbb{R}$ and $e^{\alpha_i(h)} \in \mathbb{R}$ if $i \neq i_0$, hence setting $\lambda_{i_0} = \sqrt{-1} \lambda^{-1} e^{-\alpha_{i_0}(h)}$ and $\lambda_i = -\epsilon_{\alpha_i} e^{-\alpha_i(h)}$ we see that ω has the form of (4.12).

Assume now that $\eta|_{\mathfrak{g}_0} \neq Id$. Recall that η acts on \mathfrak{g}_0 as a diagram automorphism. Since, by our choice of the set of positive roots, the simple roots of \mathfrak{g}_0 are simple in \mathfrak{g} if \mathfrak{g} is of type $A(m, n)$ while, in type $D(m, n)$, η_1 is a diagram automorphism of \mathfrak{g} , η_1 induces in both cases a transposition $i \mapsto i'$ on the indices corresponding to the simple even roots. If $i \notin \pi$, then

$$\omega(e_i) = \eta e^{ad(h)} \omega_0(e_i) = -\epsilon_{\alpha_i} \eta(e^{ad(h)} f_i) = -\epsilon_{\alpha_i} e^{-\alpha_i(h)} \eta(f_i) = -\epsilon_{\alpha_i} e^{-\alpha_i(h)} s_\eta f_{i'}.$$

Hence

$$\omega^2(e_i) = \overline{e^{-\alpha_i(h)}} e^{\alpha_{i'}(h)} e_i.$$

Since $h \in \mathfrak{t}$ we have $\alpha_{i'}(h) = \alpha_i(h)$, therefore we get $e^{\alpha_i(h)} \in \mathbb{R}$.

We distinguish the following cases.

- (1) If \mathfrak{g} is of type $D(m, n)$, then $\eta(\alpha_{i_0}) = \alpha_{i_0}$ and $s_{\eta_1} = 1$, hence

$$\omega^2(e_{i_0}) = e^{\alpha_{i_0}(h)} \overline{e^{-\alpha_{i_0}(h)}} e_{i_0}.$$

Therefore $e^{\alpha_{i_0}(h)} \in \mathbb{R}$.

- (2) If \mathfrak{g} is of type $A(m, n)$ and $(\eta_1)|_{\mathfrak{g}_0} = (-st)|_{\mathfrak{g}_0}$, then $\sigma_\gamma = -1$ and $(\eta_1) = (-st)\delta_\mu$ for some μ . Hence $(\eta_1)^2 = (-st)^2$, and we see that $(\eta_1)|_{\mathfrak{g}_1}^2 = -Id$. Moreover $s_{\eta_1} = 1$.

It follows that

$$\omega^2(e_{i_0}) = |\lambda|^{-2} e^{\gamma(h)} \overline{e^{-\alpha_{i_0}(h)}} e_{i_0}.$$

Since $h \in \mathfrak{t}$, we have $\gamma(h) = \alpha_{i_0}(h)$, hence $|\lambda| = 1$ and $e^{\alpha_{i_0}(h)} \in \mathbb{R}$.

- (3) If \mathfrak{g} is of type $A(n, n)$ and $(\eta_1)|_{\mathfrak{g}_0}$ is the flip, then $(\eta_1)|_{\mathfrak{g}_0} = F|_{\mathfrak{g}_0}$. It follows that $\sigma_\gamma = -1$ and $\eta_1 = F\delta_\mu$ for some μ . Hence $(\eta_1)^2 = F^2$, so $(\eta_1)|_{\mathfrak{g}_1}^2 = Id$. Moreover $s_{\eta_1} = -1$.

It follows that

$$\omega^2(e_{i_0}) = |\lambda|^{-2} e^{\gamma(h)} \overline{e^{-\alpha_{i_0}(h)}} e_{i_0}.$$

Since $h \in \mathfrak{t}$, we have $\gamma(h) = \alpha_{i_0}(h)$, hence $|\lambda| = 1$ and $e^{\alpha_{i_0}(h)} \in \mathbb{R}$.

(4) If \mathfrak{g} is of type $A(n, n)$ and $(\eta_1)|_{\mathfrak{g}_0} = F(-st)$ then $\gamma = \alpha_{i_0}$ and $s_{\eta_1} = -1$.

It follows that

$$\omega^2(e_{i_0}) = -\lambda e^{\alpha_{i_0}(h)} \overline{\lambda^{-1} e^{-\alpha_{i_0}(h)}} e_{i_0},$$

hence $\lambda e^{\alpha_{i_0}(h)} \in \sqrt{-1}\mathbb{R}$.

Let us now return to the general case. Fix h as above and define

$$\omega_h = \begin{cases} \delta_\lambda e^{ad(h)} \omega_0 & \text{if } \eta|_{\mathfrak{g}_0} = Id_{\mathfrak{g}_0} \\ e^{ad(h)} \omega_0 & \text{in cases (1), (2), (3)} \\ \delta_{\sqrt{-1}\lambda} e^{ad(h)} \omega_0 & \text{in case (4)}. \end{cases}$$

As observed above ω_h is an antilinear involution of the form of (4.12).

We are now ready to give an explicit formula for the Cartan involution corresponding to $\omega|_{\mathfrak{g}_0}$. Define the map $\zeta_{\omega_h} : \mathfrak{g} \rightarrow \mathfrak{g}$ by

$$(4.32) \quad \zeta_{\omega_h}(x) = x, \quad x \in \mathfrak{h}, \quad \zeta_{\omega_h}(X_\alpha) = sgn(\lambda_\alpha^h) X_\alpha,$$

where λ_α^h is the multiplier defined in (4.13). It is easy to check using (4.14) that ζ_{ω_h} is an involution of \mathfrak{g}_0 . We claim that

$$\zeta_\omega = \eta_1 \circ \zeta_{\omega_h}.$$

is the Cartan involution corresponding to $\omega|_{\mathfrak{g}_0}$.

First we prove that $sgn(\lambda_\alpha^h) = sgn(\lambda_{\eta_1(\alpha)}^h)$ for $\alpha \in \Delta_0^+$. We prove in fact that $\lambda_\alpha^h = \lambda_{\eta_1(\alpha)}^h$. A direct computation shows that $\lambda_\alpha^h = c_\alpha e^{-\alpha(h)}$ for any $\alpha \in \Delta_0^+$,

where $c_\alpha = \begin{cases} 1 & \text{if } X_\alpha \in \mathfrak{q}_0 \\ -\lambda^{\mp 2} & \text{if } X_\alpha \in \mathfrak{q}_{\pm 2} \end{cases}$. The claim follows since our choice of h forces

$$\alpha(h) = \eta_1(\alpha)(h).$$

Clearly $\eta_1(X_\alpha) = d_\alpha X_{\eta_1(\alpha)}$. We now prove that $|d_\alpha| = 1$ for any $\alpha \in \Delta_0$. For this observe that $d_{-\alpha} = s_{\eta_1} d_\alpha^{-1}$ and $d_{\eta_1(\alpha)} = d_\alpha^{-1}$. Hence, since $\omega^2(X_\alpha) = (\eta_1 \omega_h)^2(X_\alpha) = X_\alpha$, it follows that

$$1 = \epsilon_\alpha \epsilon_{-\eta_1(\alpha)} \lambda_\alpha^h \lambda_{-\eta_1(\alpha)}^h \bar{d}_{-\alpha} d_{\eta_1(\alpha)},$$

so $1 = \epsilon_\alpha \epsilon_{-\eta_1(\alpha)} s_{\eta_1} |d_\alpha|^{-2} = |d_\alpha|^{-2}$ as wished.

In order to prove that ζ_ω is the Cartan involution corresponding to $\omega|_{\mathfrak{g}_0}$, we need only to check that ω and ζ_ω commute and that $\omega|_{\mathfrak{g}_0} \circ \zeta_\omega$ is the conjugation of a compact real form of \mathfrak{g}_0 . The first claim is a simple computation:

$$\omega|_{\mathfrak{g}_0} \zeta_\omega(X_\alpha) = -\epsilon_{\eta_1(\alpha)} sgn(\lambda_\alpha^h) \lambda_{\eta_1(\alpha)}^h \bar{d}_\alpha d_{-\eta_1(\alpha)} X_{-\alpha} = -s_{\eta_1} \epsilon_{\eta_1(\alpha)} sgn(\lambda_\alpha^h) \lambda_\alpha^h X_{-\alpha},$$

while

$$\zeta_\omega \omega|_{\mathfrak{g}_0}(X_\alpha) = -\epsilon_\alpha sgn(\lambda_{-\eta_1(\alpha)}^h) \lambda_{-\eta_1(\alpha)}^h d_{-\alpha} d_{-\eta_1(\alpha)} X_{-\alpha} = -\epsilon_\alpha sgn(\lambda_\alpha^h) \lambda_\alpha^h X_{-\alpha},$$

so the claim follows from the observation that $s_{\eta_1} \epsilon_{\eta_1(\alpha)} = \epsilon_\alpha$.

For the second claim it is enough to check that $\epsilon_\alpha(X_\alpha, \omega \zeta_\omega(X_\alpha)) < 0$ for any $\alpha \in \Delta_0$. Indeed

$$\epsilon_\alpha(X_\alpha, \omega \zeta_\omega(X_\alpha)) = -\epsilon_\alpha sgn(\lambda_\alpha^h) \lambda_{\eta_1(\alpha)}^h \epsilon_{\eta_1(\alpha)} s_{\eta_1} = -sgn(\lambda_\alpha^h) \lambda_\alpha^h,$$

and this number is always negative.

Recall that we are excluding the case when \mathfrak{g} is the simple Lie superalgebra of type $A(1, 1)$. It follows from [9, Prop. 5.3.2] that, if ω and ω' are antilinear

involutions of \mathfrak{g} such that \mathfrak{g}^ω is isomorphic to $\mathfrak{g}^{\omega'}$, then ω' is conjugated by an isomorphism of \mathfrak{g} to either ω or $\omega \circ \delta_{-1}$. Consider now the antilinear involution $\omega_{\mathbb{R}}$ on \mathfrak{g} defined by $\omega_{\mathbb{R}}(e_i) = e_i$, $\omega_{\mathbb{R}}(f_i) = f_i$, and $\omega_{\mathbb{R}}(h) = h$ for $h \in \sqrt{-1}\mathfrak{h}^\omega$. Then, since ω_h is an antilinear involution of the type described in (4.12), we have that $\omega_{\mathbb{R}}\omega_h\omega_{\mathbb{R}} = \omega_h\delta_{-1}$. Moreover, if $\eta_1(\mathfrak{g}_1^+) \subset \mathfrak{g}_1^+$, we have that $\omega_{\mathbb{R}}\eta_1 = \eta_1\omega_{\mathbb{R}}$, so $\omega_{\mathbb{R}}$ provides a real isomorphism between \mathfrak{g}^ω and $\mathfrak{g}^{\omega\delta_{-1}}$. If instead $\eta_1(\mathfrak{g}_1^+) \subset \mathfrak{g}_1^-$, then \mathfrak{g} is of type $A(m, n)$, and $\delta_{\sqrt{-1}}$ provides an isomorphism between \mathfrak{g}^ω and $\mathfrak{g}^{\omega\delta_{-1}}$. It follows that we need only to classify the symmetric pairs $(\mathfrak{g}_0, \mathfrak{k})$ up to isomorphism. First of all we consider cases (3) and (4). In both cases $(\zeta_\omega)_{|\mathfrak{g}_0^1}$ gives an isomorphism from $\mathfrak{g}_0^1 \rightarrow \mathfrak{g}_0^2$, so the map $(X, Y) \mapsto X + \zeta_\omega(Y)$ gives an isomorphism between $\mathfrak{g}_0^1 \times \mathfrak{g}_0^1$ and \mathfrak{g}_0 that maps the diagonal copy of \mathfrak{g}_0^1 to the set of fixed points of ζ_ω . Thus the real form \mathfrak{g}_0^ω is isomorphic to \mathfrak{g}_0^1 seen as a real Lie algebra.

Let us now turn to the other cases. Choose $t \in \mathfrak{h}$ in such a way that $e^{\alpha_i(t)} = \text{sgn}(\lambda_{\alpha_i}^h)$ for $i \neq i_0$. We claim that we can choose $t \in \mathfrak{t}$. Since $\text{sgn}(\lambda_{\alpha_i}^h) = \text{sgn}(\lambda_{\eta_1(\alpha_i)}^h)$, it is clear that we can choose t so that $\eta_1(\alpha_i)(t) = \alpha_i(t)$ for $i \neq i_0$. It remains to check that we can choose t so that $\alpha_{i_0}(t) = \eta_1(\alpha_{i_0})(t)$. This is clear if $\eta_1(\alpha_{i_0}) = \alpha_{i_0}$, so we are left only with case (3). In this case we know that

$$e^{\alpha_{i_0}(h)} = e^{\eta_1(\alpha_{i_0})(h)} = e^{-\alpha_{i_0}(h)} \prod_{i \neq i_0} e^{-\alpha_i(h)}.$$

This implies that $(e^{\alpha_{i_0}(h)})^2 = \prod_{i \neq i_0} e^{-\alpha_i(h)}$. It follows that $\prod_{i \neq i_0} e^{-\alpha_i(h)} > 0$ so that

$$e^{\alpha_{i_0}(t)} = e^{-\alpha_{i_0}(t)} \prod_{i \neq i_0} \text{sgn}(e^{-\alpha_i(h)}) = e^{\eta_1(\alpha_{i_0})(t)}$$

hence we can choose $t \in \mathfrak{t}$ as wished.

If \mathfrak{g} is not of type $A(m, n)$, then the simple roots of \mathfrak{g} are linearly independent, so we can further assume that $e^{\alpha_{i_0}(t)} \in \mathbb{R}$. If \mathfrak{g} is of type $A(m, n)$ and $\eta_1 = Id_{\mathfrak{g}}$, then we can choose λ so that $\lambda e^{\alpha_{i_0}(t)} \in \mathbb{R}$. If, instead, we are in case (3), then, as shown above, $(e^{\alpha_{i_0}(t)})^2 = 1$, so, again, $e^{\alpha_{i_0}(t)} \in \mathbb{R}$. This implies that we can define an antilinear involution $\omega' = \eta_1 \delta_\lambda e^{ad(t)} \omega_0$. Observe that $\zeta_\omega = \zeta_{\omega'}$ so ω and ω' define the same real form. Moreover $\zeta_\omega = \eta_1 e^{ad(t')}$ with $t = t'$ if \mathfrak{g} is of type $C(n+1)$ or $A(n, m)$ and t' defined by setting $\alpha_i(t) = \alpha_i(t')$ for $i \neq i_0$, $\alpha_{i_0}(t') = \frac{\pi}{2}\sqrt{-1} + \alpha_{i_0}(t)$ in the other cases.

We will now use Kac classification of Lie algebra involutions [11] to classify ζ_ω . Let $\{\beta_1^i, \dots, \beta_k^i\}$ be the simple roots of $(\mathfrak{g}_0^i)^{\eta_1}$ corresponding to the set of positive roots $\Delta_{|\mathfrak{t}}^+$. Recall that $\beta_j^i = \alpha_{|\mathfrak{t}}$ for some simple root of \mathfrak{g}_0^i corresponding to Δ_0^+ . Let X_N^i be the diagram of \mathfrak{g}_0^i and let $(X_N^i)^{(r_i)}$ be the corresponding affinization with $r_i = 1$ if $(\eta_1)_{|\mathfrak{g}_0^i} = Id$ and $r_i = 2$ otherwise. Let a_j^i be the labels of $(X_N^i)^{(r_i)}$ and set $\theta_i = \sum_{j \geq 1} a_j^i \beta_j^i$. Using the analysis done in [12, §3] we find that there is $\tau \in \mathfrak{t}$ and w in the Weyl group of $\mathfrak{g}_0^{\eta_1}$ such that $2\pi\sqrt{-1}t'' = w(t') + \tau$ with $e^{ad(\tau)} = \delta_\mu$ and $t'' \in \mathfrak{t} \cap [\mathfrak{g}_0, \mathfrak{g}_0]$ having the following property: if $s_j^i = \beta_j^i(t'')$ and $s_0^i = \frac{1}{r_i} - \theta_i(t'')$, then

$$(4.33) \quad s_j^i \in \{0, \frac{1}{2}, 1\} \text{ and } \sum_{j \geq 0} a_j^i s_j^i = \frac{1}{r_i}.$$

It follows that $e^{2\pi\sqrt{-1}w(\alpha_i)(t'')} = e^{\alpha_i(t')}$ for all $i \neq i_0$ and that $\lambda\mu^{-1}e^{w(\alpha_{i_0})(2\pi\sqrt{-1}t'')} = \lambda e^{\alpha_{i_0}(t')}$. This implies that $e^{2\pi\sqrt{-1}\alpha_i(t'')} \in \mathbb{R}$ for all $i \neq i_0$. Moreover, in type $A(m, n)$ and $C(n+1)$ with $\eta_1 = Id_{\mathfrak{g}}$, $\lambda\mu^{-1}e^{\alpha_{i_0}(2\pi\sqrt{-1}t'')} \in \mathbb{R}$. In type $A(m, n)$ with η_1 as in case (3), since $\tau \in \mathfrak{t}$ we must have $\mu^2 = 1$ so $e^{\alpha_{i_0}(2\pi\sqrt{-1}t'')} \in \mathbb{R}$. In the other cases we have $e^{2\pi\sqrt{-1}\alpha_{i_0}(t'')} \in \sqrt{-1}\mathbb{R}$. In turn, this implies that there is an antilinear involution ω'' such that $\zeta_{\omega''} = \eta_1 e^{2\pi\sqrt{-1}ad(t'')}$. Since $\zeta_{\omega''}$ is conjugated to ζ_{ω} , we see that ω and ω'' define isomorphic real forms. The outcome is the following:

(1) If \mathfrak{g} is of type $C(n+1)$ or $A(m, n)$ and $\eta_1 = Id_{\mathfrak{g}}$, then the list of all $t'' \in \mathfrak{h} \cap [\mathfrak{g}_0, \mathfrak{g}_0]$ that satisfy (4.33) and such that there is $\lambda \in \mathbb{C}$ with $\lambda e^{\alpha_{i_0}(2\pi\sqrt{-1}t'')} \in \mathbb{R}$ gives a list of all real forms. Since the latter is an empty condition, we see that the list of all real forms is given by all the $t'' \in \mathfrak{h} \cap [\mathfrak{g}_0, \mathfrak{g}_0]$ that satisfy (4.33).

(2) If \mathfrak{g} is of type $A(m, n)$ and η_1 is as in case (3), then the list of all $t'' \in \mathfrak{t} \cap [\mathfrak{g}_0, \mathfrak{g}_0]$ that satisfy (4.33) and such that $e^{2\pi\sqrt{-1}\alpha_{i_0}(t'')} \in \mathbb{R}$ gives a list of all real forms.

(3) In all the remaining cases the list of all $t'' \in \mathfrak{t} \cap [\mathfrak{g}_0, \mathfrak{g}_0]$ that satisfy (4.33) and such that $e^{2\pi\sqrt{-1}\alpha_{i_0}(t'')} \in \sqrt{-1}\mathbb{R}$ gives the list of all real forms.

The set of $t'' \in \mathfrak{t} \cap [\mathfrak{g}_0, \mathfrak{g}_0]$ that satisfy (4.33) is finite, so we need only to check the above conditions on this finite set. The outcome of this computation is given in the following tables. In all cases except $A(m, n)$ and $C(n+1)$ define $\varpi_i \in \mathfrak{h}$ by $\alpha_j(\varpi_i) = \delta_{ij}$ where the simple roots of \mathfrak{g} are enumerated from left to right. If \mathfrak{g} is of type $A(m, n)$ or $C(n+1)$ we define ϖ_i for $i \neq i_0$ to be the unique element in $\mathfrak{h} \cap [\mathfrak{g}_0, \mathfrak{g}_0]$ such that $\alpha_i(\varpi_j) = \delta_{ij}$ for all $i \neq i_0$. In the following tables we assume that Δ^+ is a very distinguished set of positive roots. For types D, C, F we choose $\Delta_{D1}^+, \Delta_{C2}^+, \Delta_{F1}^+$, respectively. Table I displays the real forms corresponding to antilinear involutions with $\eta|_{\mathfrak{g}_0} = Id_{\mathfrak{g}_0}$ and Table II covers the remaining cases. In Table I we also list the parameters $(\lambda_1, \dots, \lambda_n)$ occurring in (4.12), setting, only once in the paper, $i = \sqrt{-1}$; the λ_j which are not listed are equal to 1.

\mathfrak{g}	real form of \mathfrak{g}_0	t''	$(\lambda_1, \dots, \lambda_n)$
$gl(m, n)$	$u(p, m-p) \times u(q, n-q)$	$\frac{1}{2}\varpi_p + \frac{1}{2}\varpi_{n+q},$ $(0 \leq p \leq m-1, 1 \leq q \leq n)$	$\lambda_m = i$ $\lambda_j = \lambda_{m+q} = -1$ $(1 \leq j \leq m, j \neq p)$
$A(m-1, n-1),$ $m \neq n$	$su(p, m-p) \times su(q, n-q) \times \mathbb{R}$	$\frac{1}{2}\varpi_p + \frac{1}{2}\varpi_{n+q},$ $(0 \leq p \leq m-1, 1 \leq q \leq n)$	same as above;
$A(n-1, n-1)$	$su(p, n-p) \times su(q, n-q)$	$\frac{1}{2}\varpi_p + \frac{1}{2}\varpi_{n+q},$ $(0 \leq p \leq n-1, 1 \leq q \leq n)$	same as above with $m = n$;
$B(0, n)$	$sp(2n, \mathbb{R})$	$\frac{1}{4}\varpi_n$	$\lambda_n = i$ $\lambda_j = -1$ $(1 \leq j \leq n-1)$
$B(m, n)$	$so(2m+1) \times sp(2n, \mathbb{R})$	$\frac{1}{4}\varpi_n$	$\lambda_n = i$ $\lambda_{n+k} = -1$ $(1 \leq k \leq m)$
$B(m, n)$	$so(2p, 2m+1-2p) \times sp(2n, \mathbb{R})$	$-\frac{1}{4}\varpi_n + \frac{1}{2}\varpi_{n+p},$ $(1 \leq p \leq m)$	$\lambda_n = -i$ $\lambda_{n+k} = -1$ $(1 \leq k \leq m, k \neq p)$
$D(m, n)$	$so(2m) \times sp(2n, \mathbb{R})$	$\frac{1}{4}\varpi_n,$	$\lambda_n = i$ $\lambda_{n+k} = -1$ $(1 \leq k \leq m)$
$D(m, n)$	$so(2p, 2m-2p) \times sp(2n, \mathbb{R})$	$-\frac{1}{4}\varpi_n + \frac{1}{2}\varpi_{n+p},$ $(1 \leq p \leq m-2)$	$\lambda_n = -i$ $\lambda_{n+k} = -1$ $(1 \leq k \leq m, k \neq p)$
$D(m, n)$	$sp(n) \times so^*(2m)$	$\frac{1}{4}\varpi_n + \frac{1}{2}\varpi_{m+n}$	$\lambda_n = i$ $\lambda_{n+k} = -1$ $(1 \leq k \leq m-1)$
$D(m, n)$	$sp(q, n-q) \times so^*(2m)$	$\frac{1}{2}\varpi_q - \frac{1}{4}\varpi_n + \frac{1}{2}\varpi_{m+n},$ $(1 \leq q \leq n-1)$	$\lambda_n = -i$ $\lambda_q = \lambda_{n+k} = -1$ $(1 \leq k \leq m-1)$
$C(n+1)$	$sp(n) \times \mathbb{R}$	0	$\lambda_1 = i$
$C(n+1)$	$sp(2n, \mathbb{R}) \times \mathbb{R}$	$\frac{1}{2}\varpi_{n+1}$	$\lambda_1 = i, \lambda_{n+1} = -1$
$C(n+1)$	$sp(q, n-q) \times \mathbb{R}$	$\frac{1}{2}\varpi_{q+1}, 1 \leq q \leq n-1$	$\lambda_1 = i, \lambda_q = -1$
$D(2, 1, \alpha)$	$(sl(2, \mathbb{R}))^{\times 3}$	$-\frac{1}{4}\varpi_1 + \frac{1}{2}\varpi_2 + \frac{1}{2}\varpi_3$	$(-i, 1, 1)$
$D(2, 1, \alpha)$	$su(2) \times su(2) \times sl(2, \mathbb{R})$	$\frac{1}{4}\varpi_1$	$(i, -1, -1)$
$F(4)$	$so(7) \times sl(2, \mathbb{R})$	$\frac{1}{4}\varpi_1$	$(i, -1, -1, -1)$
$F(4)$	$su(2) \times so(1, 6)$	$\frac{1}{2}\varpi_2 - \frac{3}{4}\varpi_1$	$(i, 1, -1, -1)$
$F(4)$	$su(2) \times so(2, 5)$	$\frac{1}{2}\varpi_4 - \frac{1}{4}\varpi_1$	$(-i, -1, 1, -1)$
$F(4)$	$so(3, 4) \times sl(2, \mathbb{R})$	$\frac{1}{2}\varpi_3 - \frac{1}{4}\varpi_1$	$(-i, -1, -1, 1)$
$G(3)$	$\mathcal{G}_{2,0} \times sl(2, \mathbb{R})$	$\frac{1}{4}\varpi_1$	$(i, -1)$
$G(3)$	$\mathcal{G}_{2,2} \times sl(2, \mathbb{R})$	$-\frac{1}{4}\varpi_1 + \frac{1}{2}\varpi_3$	$(-i, 1)$

Table I

\mathfrak{g}	$\eta _{\mathfrak{g}_0}$	real form of \mathfrak{g}_0	t''
$gl(m, n)$	$-st$	$gl(m, \mathbb{R}) \times gl(n, \mathbb{R})$	0
$A(m, n), m \neq n$	$-st$	$sl(m, \mathbb{R}) \times sl(n, \mathbb{R}) \times \mathbb{R}$	0
$A(n, n)$	$-st$	$sl(m, \mathbb{R}) \times sl(n, \mathbb{R})$	0
$gl(2m, 2n)$	$-st$	$u^*(2m) \times u^*(2n)$	$\frac{1}{2}\varpi_m + \frac{1}{2}\varpi_{2m+n}$
$A(2m, 2n), m \neq n$	$-st$	$su^*(2m) \times su^*(2n) \times \mathbb{R}$	$\frac{1}{2}\varpi_m + \frac{1}{2}\varpi_{2m+n}$
$A(2n, 2n)$	$-st$	$su^*(2m) \times su^*(2n)$	$\frac{1}{2}\varpi_m + \frac{1}{2}\varpi_{2m+n}$
$gl(n, n)$	F	$gl(n, \mathbb{C})$	0
$A(n, n)$	F	$sl(n, \mathbb{C})$	0
$D(m, n)$	$Ad(J)$	$sp(2n, \mathbb{R}) \times so(2m-2p-1, 2p+1)$	$-\frac{1}{4}\varpi_n + \frac{1}{2}\varpi_{n+p}$ $(0 \leq p \leq m-2)$

Table II

In Table II, we have not considered the case when \mathfrak{g} is of type $D(2, 1, -\frac{1}{2})$ with non trivial η_1 , because $D(2, 1, -\frac{1}{2})$ is isomorphic to $D(2, 1)$. The corresponding real form is, from Table II, $sp(2, \mathbb{R}) \times so(3, 1)$. The special isomorphism $so(3, 1) \cong sl(2, \mathbb{C})$ allows to recover the classification as stated in [18, Theorem 2.5].

Suppose now that \mathfrak{g} is simple of type $A(1, 1)$. It needs special care since in this case there is a large group of automorphisms acting identically on the even part (this was apparently missed in previous discussions on real forms in the literature). First observe that $\mathfrak{g}_0 \cong sl(2, \mathbb{C}) \times sl(2, \mathbb{C})$, hence there are four non isomorphic real forms of \mathfrak{g}_0 : $sl(2, \mathbb{C}), sl(2, \mathbb{R}) \times sl(2, \mathbb{R}), su(2) \times sl(2, \mathbb{R}), su(2) \times su(2)$. All these real forms are obtained from real forms of \mathfrak{g} by restricting an antilinear involution ω of $gl(2, 2)$ to $sl(2, 2)$. Since this restriction clearly stabilizes the center of $sl(2, 2)$, it can be pushed down to $A(1, 1)$. Hence for any real form of \mathfrak{g}_0 there is at least one real form of $A(1, 1)$ which induces it. The only non-trivial thing to prove is that also in this case the real form of \mathfrak{g}_0 determines the real form of \mathfrak{g} . Suppose that ω and ω' are antilinear involutions that restrict in the same way to \mathfrak{g}_0 . Then there is an automorphism g of \mathfrak{g} such that $g|_{\mathfrak{g}_0} = Id_{\mathfrak{g}_0}$ and $\omega' = \omega g$. We now identify the group \mathcal{S} of automorphisms of \mathfrak{g} that restrict to the identity of \mathfrak{g}_0 with $SL(2, \mathbb{C})$: indeed, if g is such an automorphism, then necessarily

$$(4.34) \quad g(X_{\alpha_2}) = aX_{\alpha_2} + bX_{-\theta}, \quad g(X_{-\alpha_2}) = cX_{\theta} + dX_{-\alpha_2}$$

$$(4.35) \quad g(X_{\theta}) = aX_{\theta} + bX_{-\alpha_2}, \quad g(X_{-\theta}) = cX_{\alpha_2} + dX_{-\theta}.$$

The fact that g is an homomorphism implies that $M_g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{C})$.

Hence we have a map to $SL(2, \mathbb{C})$. To prove that this map is bijective, we consider the local Lie superalgebra $G_{-1} \oplus G_0 \oplus G_1$ with $G_1 = \mathbb{C}X_{\alpha_1} \oplus \mathbb{C}X_{\alpha_2} \oplus \mathbb{C}X_{\alpha_3} \oplus \mathbb{C}X_{-\theta}$, $G_{-1} = \mathbb{C}X_{-\alpha_1} \oplus \mathbb{C}X_{-\alpha_2} \oplus \mathbb{C}X_{-\alpha_3} \oplus \mathbb{C}X_{\theta}$, $G_0 = \mathfrak{h}$ and whose bracket is obtained by restricting the bracket of $A(1, 1)$. The corresponding minimal \mathbb{Z} -graded Lie superalgebra is \mathfrak{g} . A direct check shows that the map g defined by (4.34), (4.35) and such that $g|_{\mathfrak{g}_0} = Id$ defines an automorphism of the local algebra, hence it extends uniquely to an automorphism of \mathfrak{g} .

Suppose now that $\omega = \eta_1 e^{ad(t)} \omega_0$. It is easy to see that for any $g \in \mathcal{S}$ there exists a unique $g' \in \mathcal{S}$ such that

$$M_{g'} = \overline{AM_g}A,$$

with $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ if $\mathfrak{g}_0^\omega = su(2) \times su(2)$ or $sl(2, \mathbb{R}) \times sl(2, \mathbb{R})$, with $A = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$

if $\mathfrak{g}_0^\omega = sl(2, \mathbb{R}) \times su(2)$, and with $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ if $\mathfrak{g}_0^\omega = sl(2, \mathbb{C})$. If $\omega' = \omega g$, then the condition that $(\omega')^2 = Id_{\mathfrak{g}}$ is equivalent to $\omega = g\omega g$, hence we must have

$$\overline{AM_g}AM_g = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Moreover we have that $(g')^{-1}\omega g g' = \omega g''$ with $M_{g''} = \overline{AM_{g'}^{-1}}AM_g M_{g'}$.

Consider the antilinear antiinvolution σ on $SL(2, \mathbb{C})$ defined by $\sigma(M) = \overline{AM}^{-1}A$. We have just shown that the set of M_g such that $(\omega \circ g)^2 = Id$ is the fixed point set S of σ . Let us also consider the action of $SL(2, \mathbb{C})$ on S by $M \cdot s = \sigma(M)sM$. If this action has a unique orbit then, as shown above, ω and $\omega \circ g$ are conjugated, so

the real forms \mathfrak{g}^ω and $\mathfrak{g}^{\omega'}$ are isomorphic. To check when this action has a unique orbit we observe that the stabilizer of any point $s \in S$ is the fixed point set of the antilinear involution on $SL(2, \mathbb{C})$ given by $g \mapsto s^{-1}\sigma(g)^{-1}s$ hence it is a real form of $SL(2, \mathbb{C})$. In particular all the orbits have the same dimension so the orbits are the connected components of S . More explicitly

$$S = \left\{ \begin{pmatrix} x & z \\ -\bar{z} & y \end{pmatrix} \in SL(2, \mathbb{C}) \mid x, y \in \mathbb{R}, z \in \mathbb{C} \right\}$$

if $\mathfrak{g}_0^\omega = su(2) \times su(2)$ or $\mathfrak{g}_0^\omega = sl(2, \mathbb{R}) \times sl(2, \mathbb{R})$, while

$$S = \left\{ \begin{pmatrix} z & \sqrt{-1}x \\ \sqrt{-1}y & \bar{z} \end{pmatrix} \in SL(2, \mathbb{C}) \mid x, y \in \mathbb{R}, z \in \mathbb{C} \right\}$$

if $\mathfrak{g}_0^\omega = sl(2, \mathbb{C})$. In both cases S is the quadric in \mathbb{R}^4 of equation $xy + a^2 + b^2 = 1$ where $z = a + \sqrt{-1}b$, which is homeomorphic to $\mathbb{R} \times S^2$.

It remains to deal with the last case, i.e. $\mathfrak{g}_0^\omega = sl(2, \mathbb{R}) \times su(2)$, where

$$S = \left\{ \begin{pmatrix} x & z \\ \bar{z} & y \end{pmatrix} \in SL(2, \mathbb{C}) \mid x, y \in \mathbb{R}, z \in \mathbb{C} \right\},$$

which is the quadric of equation $xy - a^2 - b^2 = 1$. This is homeomorphic to $\mathbb{R}^3 \times S^0$, hence disconnected. However, the two orbits are the orbit of M_{Id} and of $M_{\delta_{-1}}$ (cf. (4.31)), so we need only to check if \mathfrak{g}^ω and $\mathfrak{g}^{\omega^{\delta_{-1}}}$ are isomorphic, but the argument used in the other cases shows that $\omega_{\mathbb{R}}$ provides an isomorphism between \mathfrak{g}^ω and $\mathfrak{g}^{\omega^{\delta_{-1}}}$. Thus, $A(1, 1)$ has no real forms other than those listed in Tables I and II.

Remark 4.1. Our treatment doesn't cover the cases when the Cartan matrix is not real, which happens only if \mathfrak{g} is of type $D(2, 1, \alpha)$, where $\alpha \in \mathbb{C} \setminus \mathbb{R}$, $\alpha + \bar{\alpha} = -1$. In this case one has one extra real form, with even part $sl(2, \mathbb{C}) \times sl(2, \mathbb{R})$ (see [18]).

Remark 4.2. Our approach to classify real forms started by fixing a suitable set of positive roots and henceforth the antilinear involution ω_0 . All antilinear involutions are then gotten as $\eta_1 e^{ad(h)} \omega_0$, by letting h and η_1 vary.

One can, in a different perspective, consider just the antilinear involutions $\eta_1 \omega_0$ and let the choice of Δ^+ vary. In this way one can associate to any set of positive roots a set of real forms indexed by the outer automorphism η_1 . It can be checked that all real forms can be obtained in this way. The real forms discussed in Section 4.5 are precisely those corresponding to the distinguished sets of positive roots with $\eta_1 = Id_{\mathfrak{g}}$. In Section 4.8 we will discuss the real forms corresponding to the sets of positive roots of depth 4 when \mathfrak{g} is $gl(m, n)$ or of type $B(m, n)$ and $D(m, n)$. It will actually turn out that, in these cases, all real forms of \mathfrak{g} already appear as real forms corresponding to positive sets of roots of depth at most 4.

Remark 4.3. It is possible to deal with the problem of classifying antilinear involutions in a slightly different way. Consider marked sets of positive roots. For this we mean the datum (Π, L) where $\Pi = \{\alpha_1, \dots, \alpha_n\}$ is the set of simple roots of a set Δ^+ of positive roots and $L = \{l_1, \dots, l_n\}$ is a set of labels satisfying $l_i \in \{\pm 1\}$ if $p(\alpha_i) = 0$, $l_i \in \{\pm\sqrt{-1}\}$ if $p(\alpha_i) = 1$. Given a marked set of positive roots, we can define an antilinear involution ω_L using formulas (4.12)–(4.15) with $\lambda_i = -\xi_{\alpha_i} l_i$, so that $\lambda_{\alpha_i} = l_i$. We have a natural action of odd reflections on marked set of positive roots: if r_{α_i} denotes the odd reflection w.r.t. the simple root α_i (see (2.5)), we set $r_{\alpha_i}(\Pi, L) = (r_{\alpha_i}(\Pi), L')$, with $L' = \{l'_1, \dots, l'_n\}$, $l'_j = l_j$

if $(\alpha_i, \alpha_j) = 0$, $i \neq j$, $l'_i = l_i^{-1}$ and $l'_j = l_j l_i$ if $(\alpha_i, \alpha_j) \neq 0$. This action has the following property: if $r_\alpha(\Pi, L) = (\Pi', L')$, then, since $\zeta_{\omega_L} = \zeta_{\omega_{L'}}$, ω_L and $\omega_{L'}$ define isomorphic real forms. Using this action, one has a combinatorial recipe to detect the real form corresponding to a given marked set of positive roots: since any two sets of positive roots can be obtained from each other through a sequence of odd reflections, it suffices to consider the marked sets (Π_{vd}, L_{vd}) , where Π_{vd} is one of the sets of simple roots corresponding to a very distinguished set of positive roots we used for the classification. Given any marked set (Π, L) one can compute combinatorially the marked set (Π_{vd}, L_{vd}) in the same equivalence class and apply our classification to $\omega_{L_{vd}}$.

4.7. Compact dual pairs coming from distinguished sets of positive roots.

Let $\mathfrak{g} = gl(m, n)$ or a Lie superalgebra of type B, C, D . Fix a distinguished set Δ^+ of positive roots. Let ω be as in Subsection 4.5 and let V be the set of ω -fixed points in \mathfrak{g}_1 . Also recall from (4.29) the definition of $\mathfrak{s}_1, \mathfrak{s}_2$.

In this Subsection we prove the following proposition.

Proposition 4.8. *Let \mathfrak{g} be $gl(m, n)$ or a Lie superalgebra of type B, C, D . If Δ^+ is a distinguished set of positive roots, then there is a compact dual pair (G_1, G_2) in $Sp(V, \langle \cdot, \cdot \rangle)$ with $Lie(G_i) = \mathfrak{s}_i$, $i = 1, 2$, such that the action of \mathfrak{g}_0 on $M^{\Delta^+}(\mathfrak{g}_1)$ gives the Theta correspondence for (G_1, G_2) at the level of Lie algebras. The compact dual pairs (G_1, G_2) are listed in the following table (in which m, n are positive integers):*

\mathfrak{g}	Δ^+	(G_1, G_2)
$gl(m, n)$	$\Delta_{gl}^{(p, q)}$	$(U(n), U(p, q))$
$B(0, n)$	Δ_B^+	$(O(1), Sp(2n, \mathbb{R}))$
$B(m, n)$	Δ_B^+	$(O(2m+1), Sp(2n, \mathbb{R}))$
$D(m, n)$	Δ_{D1}^+	$(O(2m), Sp(2n, \mathbb{R}))$
$D(m, n)$	Δ_{D2}^+	$(Sp(n), SO^*(2m))$
$D(m, n)$	$\Delta_{D2'}^+$	$(Sp(n), SO^*(2m))$
$C(n+1)$	Δ_{C1}^+	$(O(2), Sp(2n, \mathbb{R}))$
$C(n+1)$	Δ_{C2}^+	$(Sp(n), SO^*(2))$
$C(n+1)$	$\Delta_{C2'}^+$	$(Sp(n), SO^*(2))$

We shall prove Proposition 4.8 by realizing explicitly the superalgebras as subalgebras of some $gl(r, s)$ and then checking the conditions of Proposition 4.2 in a case by case fashion. For shortness we will give the details only in type $D(m, n)$.

4.7.1. Δ_{D1}^+ . Endow the superspace $\mathbb{C}^{2m|2n}$ with the bilinear form given by the matrix $\begin{pmatrix} I_{2m} & 0 \\ 0 & J_{2n} \end{pmatrix}$, where $J_{2n} = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$.

The Lie superalgebra $osp(2m, 2n)$ of type $D(m, n)$ is the set of linear transformations which are skewsupersymmetric w.r.t. a supersymmetric nondegenerate bilinear form. If we choose the form as above, then $osp(2m, 2n)$ can be realized as the set of matrices

$$\begin{pmatrix} A & B_1 & B_2 \\ -B_2^t & C_1 & C_2 \\ B_1^t & C_3 & -C_1^t \end{pmatrix}$$

with A skewsymmetric $2m \times 2m$ and C_2, C_3 symmetric $n \times n$ matrices. Denoting by $E_{i,j}$ the matrix units, set

$$h_i = \begin{cases} -\sqrt{-1}E_{i,m+i} + \sqrt{-1}E_{m+i,i}, & 1 \leq i \leq m, \\ -\sqrt{-1}E_{2m+i,2m+n+i} + \sqrt{-1}E_{2m+n+i,2m+i}, & m+1 \leq i \leq m+n. \end{cases}$$

Then $\mathfrak{h} = \bigoplus_{i=1}^{m+n} \mathbb{C}h_i$ is a Cartan subalgebra of \mathfrak{g} .

Set $(X, Y) = \frac{1}{2} \text{str}(XY)$; then

$$(h_i, h_j) = \begin{cases} \delta_{ij} & 1 \leq i \leq m, \\ -\delta_{ij} & m+1 \leq i \leq m+n. \end{cases}$$

In the chosen distinguished set of positive roots we have $\Delta_1^+ = \{\delta_i \pm \epsilon_j \mid 1 \leq j \leq m, 1 \leq i \leq n\}$. We can choose for each $\alpha \in \Delta_1^+$ root vectors $X_\alpha, X_{-\alpha}$ in such a way that

$$V = \left\{ \begin{pmatrix} 0 & \sqrt{-1}A \\ -\sqrt{-1}J_{2n}{}^t A & 0 \end{pmatrix} \mid A \in M_{2m,2n}(\mathbb{R}) \right\}.$$

The map $\Psi : \mathfrak{g}_1 \rightarrow M_{2m,2n}(\mathbb{C})$,

$$(4.37) \quad \Psi \left(\begin{pmatrix} 0 & A \\ -J_{2n}{}^t A & 0 \end{pmatrix} \right) = A$$

intertwines the adjoint action of \mathfrak{g}_0 on \mathfrak{g}_1 with its action on $M_{2m,2n}(\mathbb{C})$ given by

$$(4.38) \quad \begin{pmatrix} E & 0 \\ 0 & F \end{pmatrix} \cdot A = EA - AF.$$

It is then clear that $\text{ad} \begin{pmatrix} E & 0 \\ 0 & F \end{pmatrix} (V) \subset V$ if and only if E and F are real matrices, i. e. $E \in \mathfrak{so}(2m)$ and $F \in \mathfrak{sp}(2n, \mathbb{R})$.

Consider now the map $\Phi : V \rightarrow \mathbb{R}^{2m} \otimes (\mathbb{R}^{2n})^*$ given by

$$\begin{aligned} \Phi(\sqrt{-1}(E_{i,2m+j} + E_{2m+n+j,i})) &= e_i \otimes f^j & 0 \leq i \leq 2m, 1 \leq j \leq n \\ \Phi(\sqrt{-1}(E_{i,2m+n+j} - E_{2m+j,i})) &= e_i \otimes f^{n+j} & 0 \leq i \leq 2m, 1 \leq j \leq n \end{aligned}$$

where $\{e_i\}$ is the standard basis of \mathbb{R}^{2m} and $\{f^j\}$ is the standard basis of $(\mathbb{R}^{2n})^*$. Note that Φ intertwines the action of $\text{ad}_{|\mathfrak{g}_1}(\mathfrak{g}_0) \cap \mathfrak{sp}(V)$ on V with the standard action of $\mathfrak{so}(2m) \times \mathfrak{sp}(2n, \mathbb{R})$ on $\mathbb{R}^{2m} \otimes (\mathbb{R}^{2n})^*$.

Let $(\cdot, \cdot)_{2m}$ be the standard symmetric bilinear form on \mathbb{R}^{2m} , $(\cdot, \cdot)_{2n}$ the standard symplectic form on $(\mathbb{R}^{2n})^*$ and let $\langle \cdot, \cdot \rangle$ be the tensor product $(\cdot, \cdot)_{2m} \otimes (\cdot, \cdot)_{2n}$. It is easy to check that, for $j, k \in \{0, \dots, 2m\}$, $i, r \in \{1, \dots, n\}$

$$\begin{aligned} \langle E_{j,2m+i} + E_{2m+n+i,j}, E_{k,2m+r} + E_{2m+n+r,k} \rangle &= 0, \\ \langle E_{j,2m+n+i} - E_{2m+n+i,j}, E_{k,2m+n+r} + E_{2m+r,k} \rangle &= 0, \\ \langle E_{j,2m+i} + E_{2m+n+i,j}, E_{k,2m+n+r} - E_{2m+r,k} \rangle &= -\delta_{jk} \delta_{ir}. \end{aligned}$$

It follows that $\langle \Phi(X), \Phi(Y) \rangle = \langle X, Y \rangle$ for any $X, Y \in V$. According to the classification given in [5, § 5], the pair $(O(2m), Sp(2n, \mathbb{R}))$ is a type I dual pair in $Sp(\mathbb{R}^{2m} \otimes (\mathbb{R}^{2n})^*, \langle \cdot, \cdot \rangle)$. Since Φ maps $\text{ad}_{|\mathfrak{g}_1}(\mathfrak{g}_0) \cap \mathfrak{sp}(V) \subset \mathfrak{sp}(V)$ exactly on the Lie algebras of this dual pair, we have proven Proposition 4.8 in this case.

4.7.2. $\Delta_{D_2}^+$. Endow the superspace $\mathbb{C}^{2m|2n}$ with the bilinear form given by the matrix $\begin{pmatrix} I_{m,m} & 0 \\ 0 & J_{2n} \end{pmatrix}$, where $I_{m,m} = \begin{pmatrix} 0 & I_m \\ I_m & 0 \end{pmatrix}$.

With this choice of the form, the superalgebra $osp(2m, 2n)$ can be realized as the set of matrices

$$\begin{pmatrix} A_1 & A_2 & B_1 & B_2 \\ A_3 & -{}^tA_1 & B_3 & B_4 \\ {}^tB_4 & {}^tB_2 & C_1 & C_2 \\ -{}^tB_3 & -{}^tB_1 & C_3 & -{}^tC_1 \end{pmatrix}$$

with A_2, A_3 skewsymmetric $m \times m$ and C_2, C_3 symmetric $n \times n$ matrices. In this case we set

$$h_i = \begin{cases} E_{i,i} - E_{m+i,m+i}, & 1 \leq i \leq m, \\ E_{m+i,m+i} - E_{m+n+i,m+n+i}, & m+1 \leq i \leq m+n. \end{cases}$$

Then $\mathfrak{h} = \bigoplus_{i=1}^{m+n} \mathbb{C}h_i$ is a Cartan subalgebra of \mathfrak{g} and

$$(h_i, h_j) = \begin{cases} \delta_{ij} & 1 \leq i \leq m, \\ -\delta_{ij} & m+1 \leq i \leq m+n. \end{cases}$$

In the chosen distinguished set of positive roots we have $\Delta_1^+ = \{\epsilon_i \pm \delta_j \mid 1 \leq i \leq m, 1 \leq j \leq n\}$. We can choose for each $\alpha \in \Delta_1^+$ root vectors $X_\alpha, X_{-\alpha}$ in such a way that

$$(4.39) \quad V = \left\{ \begin{pmatrix} 0 & A \\ -J_{2n} {}^tA & 0 \end{pmatrix} \mid A = \begin{pmatrix} A_1 & A_2 \\ \sqrt{-1}\bar{A}_2 & -\sqrt{-1}\bar{A}_1 \end{pmatrix}, A_1, A_2 \in M_{m,n}(\mathbb{C}) \right\}.$$

Again, the map Ψ , given by (4.37), intertwines the adjoint action of \mathfrak{g}_0 on \mathfrak{g}_1 with its action on $M_{2m,2n}(\mathbb{C})$, given by (4.38). It is then clear that $ad \begin{pmatrix} E & 0 \\ 0 & F \end{pmatrix} (V) \subset V$ if and only if

$$(4.40) \quad E = \begin{pmatrix} A_1 & A_2 \\ -\bar{A}_2 & \bar{A}_1 \end{pmatrix}, \quad F = \begin{pmatrix} B_1 & B_2 \\ -\bar{B}_2 & \bar{B}_1 \end{pmatrix}$$

with $A_1, A_2 \in M_{m,m}(\mathbb{C})$, $B_1, B_2 \in M_{n,n}(\mathbb{C})$, A_1, B_1 skew-Hermitian, A_2 antisymmetric, and B_2 symmetric.

Let \mathbb{H} be the skew field of real quaternions. Set $(\mathbb{H}^n)^* = End_{\mathbb{H}}(\mathbb{H}^n, \mathbb{H})$ where \mathbb{H}^n, \mathbb{H} are viewed as left \mathbb{H} -spaces. Endow $(\mathbb{H}^n)^*$ with a right \mathbb{H} -action by setting $(\lambda q)(v) = \lambda(v)q$. Identify V with $(\mathbb{H}^n)^* \otimes_{\mathbb{H}} \mathbb{H}^m$ as follows. The map Ψ above followed by left multiplication by $L_m = \begin{pmatrix} I_m & 0 \\ 0 & \sqrt{-1}I_m \end{pmatrix}$ maps V into the subspace of $M_{2m,2n}(\mathbb{C})$ of matrices of the form

$$(4.41) \quad \begin{pmatrix} A & -\bar{B} \\ B & \bar{A} \end{pmatrix}.$$

Identifying \mathbb{H}^r with $\mathbb{C}^r \times \mathbb{C}^r = \mathbb{C}^{2r}$ by $x + yj \leftrightarrow (x, y)$ we see that the matrices as in (4.41) are precisely those commuting with the left action of j . Thus V gets identified with $End_{\mathbb{H}}(\mathbb{H}^n, \mathbb{H}^m)$ (here $\mathbb{H}^n, \mathbb{H}^m$ are seen as left \mathbb{H} -spaces). The natural map $\lambda \otimes v \mapsto T_{\lambda,v}$ with $T_{\lambda,v}(u) = \lambda(u)v$ identifies $End_{\mathbb{H}}(\mathbb{H}^n, \mathbb{H}^m)$ and $(\mathbb{H}^n)^* \otimes_{\mathbb{H}} \mathbb{H}^m$.

Let now $\Phi : V \rightarrow (\mathbb{H}^n)^* \otimes_{\mathbb{H}} \mathbb{H}^m$ be the (\mathbb{R} -linear) isomorphism described above. Note that, if E, F are as in (4.40), then both $L_m E L_m^{-1}$ and F are of the form (4.41). Thus we can identify them with elements T_E, T_F in $End_{\mathbb{H}}(\mathbb{H}^m, \mathbb{H}^m)$ and $End_{\mathbb{H}}(\mathbb{H}^n, \mathbb{H}^n)$ respectively. Unwinding all the identifications we see that if $X = \begin{pmatrix} E & 0 \\ 0 & F \end{pmatrix} \in \mathfrak{g}_0$ and $ad(X)(V) \subset V$ then $\Phi ad(X)\Phi^{-1}$ acts on $(\mathbb{H}^n)^* \otimes_{\mathbb{H}} \mathbb{H}^m$ via the natural action of $T_F \otimes I + I \otimes T_E$.

Let $\tau : \mathbb{H} \rightarrow \mathbb{H}$ be the quaternionic conjugation: $\tau(a + bj) = \bar{a} - bj$ ($a, b \in \mathbb{C}$). It induces the complex conjugation in the above identification $\mathbb{H}^r = \mathbb{C}^{2r}$. Let $(\cdot, \cdot)'_m$ be the skew-Hermitian form on \mathbb{H}^m given by $(v, w)'_m = \sum_i v_i \sqrt{-1} \tau(w_i)$ and $(\cdot, \cdot)_n$ the Hermitian form on \mathbb{H}^n given by $(v, w)_n = \sum_i v_i \tau(w_i)$. Let $(\cdot, \cdot)_n$ denote also the form induced on $(\mathbb{H}^n)^*$. Note that, since E, F are as in (4.40), then $T_E \in u(\mathbb{H}^m, (\cdot, \cdot)'_m) \simeq so^*(2m)$ and $T_F \in u((\mathbb{H}^n)^*, (\cdot, \cdot)_n) \simeq sp(n)$.

Let $\langle \cdot, \cdot \rangle$ be the real symplectic bilinear form on $(\mathbb{H}^n)^* \otimes_{\mathbb{H}} \mathbb{H}^m$ given by

$$\langle \cdot, \cdot \rangle = Tr_{\mathbb{H}/\mathbb{R}}(\cdot, \cdot)_n \otimes \tau \circ (\cdot, \cdot)'_m.$$

For $1 \leq i \leq m, 1 \leq j \leq n$, set

$$(4.42) \quad X_{ij} = \Psi^{-1}(\sqrt{-1}E_{ij} - E_{m+i, n+j}),$$

$$(4.43) \quad Y_{ij} = \Psi^{-1}(E_{ij} - \sqrt{-1}E_{m+i, n+j}),$$

$$(4.44) \quad Z_{ij} = \Psi^{-1}(\sqrt{-1}E_{i, n+j} + E_{m+i, j}),$$

$$(4.45) \quad W_{ij} = \Psi^{-1}(E_{i, n+j} + \sqrt{-1}E_{m+i, j}).$$

If $X, Y \in \{X_{ij}, Y_{ij}, Z_{ij}, W_{ij}\}$, then it is easy to check that $\langle X, Y \rangle = 0$ except in the following cases:

$$2 = \langle X_{ij}, Y_{ij} \rangle = -\langle Y_{ij}, X_{ij} \rangle = \langle Z_{ij}, W_{ij} \rangle = -\langle W_{ij}, Z_{ij} \rangle.$$

On the other hand, letting $\{e_i\}$ be the canonical basis of \mathbb{H}^m and $\{e^i\}$ the basis dual to the canonical basis of \mathbb{H}^n , we see that

$$(4.46) \quad \Phi(X_{ij}) = e^j \otimes \sqrt{-1}e_i,$$

$$(4.47) \quad \Phi(Y_{ij}) = e^j \otimes e_i,$$

$$(4.48) \quad \Phi(Z_{ij}) = e^j \otimes \sqrt{-1}je_i,$$

$$(4.49) \quad \Phi(W_{ij}) = -e^j \otimes je_i.$$

It follows that $\langle \Phi(X), \Phi(Y) \rangle = -2\langle X, Y \rangle$ for any $X, Y \in V$.

According to the classification given in [5, § 5], the pair

$$(U((\mathbb{H}^n)^*, (\cdot, \cdot)_n), U(\mathbb{H}^m, (\cdot, \cdot)'_m)) = (Sp(n), SO^*(2m))$$

is a Type I dual pair in $Sp((\mathbb{H}^n)^* \otimes_{\mathbb{H}} \mathbb{H}^m, \langle \cdot, \cdot \rangle)$. Since the map Φ maps $ad_{|\mathfrak{g}_1}(\mathfrak{g}_0) \cap sp(V) \subset sp(V)$ exactly on the Lie algebras of this dual pair, we have proven Proposition 4.8 in this case.

4.7.3. $\Delta_{D_2}^+$. This case can be reduced to the $\Delta_{D_2}^+$ case as follows: by a careful choice of the root vectors, the space V is the set of matrices

$$\begin{pmatrix} 0 & A \\ -J_{2n} {}^t A & 0 \end{pmatrix}$$

such that

$$A = \left(\begin{array}{c|c} A_1 & A_2 \\ \hline v & w \\ \sqrt{-1}A_2 & -\sqrt{-1}A_1 \\ -\sqrt{-1}\bar{w} & \sqrt{-1}\bar{v} \end{array} \right),$$

with $A_1, A_2 \in M_{m-1,n}(\mathbb{C})$, $v, w \in \mathbb{C}^n$. Define $S_{m,2m} \in M_{2m,2m}(\mathbb{C})$ as

$$S_{m,2m} = \left(\begin{array}{cc|cc} I_{m-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ \hline 0 & 0 & I_{m-1} & 0 \\ 0 & 1 & 0 & 0 \end{array} \right).$$

Let $\sigma : \mathfrak{g} \rightarrow \mathfrak{g}$ be the map

$$\begin{pmatrix} A & B \\ -J_{2n} {}^t B & C \end{pmatrix} \mapsto \begin{pmatrix} S_{m,2m} A S_{m,2m} & S_{m,2m} B \\ -J_{2n} {}^t B S_{m,2m} & C \end{pmatrix}.$$

It is easy to check that σ is an automorphism of \mathfrak{g} . Since $\sigma(V)$ is the space described in (4.39) we see that the map $X \mapsto \Phi \circ ad(\sigma(X)) \circ \Phi^{-1}$ identifies $ad_{|\mathfrak{g}_1}(\mathfrak{g}_0) \cap sp(V)$ with $sp(n) \times so^*(2m)$. Finally observe that $str(\sigma(A)\sigma(B)) = str(AB)$ hence

$$\langle \Phi(\sigma(X)), \Phi(\sigma(Y)) \rangle = -2(X, Y)$$

for any $X, Y \in V$. This concludes the proof of Proposition 4.8 in this case.

4.8. Noncompact dual pairs and gradings of depth 4. In this subsection we classify, up to $W_{\mathfrak{g}}$ -action, the sets of positive roots such that the grading (4.16) has depth at most 4 (i.e., $\mathfrak{q}_i = \{0\}$ for $|i| > 4$) and we show that these sets of positive roots are related to the noncompact dual pairs $(U(p, q), U(r, s))$, $(O(p, q), Sp(2n, \mathbb{R}))$, $(Sp(p, q), SO^*(2n))$. These pairs exhaust the type I dual pairs, with the exception of the pair $(O(m, \mathbb{C}), Sp(2n, \mathbb{C}))$. The latter pair cannot arise in our picture, since $O(m, \mathbb{C}) \times Sp(2n, \mathbb{C})$ is not a real form of \mathfrak{g}_0 for any choice of \mathfrak{g} . It occurs as the real form $\mathfrak{g}_0 \cap \mathfrak{g}_{\mathbb{R}}$ with $\mathfrak{g} = osp(m, 2n) \times osp(m, 2n)$ and $\mathfrak{g}_{\mathbb{R}}$ the diagonal copy of $osp(m, 2n)$ in \mathfrak{g} . Since we are chiefly interested in the application to dual pairs, we confine to study $gl(m, n)$, $B(m, n)$, $D(m, n)$, $C(n+1)$.

In the following we choose ω_0 (see (4.30)) as antilinear involution and let λ_{α}^0 be the corresponding multipliers as defined in (4.13). Recall that the corresponding involution ζ_{ω_0} of \mathfrak{g}_0 is defined by setting $\zeta_{\omega_0}(X_{\alpha}) = sgn(\lambda_{\alpha}^0)X_{\alpha}$.

Case $\mathfrak{g} = gl(m, n)$. It is clear that diagrams of type A with three or four grey nodes support sets of positive roots whose associated grading has depth 3 or 4, respectively. These sets of positive roots correspond to the following sets of simple roots:

$$(4.50) \quad \{\epsilon_1 - \epsilon_2, \dots, \epsilon_p - \delta_1, \delta_1 - \delta_2, \dots, \delta_r - \epsilon_{p+1}, \epsilon_{p+1} - \epsilon_{p+2}, \dots, \epsilon_{p+1} - \delta_{r+1}, \delta_{r+1} - \delta_{r+2}, \dots, \delta_{r+s-1} - \delta_{r+s}\},$$

$$(4.51) \quad \{\epsilon_1 - \epsilon_2, \dots, \epsilon_h - \delta_1, \delta_1 - \delta_2, \dots, \delta_r - \epsilon_{h+1}, \epsilon_{h+1} - \epsilon_{h+2}, \dots, \epsilon_{h+q} - \delta_{r+1}, \delta_{r+1} - \delta_{r+2}, \dots, \delta_{r+s-1} - \delta_{r+s}, \delta_{r+s} - \epsilon_{h+q+1}, \dots, \epsilon_{h+k+q-1} - \epsilon_{h+k+q}\},$$

and are clearly the only ones with the required property about the grading, up to switching the role of ϵ and δ . Here $p+q = m, r+s = n$ in (4.50) and $h+k = p, p+q = m, r+s = n$ in (4.51). Fix (4.51) as a set of simple roots. To detect

the real form corresponding to ω_0 , we need only to classify $(\zeta_{\omega_0})|_{\mathfrak{g}_0^i}$, $i = 1, 2$ via the parameters s_j^i , $i = 1, 2$, $1 \leq j \leq k_i$, $k_1 = m - 1$, $k_2 = n - 1$ as explained in Section 4.6. In this case the roots β_j^i are the simple roots of \mathfrak{g}_0^i :

$$\begin{aligned} \beta_j^1 &= \alpha_j, \quad 1 \leq j \leq h - 1, & \beta_j^2 &= \alpha_{h+j}, \quad 1 \leq j \leq r - 1, \\ \beta_h^1 &= \alpha_h + \alpha_{h+1} + \dots + \alpha_{h+r}, & \beta_r^2 &= \alpha_{h+r} \dots + \alpha_{h+r+q}, \\ \beta_{h+j}^1 &= \alpha_{h+r+j}, \quad 1 \leq j \leq q - 1, & \beta_{r+j}^2 &= \alpha_{h+r+q+j}, \quad 1 \leq j \leq s - 1, \\ \beta_{h+q}^1 &= \alpha_{h+r+q} + \dots + \alpha_{h+r+q+s}, \\ \beta_{h+q+j}^1 &= \alpha_{h+r+q+s+j}, \quad 1 \leq j \leq k - 1. \end{aligned}$$

while θ_i is the highest root of \mathfrak{g}_0^i . In this case $\zeta_{\omega_0} = e^{2\pi\sqrt{-1}t}$ with $\beta_j^i(t) = \frac{1}{2}$ if $\text{sgn}(\lambda_{\beta_j^i}^0) = -1$ and $\beta_j^i(t) = 0$ otherwise. Note that, letting $s_j^i = \beta_j^i(t)$ and $s_0^i = 1 - \theta_i(t)$, then t satisfies (4.33). It follows that, since $s_j^1 = 0$ for $j \neq h, h + q$, $s_h^1 = s_{h+q}^1 = \frac{1}{2}$, $s_i^2 = 0$ for $i \neq 0, r$, $s_0^2 = s_r^2 = \frac{1}{2}$, the corresponding real form of \mathfrak{g}_0 is $u(p, q) \times u(r, s)$.

Case $\mathfrak{g} = B(m, n)$. Recall that in type B all diagrams have shape (4.20). Assume $|\pi| = 2$. Suppose first $\alpha_{n+m} \in \pi$: then α_{n+m} is odd and non isotropic. The grading has depth 4. The simple roots of \mathfrak{g}_0 that are not simple in \mathfrak{g} are both of degree 2, while the largest roots are of degree 4. Arguing as for $gl(m, n)$ it is easy to see that the corresponding real form is $so(2m, 1) \times sp(2n, \mathbb{R})$. The corresponding dual pair is $(O(2m, 1), Sp(2n, \mathbb{R}))$. Otherwise the set of positive roots is necessarily of the form

$$\{\epsilon_1 - \epsilon_2, \dots, \epsilon_p - \delta_1, \delta_1 - \delta_2, \dots, \delta_n - \epsilon_{p+1}, \dots, \epsilon_{m-1} - \epsilon_m, \epsilon_m\}$$

with $p + q = m$. This grading has depth 4, since the highest root is $\epsilon_1 + \epsilon_2$. Arguing as above, we can show that it gives rise to the dual pair $(O(2p, 2q + 1), Sp(2n, \mathbb{R}))$. It is easily seen that if $|\pi| \geq 3$ then the grading has depth strictly greater than 4.

Case $\mathfrak{g} = D(m, n)$. If the diagram of the set of positive roots is like (4.21), then arguing as in the previous case we deduce that the only sets of positive roots with $|\pi| = 2$ and depth at most 4 (indeed exactly 4) are

$$\begin{aligned} &\{\delta_1 - \delta_2, \dots, \delta_p - \epsilon_1, \epsilon_1 - \epsilon_2, \dots, \epsilon_{m-1} - \epsilon_m, \epsilon_m - \delta_{p+1}, \dots, \delta_{p+q-1} - \delta_{p+q}, 2\delta_{p+q}\}, \\ &\{\delta_1 - \delta_2, \dots, \delta_p - \epsilon_1, \epsilon_1 - \epsilon_2, \dots, \epsilon_{m-1} + \epsilon_m, -\epsilon_m - \delta_{p+1}, \dots, \delta_{p+q-1} - \delta_{p+q}, 2\delta_{p+q}\}. \end{aligned}$$

with $p + q = n$, which gives rise to the dual pair $(Sp(2p, 2q), SO^*(2m))$. If instead the diagram is like (4.23), we again obtain a unique set of positive roots:

$$(4.52) \quad \{\epsilon_1 - \epsilon_2, \dots, \epsilon_p - \delta_1, \delta_1 - \delta_2, \dots, \delta_n - \epsilon_{p+1}, \dots, \epsilon_{m-1} - \epsilon_m, \epsilon_{m-1} + \epsilon_m\}$$

with $p + q = m$, which gives rise to the dual pair $(O(2p, 2q), Sp(2n, \mathbb{R}))$. In both cases, if $|\pi| \geq 3$ then the grading has depth strictly greater than 4.

The pairs $(O(2p + 1, 2q - 1), Sp(2n, \mathbb{R}))$ are gotten by considering the antilinear involution $\omega = \eta_1 \omega_0$ when the positive sets of roots are those in (4.52) and $\eta_1 = Ad(J)$. We need to classify $\zeta_\omega = \eta_1 \circ \zeta_{\omega_0}$. The extended Dynkin diagrams of $\mathfrak{g}_0^1, \mathfrak{g}_0^2$

are of type $D_m^{(2)}$, $C_n^{(1)}$ respectively. The corresponding roots β_j^i are

$$\begin{aligned}\beta_j^1 &= \alpha_j, \quad 1 \leq j \leq p-1, & \beta_j^2 &= \alpha_{p+j}, \quad 1 \leq j \leq n-1, \\ \beta_p^1 &= \alpha_p + \alpha_{p+1} + \dots + \alpha_{p+n}, & \beta_{2n}^2 &= 2(\alpha_{p+n} \dots + \alpha_{m+n-2}) + \alpha_{m+n-1} + \alpha_{m+n}, \\ \beta_{p+j}^1 &= \alpha_{p+n+j}, \quad 1 \leq j \leq q-2, \\ \beta_{m-1}^1 &= \frac{1}{2}(\alpha_{m+n} + \alpha_{m+n-1}).\end{aligned}$$

The parameters for ζ_ω are $s_p^1 = \frac{1}{2}$ and $s_j^1 = 0$ for $j \neq p$, $s_n^2 = \frac{1}{2}$ and $s_j^2 = 0$ for $j \neq n$. It follows that the real forms of \mathfrak{g}_0^i are respectively $so(2p+1, 2q-1)$ and $sp(2n, \mathbb{R})$, which give rise to the dual pairs $(O(2p+1, 2q-1), Sp(2n, \mathbb{R}))$.

Note that the pair $(O(1, 2m-1), Sp(2n, \mathbb{R}))$ is obtained from the distinguished set of positive roots whose set of simple roots is

$$(4.53) \quad \{\delta_1 - \delta_2, \dots, \delta_n - \epsilon_1, \epsilon_1 - \epsilon_2, \dots, \epsilon_{m-1} - \epsilon_m, \epsilon_{m-1} + \epsilon_m\}.$$

Case $\mathfrak{g} = C(n+1)$. According to [9, page 52], the only sets of simple roots which correspond to non-distinguished sets of positive roots are

$$\begin{aligned}\{\delta_1 - \delta_2, \dots, \delta_i - \epsilon_1, \epsilon_1 - \delta_{i+1}, \delta_{i+1} - \delta_{i+2}, \dots, 2\delta_n\}, \\ \{\delta_1 - \delta_2, \dots, \delta_i + \epsilon_1, -\epsilon_1 - \delta_{i+1}, \delta_{i+1} - \delta_{i+2}, \dots, 2\delta_n\}.\end{aligned}$$

where i ranges from 1 to $n-1$. The associated gradings have depth 4.

5. THETA CORRESPONDENCE FOR THE PAIR $(O(2m+1), Sp(2n, \mathbb{R}))$

In this section we use the denominator identity developed in the previous sections to derive the Theta correspondence for the compact dual pair $(O(2m+1), Sp(n, \mathbb{R}))$. This pair, according to Proposition 4.8, corresponds to the distinguished set of positive roots Δ_B^+ in a superalgebra \mathfrak{g} of type $B(m, n)$. Recall that

$$(5.1) \quad \Delta_0 = \pm\{\epsilon_i \pm \epsilon_j, \epsilon_i, \delta_k \pm \delta_l, 2\delta_k \mid 1 \leq i \neq j \leq m, 1 \leq k \neq l \leq n\},$$

$$\Delta_1 = \pm\{\delta_k \pm \epsilon_i, \delta_k \mid 1 \leq i \leq m, 1 \leq k \leq n\};$$

$$(5.2) \quad \overline{\Delta}_0 = \pm\{\epsilon_i \pm \epsilon_j, \epsilon_i, \delta_k \pm \delta_l \mid 1 \leq i \neq j \leq m, 1 \leq k \neq l \leq n\},$$

$$\overline{\Delta}_1 = \pm\{\delta_k \pm \epsilon_i \mid 1 \leq i \leq m, 1 \leq k \leq n\}.$$

Also recall that the defect of \mathfrak{g} is $d = \min(n, m)$. In this Section we will use the following notation

$$(5.3) \quad \Delta(B_r) = \pm\{\epsilon_i \pm \epsilon_j, \epsilon_i \mid 1 \leq i \neq j \leq r\},$$

$$(5.4) \quad \Delta(C_r) = \pm\{\delta_k \pm \delta_l, 2\delta_k \mid n-r+1 \leq k \neq l \leq n\},$$

$$(5.5) \quad \Delta(A_{r-1}) = \pm\{\delta_k - \delta_l \mid n-r+1 \leq k \neq l \leq n\},$$

$$(5.6) \quad W(A_{r-1}) = \text{Weyl group of } \Delta(A_{r-1}),$$

$$(5.7) \quad W(B_r) = \text{Weyl group of } \Delta(B_r),$$

$$(5.8) \quad \mathcal{W}_r = \text{subgroup of } W_{\mathfrak{g}} \text{ generated by } \{s_{2\delta_i} \mid i = 1, \dots, r\},$$

$$(5.9) \quad \mathcal{W}_r^+ = \{w \in \mathcal{W}_r \mid w = s_{2\delta_{i_1}} \dots s_{2\delta_{i_k}}, k \text{ even}\},$$

$$(5.10) \quad \mathcal{W}_r^- = \{w \in \mathcal{W}_r \mid w = s_{2\delta_{i_1}} \dots s_{2\delta_{i_k}}, k \text{ odd}\}.$$

Now fix the set of positive roots Δ_B^+ (cf. (4.25)). Then $2\rho_1 = (2m+1)(\delta_1 + \dots + \delta_n)$. Set $\Delta^+(C_n) = \Delta_0^+ \cap \Delta(C_n)$, $\Delta^+(B_m) = \Delta_0^+ \cap \Delta(B_m)$, and $\Delta^+(A_{n-1}) = \Delta_0^+ \cap$

$\Delta(A_{n-1})$. We denote by ρ^C, ρ^B, ρ^A the corresponding ρ -vectors. With notation as in Section 3.1, the total order corresponding to this choice of Δ^+ is

$$\delta_1 > \cdots > \delta_n > \epsilon_1 > \cdots > \epsilon_m.$$

There is only one arc diagram associated to this total order, namely

$$X = \{\widehat{\delta_n \epsilon_1}, \dots, \widehat{\delta_{n-d+1} \epsilon_d}\}.$$

The corresponding maximal isotropic set of roots is $S(X) = \{\gamma_1, \dots, \gamma_d\}$, where

$$(5.11) \quad \gamma_1 = \delta_n - \epsilon_1, \gamma_2 = \delta_{n-1} - \epsilon_2, \dots, \gamma_d = \delta_{n-d+1} - \epsilon_d.$$

We apply Proposition 3.8 with $\mathcal{B}' = \text{Supp}(X)$. Note that, with notation as in 3.2.1, $\Delta(\text{Supp}(X)) = \Delta(B_d) \times \Delta(C_d)$, and we can choose $\Delta^\sharp(\text{Supp}(X)) = \Delta(B_d)$, so that $W_{\text{Supp}(X)} W^\sharp(\text{Supp}(X)) = W(A_{d-1}) W(B_d)$. With notation as in (5.8), set

$$W = W(A_{n-1}) \mathcal{W}_{n-d} W(B_m).$$

We can choose

$$Z = (W(A_{n-1})/W(A_{d-1})) \mathcal{W}_{n-d} (W(B_m)/W(B_d)),$$

so that $W_0 = W$. Since $\prod_{i=1}^d \frac{ht(\gamma_i)+1}{2} = d!$, Proposition 3.8 gives

$$(5.12) \quad e^\rho \check{R} = \check{\mathcal{F}}_W \left(\frac{e^\rho}{\prod_{i=1}^d (1 - e^{-\llbracket \gamma_i \rrbracket})} \right).$$

Dividing (5.12) by $\mathcal{D}_0 = e^{\rho_0} \prod_{\alpha \in \Delta_0^+} (1 - e^{-\alpha})$, we find

$$(5.13) \quad chM^{\Delta^+}(\mathfrak{g}_1) = \frac{e^{-\rho_1}}{\prod_{\alpha \in \Delta_1^+} (1 - e^{-\alpha})} = \frac{1}{\mathcal{D}_0} \check{\mathcal{F}}_W \left(\frac{e^\rho}{\prod_{i=1}^d (1 - e^{-\llbracket \gamma_i \rrbracket})} \right).$$

Let $\mathbb{P}_d = \{\mathbf{a} = (a_1, \dots, a_d) \in (\mathbb{Z}^+)^d \mid a_1 \geq a_2 \geq \cdots \geq a_d\}$ the set of partitions with at most d parts. Recall from Section 4 that there is an element $H \in \mathfrak{h}$ such that $\alpha(H) = 1$ for all $\alpha \in \Delta_1^+(\mathfrak{g})$, thus the domain $\mathfrak{h}^+ = \{h \in \mathfrak{h} \mid \alpha(h) > 0 \forall \alpha \in \Delta_1^+\}$ is nonempty. Since, by our choice of W , in the denominators of (5.13) only sums of roots in $-\Delta_1^+$ occur, we can expand the r.h.s. of (5.13) in a product of geometric series on \mathfrak{h}^+ , thus obtaining the following equality of formal power series:

$$(5.14) \quad chM^{\Delta^+}(\mathfrak{g}_1) = \sum_{\mathbf{a} \in \mathbb{P}_d} \frac{1}{\mathcal{D}_0} \check{\mathcal{F}}_W (e^{\rho - \sum_{i=1}^d a_i \gamma_i}).$$

We can write $\mathcal{D}_0 = e^{\rho^C} \prod_{\alpha \in \Delta^+(C_n)} (1 - e^{-\alpha}) e^{\rho^B} \prod_{\alpha \in \Delta^+(B_m)} (1 - e^{-\alpha})$. For $\mathbf{a} \in \mathbb{P}_d$, define

$$\mu(\mathbf{a}) = - \sum_{r=1}^d (a_{d+1-r}) \delta_{n-d+r}, \quad \varepsilon(\mathbf{a}) = \sum_{r=1}^m a_r \epsilon_r.$$

Using (5.14), we obtain that

$$\begin{aligned} chM^{\Delta^+}(\mathfrak{g}_1) &= \sum_{\mathbf{a} \in \mathbb{P}_d} \left(\sum_{w \in W(A_{n-1}) \mathcal{W}_{n-d}} sgn'(w) \frac{e^{w(\rho^C - \rho_1 + \mu(\mathbf{a})) - \rho^C}}{\prod_{\alpha \in \Delta^+(C_n)} (1 - e^{-\alpha})} \right) \\ &\quad \times \left(\sum_{w \in W(B_m)} sgn(w) \frac{e^{w(\rho^B + \varepsilon(\mathbf{a})) - \rho^B}}{\prod_{\alpha \in \Delta^+(B_m)} (1 - e^{-\alpha})} \right). \end{aligned}$$

Here we used the fact that $\text{sgn}(w) = \text{sgn}'(w)$ if $w \in W(B_m)$. By the Weyl character formula, we see that

$$\sum_{w \in W(B_m)} \text{sgn}(w) \frac{e^{w(\rho^B + \varepsilon(\mathbf{a})) - \rho^B}}{\prod_{\alpha \in \Delta^+(B_m)} (1 - e^{-\alpha})} = \text{ch}F_B(\varepsilon(\mathbf{a})),$$

$F_B(\varepsilon(\mathbf{a}))$ being the finite-dimensional irreducible $so(n)$ -module with highest weight $\varepsilon(\mathbf{a})$. We therefore obtain that

$$(5.15) \quad \text{ch}M^{\Delta^+}(\mathfrak{g}_1) = \sum_{\mathbf{a} \in \mathbb{P}_d} \sum_{w \in W(A_{n-1})\mathcal{W}_{n-d}} \text{sgn}'(w) \frac{e^{w(-\rho_1 + \mu(\mathbf{a}) + \rho^C) - \rho^C}}{\prod_{\alpha \in \Delta^+(C_n)} (1 - e^{-\alpha})} \text{ch}F_B(\varepsilon(\mathbf{a})).$$

If $\lambda \in \text{span}(\delta_i)$, set

$$(5.16) \quad \text{ch}F_A(\lambda) = \sum_{w \in W(A_{n-1})} \text{sgn}(w) \frac{e^{w(\lambda + \rho^A) - \rho^A}}{\prod_{\alpha \in \Delta^+(A_{n-1})} (1 - e^{-\alpha})}.$$

As the notation is suggesting, when $\lambda + \rho^A$ is regular and dominant integral for $\Delta(A_{n-1})$, then $\text{ch}F_A(\lambda)$ is the character of a finite-dimensional irreducible representation of $sl(n, \mathbb{C})$, while, if $\lambda + \rho^A$ is singular, then $\text{ch}F_A(\lambda) = 0$.

Using the fact that $\text{sgn}(w) = 1$ if $w \in \mathcal{W}_{n-d}$, we can rewrite (5.15) as

$$(5.17) \quad \text{ch}M^{\Delta^+}(\mathfrak{g}_1) = \sum_{\mathbf{a} \in \mathbb{P}_d} \sum_{w \in \mathcal{W}_{n-d}} \frac{\text{ch}F_A(w(-\rho_1 + \mu(\mathbf{a}) + \rho^C) - \rho^C)}{\prod_{\alpha \in \Delta^+(C_n) \setminus \Delta^+(A_{n-1})} (1 - e^{-\alpha})} \text{ch}F_B(\varepsilon(\mathbf{a})).$$

Recall from the previous section that we constructed a real symplectic subspace V of \mathfrak{g}_1 and a map $\Phi : V \rightarrow \mathbb{R}^{2m+1} \otimes (\mathbb{R}^{2n})^*$ such that there is a dual pair (G_1, G_2) in $sp(V, (\cdot, \cdot))$ having the properties described in Proposition 4.8. We now want to describe explicitly the set Σ and the map τ occurring in Theorem 4.1. In what follows we show that all these information can be read off from formula (5.17).

First of all we need to parametrize the finite-dimensional irreducible representations of \tilde{G}_1 . To accomplish this we start by describing \tilde{G}_1 . With notation as in the previous section, let $J' : (\mathbb{R}^{2n})^* \rightarrow (\mathbb{R}^{2n})^*$ be the complex structure such that $J'(f_i) = -f_{n+i}$ and $J'(f_{n+i}) = f_i$ for $i = 1, \dots, n$. A direct computation shows that, if J is the compatible complex structure on V introduced in the previous section, then $\Phi J \Phi^{-1} = Id \otimes J'$. It follows that, if we let W' be the space $(\mathbb{R}^{2n})^*$ seen as a complex space via the complex structure J' , then, obviously, Φ is a \mathbb{C} -linear isomorphism between W and the complex space $\mathbb{R}^{2m+1} \otimes W'$. We look upon $O(2m+1)$ as a subgroup of $Sp(\mathbb{R}^{2m+1} \otimes (\mathbb{R}^{2n})^*)$ via its action on the first factor. Then $G_1 = \{\Phi^{-1}g\Phi \mid g \in O(2m+1)\}$. It follows that, if $\Phi^{-1}g\Phi \in G_1$, then

$$\det_W(\Phi^{-1}g\Phi) = \det_{\mathbb{R}^{2m+1} \otimes W'}(g) = (\det_{\mathbb{R}^{2m+1}}(g))^n.$$

Therefore

$$\tilde{G}_1 = \begin{cases} \{(\Phi^{-1}g\Phi, \pm 1) \mid g \in O(2m+1)\} & \text{if } n \text{ is even,} \\ \{(\Phi^{-1}g\Phi, \pm 1) \mid g \in SO(2m+1)\} \\ \cup \{(\Phi^{-1}g\Phi, \pm \sqrt{-1}) \mid -g \in SO(2m+1)\} & \text{if } n \text{ is odd.} \end{cases}$$

Let $\tilde{G}_1^0 \simeq SO(2m+1)$ be the connected component of the identity of \tilde{G}_1 . From the description of \tilde{G}_1 we see that $\tilde{G}_1 = \tilde{G}_1^0 \times Z$ with Z isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ if n is even and isomorphic to $\mathbb{Z}/4\mathbb{Z}$ if n is odd. The generators of Z are $(-Id, 1)$, $(Id, -1)$

in the first case, while Z is generated by $(-Id, \sqrt{-1})$ in the second case. It follows that the finite-dimensional irreducible representations of \tilde{G}_1 are pairs (F, χ) where F is a finite-dimensional irreducible representation of \tilde{G}_1^0 (hence of its complexified Lie algebra $\mathfrak{s}_1^{\mathbb{C}}$) and χ is a character of Z .

If n is even and $\epsilon \in \{1, -1\}$, we let χ_ϵ be the character of Z such that $\chi_\epsilon(-Id, 1) = \epsilon$ and $\chi_\epsilon(Id, -1) = -1$. If n is odd we define χ_ϵ to be the character of Z such that $\chi_\epsilon(-Id, \sqrt{-1}) = -\sqrt{-1}\epsilon$. If $\mathbf{a} \in \mathbb{P}_d$ set $\epsilon(\mathbf{a}) = (-1)^{\sum_{i=1}^d a_i}$.

In our case the set of roots of $\mathfrak{s}_2^{\mathbb{C}}$ is $\Delta(C_n)$. In the identification of \mathfrak{h} with \mathfrak{h}^* given by $(\ , \)$, we see that the element $H \in \mathfrak{h}$ that corresponds to $-\sum_{i=1}^n \delta_i$ has the property that $H|_{V_{\mathbb{C}}^\pm} = \pm I$. Thus the parabolic subalgebra \mathfrak{p}_2 defined by H is

$$\mathfrak{p}_2 = \mathfrak{h} \oplus \sum_{\alpha \in \Delta(A_{n-1})} (\mathfrak{g}_0)_\alpha \oplus \mathfrak{n},$$

where $\mathfrak{n} = \sum_{\alpha \in \Delta^+(C_n) \setminus \Delta(A_{n-1})} (\mathfrak{g}_0)_\alpha$ is the nilradical.

Notation 5.1. *If $\lambda \in (\mathfrak{h} \cap \mathfrak{s}_2^{\mathbb{C}})^*$ is such that $\lambda + \rho^A$ is regular and integral for $\Delta(A_{n-1})$, we let $L^2(\lambda)$ be the irreducible quotient of the \mathfrak{p}_2 -parabolic Verma module $V^2(\lambda)$ for $\mathfrak{s}_2^{\mathbb{C}}$.*

If $\mu \in (\mathfrak{h} \cap \mathfrak{s}_2^{\mathbb{C}})^$ is such that μ is regular for $\Delta(A_{n-1})$, we let $\{\mu\}$ be the unique element in the $W(A_{n-1})$ -orbit of μ that is dominant with respect to $\Delta^+(A_{n-1})$.*

Finally, we set $c_\mu = \text{sgn}(v)$, where v is the unique element of $W(A_{n-1})$ such that $v(\mu) = \{\mu\}$.

Note that

$$\text{ch}V^2(\lambda) = c_{\lambda+\rho^A} \frac{\text{ch}F_A(\lambda)}{\prod_{\alpha \in \Delta^+(C_n) \setminus \Delta^+(A_{n-1})} (1 - e^{-\alpha})}.$$

Proposition 5.1. *With notation as in Theorem 4.1, we have that, if $(F, \chi) \in \Sigma$, then there exists $\mathbf{a} \in \mathbb{P}_d$ and $\epsilon \in \{1, -1\}$ such that*

$$F = F_B(\varepsilon(\mathbf{a})), \quad \chi = \chi_\epsilon.$$

Furthermore, the \mathfrak{h} -character of the isotypic component of $(F_B(\varepsilon(\mathbf{a})), \chi_{\pm\epsilon(\mathbf{a})})$ in $M^{\Delta^+}(\mathfrak{g}_1)$ is

$$(5.18) \quad \sum_{w \in \mathcal{W}_{n-d}^\pm} \frac{\text{ch}F_A(w(\rho^C - \rho_1 + \mu(\mathbf{a})) - \rho^C)}{\prod_{\alpha \in \Delta^+(C_n) \setminus \Delta^+(A_{n-1})} (1 - e^{-\alpha})} \text{ch}F_B(\varepsilon(\mathbf{a})).$$

(\mathcal{W}_{n-d}^\pm are defined in (5.9), (5.10)).

Proof. If (F, χ) occurs in $M^{\Delta^+}(\mathfrak{g}_1)$ then, by (4.1), $F = F_B(\varepsilon(\mathbf{a}))$ for some $\mathbf{a} \in \mathbb{P}_d$. As in the previous section, we identify $M^{\Delta^+}(\mathfrak{g}_1)$ with $P(W)$. Let $P(W)^+$ and $P(W)^-$ be the subspaces of homogeneous polynomials of even and odd degree respectively. By the explicit expression for the action of \tilde{K} given in (4.2), we see that Z acts by χ_1 on $P(W)^+$ and by χ_{-1} on $P(W)^-$. This proves the first assertion.

Let $M(\mathbf{a}, \chi_\epsilon)$ be the isotypic component of $(F_B(\varepsilon(\mathbf{a})), \chi_\epsilon)$. By Theorem 4.1, if $M(\mathbf{a}, \chi_\epsilon) \neq \{0\}$, then

$$\text{ch}M(\mathbf{a}, \chi_\epsilon) = \text{ch}L^2(\lambda) \text{ch}F_B(\varepsilon(\mathbf{a}))$$

for some $\lambda \in (\mathfrak{h} \cap \mathfrak{s}_2)^*$.

Observe that H acts on a homogeneous polynomial p as $H \cdot p = (-\rho_1(H) - \deg(p))p$. Since $M(\mathbf{a}, \chi_\epsilon) = L^2(\lambda) \otimes F_B(\epsilon(\mathbf{a}))$, we see that we must have

$$\epsilon = (-1)^{(\lambda + \epsilon(\mathbf{a}) + \rho_1)(H)}.$$

It is well known that $chL^2(\lambda) = \sum_{\mu} c_{\lambda, \mu} chV^2(\mu)$, where $\lambda - \mu = \sum_{\alpha \in \Delta(C_n)} n_{\alpha} \alpha$. Since $\alpha(H)$ is even for any root in $\Delta(C_n)$ we deduce that

$$(5.19) \quad chM(\mathbf{a}, \chi_\epsilon) = \sum_{\mu} c_{\lambda, \mu} chV^2(\mu) chF_B(\epsilon(\mathbf{a}))$$

with $\epsilon = (-1)^{(\mu + \epsilon(\mathbf{a}) + \rho_1)(H)}$.

Hence, if we set

$$(5.20) \quad \mathcal{W}_{n-d}^{reg} = \{w \in \mathcal{W}_{n-d} \mid w(-\rho_1 + \mu(\mathbf{a}) + \rho^C) \text{ is regular for } \Delta(A_{n-1})\},$$

we derive from (5.17) that

$$(5.21) \quad chM(\mathbf{a}, \chi_1) + chM(\mathbf{a}, \chi_{-1}) = \sum_{w \in \mathcal{W}_{n-d}^{reg}} c_w chV^2(\{w(-\rho_1 + \mu(\mathbf{a}) + \rho^C)\} - \rho^C) chF_B(\epsilon(\mathbf{a})),$$

where $c_w = c_{w(-\rho_1 + \mu(\mathbf{a}) + \rho^C)}$.

We need to compute

$$(-1)^{(\{w(-\rho_1 + \mu(\mathbf{a}) + \rho^C)\} - \rho^C + \epsilon(\mathbf{a}) + \rho_1)(H)}.$$

For this observe that $\epsilon(\mathbf{a})(H) = 0$ and that $w(\mu(\mathbf{a})) = \mu(\mathbf{a})$ for all $\mathbf{a} \in \mathbb{P}_d$ and $w \in \mathcal{W}_{n-d}$, hence

$$\begin{aligned} & (w(-\rho_1 + \mu(\mathbf{a}) + \rho^C) - \rho^C + \epsilon(\mathbf{a}) + \rho_1)(H) \\ &= \mu(\mathbf{a})(H) + (-w(\rho_1) + \rho_1)(H) + (w(\rho^C) - \rho^C)(H) \\ &\equiv \mu(\mathbf{a})(H) + (-w(\rho_1) + \rho_1)(H) \pmod{2}. \end{aligned}$$

Since $(-1)^{\mu(\mathbf{a})(H)} = \epsilon(\mathbf{a})$, $\rho_1 - s_{2\delta_i}(\rho_1) = (2m+1)\delta_i$, and $v(H) = H$ for any $v \in W(A_{n-1})$, we see that

$$(-1)^{(\{w(-\rho_1 + \mu(\mathbf{a}) + \rho^C)\} - \rho^C + \epsilon(\mathbf{a}) + \rho_1)(H)} = \pm \epsilon(\mathbf{a}) \text{ if } w \in \mathcal{W}_{n-d}^{\pm}.$$

If $w, w' \in \mathcal{W}_{n-d}$ with $w \neq w'$ then $W(A_{n-1})w \cap W(A_{n-1})w' = \emptyset$, hence the set of characters $\{chV^2(\{w(-\rho_1 + \mu(\mathbf{a}) + \rho^C)\} - \rho^C) \mid w \in \mathcal{W}_{n-d}^{reg}\}$ is linearly independent. It follows that, if $M(F_B(\epsilon(\mathbf{a})), \chi_{\epsilon(\mathbf{a})}) \neq \{0\}$, then, comparing (5.19) with (5.21), we find

$$\begin{aligned} & chM(F_B(\epsilon(\mathbf{a})), \chi_{\epsilon(\mathbf{a})}) \\ &= \sum_{w \in \mathcal{W}_{n-d}^{reg} \cap \mathcal{W}_{n-d}^+} c_w chV^2(\{w(-\rho_1 + \mu(\mathbf{a}) + \rho^C)\} - \rho^C) chF_B(\epsilon(\mathbf{a})). \end{aligned}$$

Likewise, if $M(F_B(\epsilon(\mathbf{a})), \chi_{-\epsilon(\mathbf{a})}) \neq \{0\}$, then

$$\begin{aligned} & chM(F_B(\epsilon(\mathbf{a})), \chi_{-\epsilon(\mathbf{a})}) \\ &= \sum_{w \in \mathcal{W}_{n-d}^{reg} \cap \mathcal{W}_{n-d}^-} c_w chV^2(\{w(-\rho_1 + \mu(\mathbf{a}) + \rho^C)\} - \rho^C) chF_B(\epsilon(\mathbf{a})). \end{aligned}$$

Since $chF_A(\lambda) = 0$ if $\lambda + \rho^A$ is singular, (5.18) follows.

Finally if $M(F_B(\varepsilon(\mathbf{a})), \chi_{\pm\epsilon(\mathbf{a})}) = \{0\}$, then the linear independence of the characters involved implies that $\mathcal{W}_{n-d}^{reg} \cap \mathcal{W}_{n-d}^{\pm} = \emptyset$, hence (5.18) holds as well. \square

Let \mathbb{P}_j^* denote the set of all partitions with exactly j parts and set

$$(5.22) \quad \mathbb{P} = \bigcup_{j=\max(0, m+1-(n-d))}^m \mathbb{P}_j^*.$$

For $\max(0, m+1-(n-d)) \leq j \leq m$ and $\mathbf{a} \in \mathbb{P}_j^*$, we set

$$(5.23) \quad \nu(\mathbf{a}) = - \sum_{r=n-d-m+j}^{n-j} \delta_r - \sum_{r=n-j+1}^n a_{n-r+1} \delta_r.$$

Corollary 5.2 (Theta correspondence). *With notation as in Theorem 4.1, we have that*

$$\Sigma = \{(F_B(\varepsilon(\mathbf{a})), \chi_{\epsilon(\mathbf{a})}) \mid \mathbf{a} \in \mathbb{P}_d\} \cup \{(F_B(\varepsilon(\mathbf{a})), \chi_{-\epsilon(\mathbf{a})}) \mid \mathbf{a} \in \mathbb{P}\}.$$

Moreover

$$\tau(F_B(\varepsilon(\mathbf{a})), \chi_{\epsilon(\mathbf{a})}) = L^2(-\rho_1 + \mu(\mathbf{a}))$$

and, if $\mathbf{a} \in \mathbb{P}$, then

$$\tau(F_B(\varepsilon(\mathbf{a})), \chi_{-\epsilon(\mathbf{a})}) = L^2(-\rho_1 + \nu(\mathbf{a})).$$

Proof. By Proposition 5.1, $(F_B(\varepsilon(\mathbf{a})), \chi_{\pm\epsilon(\mathbf{a})}) \in \Sigma$ if and only if $\mathcal{W}_{n-d}^{\pm} \cap \mathcal{W}_{n-d}^{reg} \neq \emptyset$. Clearly $1 \in \mathcal{W}_{n-d}^+$ and $-\rho_1 + \mu(\mathbf{a}) = 1(-\rho_1 + \mu(\mathbf{a}) + \rho^C) - \rho^C$ is dominant for $\Delta^+(A_{n-1})$, hence $-\rho_1 + \mu(\mathbf{a}) + \rho^A$ is regular. This implies that $(F_B(\varepsilon(\mathbf{a})), \chi_{\epsilon(\mathbf{a})}) \in \Sigma$ for all $\mathbf{a} \in \mathbb{P}_d$.

We now show that, if $\mathbf{a} \in \mathbb{P}$, then $(F_B(\varepsilon(\mathbf{a})), \chi_{-\epsilon(\mathbf{a})}) \in \Sigma$. First observe that, if $\mathbb{P} \neq \emptyset$, then $d = m < n$. Consider $s = s_{2\delta_{n-d-m+j}}$. Then

$$\begin{aligned} s(-\rho_1 + \mu(\mathbf{a}) + \rho^C) &= \\ & \sum_{i=1}^{n-d-m+j-1} (n-m-i + \frac{1}{2})\delta_i + (-m - \frac{1}{2} + j)\delta_{n-d-m+j} \\ & + \sum_{i=n-d-m+j+1}^{n-j} (n-m-i + \frac{1}{2})\delta_i + \sum_{i=n-j+1}^n (n-m-i + \frac{1}{2} - a_{n-i+1})\delta_i. \end{aligned}$$

Let σ be the permutation (in cycle notation) $\sigma = (n-j \ n-j-1 \ \dots \ n-d-m+j)$, and let w be the element of $W(A_{n-1})$ such that $w(\delta_i) = \delta_{\sigma(i)}$. Then

$$\begin{aligned} ws(-\rho_1 + \mu(\mathbf{a}) + \rho^C) &= \\ & \sum_{i=1}^{n-d-m+j-1} (n-m-i + \frac{1}{2})\delta_i + \sum_{i=n-d-m+j}^{n-j} (n-m-i - \frac{1}{2})\delta_i + \\ & \sum_{i=n-j+1}^n (n-m-i - \frac{1}{2} - (a_{n-i+1} - 1))\delta_i = -\rho_1 + \nu(\mathbf{a}) + \rho^C, \end{aligned}$$

and, since $a_r \geq 1$ for $r \geq j$, we see that $ws(-\rho_1 + \mu(\mathbf{a}) + \rho^C)$ is regular for $\Delta(A_{n-1})$. It follows that $s \in \mathcal{W}_{n-d}^{reg} \cap \mathcal{W}_{n-d}^-$.

Suppose now that $\mathbf{a} \in \mathbb{P}_d \setminus \mathbb{P}$. Then $a_r = 0$ for $r > m - (n - d)$. In this case

$$\begin{aligned} -\rho_1 + \mu(\mathbf{a}) + \rho^C &= \sum_{i=1}^{2(n-d)} (n - m - i + \frac{1}{2})\delta_i \\ &+ \sum_{i=2(n-d)+1}^n (n - m - i + \frac{1}{2} - a_{n-i+1})\delta_i. \end{aligned}$$

If $w \in \mathcal{W}_{n-d}$, $w \neq 1$, then write $w = s_{2\delta_{i_1}} \dots s_{2\delta_{i_k}}$ with $1 \leq i_1 < \dots < i_k \leq n - d$. It is clear that

$$\begin{aligned} (w(-\rho_1 + \mu(\mathbf{a}) + \rho^C), \delta_{i_1} - \delta_{2(n-d)-i_1+1}) &= \\ (-\rho_1 + \mu(\mathbf{a}) + \rho^C, -\delta_{i_1} - \delta_{2(n-d)-i_1+1}) &= 0, \end{aligned}$$

showing that $\mathcal{W}_{n-d}^{reg} = \{1\}$ in this case. This proves the first statement.

For the second statement recall that, if $\mathbf{a} \in \mathbb{P}_j^*$, then $-\rho_1 + \nu(\mathbf{a}) = \{s(-\rho_1 + \mu(\mathbf{a}) + \rho^C)\} - \rho^C$. Note that, if $\tau(F_B(\varepsilon(\mathbf{a})), \chi_{\pm\varepsilon(\mathbf{a})}) = L^2(\lambda)$ then, by (5.18), $\lambda(H) \geq (\{w(-\rho_1 + \mu(\mathbf{a}) + \rho^C)\} - \rho^C)(H)$ for all $w \in \mathcal{W}_{n-d}^\pm \cap \mathcal{W}_{n-d}^{reg}$. Hence if we show that, for $w \in \mathcal{W}_{n-d}^+ \cap \mathcal{W}_{n-d}^{reg}$, $w \neq 1$,

$$(-\rho_1 + \mu(\mathbf{a}))(H) > (\{w(-\rho_1 + \mu(\mathbf{a}) + \rho^C)\} - \rho^C)(H),$$

and that, for $\mathbf{a} \in \mathbb{P}_j^*$, $w \in \mathcal{W}_{n-d}^- \cap \mathcal{W}_{n-d}^{reg}$, $w \neq s_{2\delta_{n-d-m+j}}$,

$$(-\rho_1 + \nu(\mathbf{a}))(H) > (\{w(-\rho_1 + \mu(\mathbf{a}) + \rho^C)\} - \rho^C)(H),$$

we are done.

Since $v(H) = H$ if $v \in W(A_{n-1})$, it is enough to check that, for $w \in \mathcal{W}_{n-d}^+ \cap \mathcal{W}_{n-d}^{reg}$, $w \neq 1$,

$$(5.24) \quad (-\rho_1 + \mu(\mathbf{a}) + \rho^C)(H) > w(-\rho_1 + \mu(\mathbf{a}) + \rho^C)(H),$$

and that, for $\mathbf{a} \in \mathbb{P}_j^*$, $w \in \mathcal{W}_{n-d}^- \cap \mathcal{W}_{n-d}^{reg}$, $w \neq s_{2\delta_{n-d-m+j}}$,

$$(5.25) \quad s_{2\delta_{n-d-m+j}}(-\rho_1 + \mu(\mathbf{a}) + \rho^C)(H) > w(-\rho_1 + \mu(\mathbf{a}) + \rho^C)(H).$$

If $w = s_{2\delta_{i_1}} \dots s_{2\delta_{i_k}}$, then $(\lambda, H - w(H)) = (\lambda, -2\delta_{i_1} - \dots - 2\delta_{i_k})$, hence

$$(-\rho_1 + \mu(\mathbf{a}) + \rho^C)(H - w(H)) = 2(n - d - i_1 + \frac{1}{2}) + \dots + 2(n - d - i_k + \frac{1}{2}) > 0,$$

so, using the fact that $w = w^{-1}$, we obtain (5.24). To obtain (5.25) we set $j_0 = n - d - m + j$ and observe that, since $\mathbf{a} \in \mathbb{P}_j^*$, if $j_0 < i_k \leq n - d$, then $w(-\rho_1 + \mu(\mathbf{a}) + \rho^C)$ is singular for $\Delta(A_{n-1})$. We can therefore assume that $i_k \leq j_0$. In this case we need to evaluate $(-\rho_1 + \mu(\mathbf{a}) + \rho^C)(s_{2\delta_{j_0}}(H) - w(H))$, which is equal to

$$(-\rho_1 + \mu(\mathbf{a}) + \rho^C, 2\delta_{j_0} - 2\delta_{i_1} - \dots - 2\delta_{i_k}).$$

If $i_k < j_0$ then this is $2(n - d - i_1 + \frac{1}{2}) + \dots + 2(n - d - i_k + \frac{1}{2}) - 2(n - d - j_0 + \frac{1}{2}) > 0$. If $i_k = j_0$ then, since $w \neq s_{2\delta_{j_0}}$, we have $k > 1$ so $(-\rho_1 + \mu(\mathbf{a}) + \rho^C)(s_{2\delta_{j_0}}(H) - w(H))$ is equal to $2(n - d - i_1 + \frac{1}{2}) + \dots + 2(n - d - i_{k-1} + \frac{1}{2}) > 0$. The proof is now complete. \square

We conclude this section by observing that, as a byproduct, we have computed the character of $\tau(\eta)$ for $\eta \in \Sigma$. In fact, combining Corollary 5.2 with Proposition 5.1, we find (cf. (5.20) for notation)

$$(5.26) \quad ch(L^2(-\rho_1 + \mu(\mathbf{a}))) = \sum_{w \in \mathcal{W}_{n-d}^{reg} \cap \mathcal{W}_{n-d}^+} c_w chV^2(\{w(-\rho_1 + \mu(\mathbf{a}) + \rho^C)\} - \rho^C)$$

and

$$(5.27) \quad ch(L^2(-\rho_1 + \nu(\mathbf{a}))) = \sum_{w \in \mathcal{W}_{n-d}^{reg} \cap \mathcal{W}_{n-d}^-} c_w chV^2(\{w(-\rho_1 + \mu(\mathbf{a}) + \rho^C)\} - \rho^C),$$

where $c_w = c_{w(-\rho_1 + \mu(\mathbf{a}) + \rho^C)}$.

If $\lambda \in (\mathfrak{h} \cap \mathfrak{s}_2^{\mathbb{C}})^*$, let W_λ be the subgroup of $W(C_n)$ defined by Enright in [2, Definition 2.1]. In our context, W_λ is the subgroup, generated by reflections in roots $\alpha = \delta_i + \delta_j$, such that:

- (i) $(\lambda, \alpha^\vee) \in \mathbb{Z}^{>0}$;
- (ii) $\beta \in \Delta, (\lambda, \beta) = 0 \implies (\alpha, \beta) = 0$.

Following [2, Definition 2.1], we introduce the root system Δ_λ as the set of the roots α such that $s_\alpha \in W_\lambda$. Set $\Delta_\lambda(A_{n-1}) = \Delta_\lambda \cap \Delta(A_{n-1})$, $\Delta_\lambda^+ = \Delta^+ \cap \Delta_\lambda$. Let also $W_\lambda(A_{n-1})$ be the Weyl group of $\Delta_\lambda(A_{n-1})$. We denote by ℓ_λ the length function in W_λ with respect to Δ_λ^+ and let W_λ^A be the set of minimal length coset representatives for $W_\lambda(A_{n-1}) \backslash W_\lambda$. In [2, Corollary 2.3] Enright proved a character formula that holds for any unitary highest weight module of a classical real reductive group. This formula is also implied by a stronger result proved in [3]. In the following we show how Enright's formula for $L^2(-\rho_1 + \mu(\mathbf{a}))$ and $L^2(-\rho_1 + \nu(\mathbf{a}))$ can be derived combinatorially from (5.26) and (5.27) respectively.

Corollary 5.3. *If $\mathbf{a} \in \mathbb{P}_d$ and $\lambda_0 = -\rho_1 + \mu(\mathbf{a}) + \rho^C$, then*

$$ch(L^2(-\rho_1 + \mu(\mathbf{a}))) = \sum_{w \in W_{\lambda_0}^A} (-1)^{\ell_{\lambda_0}(w)} V^2(\{w(\lambda_0)\} - \rho^C).$$

Proof. Fix $\mathbf{a} \in A$ and consider λ_0 , whose coordinates in the basis $\{\delta_i\}$ are of the form

$$v = (v_1, \dots, v_n) = (n - m + \frac{1}{2}, n - m - \frac{1}{2}, \dots, \frac{1}{2}, -\frac{1}{2} - b_1, \dots, -\frac{1}{2} - b_m),$$

where $b_i = a_{m-i+1} + i - 1$. We say that a positive entry a in v is singular or regular according to whether $-a$ appears in v or not. The group W_{λ_0} acts on λ_0 as follows: it is the identity if the number of regular entries is less or equal than one; if there are exactly two regular entries, then the only nontrivial action is given by exchanging positions and changing signs of both regular entries; if there are more than two regular entries, it acts by an even number of sign changes of the regular entries followed by a permutation of the result. All other entries are fixed.

If $x \in \mathcal{W}_{n-d}^{reg}$, then it acts on λ_0 by changing signs of some regular entries. Let w_x be the permutation that arranges the result in decreasing order. Since the set $\mathcal{W}_{n-d}^+ \cap \mathcal{W}_{n-d}^{reg}$ is a set of coset representatives for $W_{\lambda_0}(A_{n-1})$ in W_{λ_0} , we see that $W_{\lambda_0}^A = \{w_x x \mid x \in \mathcal{W}_{n-d}^+ \cap \mathcal{W}_{n-d}^{reg}\}$ (notice that this is true also when there are only two regular entries). Let w'_x be the unique element of $W(A_{n-1})$ such that $w'_x w_x x(\lambda_0)$

is $\Delta^+(A_{n-1})$ -dominant. Since $\text{sgn}(x) = 1$, it follows from (5.26) that

$$\text{ch}(L^2(-\rho_1 + \mu(\mathbf{a}))) = \sum_{x \in \mathcal{W}_{n-d}^+ \cap \mathcal{W}_{n-d}^{\text{reg}}} \text{sgn}(w'_x w_x x) V^2(\{w_x x(\lambda_0)\} - \rho^C).$$

To obtain our result we need only to prove that $\text{sgn}(w'_x) = 1$. We shall prove this claim by induction on $n - m$. Let s be the number of singular entries in v and let $t \geq 0$ be maximal for which $v_{n-m-t}, \dots, v_{n-m}$ are singular. Set $v' = w_x x(v)$. We want to prove that arranging v' in decreasing order involves an even number of simple transpositions. If x does not change the sign to v_1 , then induction applies in a straightforward way. Otherwise we have $v' = (v'_1, \dots, -v_1, v_{n-m-t}, \dots, v_{n-m}, \dots)$. In making v' dominant for A_{n-1} we should pass $-v_1$ through the t singular values and the s negative values corresponding to negatives of the singular entries. So we obtain after $s + t$ simple transpositions the element

$$v'' = (v'_1, \dots, v_{n-m-t}, \dots, v_{n-m}, \underbrace{\dots}_{s \text{ entries}}, -v_1, \dots).$$

Consider now the vector

$$u = (u_1, \dots, u_n) = (n - m - \frac{1}{2}, n - m - \frac{3}{2}, \dots, \frac{1}{2}, -\frac{1}{2} - b'_1, \dots, -\frac{1}{2} - b'_{m+1})$$

with $b_i = b'_i$ for $i = 1, \dots, s$, $b'_{s+1} = -x_1$, and $b'_i = b_{i-1}$ for $i = s + 2, \dots, m + 1$. Let $y = s_{2\delta_1} x$ and set $u' = w_y(u)$. Observe that v'' is obtained from u' by performing exactly $s - t$ simple transpositions. Hence v' differs from u' by an even number ($2s$) of simple transpositions, and we can apply induction. \square

Corollary 5.4. *If $\mathbf{a} \in \mathbb{P}$ and $\lambda_1 = -\rho_1 + \nu(\mathbf{a}) + \rho^C$, then*

$$\text{ch}(L^2(-\rho_1 + \nu(\mathbf{a}))) = \sum_{w \in W_{\lambda_1}^A} (-1)^{\ell_{\lambda_1}(w)} V^2(\{w(\lambda_1)\} - \rho^C).$$

Proof. Assume $\mathbf{a} \in A_j^+$ and let σ be the permutation described in the proof of Corollary 5.2. Let w be the element of $W_{A_{n-1}}$ such that $w(\delta_i) = \delta_{\sigma(i)}$ and set $s = s_{2\delta_{n-d-m+j}}$. Recall that $ws(\lambda_0) = \lambda_1$.

We now show that $W_{\lambda_1} = wW_{\lambda_0}w^{-1}$. For this it is enough to check that α satisfies conditions (i) and (ii) for λ_0 if and only if $w(\alpha)$ satisfies both conditions for λ_1 . This is clear if $(\alpha, \delta_{n-d-m+j}) = 0$, so that $s(\alpha) = \alpha$. If $\alpha = \delta_{n-d-m+j} + \delta_i$, with $n-d-m+j < i \leq n-j$, then α does not satisfy condition (ii) and, if $i > n-j$ then it does not satisfy condition (i). On the other hand $(w(\alpha)^\vee, ws(\lambda_0)) = (s(\alpha)^\vee, \lambda_0) < 0$ so $w(\alpha)$ does not satisfy condition (i). It remains to check the case when $\alpha = \delta_i + \delta_{n-d-m+j}$ with $i < n-d-m+j$. In this case one checks readily that α satisfies condition (i). On the other hand $s(\alpha) \in \Delta^+(A_{n-1})$, so $(s(\alpha)^\vee, \lambda_0) > 0$, hence $(w(\alpha), ws(\lambda_0)) > 0$. If β is a root such that $(\beta, \lambda_1) = 0$, then $(sw^{-1}(\beta), \lambda_0) = 0$, so $(\alpha, sw^{-1}(\beta)) = 0$. If $(\alpha, w^{-1}(\beta)) \neq 0$ the only possibility is that $w^{-1}(\beta) = \pm\alpha$, but this implies $(s(\alpha), \lambda_0) = 0$. This contradiction implies $(\alpha, w^{-1}(\beta)) = (w(\alpha), \beta) = 0$. This shows that if α satisfies condition (ii) then also $w(\alpha)$ does. Reversing this argument we obtain that if $w(\alpha)$ satisfies condition (ii), then also α does. This proves our claim.

Notice now that $w(\Delta_{\lambda_0}^+) = \Delta_{\lambda_1}^+$, thus $W_{\lambda_1}^A = wW_{\lambda_0}^A w^{-1}$. By applying (5.27) we can write

$$ch(L^2(-\rho_1 + \nu(\mathbf{a}))) = \sum_{x \in \mathcal{W}_{n-d}^{reg} \cap \mathcal{W}_{n-d}^-} sgn(w'_x w_x) chV^2(\{ww_{x_s} x s w^{-1}(\lambda_1)\} - \rho^C).$$

As shown in Corollary 5.3, $sgn(w'_x) = 1$, so we need only to show that $sgn(w_x) = sgn(ww_{x_s} x s w^{-1})$, or, equivalently, that $sgn(w_x) = sgn(w_{x_s})$. Recall that, for $y \in \mathcal{W}^{reg}$, w_y is the unique element of $W_{\lambda_0}(A_{n-1})$ such that $w_y y(\lambda_0)$ is dominant for $\Delta_{\lambda_0}^+(A_{n-1})$. Since $s(\lambda_0)$ is dominant for $\Delta_{\lambda_0}^+$, it follows that $w_x = w_{x_s}$ and we are done. \square

Remark 5.1. It should be noted that, even though the proofs of Proposition 5.1 and its corollaries do not use classical invariant theory, they depend on Theorem 4.1, which uses in a crucial way the first fundamental theorem of classical invariant theory for $O(2m+1)$.

6. THETA CORRESPONDENCE FOR THE PAIR $(Sp(n), SO^*(2m))$

In this section we use the denominator identity developed in the previous sections to derive the Theta correspondence for the compact dual pair $(Sp(n), SO^*(2m))$. This dual pair, according to Proposition 4.8, corresponds to the distinguished sets of positive roots $\Delta_{D_2}^+$ and $\Delta_{D_2'}^+$ in a superalgebra of type $D(m, n)$.

We will develop the theory only for $\Delta_{D_2}^+$: the formulas corresponding to $\Delta_{D_2'}^+$ are obtained by simply applying the reflection s_{ϵ_m} to the formulas corresponding to $\Delta_{D_2}^+$.

We have, for $1 \leq i \neq j \leq m$, $1 \leq k \neq l \leq n$,

$$(6.1) \quad \begin{aligned} \Delta_0^+ &= \{\epsilon_i + \epsilon_j, \delta_k + \delta_l, 2\delta_k \mid 1 \leq i \neq j \leq m, 1 \leq k \neq l \leq n\} \cup \\ &\quad \{\epsilon_i - \epsilon_j, \delta_k - \delta_l, \mid 1 \leq i < j \leq m, 1 \leq k < l \leq n\}, \\ \Delta_1^+ &= \{\epsilon_i \pm \delta_k \mid 1 \leq i \leq m, 1 \leq k \leq n\}. \end{aligned}$$

We have $2\rho_1 = (2n)(\epsilon_1 + \dots + \epsilon_m)$ and, as in type $B(m, n)$, the defect of \mathfrak{g} is $d = \min(m, n)$. Set also

$$\begin{aligned} \Delta(D_r) &= \pm\{\epsilon_i \pm \epsilon_j \mid m - r + 1 \leq i \neq j \leq m\}, \\ \Delta(C_r) &= \pm\{\delta_k \pm \delta_l, 2\delta_k \mid 1 \leq k \neq l \leq r\}, \\ \Delta(A_{r-1}) &= \pm\{\epsilon_k - \epsilon_l \mid m - r + 1 \leq k \neq l \leq m\}. \end{aligned}$$

Let $W(A_{r-1}), W(D_r), W(C_r)$ be the Weyl groups of $\Delta(A_{r-1}), \Delta(D_r), \Delta(C_r)$, respectively. Set $\Delta^+(C_n) = \Delta_0^+ \cap \Delta(C_n)$, $\Delta^+(D_m) = \Delta_0^+ \cap \Delta(D_m)$, and $\Delta^+(A_{m-1}) = \Delta_0^+ \cap \Delta(A_{m-1})$. We denote by ρ^C, ρ^D, ρ^A the corresponding ρ -vectors. Set

$$(6.2) \quad \mathcal{W}_r = \text{subgroup of } W_{\mathfrak{g}} \text{ generated by } \{s_{\epsilon_i} s_{\epsilon_j} \mid 1 \leq i < j \leq r\}.$$

With notation as in Section 3.1, the total order corresponding to this choice of Δ^+ is

$$\epsilon_1 > \dots > \epsilon_m > \delta_1 > \dots > \delta_n.$$

There is only one arc diagram associated to this total order, namely

$$X = \{\widehat{\epsilon_m \delta_1}, \dots, \widehat{\epsilon_{m-d+1} \delta_d}\}.$$

The corresponding maximal isotropic set of roots is $S(X) = \{\gamma_1, \dots, \gamma_d\}$, where

$$(6.3) \quad \gamma_1 = \epsilon_m - \delta_1, \gamma_2 = \epsilon_{m-1} - \delta_2, \dots, \gamma_d = \epsilon_{m-d+1} - \delta_d.$$

We want to apply Proposition 3.8 with $\mathcal{B}' = \text{Supp}(X)$. Note that, with notation as in 3.2.1, $\Delta(\mathcal{B}') = \Delta(D_d) \times \Delta(C_d)$, and we have that $\Delta^\sharp(\mathcal{B}') = \Delta(C_d)$, so that $W_{\mathcal{B}'}W^\sharp(\mathcal{B}') = W(A_{d-1})W(C_d)$. With notation as in (6.2), set

$$Z_0 = (W(A_{n-1})/W(A_{d-1}))\mathcal{W}_{n-d}(W(C_m)/W(C_d))$$

and, if $m > n$,

$$Z_1 = (W(A_{n-1})/W(A_{d-1}))\mathcal{W}_{m-d}S_{\epsilon_{m-d}}S_{\epsilon_{m-d+1}}(W(C_m)/W(C_d)).$$

Then we can choose $Z = Z_0$ if $m \leq n$ and $Z = Z_0 \cup Z_1$ otherwise.

We define

$$W = W(A_{m-1})\mathcal{W}_{m-d}W(C_n)$$

so that we have $W_0 = W$ if $m \leq n$ and while $W_0 = W \cup W_1$ with

$$W_1 = (W(A_{m-1})/W(A_{d-1}))\mathcal{W}_{m-d}S_{\epsilon_{m-d}}S_{\epsilon_{m-d+1}}W(A_{d-1})W(C_n)$$

otherwise.

If $m \leq n$, by Proposition 3.8, we have

$$(6.4) \quad e^\rho \check{R} = \check{F}_W\left(\frac{e^\rho}{\prod_{j=1}^d (1 - e^{-\llbracket \gamma_j \rrbracket})}\right).$$

If $m > n$, Proposition 3.8 gives

$$2e^\rho \check{R} = \check{F}_W\left(\frac{e^\rho}{\prod_{j=1}^d (1 - e^{-\llbracket \gamma_j \rrbracket})}\right) + \check{F}_{W_1}\left(\frac{e^\rho}{\prod_{j=1}^d (1 - e^{-\llbracket \gamma_j \rrbracket})}\right).$$

Lemma 6.1. *If $m > n$ then*

$$(6.5) \quad \check{F}_W\left(\frac{e^\rho}{\prod_{j=1}^d (1 - e^{-\llbracket \gamma_j \rrbracket})}\right) = \check{F}_{W_1}\left(\frac{e^\rho}{\prod_{j=1}^d (1 - e^{-\llbracket \gamma_j \rrbracket})}\right).$$

In particular (6.4) holds for any m, n .

Proof. We need only to check that $\check{F}_W(\mathcal{P}(X)) = \check{F}_{W_1}(\mathcal{P}(X))$. Since right multiplication by $W_{\text{Supp}(X)}$ stabilizes both W and W_1 , we can apply Corollary 3.6 to both sides of (6.5). Since odd reflections only change the sign of $\mathcal{P}(X)$, by applying a series of odd and interval reflection to both sides of (6.5), we are reduced to checking that $\check{F}_W(\mathcal{P}(X')) = \check{F}_{W_1}(\mathcal{P}(X'))$, where X' is the arc diagram $X' = \{\widehat{\epsilon_{m-n+i}\delta_i} \mid i = 1, \dots, n\}$ whose underlying order on \mathcal{B} is $\epsilon_1 > \dots > \epsilon_{m-d+1} > \delta_1 > \epsilon_{m-d+2} > \delta_2 > \dots > \epsilon_m > \delta_n$. Let Y' be the arc diagram X' seen as a diagram on $\mathcal{B}'' = \{\epsilon_{m-d+i} \mid 1 \leq i \leq d\} \cup \{\delta_i \mid i = 1, \dots, d\}$. Since all simple roots of \mathcal{B}'' are isotropic, we have $(\rho_{X'}, \alpha) = (\rho_{Y'}, \alpha) = 0$ for all $\alpha \in \Delta(\mathcal{B}'')$ so

$$\check{F}_W(\mathcal{P}(X)) = \check{F}_{(W(A_{m-1})/W(A_{d-1}))\mathcal{W}_{m-d}}(e^{\rho_{X'} - \rho_{Y'}} \check{F}_{W(A_{d-1})W(C_d)}(\mathcal{P}(Y')))$$

Since all simple roots of $\mathcal{B}'' \cup \{\epsilon_{m-d}\}$ are all isotropic, we see that $(\rho_{X'}, \epsilon_{m-d}) = 0$. Clearly $(\rho_{Y'}, \epsilon_{m-d}) = (\gamma, \epsilon_{m-d}) = 0$ for all $\gamma \in S(Y')$. Thus, setting $s = S_{\epsilon_{m-d}}S_{\epsilon_{m-d+1}}$, we have

$$\check{F}_{W_1}(\mathcal{P}(X)) = \check{F}_{(W(A_{m-1})/W(A_{d-1}))\mathcal{W}_{m-d}}(e^{\rho_{X'} - \rho_{Y'}} \check{F}_{sW(A_{d-1})W(C_d)}(\mathcal{P}(Y'))),$$

hence it is enough to check that $\check{\mathcal{F}}_{W(A_{d-1})W(C_d)}(\mathcal{P}(Y')) = \check{\mathcal{F}}_{sW(A_{d-1})W(C_d)}(\mathcal{P}(Y'))$. It is enough to verify the $s_{\epsilon_{m-d+1}}$ -invariance. However,

$$\check{\mathcal{F}}_{W(A_{d-1})W(C_d)}(\mathcal{P}(Y')) = d! \check{R}_{Y'} e^{\rho_{Y'}}$$

and this, up to a sign, does not depend on the choice of simple roots for $\Delta(Y')$. Taking a system of simple roots containing $\delta_1 \pm \epsilon_{m-d+1}$, we realize $s_{\epsilon_{m-d+1}}$ as an automorphism of the corresponding Dynkin diagram so $\check{R}_{Y'} e^{\rho_{Y'}}$ is $s_{\epsilon_{m-d+1}}$ -invariant. \square

For $\mathbf{a} \in \mathbb{P}_d$, define

$$\mu(\mathbf{a}) = - \sum_{r=1}^d (a_{d+1-r}) \epsilon_{m-d+r}, \quad \varepsilon(\mathbf{a}) = \sum_{r=1}^m a_r \delta_r.$$

If $\lambda \in \text{span}(\epsilon_i)$, set

$$(6.6) \quad \text{ch}F_A(\lambda) = \sum_{w \in W(A_{m-1})} \text{sgn}(w) \frac{e^{w(\lambda + \rho^A) - \rho^A}}{\prod_{\alpha \in \Delta^+(A_{m-1})} (1 - e^{-\alpha})}.$$

Arguing as in the previous section, it follows from (6.4) that

$$(6.7) \quad \text{ch}M^{\Delta^+}(\mathfrak{g}_1) = \sum_{\mathbf{a} \in \mathbb{P}_d} \sum_{w \in W_{m-d}} \frac{\text{ch}F_A(w(-\rho_1 + \mu(\mathbf{a}) + \rho^D) - \rho^D)}{\prod_{\alpha \in \Delta^+(D_m) \setminus \Delta^+(A_{m-1})} (1 - e^{-\alpha})} \text{ch}F_C(\varepsilon(\mathbf{a})),$$

$F_C(\varepsilon(\mathbf{a}))$ being the finite-dimensional irreducible $sp(n)$ -module with highest weight $\varepsilon(\mathbf{a})$.

Recall from section 4 that we constructed a real symplectic subspace V of \mathfrak{g}_1 and a map $\Phi : V \rightarrow (\mathbb{H}^n)^* \otimes_{\mathbb{H}} \mathbb{H}^m$, such that there is a dual pair (G_1, G_2) in $sp(V, \langle \cdot, \cdot \rangle)$ having the properties described in Proposition 4.8. We look upon $Sp(n)$ as a subgroup of $Sp((\mathbb{H}^n)^* \otimes_{\mathbb{H}} \mathbb{H}^m)$ via its action on the first factor. Then $G_1 = \{\Phi^{-1}g\Phi \mid g \in Sp(n)\}$. It follows that, if $\Phi^{-1}g\Phi \in G_1$, then, since $Sp(n)$ is compact, connected and simple,

$$\det_W(\Phi^{-1}g\Phi) = 1.$$

Therefore

$$\tilde{G}_1 = G_1 \times \{\pm 1\}.$$

Given a representation (π, V_π) of $G_1 = \Phi^{-1}Sp(n)\Phi$, we let $(\tilde{\pi}, V_\pi)$ be the representation of \tilde{G}_1 on the same representation space such that

$$(6.8) \quad \tilde{\pi}(g_1, z)(v) = z^{-1} \pi(g_1)(v).$$

If Ω is the set of finite-dimensional irreducible representations of G_1 , then we know from (4.2) that the set Σ , occurring in Theorem 4.1, is a subset of $\tilde{\Omega} = \{\tilde{\pi} \mid \pi \in \Omega\}$.

Proposition 6.2. *With notation as in Theorem 4.1, we have that, if $\tilde{\pi} \in \Sigma$, then there is $\mathbf{a} \in \mathbb{P}_d$ such that*

$$\pi = F_C(\varepsilon(\mathbf{a})).$$

Furthermore, the \mathfrak{h} -character of the isotypic component of $\tilde{F}_C(\varepsilon(\mathbf{a}))$ in $M^{\Delta^+}(\mathfrak{g}_1)$ is

$$(6.9) \quad \sum_{w \in W_{m-d}} \frac{\text{ch}F_A(w(\rho^D - \rho_1 + \mu(\mathbf{a})) - \rho^D)}{\prod_{\alpha \in \Delta^+(D_m) \setminus \Delta^+(A_{m-1})} (1 - e^{-\alpha})} \text{ch}F_C(\varepsilon(\mathbf{a})).$$

Proof. Clear from (6.7). \square

In the case in question the set of roots of $\mathfrak{s}_2^{\mathbb{C}}$ is $\Delta(D_m)$. In the identification of \mathfrak{h} with \mathfrak{h}^* , given by (,), we see that the element $H \in \mathfrak{h}$ that corresponds to $\sum_{i=1}^m \epsilon_i$ has the property that $H|_{V_{\mathbb{C}}^{\pm}} = \pm I$. Thus the parabolic subalgebra \mathfrak{p}_2 , defined by H , is

$$\mathfrak{p}_2 = \mathfrak{h} \oplus \sum_{\alpha \in \Delta(A_{m-1})} (\mathfrak{g}_0)_{\alpha} \oplus \mathfrak{n},$$

where $\mathfrak{n} = \sum_{\alpha \in \Delta^+(D_m) \setminus \Delta(A_{m-1})} (\mathfrak{g}_0)_{\alpha}$ is the nilradical.

We use the notation 5.1. Again we remark that

$$chV^2(\lambda) = c_{\lambda+\rho^A} \frac{chF_A(\lambda)}{\prod_{\alpha \in \Delta^+(D_m) \setminus \Delta^+(A_{m-1})} (1 - e^{-\alpha})}.$$

Corollary 6.3 (Theta correspondence). *With notation as in Theorem 4.1, we have that*

$$\Sigma = \{\tilde{F}_C(\varepsilon(\mathbf{a})) \mid \mathbf{a} \in \mathbb{P}_d\}$$

Moreover, if $\mathbf{a} \in \mathbb{P}_d$, then

$$\tau(\tilde{F}_C(\varepsilon(\mathbf{a}))) = L^2(-\rho_1 + \mu(\mathbf{a})).$$

Proof. By Proposition 6.2, we need only to check that $\tilde{F}_C(\varepsilon(\mathbf{a})) \in \Sigma$ for all $\mathbf{a} \in \mathbb{P}_d$. Clearly $1 \in \mathcal{W}_{m-d}$ and $-\rho_1 + \mu(\mathbf{a}) = 1(-\rho_1 + \mu(\mathbf{a}) + \rho^D) - \rho^D$ is dominant for $\Delta^+(A_{m-1})$, hence $-\rho_1 + \mu(\mathbf{a}) + \rho^A$ is regular. By Proposition 6.2, this implies that the isotypic component of $\tilde{F}_C(\varepsilon(\mathbf{a}))$ has a nonzero character, thus $\tilde{F}_C(\varepsilon(\mathbf{a}))$ occurs in Σ as wished.

For the second statement, arguing as in the proof of Corollary 5.2, we need only to show that, for $w \in \mathcal{W}_{m-d}$ with $w(-\rho_1 + \mu(\mathbf{a}) + \rho^D)$ regular for $\Delta^+(A_{m-1})$ and $w \neq 1$, we have that

$$(-\rho_1 + \mu(\mathbf{a}))(H) > (\{w(-\rho_1 + \mu(\mathbf{a}) + \rho^D)\} - \rho^D)(H).$$

Since $v(H) = H$ if $v \in W(A_{m-1})$, it is enough to check that, for $w \in \mathcal{W}_{m-d}$, $w \neq 1$,

$$(-\rho_1 + \mu(\mathbf{a}) + \rho^D)(H) > w(-\rho_1 + \mu(\mathbf{a}) + \rho^D)(H).$$

This is clear because, if $w = s_{\epsilon_{i_1}} \dots s_{\epsilon_{i_k}}$, then $(\lambda, H - w^{-1}(H)) = (\lambda, 2\epsilon_{i_1} + \dots + 2\epsilon_{i_k})$, hence

$$(-\rho_1 + \mu(\mathbf{a}) + \rho^D)(H - w^{-1}(H)) = 2(m - d - i_1) + \dots + 2(m - d - i_k) > 0.$$

The proof is now complete. \square

As already observed in Section 5, we have also computed the character of $\tau(\tilde{F}_C(\varepsilon(\mathbf{a})))$. In fact, letting \mathcal{W}_{m-d}^{reg} be the set of $w \in \mathcal{W}_{m-d}$ such that $w(-\rho_1 + \mu(\mathbf{a}) + \rho^D)$ is regular for $\Delta(A_{m-1})$, then, combining Corollary 6.3 with Proposition 6.2, we find

$$(6.10) \quad ch(L^2(-\rho_1 + \mu(\mathbf{a}))) = \sum_{w \in \mathcal{W}_{m-d}^{reg}} c_w \operatorname{sgn}(w) chV^2(\{w(-\rho_1 + \mu(\mathbf{a}) + \rho^D)\} - \rho^D),$$

where $c_w = c_{w(-\rho_1 + \mu(\mathbf{a}) + \rho^D)}$.

The argument given in Corollary 5.3 works also in the present case thus, with notation as in Section 5, we obtain

Corollary 6.4. *If $\mathbf{a} \in \mathbb{P}_d$ and $\lambda_0 = -\rho_1 + \mu(\mathbf{a}) + \rho^D$, then*

$$ch(L^2(-\rho_1 + \mu(\mathbf{a}))) = \sum_{w \in W_{\lambda_0}^A} (-1)^{\ell_{\lambda_0}(w)} V^2(\{w(\lambda_0)\} - \rho^D).$$

7. THETA CORRESPONDENCE FOR THE PAIR $(O(2m), Sp(2n, \mathbb{R}))$

This is the most difficult case. Complications arise from the representation theory of even orthogonal groups, which we now discuss. Let V be the real symplectic subspace of \mathfrak{g}_1 and $\Phi : V \rightarrow \mathbb{R}^{2m} \otimes (\mathbb{R}^{2n})^*$ be the map such that there is a dual pair (G_1, G_2) in $sp(V, \langle \cdot, \cdot \rangle)$ having the properties described in Proposition 4.8. We first parametrize the representations of \tilde{G}_1 : the same argument given in Section 5 implies that \tilde{G}_1 is the group of pairs $(\Phi^{-1}g\Phi, z)$ with $g \in O(2m)$ and $z^2 = (\det_{\mathbb{R}^{2m}}(g))^n$. If Ω is the set of finite-dimensional irreducible representations of G_1 , then, by (4.2), we know that $\Sigma \subset \{\tilde{\pi} \mid \pi \in \Omega\}$ where $\tilde{\pi}$ is as in (6.8).

To describe Ω we need to recall that $G_1 \simeq O(2m)$ is isomorphic to the semidirect product of $\mathbb{Z}/2\mathbb{Z}$ with $SO(2m)$. The generator x of $\mathbb{Z}/2\mathbb{Z}$ in G_1 can be chosen so that $Ad(x)$ induces on $\mathfrak{h} \cap \mathfrak{s}_1^{\mathbb{C}}$ the reflection s_{ϵ_m} . The finite-dimensional irreducible representations of G_1 are determined by their restriction to the connected component of the identity $G_1^0 \simeq SO(2m)$ of G_1 and by the action of x . First of all, we parametrize the irreducible finite-dimensional representations of G_1^0 as irreducible finite-dimensional representations of $\mathfrak{s}_1^{\mathbb{C}}$: having chosen $\mathfrak{h} \cap \mathfrak{s}_1^{\mathbb{C}}$ as a Cartan subalgebra, we choose $\Delta_0^+ \cap \Delta(\mathfrak{s}_1^{\mathbb{C}})$ as a set of positive roots for $\mathfrak{s}_1^{\mathbb{C}}$, where Δ_0^+ is as in (6.1), and let ρ^D be the corresponding ρ -vector. Given $\lambda \in (\mathfrak{h} \cap \mathfrak{s}_1^{\mathbb{C}})^*$ dominant integral, we let $F_D(\lambda)$ be the irreducible finite-dimensional $\mathfrak{s}_1^{\mathbb{C}}$ -module with highest weight λ .

If $\lambda = \sum_{i=1}^n a_i \epsilon_i$ with $a_i \in \mathbb{Z}$ and $a_1 \geq a_2 \geq \dots \geq a_m > 0$, then there is a unique irreducible representation of G_1 that restricted to G_1^0 contains $F_D(\lambda)$. We denote this representation by $F^+(\lambda)$; it can be checked that the restriction of $F^+(\lambda)$ to G_1^0 is

$$F_D^+(\lambda) = F_D(\lambda) \oplus F_D(s_{\epsilon_m}(\lambda)).$$

If instead $a_m = 0$, then there are two irreducible representations of G_1 whose restriction to G_1^0 contains $F_D(\lambda)$. Both restrict to G_1^0 as $F_D(\lambda)$, and we let $F^+(\lambda)$ be the one such that x acts trivially on the highest weight vector, and $F^-(\lambda)$ the one such that x acts by $\det_{\mathbb{R}^{2m}}(x) = -1$. Note that this parametrization depends on the choice of x .

We also need Kostant's generalization to nonconnected groups of the Weyl character formula, given in [16, Theorem 7.5]. In the particular case at hand, we let T be the torus in G_1 such that the Lie algebra of T is contained in \mathfrak{h} . Set $H^+ = T \cup xT$. The group H^+ is the normalizer of the torus T and of the Borel subgroup of G_1^0 corresponding to our choice of positive roots. Then Kostant's formula in this case reads

$$(7.1) \quad ch_{F^+(\lambda)}(x^s t) = \frac{1 + (-1)^s}{2} ch_{F_D^+(\lambda)}(t)$$

if $\lambda = \sum_{i=1}^m a_i \epsilon_i$ with $a_m > 0$, while, if $a_m = 0$,

$$(7.2) \quad \begin{aligned} ch_{F^\pm(\lambda)}(x^s t) &= \frac{1 + (-1)^s}{2} ch_{F_D}(\lambda)(t) \\ &\pm \frac{1 - (-1)^s}{2} \frac{\sum_{w \in W(C_{m-1})} sgn(w) e^{w(\lambda + \rho^D)}}{\sum_{w \in W(C_{m-1})} sgn(w) e^{w(\rho^D)}}(t), \end{aligned}$$

where $W(C_{m-1})$ is the Weyl group of the root system

$$\begin{aligned} \Delta(C_{m-1}) &= \{\pm(\epsilon_i - \epsilon_j) \mid 1 \leq i < j \leq m-1\} \\ &\cup \{\pm(\epsilon_i + \epsilon_j) \mid 1 \leq i \leq j \leq m-1\}. \end{aligned}$$

Note that, since $\rho^D = \sum_{i=1}^{m-1} (m-i)\epsilon_i$,

$$\sum_{w \in W(C_{m-1})} sgn(w) e^{w(\rho^D)} = e^{\rho^D} \prod_{\alpha \in \Delta^+(C_{m-1})} (1 - e^{-\alpha}),$$

where

$$\begin{aligned} \Delta^+(C_{m-1}) &= \{(\epsilon_i - \epsilon_j) \mid 1 \leq i < j \leq m-1\} \\ &\cup \{(\epsilon_i + \epsilon_j) \mid 1 \leq i \leq j \leq m-1\}. \end{aligned}$$

Set $\mathfrak{h}_2 = \mathfrak{h} \cap s_2^{\mathbb{C}}$. We need to compute the $H^+ \times \mathfrak{h}_2$ -character of $M^{\Delta^+}(\mathfrak{g}_1)$. In this case

$$\Delta_1^+ = \{\delta_k \pm \epsilon_i \mid 1 \leq i \leq m, 1 \leq k \leq n\},$$

and $2\rho_1 = (2m)(\delta_1 + \dots + \delta_n)$.

Let ad be the adjoint action of \mathfrak{g}_0 on \mathfrak{g}_1 . Recall from Section 4 that, as \tilde{G}_1 -module, $M^{\Delta^+}(\mathfrak{g}_1)$ is given, up to the action of the center, by the action of G_1 on a polynomial algebra $P(W)$. The differential of this action on W coincides with $ad_{|s_1^{\mathbb{C}}}^*$ on $(\mathfrak{g}_1^-)^*$. Hence we can identify $P(W)$ with the symmetric algebra $S(\mathfrak{g}_1^-)$ with $s_1^{\mathbb{C}}$ acting by ad . Let us denote by Ad the action of G_1 on $S(\mathfrak{g}_1^-)$ coming from its action on $P(W)$. We normalize the choice of x by assuming that we can choose the root vectors $X_{-\delta_i \pm \epsilon_j}$ in \mathfrak{g}_1^- in such a way that

$$Ad(x)(X_{-\delta_i \pm \epsilon_j}) = X_{-\delta_i \pm \epsilon_j}$$

if $j < m$, and

$$Ad(x)(X_{-\delta_i + \epsilon_m}) = X_{-\delta_i - \epsilon_m}.$$

Clearly, if $t \in T \times \mathfrak{h}_2$, then

$$ch_{H^+ \times \mathfrak{h}_2} M^{\Delta^+}(\mathfrak{g}_1)(t) = \frac{e^{-\rho_1}}{\prod_{\alpha \in \Delta_1^+} (1 - e^{-\alpha})}(t).$$

In order to compute $ch_{H^+ \times \mathfrak{h}_2} M^{\Delta^+}(\mathfrak{g}_1)(xt)$ we simply compute the trace of the matrix of the action of xt in the basis given by monomials

$$\prod_{i,j} X_{-\delta_i + \epsilon_j}^{a_{ij}} X_{-\delta_i - \epsilon_j}^{b_{ij}}.$$

If λ is the \mathfrak{h} -weight of this monomial, then

$$Ad(xt) \left(\prod_{i,j} X_{-\delta_i + \epsilon_j}^{a_{ij}} X_{-\delta_i - \epsilon_j}^{b_{ij}} \right) = e^\lambda(t) \left(\prod_{\substack{1 \leq i \leq n \\ 1 \leq j < m}} X_{-\delta_i + \epsilon_j}^{a_{ij}} X_{-\delta_i - \epsilon_j}^{b_{ij}} \prod_{1 \leq i \leq n} X_{-\delta_i + \epsilon_m}^{b_{im}} X_{-\delta_i - \epsilon_m}^{a_{im}} \right).$$

Thus the only contribution to the trace is given by the monomials

$$\prod_{\substack{1 \leq i \leq n \\ 1 \leq j < m}} X_{-\delta_i + \epsilon_j}^{a_{ij}} X_{-\delta_i - \epsilon_j}^{b_{ij}} \prod_{1 \leq i \leq n} (X_{-\delta_i + \epsilon_m} X_{-\delta_i - \epsilon_m})^{a_{im}}.$$

It follows that

$$ch_{H^+ \times \mathfrak{h}_2} M^{\Delta^+}(\mathfrak{g}_1^-)(xt) = \frac{e^{-\rho_1}}{\prod_{\substack{1 \leq i \leq n \\ 1 \leq j < m}} (1 - e^{-\delta_i \pm \epsilon_j}) \prod_{1 \leq i \leq n} (1 - e^{-2\delta_i})}(t).$$

Putting everything together we find that

$$(7.3) \quad ch_{H^+ \times \mathfrak{h}_2} M^{\Delta^+}(\mathfrak{g}_1^-)(x^s t) = \frac{1 + (-1)^s}{2} \frac{e^{-\rho_1}}{\prod_{\alpha \in \Delta_1^+} (1 - e^{-\alpha})}(t) \\ + \frac{1 - (-1)^s}{2} \frac{e^{-\rho_1}}{\prod_{\substack{1 \leq i \leq n \\ 1 \leq j < m}} (1 - e^{-\delta_i \pm \epsilon_j}) \prod_{1 \leq i \leq n} (1 - e^{-2\delta_i})}(t).$$

We now apply our denominator formulas to the two summands above. We start with the first one: again the defect of \mathfrak{g} is $d = \min(n, m)$. We set

$$\Delta(D_r) = \pm\{\epsilon_i \pm \epsilon_j, \mid 1 \leq i \neq j \leq r\}, \\ \Delta(C_r) = \pm\{\delta_k \pm \delta_l, 2\delta_k \mid n - r + 1 \leq k \neq l \leq n\}, \\ \Delta(A_{r-1}) = \pm\{\delta_k - \delta_l \mid n - r + 1 \leq k \neq l \leq n\}.$$

Let $W(A_{r-1})$, $W(D_r)$, $W(C_r)$ be the corresponding Weyl groups. In this case $2\rho_1 = 2m \sum_{i=1}^n \delta_i$. Set $\Delta^+(C_n) = \Delta_0^+ \cap \Delta(C_n)$, $\Delta^+(B_m) = \Delta_0^+ \cap \Delta(B_m)$, and $\Delta^+(A_{n-1}) = \Delta_0^+ \cap \Delta(A_{n-1})$. We denote by ρ^C , ρ^B , ρ^A the corresponding ρ -vectors.

With notation as in Section 3.1, the total order corresponding to this choice of Δ^+ is

$$\delta_1 > \cdots > \delta_n > \epsilon_1 > \cdots > \epsilon_m.$$

There is only one arc diagram associated to this total order, namely

$$X = \{\widehat{\delta_n \epsilon_1}, \dots, \widehat{\delta_{n-d+1} \epsilon_d}\}.$$

The corresponding maximal isotropic set of roots is $S = \{\gamma_1, \dots, \gamma_d\}$, where

$$(7.4) \quad \gamma_1 = \delta_n - \epsilon_1, \gamma_2 = \delta_{n-1} - \epsilon_2, \dots, \gamma_d = \delta_{n-d+1} - \epsilon_d.$$

Applying formula (1.15) we can write

$$(7.5) \quad e^\rho \check{R} = \check{F}_{W^\#} \left(\frac{e^{\rho + \sum_{i=1}^d \gamma_i}}{\prod_{i=1}^d (1 - e^{-\gamma_i})} \right).$$

Recall from (5.8) that \mathcal{W}_r is the subgroup of $W_{\mathfrak{g}}$ generated by $\{s_{2\delta_i} \mid i = 1, \dots, r\}$. Define

$$W_{\mathfrak{g}}^{ext} = \text{reflection group generated by } W_{\mathfrak{g}} \text{ and the reflections } s_{\epsilon_i}, \\ W(B_m) = \text{subgroup of } W_{\mathfrak{g}}^{ext} \text{ generated by } W(D_m) \text{ and } s_{\epsilon_i}.$$

Extend sgn' to $W_{\mathfrak{g}}^{ext}$ by setting $sgn'(s_{\epsilon_i}) = 1$. We define

$$W = W(A_{n-1})W_{n-d}W(B_m), \quad W_D = W(A_{n-1})W_{n-d}W(D_m)$$

If $n < m$ then $W^\sharp = W(D_m)$ hence, summing over $W(A_{n-1})$, we find

$$(7.6) \quad d!e^\rho \check{R} = \check{\mathcal{F}}_{W_D} \left(\frac{e^{\rho + \sum_{i=1}^d \llbracket \gamma_i \rrbracket}}{\prod_{i=1}^d (1 - e^{-\gamma_i})} \right).$$

Since s_{ϵ_m} fixes $\frac{e^{\rho + \sum_{i=1}^d \llbracket \gamma_i \rrbracket}}{\prod_{i=1}^d (1 - e^{-\gamma_i})}$ we can rewrite (7.6) as

$$d!e^\rho \check{R} = \check{\mathcal{F}}_W \left(\frac{e^{\rho + \sum_{i=1}^d \llbracket \gamma_i \rrbracket}}{\prod_{i=1}^d (1 - e^{-\gamma_i})} \right) - \check{\mathcal{F}}_{W_D} \left(\frac{e^{\rho + \sum_{i=1}^d \llbracket \gamma_i \rrbracket}}{\prod_{i=1}^d (1 - e^{-\gamma_i})} \right).$$

The same argument given to prove (5.12) gives

$$(7.7) \quad e^\rho \check{R} = \check{\mathcal{F}}_W \left(\frac{e^\rho}{\prod_{i=1}^d (1 - e^{-\llbracket \gamma_i \rrbracket})} \right) - \check{\mathcal{F}}_{W_D} \left(\frac{e^\rho}{\prod_{i=1}^d (1 - e^{-\llbracket \gamma_i \rrbracket})} \right).$$

If $n \geq m$ then $W^\sharp = W(C_n)$. By an explicit computation we see that $\rho + \sum_{i=1}^d \llbracket \gamma_i \rrbracket = \sum_{i=1}^{n-m} (n - m - i + 1) \delta_i$.
Set

$$(7.8) \quad \mathcal{Y}_d = \text{subgroup of } W_{\mathfrak{g}}^{\text{ext}} \text{ generated by } \{s_{2\delta_{n-i+1}} s_{\epsilon_i} \mid 1 \leq i \leq d\}.$$

We have that $s_{2\delta_{n-i+1}} s_{\epsilon_i} (\rho + \sum_{i=1}^d \llbracket \gamma_i \rrbracket) = \rho + \sum_{i=1}^d \llbracket \gamma_i \rrbracket$ hence

$$\text{sgn}'(s_{2\delta_{n-i+1}} s_{\epsilon_i}) s_{2\delta_{n-i+1}} s_{\epsilon_i} \frac{e^{\rho + \sum_{i=1}^d \llbracket \gamma_i \rrbracket}}{\prod_{j=1}^d (1 - e^{-\gamma_j})} = \frac{e^{\rho + \sum_{i=1}^d \llbracket \gamma_i \rrbracket}}{\prod_{j=1}^d (1 - e^{-\gamma_j})} - \frac{e^{\rho + \sum_{i=1}^d \llbracket \gamma_i \rrbracket}}{\prod_{j \neq i}^d (1 - e^{-\gamma_j})}.$$

It follows that, if $w = \prod_{r=1}^k s_{2\delta_{n-d+i_r}} s_{\epsilon_{i_r}} \in \mathcal{Y}_d$, then

$$\text{sgn}'(w) w \frac{e^{\rho + \sum_{i=1}^d \llbracket \gamma_i \rrbracket}}{\prod_{j=1}^d (1 - e^{-\gamma_j})} = \sum_{J \subset \{i_1, \dots, i_k\}} (-1)^{|J|} \frac{e^{\rho + \sum_{i=1}^d \llbracket \gamma_i \rrbracket}}{\prod_{j \notin J} (1 - e^{-\gamma_j})}.$$

Note that if $|J| \geq 2$ then there is a reflection in $W(A_{m-1})$ that fixes the element $\frac{e^{\rho + \sum_{i=1}^d \llbracket \gamma_i \rrbracket}}{\prod_{j \notin J} (1 - e^{-\gamma_j})}$, hence

$$\check{\mathcal{F}}_W \left(\text{sgn}'(w) w \frac{e^{\rho + \sum_{i=1}^d \llbracket \gamma_i \rrbracket}}{\prod_{j=1}^d (1 - e^{-\gamma_j})} \right) = \check{\mathcal{F}}_W \left(\frac{e^{\rho + \sum_{i=1}^d \llbracket \gamma_i \rrbracket}}{\prod_{j=1}^d (1 - e^{-\gamma_j})} - \sum_{h=0}^k \frac{e^{\rho + \sum_{i=1}^d \llbracket \gamma_i \rrbracket}}{\prod_{j \neq j_h} (1 - e^{-\gamma_j})} \right).$$

Note that

$$s_{\delta_{n-j_h+1} - \delta_{n-d+1}} s_{\epsilon_{j_h} - \epsilon_d} \left(\frac{e^{\rho + \sum_{i=1}^d \llbracket \gamma_i \rrbracket}}{\prod_{j \neq j_h} (1 - e^{-\gamma_j})} \right) = \frac{e^{\rho + \sum_{i=1}^d \llbracket \gamma_i \rrbracket}}{\prod_{j \neq m} (1 - e^{-\gamma_j})},$$

hence, if $w \in \mathcal{Y}_d$, we have

$$\begin{aligned} & \check{\mathcal{F}}_W \left(\text{sgn}'(w) w \frac{e^{\rho + \sum_{i=1}^d \llbracket \gamma_i \rrbracket}}{\prod_{j=1}^d (1 - e^{-\gamma_j})} \right) \\ &= \check{\mathcal{F}}_W \left(\frac{e^{\rho + \sum_{i=1}^d \llbracket \gamma_i \rrbracket}}{\prod_{j=1}^d (1 - e^{-\gamma_j})} - k \frac{e^{\rho + \sum_{i=1}^d \llbracket \gamma_i \rrbracket}}{\prod_{j=1}^{d-1} (1 - e^{-\gamma_j})} \right). \end{aligned}$$

Using the fact that $W_{\mathfrak{g}}^{ext} = W\mathcal{Y}_d$ and that $s_{\epsilon_i}e^\rho\check{R} = e^\rho\check{R}$, it follows that

$$\begin{aligned} 2^d d! e^\rho \check{R} &= \\ \check{F}_{W_{\mathfrak{g}}^{ext}}\left(\frac{e^{\rho+\sum_{i=1}^d \llbracket \gamma_i \rrbracket}}{\prod_{j=1}^d (1 - e^{-\gamma_j})}\right) &= \check{F}_{W\mathcal{Y}_d}\left(\frac{e^{\rho+\sum_{i=1}^d \llbracket \gamma_i \rrbracket}}{\prod_{j=1}^d (1 - e^{-\gamma_j})}\right) = \\ 2^d \check{F}_W\left(\frac{e^{\rho+\sum_{i=1}^d \llbracket \gamma_i \rrbracket}}{\prod_{j=1}^d (1 - e^{-\gamma_j})}\right) &- 2^{d-1} d \check{F}_W\left(\frac{e^{\rho+\sum_{i=1}^d \llbracket \gamma_i \rrbracket}}{\prod_{j=1}^{d-1} (1 - e^{-\gamma_j})}\right). \end{aligned}$$

Hence

$$(7.9) \quad d! e^\rho \check{R} = \check{F}_W\left(\frac{e^{\rho+\sum_{i=1}^d \llbracket \gamma_i \rrbracket}}{\prod_{j=1}^d (1 - e^{-\gamma_j})}\right) - \frac{1}{2} d \check{F}_W\left(\frac{e^{\rho+\sum_{i=1}^d \llbracket \gamma_i \rrbracket}}{\prod_{j=1}^{d-1} (1 - e^{-\gamma_j})}\right).$$

Set $W' = W_{[\delta_{n-d+1}, \epsilon_d]}$. Using the fact that \mathcal{W}_{n-d} and W' commute, we can choose

$$W/W' = ((W(A_{n-1}) \times W(B_m))/W') \mathcal{W}_{n-d}.$$

Let Y be the arc subdiagram corresponding to the interval $[\delta_{n-d+1}, \epsilon_d]$. Applying (1.15), we have

$$\begin{aligned} \check{F}_W\left(\frac{e^{\rho+\sum_{i=1}^d \llbracket \gamma_i \rrbracket}}{\prod_{j=1}^d (1 - e^{-\gamma_j})}\right) &= \check{F}_{W/W'}(e^{\rho_X - \rho_Y} \mathcal{F}_{W'}\left(\frac{e^{\rho_Y + \sum_{i=1}^d \llbracket \gamma_i \rrbracket}}{\prod_{i=1}^d (1 - e^{-\gamma_i})}\right)) \\ &= \check{F}_{W/W'}(e^{\rho_X - \rho_Y} d! e^{\rho_Y} \check{R}_Y) \end{aligned}$$

Observe that $ht(\gamma_i) = 2i - 1$ so $\prod_{i=1}^d \frac{ht(\gamma_i)+1}{2} = d!$. Applying (3.9) to Y we find

$$\check{F}_W\left(\frac{e^{\rho+\sum_{i=1}^d \llbracket \gamma_i \rrbracket}}{\prod_{j=1}^d (1 - e^{-\gamma_j})}\right) = \check{F}_{W/W'}(e^{\rho_X - \rho_Y} d! \mathcal{F}_{W'}\left(\frac{e^{\rho_Y}}{\prod_{i=1}^d (1 - e^{-\llbracket \gamma_i \rrbracket})}\right)).$$

hence

$$(7.10) \quad \check{F}_W\left(\frac{e^{\rho+\sum_{i=1}^d \llbracket \gamma_i \rrbracket}}{\prod_{j=1}^d (1 - e^{-\gamma_j})}\right) = d! \check{F}_W\left(\frac{e^\rho}{\prod_{j=1}^d (1 - e^{-\llbracket \gamma_j \rrbracket})}\right).$$

Analogously one checks that

$$(7.11) \quad \check{F}_W\left(\frac{e^{\rho+\sum_{i=1}^d \llbracket \gamma_i \rrbracket}}{\prod_{j=1}^{d-1} (1 - e^{-\gamma_j})}\right) = (d-1)! \check{F}_W\left(\frac{e^{\rho+\llbracket \gamma_d \rrbracket}}{\prod_{j=1}^{d-1} (1 - e^{-\llbracket \gamma_j \rrbracket})}\right).$$

Observe in (7.11) that $\llbracket \gamma_d \rrbracket = \llbracket \gamma_{d-1} \rrbracket$, hence

$$(7.12) \quad \check{F}_W\left(\frac{e^{\rho+\llbracket \gamma_d \rrbracket}}{\prod_{j=1}^{d-1} (1 - e^{-\llbracket \gamma_j \rrbracket})}\right) = \check{F}_W\left(\frac{e^\rho}{\prod_{j=1}^{d-1} (1 - e^{-\llbracket \gamma_j \rrbracket})}\right) + \check{F}_W\left(\frac{e^{\rho+\llbracket \gamma_{d-1} \rrbracket}}{\prod_{j=1}^{d-2} (1 - e^{-\llbracket \gamma_j \rrbracket})}\right)$$

and the second summand in (7.12) is zero since the reflection $s_{\epsilon_{d-1}-\epsilon_d}$ fixes it. Plugging these formulas in (7.9) we find

$$(7.13) \quad e^\rho \check{R} = \check{F}_W\left(\frac{e^\rho}{\prod_{j=1}^d (1 - e^{-\llbracket \gamma_j \rrbracket})}\right) - \frac{1}{2} \check{F}_W\left(\frac{e^\rho}{\prod_{j=1}^{d-1} (1 - e^{-\llbracket \gamma_j \rrbracket})}\right).$$

Finally, observe that $W = W_D \cup W_D s_{\epsilon_m}$ and s_{ϵ_m} fixes $\frac{e^\rho}{\prod_{j=1}^{d-1} (1 - e^{-\llbracket \gamma_j \rrbracket})}$, so

$$(7.14) \quad e^\rho \check{R} = \check{F}_W\left(\frac{e^\rho}{\prod_{j=1}^d (1 - e^{-\llbracket \gamma_j \rrbracket})}\right) - \check{F}_{W_D}\left(\frac{e^\rho}{\prod_{j=1}^{d-1} (1 - e^{-\llbracket \gamma_j \rrbracket})}\right).$$

To give a uniform treatment of the cases $n < m$ and $n \geq m$, we set $d_1 = \min(n, m - 1)$ so that (7.14) and (7.7) combine to give

$$(7.15) \quad e^\rho \check{R} = \check{F}_W \left(\frac{e^\rho}{\prod_{j=1}^d (1 - e^{-\llbracket \gamma_j \rrbracket})} \right) - \check{F}_{W_D} \left(\frac{e^\rho}{\prod_{j=1}^{d_1} (1 - e^{-\llbracket \gamma_j \rrbracket})} \right).$$

Set $\mathbb{P}_d^+ = \{\mathbf{a} \in \mathbb{P}_d \mid a_m > 0\}$ and $\mathbb{P}_d^0 = \mathbb{P}_d \setminus \mathbb{P}_d^+$. Clearly, if $d < m$ then $\mathbb{P}_d^+ = \emptyset$ and $\mathbb{P}_d^0 = \mathbb{P}_d$. Dividing (7.15) by $\mathcal{D}_0 = e^{\rho^0} \prod_{\alpha \in \Delta_0^+} (1 - e^{-\alpha})$, we find that, arguing as for type $B(m, n)$,

$$\frac{e^{-\rho_1}}{\prod_{\alpha \in \Delta_1^+} (1 - e^{-\alpha})} = \frac{1}{\mathcal{D}_0} \left(\sum_{\mathbf{a} \in \mathbb{P}_d^+} \check{F}_W(e^{\rho - \sum_{i=1}^d a_i \gamma_i}) + \sum_{\mathbf{a} \in \mathbb{P}_d^0} \check{F}_{W_D}(e^{\rho - \sum_{i=1}^d a_i \gamma_i}) \right).$$

Since $\mathcal{D}_0 = e^{\rho^C} \prod_{\alpha \in \Delta^+(C_n)} (1 - e^{-\alpha}) e^{\rho^D} \prod_{\alpha \in \Delta^+(D_m)} (1 - e^{-\alpha})$, we can rewrite the above formula as

$$(7.16) \quad \begin{aligned} & \frac{e^{-\rho_1}}{\prod_{\alpha \in \Delta_1^+} (1 - e^{-\alpha})} \\ &= \sum_{\mathbf{a} \in \mathbb{P}_d^+} \sum_{w \in \mathcal{W}_{n-d}} \operatorname{sgn}'(w) \frac{\operatorname{ch} F_A(w(-\rho_1 + \mu(\mathbf{a}) + \rho^C) - \rho^C)}{\prod_{\alpha \in \Delta^+(C_n) \setminus \Delta^+(A_{n-1})} (1 - e^{-\alpha})} \operatorname{ch} F_D^+(\varepsilon(\mathbf{a})) \\ &+ \sum_{\mathbf{a} \in \mathbb{P}_d^0} \sum_{w \in \mathcal{W}_{n-d}} \operatorname{sgn}'(w) \frac{\operatorname{ch} F_A(w(-\rho_1 + \mu(\mathbf{a}) + \rho^C) - \rho^C)}{\prod_{\alpha \in \Delta^+(C_n) \setminus \Delta^+(A_{n-1})} (1 - e^{-\alpha})} \operatorname{ch} F_D(\varepsilon(\mathbf{a})). \end{aligned}$$

Here we recall that, if $\mathbf{a} \in \mathbb{P}_d$, then $\mu(\mathbf{a}) = -\sum_{r=1}^d (a_{d+1-r}) \delta_{n-d+r}$, $\varepsilon(\mathbf{a}) = \sum_{r=1}^d a_r \varepsilon_r$, and $\operatorname{ch} F_A(\lambda)$ is given by (5.16).

We now take care of the second summand in (7.3). Note that

$$\begin{aligned} & \frac{e^{-\rho_1} e^{\rho^C} \prod_{\alpha \in \Delta^+(C_n)} (1 - e^{-\alpha}) e^{\rho^D} \prod_{\alpha \in \Delta^+(C_{m-1})} (1 - e^{-\alpha})}{\prod_{\substack{1 \leq i \leq n \\ 1 \leq j < m}} (1 - e^{-\delta_i \pm \varepsilon_j}) \prod_{1 \leq i \leq n} (1 - e^{-2\delta_i})} \\ &= \frac{e^{-\rho_1} e^{\rho^C} \prod_{\alpha \in \Delta^+(D_n)} (1 - e^{-\alpha}) e^{\rho^D} \prod_{\alpha \in \Delta^+(C_{m-1})} (1 - e^{-\alpha})}{\prod_{\substack{1 \leq i \leq n \\ 1 \leq j < m}} (1 - e^{-\delta_i \pm \varepsilon_j})}, \end{aligned}$$

and that

$$\rho^C + \rho^D - \rho_1 = \sum_{i=1}^n (n - (m - 1) - i) \delta_i + \sum_{i=1}^{m-1} (m - i) \varepsilon_i,$$

so the formula above is precisely the denominator for the distinguished Borel subalgebra of type $\Delta_{D_2}^+$ for a superalgebra of type $D(n, m - 1)$.

We can therefore apply the results of Section 6 and find that

$$\begin{aligned} & \frac{e^{-\rho_1} e^{\rho^C} \prod_{\alpha \in \Delta^+(C_n)} (1 - e^{-\alpha}) e^{\rho^D} \prod_{\alpha \in \Delta^+(C_{m-1})} (1 - e^{-\alpha})}{\prod_{\substack{1 \leq i \leq n \\ 1 \leq j < m}} (1 - e^{-\delta_i \pm \varepsilon_j}) \prod_{1 \leq i \leq n} (1 - e^{-2\delta_i})} \\ &= \check{F}_{W(m-1)} \left(\frac{e^\rho}{\prod_{i=1}^{d_1} (1 - e^{-\llbracket \gamma_i \rrbracket})} \right), \end{aligned}$$

where $W(m-1) = W(A_{n-1})\mathcal{W}_{n-d_1}W(C_{m-1})$. Recall from (5.9), (5.10), that \mathcal{W}_r^+ (resp. \mathcal{W}_r^-) is the set of $w \in \mathcal{W}_r$ such that $w = s_{2\delta_{i_1}} \dots s_{2\delta_{i_k}}$ with k even (resp. odd).

By dividing off $e^{\rho^C} \prod_{\alpha \in \Delta^+(C_n)} (1 - e^{-\alpha}) e^{\rho^D} \prod_{\alpha \in \Delta^+(C_{m-1})} (1 - e^{-\alpha})$ and expanding the denominator, we find

$$(7.17) \quad \begin{aligned} & ch_{H^+ \times \mathfrak{h}_2}(M^{\Delta^+}(\mathfrak{g}_1))(xt) = \\ & \sum_{\mathbf{a} \in \mathbb{P}_d^0} \left(\sum_{w \in \mathcal{W}_{n-d_1}^+} sgn(w) \frac{chF_A(w(-\rho_1 + \mu(\mathbf{a}) + \rho^C) - \rho^C)}{\prod_{\alpha \in \Delta^+(C_n) \setminus \Delta^+(A_{n-1})} (1 - e^{-\alpha})} \times \right. \\ & \left. \times \frac{\sum_{w \in W(C_{m-1})} sgn(w) e^{w(\varepsilon(\mathbf{a}) + \rho^D)}}{\sum_{w \in W(C_{m-1})} sgn(w) e^{w(\rho^D)}}(t) \right). \end{aligned}$$

Recall that, if π is a finite-dimensional representation of G_1 , then $\tilde{\pi}$ is a representation of \tilde{G}_1 whose definition is given in (6.8). We can now state

Proposition 7.1. *With notation as in Theorem 4.1, we have that, if $\tilde{\pi} \in \Sigma$, then there is $\mathbf{a} \in \mathbb{P}_d$ such that*

$$\pi = F^\pm(\varepsilon(\mathbf{a})).$$

Furthermore, if $\mathbf{a} \in \mathbb{P}_d^+$, then the $H^+ \times \mathfrak{h}_2$ -character of the isotypic component of $\tilde{F}^+(\varepsilon(\mathbf{a}))$ in $M^{\Delta^+}(\mathfrak{g}_1)$ is

$$(7.18) \quad \sum_{w \in \mathcal{W}_{n-d}} sgn(w) \frac{chF_A(w(\rho^C - \rho_1 + \mu(\mathbf{a})) - \rho^C)}{\prod_{\alpha \in \Delta^+(C_n) \setminus \Delta^+(A_{n-1})} (1 - e^{-\alpha})} ch_{H^+} F^+(\varepsilon(\mathbf{a})),$$

while, for $\mathbf{a} \in \mathbb{P}_d^0$, the $H^+ \times \mathfrak{h}_2$ -character of the isotypic component of $\tilde{F}^\pm(\varepsilon(\mathbf{a}))$ is

$$(7.19) \quad \sum_{w \in \mathcal{W}_{n-d}^\pm} sgn(w) \frac{chF_A(w(\rho^C - \rho_1 + \mu(\mathbf{a})) - \rho^C)}{\prod_{\alpha \in \Delta^+(C_n) \setminus \Delta^+(A_{n-1})} (1 - e^{-\alpha})} ch_{H^+} F^\pm(\varepsilon(\mathbf{a})),$$

Proof. It follows from (7.16) that $\tilde{F}^\pm(\sum_{i=1}^m a_i \varepsilon_i)$ can occur in $M^{\Delta^+}(\mathfrak{g}_1)$ only if $a_i = 0$ for $i > d$. This proves the first statement.

It is also clear from (7.16) and Kostant's formula (7.1) that, if $\mathbf{a} \in \mathbb{P}_d^+$, then the isotypic component of $\tilde{F}^+(\varepsilon(\mathbf{a}))$ is given by (7.18).

If $\mathbf{a} \in \mathbb{P}_d^0$, let $M_{\mathbf{a}}$ be the sum of the isotypic components of $\tilde{F}^+(\varepsilon(\mathbf{a}))$ and $\tilde{F}^-(\varepsilon(\mathbf{a}))$. By substituting (7.16) and (7.17) in (7.3), we find that

$$\begin{aligned} & ch_{H^+ \times \mathfrak{h}_2}(M_{\mathbf{a}})(x^s t) \\ &= \frac{1 + (-1)^s}{2} \sum_{w \in \mathcal{W}_{n-d}} sgn(w) \frac{chF_A(w(-\rho_1 + \mu(\mathbf{a}) + \rho^C) - \rho^C)}{\prod_{\alpha \in \Delta^+(C_n) \setminus \Delta^+(A_{n-1})} (1 - e^{-\alpha})} \times ch_T(F_D(\varepsilon(\mathbf{a}))(t) \\ &+ \frac{1 - (-1)^s}{2} \left(\sum_{w \in \mathcal{W}_{n-d_1}^+} sgn(w) \frac{chF_A(w(-\rho_1 + \mu(\mathbf{a}) + \rho^C) - \rho^C)}{\prod_{\alpha \in \Delta^+(C_n) \setminus \Delta^+(A_{n-1})} (1 - e^{-\alpha})} \right. \\ & \left. \times \frac{\sum_{w \in W(C_{m-1})} sgn(w) e^{w(\varepsilon(\mathbf{a}) + \rho^D)}}{\sum_{w \in W(C_{m-1})} sgn(w) e^{w(\rho^D)}}(t) \right). \end{aligned}$$

Observe now that, since $\mathbf{a} \in \mathbb{P}_d^0$, then $s_{-2\delta_{n-d_1}}$ fixes $-\rho_1 + \mu(\mathbf{a}) + \rho^C$ so, if $w = w' s_{-2\delta_{n-d_1}}$ with $w' \in \mathcal{W}_{n-d}^-$ then $sgn(w) = -sgn(w')$ and $w(-\rho_1 + \mu(\mathbf{a}) + \rho^C) =$

$w'(-\rho_1 + \mu(\mathbf{a}) + \rho^C)$. Combining this observation with Kostant's character formula (7.2), we can rewrite the formula above as

$$\begin{aligned} & ch_{H^+ \times \mathfrak{h}_2}(M_{\mathbf{a}})(x^s t) \\ &= \sum_{w \in \mathcal{W}_{n-d}^+} sgn(w) \frac{ch F_A(w(-\rho_1 + \mu(\mathbf{a}) + \rho^C) - \rho^C)}{\prod_{\alpha \in \Delta^+(C_n) \setminus \Delta^+(A_{n-1})} (1 - e^{-\alpha})} ch_{H^+}(F^+(\varepsilon(\mathbf{a}))(x^s t)) \\ &+ \sum_{w \in \mathcal{W}_{n-d}^-} sgn(w) \frac{ch F_A(w(-\rho_1 + \mu(\mathbf{a}) + \rho^C) - \rho^C)}{\prod_{\alpha \in \Delta^+(C_n) \setminus \Delta^+(A_{n-1})} (1 - e^{-\alpha})} ch_{H^+}(F^-(\varepsilon(\mathbf{a}))(x^s t)), \end{aligned}$$

from which (7.19) follows readily. \square

In our case the set of roots of $\mathfrak{s}_2^{\mathbb{C}}$ is $\Delta(C_n)$. The element $H \in \mathfrak{h}$ that corresponds to $-\sum_{i=1}^n \delta_i$ has the property that $H|_{V_{\mathbb{C}}^{\pm}} = \pm I$. Thus the parabolic subalgebra \mathfrak{p}_2 defined by H is

$$\mathfrak{p}_2 = \mathfrak{h} \oplus \sum_{\alpha \in \Delta(A_{n-1})} (\mathfrak{g}_0)_{\alpha} \oplus \mathfrak{n},$$

where $\mathfrak{n} = \sum_{\alpha \in \Delta^+(C_n) \setminus \Delta^+(A_{n-1})} (\mathfrak{g}_0)_{\alpha}$ is the nilradical.

Recall that, if \mathbb{P} is as in (5.22) and $\mathbf{a} \in \mathbb{P}$, then we can define the weight $\nu(\mathbf{a})$ as in (5.23). Using Notation 5.1, we can now state

Corollary 7.2 (Theta correspondence). *With notation as in Theorem 4.1, we have*

$$\Sigma = \{\tilde{F}^+(\varepsilon(\mathbf{a})) \mid \mathbf{a} \in \mathbb{P}_d\} \cup \{\tilde{F}^-(\varepsilon(\mathbf{a})) \mid \mathbf{a} \in \mathbb{P}\}.$$

Moreover

$$\tau(\tilde{F}^+(\varepsilon(\mathbf{a}))) = L^2(-\rho_1 + \mu(\mathbf{a})),$$

and, if $\mathbf{a} \in \mathbb{P}$, then

$$\tau(\tilde{F}^-(\varepsilon(\mathbf{a}))) = L^2(-\rho_1 + \nu(\mathbf{a})).$$

Proof. Using Proposition 7.1, the proof follows exactly as in the proof of Corollary 5.2. \square

As already observed in previous sections, we have also computed the character of $\tau(\tilde{F}^{\pm}(\varepsilon(\mathbf{a})))$. We set, as before, \mathcal{W}_{n-d}^{reg} to be the set of $w \in \mathcal{W}_{n-d}$ such that $w(-\rho_1 + \mu(\mathbf{a}) + \rho^C)$ is regular for $\Delta(A_{n-1})$ and, for $w \in \mathcal{W}_{n-d}^{reg}$, $c_w = c_{w(-\rho_1 + \mu(\mathbf{a}) + \rho^C)}$. Then, combining Corollary 7.2 with Proposition 7.1, we find that, if $\mathbf{a} \in \mathbb{P}_d^+$, then

$$(7.20) \quad ch(L^2(-\rho_1 + \mu(\mathbf{a}))) = \sum_{w \in \mathcal{W}_{n-d}^{reg}} c_w sgn(w) ch V^2(\{w(-\rho_1 + \mu(\mathbf{a}) + \rho^C)\} - \rho^C),$$

while, if $\mathbf{a} \in \mathbb{P}_0^+$, then

$$(7.21) \quad ch(L^2(-\rho_1 + \mu(\mathbf{a}))) = \sum_{w \in \mathcal{W}_{n-d}^+ \cap \mathcal{W}_{n-d}^{reg}} c_w ch V^2(\{w(-\rho_1 + \mu(\mathbf{a}) + \rho^C)\} - \rho^C),$$

and, if $\mathbf{a} \in \mathbb{P}$, then

$$(7.22) \quad ch(L^2(-\rho_1 + \nu(\mathbf{a}))) = - \sum_{w \in \mathcal{W}_{n-d}^- \cap \mathcal{W}_{n-d}^{reg}} c_w ch V^2(\{w(-\rho_1 + \mu(\mathbf{a}) + \rho^C)\} - \rho^C).$$

The argument given in Corollary 5.3 works also in the present case, thus, we obtain

Corollary 7.3. *If $\mathbf{a} \in \mathbb{P}_d$ and $\lambda_0 = -\rho_1 + \mu(\mathbf{a}) + \rho^C$, then*

$$ch(L^2(-\rho_1 + \mu(\mathbf{a}))) = \sum_{w \in W_{\lambda_0}^A} (-1)^{\ell_{\lambda_0}(w)} V^2(\{w(\lambda_0)\} - \rho^C).$$

If $\mathbf{a} \in \mathbb{P}$ and $\lambda_1 = -\rho_1 + \nu(\mathbf{a}) + \rho^C$, then

$$ch(L^2(-\rho_1 + \nu(\mathbf{a}))) = \sum_{w \in W_{\lambda_1}^A} (-1)^{\ell_{\lambda_1}(w)} V^2(\{w(\lambda_1)\} - \rho^C).$$

8. THETA CORRESPONDENCE FOR THE PAIR $(U(n), U(p, q))$

In this section we use the combinatorial machinery developed in the previous sections to derive the Theta correspondence for the compact dual pair $(U(n), U(p, q))$. This dual pair, according to Proposition 4.8, corresponds to the distinguished sets of positive roots $\Delta_{gl}^{(p,q)}$ in a superalgebra of type $gl(m, n)$, $p + q = m$.

Let $\mathfrak{g} = gl(m, n)$. Its Weyl group is

$$W_{\mathfrak{g}} = \text{group generated by } s_{\epsilon_1 - \epsilon_2}, \dots, s_{\epsilon_{m-1} - \epsilon_m}, s_{\delta_1 - \delta_2}, \dots, s_{\delta_{n-1} - \delta_n},$$

which we identify with $S_m \times S_n$. Recall that the defect of \mathfrak{g} is $d = \min(n, m)$. Set

$$\begin{aligned} \Delta(A_{m-1}) &= \{\epsilon_i - \epsilon_j \mid 1 \leq i \neq j \leq m\}, \\ \Delta(A_{n-1}) &= \{\delta_k - \delta_l \mid 1 \leq k \neq l \leq n\}, \\ \Delta(A_{p-1} \times A_{q-1}) &= \{\epsilon_k - \epsilon_l \mid 1 \leq k \neq l \leq p \text{ or } p+1 \leq k \neq l \leq m\}, \\ \Delta_c &= \Delta(A_{p-1} \times A_{q-1}) \cup \Delta(A_{n-1}). \end{aligned}$$

Let $W(A_{n-1})$ be the Weyl group of $\Delta(A_{n-1})$ and $W(A_{m-1})$ the Weyl group of $\Delta(A_{m-1})$. Let W_c be the Weyl group of Δ_c (the subscript “ c ” stands for compact). Set $\Delta^+(A_{n-1}) = \Delta_0^+ \cap \Delta(A_{n-1})$, $\Delta^+(A_{m-1}) = \Delta_0^+ \cap \Delta(A_{m-1})$, and $\Delta^+(A_{p-1} \times A_{q-1}) = \Delta_0^+ \cap \Delta(A_{p-1} \times A_{q-1})$. We denote by $\rho^{A_{n-1}}$, $\rho^{A_{m-1}}$, $\rho^{p,q}$ the corresponding ρ -vectors.

Set

$$L = \{i \in \{0, \dots, p\} \mid 0 \leq d - i \leq q\}.$$

Set $i_{max} = \min(n, p)$ and $j_{min} = d - i_{max}$. For $\sigma \in S_m$ set

$$\begin{aligned} n_1(\sigma) &= |\{\sigma(i) \mid p - i_{max} + 1 \leq i \leq p + j_{min}\} \cap \{1, \dots, p\}|, \\ U_i &= \{w \in S_m \mid n_1(\sigma) = i\}. \end{aligned}$$

Then $S_m = \coprod_{i \in L} U_i$.

With notation as in Section 3, the order corresponding to $\Delta_{gl}^{(p,q)}$ is

$$\epsilon_1 > \dots > \epsilon_p > \delta_1 > \dots > \delta_n > \epsilon_{p+1} > \dots > \epsilon_m.$$

Define, for $i \in L$, $\beta_t^i = \epsilon_{p-t+1} - \delta_t$ for $1 \leq t \leq i$ and $\gamma_t^i = \delta_{n-t+1} - \epsilon_{t+p}$ for $1 \leq t \leq d - i$. For $i \in L$, let X_i be the arc diagram

$$X_i = \{\widehat{\epsilon_{p-t+1} \delta_t} \mid 1 \leq t \leq i\} \cup \{\widehat{\delta_{n-t+1} \epsilon_{p+t}} \mid 1 \leq t \leq j\}.$$

Then

$$(8.1) \quad S(X_i) = \{\beta_t^i \mid 1 \leq t \leq i\} \cup \{\gamma_t^i \mid 1 \leq t \leq d - i\}.$$

We start by writing formula (1.15) corresponding to $S(X_{i_{max}})$, which, for simplicity, we denote from now on by \underline{S} . Let W_1^i (resp. W_2^i) be the subgroup of S_m that fixes $1, \dots, p-i$ and $p+d-i+1, \dots, m$ (resp. $p-i+1, \dots, p+d-i$). Set $W_1 = W_1^{i_{max}}$ and $W_2 = W_2^{i_{max}}$. Then U_i is stable under the right action of W_1, W_2 , and under the left action of $S_p \times S_q$. Choose $V_i \subset U_i$ such that $U_i = V_i W_1$. Let W'_i be the subgroup of W_1 stabilizing V_i and set $V'_i = V_i/W'_i$.

Assume that $n < m$. By the above, using notation of Section 1.2,

$$\check{R}e^\rho = \check{\mathcal{F}}_{S_m}(\mathcal{Q}(X_{i_{max}})) = \sum_{i \in L} \check{\mathcal{F}}_{U_i}(\mathcal{Q}(X_{i_{max}})) = \sum_{i \in L} \check{\mathcal{F}}_{V'_i}(\check{\mathcal{F}}_{W_1}(\mathcal{Q}(X_{i_{max}}))).$$

Set $v = \epsilon_{p-i_{max}+1}, w = \epsilon_{p-i_{max}+d}$; let Y be the subdiagram in $X_{i_{max}}$ which corresponds to the interval $[v, w]$; note that $S(Y) = S(X_{i_{max}})$ and each vertex of Y is an end of some arc. Moreover, $W_{[v,w]} = S_n \times W_1$ and we may choose W^\sharp for Y to be either S_n or W_1 . One has

$$\check{\mathcal{F}}_{W_1}(\mathcal{Q}(X_{i_{max}})) = e^{\rho-\rho_Y} \check{\mathcal{F}}_{W_1}(\mathcal{Q}(Y)) = e^{\rho-\rho_Y} \check{\mathcal{F}}_{S_n}(\mathcal{Q}(Y)),$$

hence

$$\begin{aligned} \check{R}e^\rho &= \sum_{i \in L} \check{\mathcal{F}}_{V'_i}(e^{\rho-\rho_Y} \check{\mathcal{F}}_{S_n}(\mathcal{Q}(Y))) = \frac{1}{|W'_i|} (\check{\mathcal{F}}_{V'_i}(\check{\mathcal{F}}_{W'_i S_n}(e^{\rho-\rho_Y} \check{\mathcal{F}}_{S_n}(\mathcal{Q}(Y)))) \\ &= \frac{1}{|W'_i|} (\check{\mathcal{F}}_{V_i}(\check{\mathcal{F}}_{S_n}(\mathcal{Q}(X_{i_{max}})))). \end{aligned}$$

Since the elements of $V_i \subset S_m$ commute with the elements of S_n we conclude that

$$(8.2) \quad \check{R}e^\rho = \frac{1}{|W'_i|} (\check{\mathcal{F}}_{S_n}(\check{\mathcal{F}}_{V_i}(\mathcal{Q}(X_{i_{max}})))).$$

Note that, if $m \leq n$, then $L = \{p\}$, $U_p = S_m = W_1$. Thus (8.2) holds as well with V_p any subset of S_m , being just (1.15) for this case.

We now make a careful choice of V_i by exploiting the action of $W_1 \times W_2$ on U_i . Set $\sigma_i = \prod_{t=1}^{i_{max}-i} s_{\epsilon_{p-i_{max}+t} - \epsilon_{p+j_{min}+t}}$. Note that σ_i is an element of U_i . Let \mathcal{O} be the orbit of $(S_p \times S_q)\sigma_i$ under the right action of W_2 in $(S_p \times S_q) \setminus U_i$. First we observe that, if x, y are distinct elements of \mathcal{O} , then $xW_1 \cap yW_1 = \emptyset$. Indeed, if for $\sigma \in W_2$ we have $(S_p \times S_q)\sigma_i\sigma \neq (S_p \times S_q)\sigma_i$, then $(S_p \times S_q)\sigma_i\eta\sigma \neq (S_p \times S_q)\sigma_i$ for any $\eta \in W_1$. This can be checked as follows: assume $\sigma_i\sigma\sigma_i \notin S_p \times S_q$, hence there is $i_0 \in \{1, \dots, p\}$ such that $\sigma_i\sigma\sigma_i(i_0) > p$. If $i_0 \leq p - i_{max}$ then, since $\sigma_i(i_0) = i_0$, we have $\sigma(i_0) > p + d - i$, so $\sigma_i\eta\sigma\sigma_i(i_0) > p$ for any $\eta \in W_1$. If $p - i < i_0 \leq p$ then $\sigma_i\sigma\sigma_i(i_0) = i_0$ so we can assume $i_0 = p - i_{max} + t$ with $t \leq i - i_{max}$. In this case $\sigma_i(i_0) = p + j_{min} + t$ and we must have that $\sigma(p + j_{min} + t) > p + d - i$ for otherwise we would have $\sigma_i\sigma\sigma_i(i_0) \leq p$. But in this case $\eta\sigma\sigma_i(i_0) > p + d - i$ for $\eta \in W_1$ and we are done.

Next we show that the action of $W_1 \times W_2$ on $(S_p \times S_q) \setminus U_i$ by right multiplication is transitive. For this it is enough to show that for any $w \in S_m \cap U_i$, there is $\sigma \in W_1$ and $\tau \in W_2, \eta \in (S_p \times S_q)$ such that $\sigma_i = \eta w \sigma \tau$. Since $|\{w(t) \mid p - i_{max} < t \leq p + j_{min}\} \cap \{1, \dots, p\}| = i$, we can find $\sigma \in W_1$ such that $w\sigma(t) \leq p$ for $p - i < t \leq p$ and $w\sigma(t) > p$ for $p - i_{max} < t \leq p - i_{max} + i$ or $p < t \leq p + j_{min}$. We can find $\eta \in (S_p \times S_q)$ so that $\eta w \sigma(t) = t$ for $p - i < t \leq p + j_{min}$. Moreover we can also assume that $\eta w \sigma(p - i_{max} + t) = p + j_{min} + t$ for $1 \leq t \leq i_{max} - i$. Finally, we can find $\tau \in W_2$ such that $\eta w \sigma \tau(t) = \eta w \sigma(t)$ for $p - i_{max} < t \leq p + j_{min}$,

$\eta w \sigma \tau(p + j_{\min} + t) = p - i_{\max} + t$ for $p + j_{\min} < t \leq p + d - i$ and $\eta w \sigma \tau(t) = t$ otherwise. This means that $\eta w \sigma \tau = \sigma_i$ as wished.

We set \mathcal{Y}_{m-d}^i to be a set of coset representatives for $Stab_i \backslash W_2$ where $Stab_i$ is the stabilizer of the coset $(S_p \times S_q)\sigma_i$ under the right action of W_2 on $(S_p \times S_q) \backslash U_i$.

From our discussion it follows that we can choose $V_i = (S_p \times S_q)\sigma_i \mathcal{Y}_{m-d}^i$. Since $\mathcal{Y}_{m-d}^i \subset W_2$, we see that W'_i is the subgroup of W_1 stabilizing $(S_p \times S_q)\sigma_i$, thus $\sigma_i W'_i \sigma_i$ is the subgroup of W_1^i stabilizing $S_p \times S_q$. It follows that $\sigma_i W'_i \sigma_i = W_1^i \cap (S_p \times S_q) \simeq S_i \times S_{d-i}$. In particular $|W'_i| = i!(d-i)!$.

We can rewrite (8.2) as

$$(8.3) \quad e^\rho \check{R} = \sum_{i \in L} \frac{1}{i!(d-i)!} \check{F}_{W_c \sigma_i \mathcal{Y}_{m-d}^i}(\mathcal{Q}(X_{i_{\max}})).$$

Set $\mathcal{W}_{m-d}^i = \sigma_i \mathcal{Y}_{m-d}^i \sigma_i$. Since $\mathcal{Y}_{m-d}^i \subset W_2$, it is clear that $\mathcal{W}_{m-d}^i \subset W_2^i$. We can therefore rewrite (8.3) as

$$(8.4) \quad e^\rho \check{R} = \sum_{i \in L} \check{F}_{W_c \mathcal{W}_{m-d}^i}(\text{sgn}(\sigma_i) \sigma_i \mathcal{Q}(X_{i_{\max}})),$$

Lemma 8.1. *Given $i \in L$, set $r_i = i_{\max} - i$. Then*

$$(8.5) \quad \text{sgn}(\sigma_i) \sigma_i(\mathcal{Q}(X_{i_{\max}})) = e^{r_i(\llbracket \beta_i^i \rrbracket - \llbracket \gamma_{d-i}^i \rrbracket)} \mathcal{Q}(X_i)$$

Proof. By an explicit computation, we see that

$$\begin{aligned} \rho + \sum_{\gamma \in S(X_i)} \llbracket \gamma \rrbracket &= \sum_{r=1}^{p-i} \frac{m-n-2r+1}{2} \epsilon_r + \sum_{r=p-i+1}^p \frac{q-p+i-j-1}{2} \epsilon_r \\ &+ \sum_{r=p+1}^{p+j} \frac{q-p+i-j+1}{2} \epsilon_r + \sum_{r=p+j+1}^m \frac{m+n-2r+1}{2} \epsilon_r \\ &+ \sum_{r=1}^i \frac{q-p+j-i+1}{2} \delta_r + \sum_{r=i+1}^n \frac{q-p+j-i-1}{2} \delta_r. \end{aligned}$$

Observe that $s_{\epsilon_{p-i+1} - \epsilon_{p+d-i+1}}$ fixes $\rho + \sum_{\gamma \in S(X_i)} \llbracket \gamma \rrbracket$, hence we have

$$\begin{aligned} & \frac{e^{\rho + \sum_{\gamma \in S(X_i)} \llbracket \gamma \rrbracket}}{s_{\epsilon_{p-i+1} - \epsilon_{p+d-i+1}} \prod_{r=1}^i (1 - e^{-\epsilon_{p-r+1} + \delta_r}) \prod_{r=1}^{d-i} (1 - e^{\epsilon_{p+r} - \delta_{n-r+1}})} = \\ & \frac{e^{\rho + \sum_{\gamma \in S(X_i)} \llbracket \gamma \rrbracket}}{(1 - e^{-\epsilon_{p+d-i+1} + \delta_i}) \prod_{r=1}^{i-1} (1 - e^{-\epsilon_{p-r+1} + \delta_r}) \prod_{r=1}^{d-i} (1 - e^{\epsilon_{p+r} - \delta_{n-r+1}})} = \\ & \frac{e^{\rho + \sum_{\gamma \in S(X_{i-1})} \llbracket \gamma \rrbracket + \llbracket \beta_i^i \rrbracket - \llbracket \gamma_{d-i+1}^{i-1} \rrbracket + \epsilon_{p+d-i+1} - \delta_i}}{\prod_{r=1}^{i-1} (1 - e^{-\epsilon_{p-r+1} + \delta_r}) \prod_{r=1}^{d-i+1} (1 - e^{\epsilon_{p+r} - \delta_{n-r+1}})} \end{aligned}$$

Since $\llbracket \beta_i^i \rrbracket = \llbracket \beta_{i-1}^{i-1} \rrbracket$ and $\llbracket \gamma_{d-i+1}^{i-1} \rrbracket + \epsilon_{p+d-i+1} - \delta_i = \llbracket \gamma_{d-i+1}^{i-1} \rrbracket$, we obtain that

$$(8.6) \quad s_{\epsilon_{p-i+1} - \epsilon_{p+d-i+1}} \frac{e^{\rho + \sum_{\gamma \in S(X_i)} \llbracket \gamma \rrbracket}}{\prod_{\gamma \in S(X_i)} (1 - e^{-\gamma})} = - \frac{e^{\rho + \sum_{\gamma \in S(X_{i-1})} \llbracket \gamma \rrbracket}}{\prod_{\gamma \in S(X_{i-1})} (1 - e^{-\gamma})} e^{\llbracket \beta_{i-1}^{i-1} \rrbracket - \llbracket \gamma_{d-i+1}^{i-1} \rrbracket}.$$

Since

$$s_{\epsilon_{p-i+1} - \epsilon_{p+d-i+1}} (\llbracket \beta_i^i \rrbracket - \llbracket \gamma_{d-i}^i \rrbracket) = \llbracket \beta_{i-1}^{i-1} \rrbracket - \llbracket \gamma_{d-i+1}^{i-1} \rrbracket$$

the lemma is proven by an obvious induction. \square

Combining (8.4) and (8.5), we get

$$(8.7) \quad e^\rho \check{R} = \sum_{i \in L} \frac{1}{i!(d-i)!} \check{\mathcal{F}}_{W_c \mathcal{W}_{m-d}^i} (e^{r_i(\llbracket \beta_i^i \rrbracket - \llbracket \gamma_{d-i}^i \rrbracket)} \mathcal{Q}(X_i)).$$

One last computation is needed:

Lemma 8.2.

$$(8.8) \quad \frac{1}{i!(d-i)!} \check{\mathcal{F}}_{W_c \mathcal{W}_{m-d}^i} (e^{r_i(\llbracket \beta_i^i \rrbracket - \llbracket \gamma_{d-i}^i \rrbracket)} \mathcal{Q}(X_i)) = \check{\mathcal{F}}_{W_c \mathcal{W}_{m-d}^i} \left(\frac{e^{\rho + r_i(\llbracket \beta_i^i \rrbracket - \llbracket \gamma_{d-i}^i \rrbracket)}}{\prod_{\gamma \in S(X_i)} (1 - e^{-\llbracket \gamma \rrbracket})} \right).$$

Proof. Set Y_i^1 the subdiagram of X_i whose support is $[\epsilon_{p-i+1}, \delta_i]$ and Y_i^2 the subdiagram having support $[\delta_{n-d+i+1}, \epsilon_{p+d-i}]$. Let $G_1 = W_{[\epsilon_{p-i+1}, \delta_i]}$ and $G_2 = W_{[\delta_{n-d+i+1}, \epsilon_{p+d-i}]}$. An easy calculation shows that

$$\llbracket \beta_i^i \rrbracket - \llbracket \gamma_{d-i}^i \rrbracket = \sum_{h=p-i+1}^{p+d-i} \epsilon_h - \sum_{h=1}^i \delta_h - \sum_{h=n-d+i+1}^n \delta_h.$$

In particular $\llbracket \beta_i^i \rrbracket - \llbracket \gamma_{d-i}^i \rrbracket$ is $G_1 \times G_2$ -invariant. Since $G_1 \times G_2 \subset W_c$ and commutes with \mathcal{W}_{m-d}^i , we can write

$$\begin{aligned} & \check{\mathcal{F}}_{W_c \mathcal{W}_{m-d}^i} (\mathcal{Q}(X_i) e^{r_i(\llbracket \beta_i^i \rrbracket - \llbracket \gamma_{d-i}^i \rrbracket)}) \\ &= \check{\mathcal{F}}_{W_c / (G_1 \times G_2) \mathcal{W}_{m-d}^i} (e^{\rho - \rho_{Y_i^1} - \rho_{Y_i^2}} e^{r_i(\llbracket \beta_i^i \rrbracket - \llbracket \gamma_{d-i}^i \rrbracket)} \check{\mathcal{F}}_{G_1} (\mathcal{Q}(Y_i^1)) \check{\mathcal{F}}_{G_2} (\mathcal{Q}(Y_i^2))). \end{aligned}$$

Combining (3.9) with (1.15), we have that, for any \mathfrak{g} , $\check{\mathcal{F}}_{W_{\mathfrak{g}}} (\mathcal{Q}(X)) = \check{\mathcal{F}}_{W_{\mathfrak{g}}} (\mathcal{P}(X))$, thus

$$\begin{aligned} & \check{\mathcal{F}}_{W_c \mathcal{W}_{m-d}^i} (\mathcal{Q}(X_i) e^{r_i(\llbracket \beta_i^i \rrbracket - \llbracket \gamma_{d-i}^i \rrbracket)}) \\ &= \check{\mathcal{F}}_{W_c / (G_1 \times G_2) \mathcal{W}_{m-d}^i} (e^{\rho - \rho_{Y_i^1} - \rho_{Y_i^2}} e^{r_i(\llbracket \beta_i^i \rrbracket - \llbracket \gamma_{d-i}^i \rrbracket)} \check{\mathcal{F}}_{G_1} (\mathcal{P}(Y_i^1)) \check{\mathcal{F}}_{G_2} (\mathcal{P}(Y_i^2))) \\ &= \check{\mathcal{F}}_{W_c \mathcal{W}_{m-d}^i} (\mathcal{P}(X_i) e^{r_i(\llbracket \beta_i^i \rrbracket - \llbracket \gamma_{d-i}^i \rrbracket)}). \end{aligned}$$

The lemma follows from the observation that $\mathcal{P}(X_i) = i!(d-i)! \frac{e^\rho}{\prod_{\gamma \in S(X_i)} (1 - e^{-\llbracket \gamma \rrbracket})}$. \square

Using (8.8), we see that (8.7) becomes

$$(8.9) \quad e^\rho \check{R} = \sum_{i \in L} \check{\mathcal{F}}_{W_c \mathcal{W}_{m-d}^i} \left(\frac{e^\rho}{\prod_{\gamma \in S(X_i)} (1 - e^{-\llbracket \gamma \rrbracket})} e^{r_i(\llbracket \beta_i^i \rrbracket - \llbracket \gamma_{d-i}^i \rrbracket)} \right).$$

Using (8.9), and expanding in geometric series in the domain $\mathfrak{h}^+ = \{h \in \mathfrak{h} \mid \alpha(h) > 0, \alpha \in \Delta_1^+\}$, formula (8.9) becomes

$$(8.10) \quad e^\rho \check{R} = \sum_{i \in L} \mathcal{F}_{W_c \mathcal{W}_{m-d}^i} \left(\sum_{\mathbf{a} \in \mathbb{P}_i, \mathbf{b} \in \mathbb{P}_{d-i}} e^{V(\mathbf{a} - r_i^i, \mathbf{b} + r_i^{d-i})} \right),$$

where

$$V(\mathbf{a}, \mathbf{b}) = \rho - \sum_{t=1}^i a_{i-t+1} \epsilon_{p-i+t} + \sum_{t=1}^{d-i} b_t \epsilon_{p+t} + \sum_{t=1}^i a_t \delta_t - \sum_{t=1}^{d-i} b_{d-i-t+1} \delta_{n-d+i+t}$$

and we denote by r^k the partition $(r, r, \dots, r) \in \mathbb{P}_k$. Note that (8.10) can be written as

$$(8.11) \quad e^\rho \check{R} = \sum_{i \in L} \mathcal{F}_{W_c \mathcal{W}_{m-d}^i} \left(\sum_{\mathbf{a} \in \mathbb{P}_i, r_i^{d-i} \subset \mathbf{b} \in \mathbb{P}_{d-i}} e^{V(\mathbf{a}-r_i^i, \mathbf{b})} \right).$$

From the formula above we can already derive an explicit form of the Theta correspondence, but, in order to have the same parametrization of [15], some extra work is needed. For $\mathbf{a} \in \mathbb{P}_i$ and $k \leq i$ define

$$\begin{aligned} f_k(\mathbf{a}) &= (a_1, \dots, a_k), \\ l_k(\mathbf{a}) &= (a_{i-k+1}, \dots, a_i). \end{aligned}$$

If $\mathbf{a} \subset r^i$, we set $r^i \setminus \mathbf{a} = (r - a_i, \dots, r - a_1)$. If $\mathbf{a} \in \mathbb{P}_i$ then, given $b \in \mathbb{P}_{d-i}$ with $a_1^{d-i} \subset \mathbf{b}$, we let $\mathbf{b} \cup \mathbf{a} = (b_1, \dots, b_{d-i}, a_1, \dots, a_i)$.

Given $r \in \mathbb{N}$ and $\mathbf{a} \in \mathbb{P}_i$, let $k \in \{0, \dots, i\}$ be such that $r^k \subset \mathbf{a}$. If $(r - a_i)^{d-i} \subset \mathbf{b}$, then we set

$$s_k(\mathbf{a}, \mathbf{b}) = (f_k(\mathbf{a}) - r^k, \mathbf{b} \cup (r^{i-k} \setminus l_{i-k}(\mathbf{a}))).$$

We let $\mathcal{W}_{m-d}^i(\mathbf{a}, \mathbf{b})$ be the set of $v \in \mathcal{W}_{m-d}^i$ such that $v(V(\mathbf{a}, \mathbf{b}))$ is regular for $\Delta(A_{p-1} \times A_{q-1})$. Given $r \in \mathbb{N}$ and $\mathbf{a} \in \mathbb{P}_i$, let $k_r(\mathbf{a})$ be maximal in the set of $k \in \{0, \dots, i\}$ such that $r^k \subset \mathbf{a}$.

Lemma 8.3. *Assume that $m < n$ and fix $i \in L$. Let $(\mathbf{a}, \mathbf{b}) \in \mathbb{P}_i \times \mathbb{P}_{d-i}$ be such that $(r_i - a_i)^{d-i} \subset \mathbf{b}$ and set $k = k_{r_i}(\mathbf{a})$.*

- (1) *If $k \notin L$, then $v(V(\mathbf{a} - r_i^i, \mathbf{b}))$ is Δ_c -singular for any $v \in \mathcal{W}_{m-d}^i$.*
- (2) *If $k \in L$ then there is a bijection $v \mapsto z$ between $\mathcal{W}_{m-d}^i(\mathbf{a} - r_i^i, \mathbf{b})$ and $\mathcal{W}_{n-d}^k(s_k(\mathbf{a}, \mathbf{b}))$ such that*

$$\mathcal{F}_{W_c}(\text{sgn}(v)e^{vV(\mathbf{a}-r_i^i, \mathbf{b})}) = \mathcal{F}_{W_c}(\text{sgn}(z)e^{zV(s_k(\mathbf{a}, \mathbf{b}))}).$$

Proof. We will prove both statements by induction on $i - k$. The case $i = k$ is obvious.

If $k < i$, set $t_0 = p - i + 1 - (r_i - a_i)$. Note that $p - i_{\max} + 1 \leq t_0 \leq p - i$.

If $p + d - i = m$ then, since $k < i$, $k \notin L$. Moreover we have that we can choose $\mathcal{W}_{m-d}^i = \{1\}$. An easy calculation shows that

$$\rho = \sum_{t=1}^p \left(\frac{m-n+1}{2} - t \right) \epsilon_t + \sum_{t=p+1}^m \left(\frac{n+m+1}{2} - t \right) \epsilon_t + \sum_{t=1}^n \left(\frac{q-p+n+1}{2} - t \right) \delta_t,$$

so $(V(\mathbf{a}, \mathbf{b}), \epsilon_{t_0} - \epsilon_{p-i+1}) = 0$. It follows that $v(V(\mathbf{a} - r_i^i, \mathbf{b}))$ is Δ_c -singular for any $v \in \mathcal{W}_{m-d}^i$.

If $p + d - i < m$, let $s \in S_m$ be defined by setting $s(t_0) = p + d - i + 1$, $s(p + d - i + 1) = p - i + 1$, $s(p - i + 1) = t_0$, and $s(t) = t$ for $t \neq t_0, p - i + 1, p + d - i + 1$. Then $s(V(\mathbf{a}, \mathbf{b})) = V(\mathbf{a}' - (r_i + 1)^{i-1}, \mathbf{b}')$ where $\mathbf{a}' = l_{i-1}(\mathbf{a}) + 1^{i-1}$ and $\mathbf{b}' = \mathbf{b} \cup (r_i - a_i)^1$. Suppose that $v \in \mathcal{W}_{m-d}^i$ and assume that $v(V(\mathbf{a} - r_i^i, \mathbf{b}))$ is Δ_c -regular. Then $v(t_0) > p + d - i$, so $vs^{-1}(p - d + i + 1) = s_0 > p + d - i$. Hence $w' = s_{\epsilon_{p-d+i+1} - \epsilon_{s_0}} vs^{-1} \in W_2^{i-1}$ and

$$\mathcal{F}_{W_c}(\text{sgn}(v)e^{vV(\mathbf{a}-r_i^i, \mathbf{b})}) = \mathcal{F}_{W_c}(\text{sgn}(w')e^{w'V(\mathbf{a}'-(r_i+1)^{i-1}, \mathbf{b}')}).$$

This implies that there is a unique $w \in \mathcal{W}_{m-d}^{i-1}(\mathbf{a}' - (r_i + 1)^{i-1}, \mathbf{b}')$ such that

$$\mathcal{F}_{W_c}(\text{sgn}(v)e^{vV(\mathbf{a}-r_i^i, \mathbf{b})}) = \mathcal{F}_{W_c}(\text{sgn}(w)e^{wV(\mathbf{a}'-(r_i+1)^{i-1}, \mathbf{b}')}).$$

Clearly the argument above can be reversed to prove that the map $v \mapsto w$ is a bijection. Since $r_i + 1 = r_{i-1}$, $(r_{i-1} - a'_{i-1})^{d-i+1} \subset (r_i - a_i)^{d-i+1} \subset \mathbf{b}'$, and $k_{r_{i-1}}(\mathbf{a}') = k_{r_i}(\mathbf{a})$, we see that we can apply the induction hypothesis. Thus, since $w(V(\mathbf{a}' - (r_i + 1)^{i-1}, \mathbf{b}'))$ is Δ_c -regular, we have $k \in L$. Moreover there is a bijection $w \mapsto z$ from $\mathcal{W}_{m-d}^{i-1}(\mathbf{a}' - (r_i + 1)^{i-1}, \mathbf{b}')$ to $\mathcal{W}_{m-d}^k(s_k(\mathbf{a}', \mathbf{b}'))$ such that

$$\mathcal{F}_{W_c}(sgn(w)e^{wV(\mathbf{a}' - (r_i + 1)^{i-1}, \mathbf{b}')})) = \mathcal{F}_{W_c}(sgn(z)e^{zV(s_k(\mathbf{a}', \mathbf{b}'))}).$$

Since it is clear that $s_k(\mathbf{a}', \mathbf{b}') = s_k(\mathbf{a}, \mathbf{b})$ we are done by composing the two bijections. \square

Proposition 8.4.

$$e^\rho \check{R} = \mathcal{F}_{W_c \mathcal{W}_{m-d}^{i_{max}}} \left(\sum_{\mathbf{a} \in \mathbb{P}_{i_{max}}, \mathbf{b} \in \mathbb{P}_{j_{min}}} e^{V(\mathbf{a}, \mathbf{b})} \right) + \sum_{i \in L, i < i_{max}} \mathcal{F}_{W_c \mathcal{W}_{m-d}^i} \left(\sum_{\mathbf{a} \in \mathbb{P}_i, \mathbf{b} \in \mathbb{P}_{d-i}^*} e^{V(\mathbf{a}, \mathbf{b})} \right).$$

Proof. The expression (8.11) for $e^\rho \check{R}$ can be written as

$$\begin{aligned} & \mathcal{F}_{W_c \mathcal{W}_{m-d}^{i_{max}}} \left(\sum_{\mathbf{a} \in \mathbb{P}_{i_{max}}, \mathbf{b} \in \mathbb{P}_{j_{min}}} e^{V(\mathbf{a}, \mathbf{b})} \right) + \sum_{i \in L, i < i_{max}} \mathcal{F}_{W_c \mathcal{W}_{m-d}^i} \left(\sum_{\mathbf{a} \in \mathbb{P}_i, r_i^{d-i} \subset \mathbf{b} \in \mathbb{P}_{d-i}} e^{V(\mathbf{a}, \mathbf{b})} \right) \\ & + \sum_{i \in L} \mathcal{F}_{W_c \mathcal{W}_{m-d}^i} \left(\sum_{r_i^i \not\subset \mathbf{a} \in \mathbb{P}_i, r_i^{d-i} \subset \mathbf{b} \in \mathbb{P}_{d-i}} e^{V(\mathbf{a} - r_i^i, \mathbf{b})} \right). \end{aligned}$$

Let III denote the third summand. By Lemma 8.3 we have

$$III = \sum_{i \in L} \sum_{k \in L, k < i} \mathcal{F}_{W_c \mathcal{W}_{m-d}^k} \left(\sum_{\substack{r_i^k \subset \mathbf{a} \in \mathbb{P}_i, r_i^{k+1} \not\subset \mathbf{a} \\ \mathbf{b} \in \mathbb{P}_j, r_i^{d-i} \subset \mathbf{b}}} e^{V(s_k(\mathbf{a}, \mathbf{b}))} \right),$$

which is

$$\sum_{k \in L, k < i_{max}} \sum_{i \in L, i > k} \mathcal{F}_{W_c \mathcal{W}_{m-d}^k} \left(\sum_{\substack{r_i^k \subset \mathbf{a} \in \mathbb{P}_i, r_i^{k+1} \not\subset \mathbf{a} \\ \mathbf{b} \in \mathbb{P}_{d-i}, r_i^{d-i} \subset \mathbf{b}}} e^{V(s_k(\mathbf{a}, \mathbf{b}))} \right).$$

We observe that the map s_k is a bijection between the set

$$\bigcup_{i \in L, i > k} \{(\mathbf{a}, \mathbf{b}) \in \mathbb{P}_i \times \mathbb{P}_{d-i} \mid k_{r_i}(\mathbf{a}) = k, r_i^{d-i} \subset \mathbf{b}\}$$

and the set

$$\{(\mathbf{a}, \mathbf{b}) \in \mathbb{P}_k \times \mathbb{P}_h^* \mid (i_{max} - k)^h \not\subset \mathbf{b}\},$$

hence we can rewrite III as

$$\sum_{k \in L, k < i_{max}} \mathcal{F}_{W_c \mathcal{W}_{m-d}^k} \left(\sum_{\substack{\mathbf{a} \in \mathbb{P}_k \\ \mathbf{b} \in \mathbb{P}_{d-k}^*, (i_{max} - k)^{d-k} \not\subset \mathbf{b}}} e^{V((\mathbf{a}, \mathbf{b}))} \right).$$

Upon substituting we get the desired statement. \square

Arguing as in the previous sections, it follows from Proposition 8.4 that

$$\begin{aligned}
 chM^{\Delta^+}(\mathfrak{g}_1) &= \\
 &\sum_{\mathbf{a} \in \mathbb{P}_{i_{max}}, \mathbf{b} \in \mathbb{P}_{j_{min}}} \sum_{v \in \mathcal{W}_{m-d}^{i_{max}}} sgn(v) \\
 &\frac{ch(F_{p,q}(v(-\rho_1 + \mu(\mathbf{a}, \mathbf{b}) + \rho^{A_{m-1}}) - \rho^{A_{m-1}}))}{\prod_{\alpha \in \Delta^+(A_{m-1}) \setminus \Delta^+(A_{p-1} \times A_{q-1})} (1 - e^{-\alpha})} ch F_n(\varepsilon(\mathbf{a}, \mathbf{b})) \\
 &+ \sum_{i \in L, i < i_{max}} \sum_{\mathbf{a} \in \mathbb{P}_i, \mathbf{b} \in \mathbb{P}_j^*} \sum_{v \in \mathcal{W}_{m-d}^i} sgn(v) \\
 &\frac{ch(F_{p,q}(v(-\rho_1 + \mu(\mathbf{a}, \mathbf{b}) + \rho^{A_{m-1}}) - \rho^{A_{m-1}}))}{\prod_{\alpha \in \Delta^+(A_{m-1}) \setminus \Delta^+(A_{p-1} \times A_{q-1})} (1 - e^{-\alpha})} ch F_n(\varepsilon(\mathbf{a}, \mathbf{b})).
 \end{aligned}$$

Notation is as follows: $\Delta^+ = \Delta_{gl}^{(p,q)}$; if $\mathbf{a} \in \mathbb{P}_i, \mathbf{b} \in \mathbb{P}_{d-i}$, we set

$$\begin{aligned}
 \mu(\mathbf{a}, \mathbf{b}) &= - \sum_{t=1}^i a_{i-t+1} \epsilon_{p-i+t} + \sum_{t=1}^{d-i} b_t \epsilon_{p+t}, \\
 \varepsilon(\mathbf{a}, \mathbf{b}) &= \sum_{t=1}^i a_t \delta_t - \sum_{t=1}^{d-i} b_{d-i-t+1} \delta_{n-d+i+t}.
 \end{aligned}$$

Here $F_n(\varepsilon(\mathbf{a}, \mathbf{b}))$ is the finite-dimensional irreducible $u(n)$ -module with highest weight $\varepsilon(\mathbf{a}, \mathbf{b})$. Finally, if $\lambda \in span(\epsilon_i)$, set

$$chF_{p,q}(\lambda) = \sum_{w \in S_p \times S_q} sgn(w) \frac{e^{w(\lambda + \rho^{p,q}) - \rho^{p,q}}}{\prod_{\alpha \in \Delta^+(A_{p-1} \times A_{q-1})} (1 - e^{-\alpha})}.$$

Note that

$$V(\mathbf{a}, \mathbf{b}) = \rho + \varepsilon(\mathbf{a}, \mathbf{b}) + \mu(\mathbf{a}, \mathbf{b}).$$

By [1], the inverse image \tilde{G}_1 of $G_1 = U(n)$ in \tilde{K} is isomorphic to the cover defined by the character $\det^{\frac{m}{2}}$. In particular, \tilde{G}_1 is connected, hence the set Σ , occurring in Theorem 4.1, is a subset of the set of finite-dimensional irreducible representations of its Lie algebra $u(n)$.

Set $\mathfrak{h}_1 = \mathfrak{h} \cap u(n)$, $\mathfrak{h}_2 = \mathfrak{h} \cap u(p, q)$.

Corollary 8.5 (Theta correspondence). *With notation as in Theorem 4.1, we have*

$$\Sigma = \bigcup_{\substack{i \in L \\ \mathbf{a} \in \mathbb{P}_i, \mathbf{b} \in \mathbb{P}_{d-i}}} F_n(-(\rho_1)|_{\mathfrak{h}_1} + \varepsilon(\mathbf{a}, \mathbf{b})).$$

Moreover

$$\tau(F_n(\varepsilon(\mathbf{a}, \mathbf{b}))) = L^2(-(\rho_1)|_{\mathfrak{h}_2} + \mu(\mathbf{a}, \mathbf{b})).$$

Proof. Similar to the proof of Corollary 6.3. \square

Let us finally display the character formula. We let $\overline{\mathcal{W}}_{m-d}^i$ be the set of $v \in \mathcal{W}_{m-d}^i$ such that $v(-(\rho_1)|_{\mathfrak{h}_2} + \mu(\mathbf{a}, \mathbf{b}) + \rho^{A_{m-1}})$ is regular for $\Delta(A_{p-1} \times A_{q-1})$. For $v \in \overline{\mathcal{W}}_{m-d}^i$, let

$$c_v = c_{v(-(\rho_1)|_{\mathfrak{h}_2} + \mu(\mathbf{a}, \mathbf{b}) + \rho^{A_{m-1}})}.$$

From Proposition 8.4 we get that, if $\mathbf{a} \in \mathbb{P}_{i_{max}}$, $\mathbf{b} \in \mathbb{P}_{j_{min}}$ then

$$\begin{aligned} ch(L^2(-(\rho_1)|_{\mathfrak{h}_2} + \mu(\mathbf{a}, \mathbf{b}))) &= \\ \sum_{v \in \overline{\mathcal{W}}_{m-d}^{i_{max}}} c_v sgn(v) chV^2(\{v(-\rho_1 + \mu(\mathbf{a}, \mathbf{b}) + \rho^{A_{m-1}})\} - \rho^{A_{m-1}}), \end{aligned}$$

while, if $i \in L$ with $i < i_{max}$ and $\mathbf{a} \in \mathbb{P}_i$, $\mathbf{b} \in \mathbb{P}_j^*$, then

$$\begin{aligned} ch(L^2(-(\rho_1)|_{\mathfrak{h}_2} + \mu(\mathbf{a}, \mathbf{b}))) &= \\ \sum_{v \in \overline{\mathcal{W}}_{m-d}^i} c_v sgn(v) chV^2(\{v(-\rho_1 + \mu(\mathbf{a}, \mathbf{b}) + \rho^{A_{m-1}})\} - \rho^{A_{m-1}}). \end{aligned}$$

However we can check that if $\mathbf{a} \in \mathbb{P}_k^*$, $\mathbf{b} \in \mathbb{P}_{d-i}^*$ with $k \leq i$, then, $v \in \overline{\mathcal{W}}_{m-d}^i$ implies that $W_c v = W_c w$ with $w \in W_2^{(i, d-k)}$, where $W_2^{(i, d-k)} = \{\sigma \in S_m \mid \sigma(\epsilon_a) = \epsilon_a, p - i + 1 \leq a \leq p + d - k\}$. It follows that the character formula can be written in a more uniform and symmetric way as follows: fix $\mathbf{a} \in \mathbb{P}_k^*$, $\mathbf{b} \in \mathbb{P}_h^*$ with $k \leq p$, $h \leq q$, and $h + k \leq d$. Choose a set $\mathcal{W}_{m-d}^{(d-h, d-k)}$ of coset representatives for $W_c \backslash W_c W_2^{(d-h, d-k)}$. Then

$$\begin{aligned} ch(L^2(-(\rho_1)|_{\mathfrak{h}_2} + \mu(\mathbf{a}, \mathbf{b}))) &= \\ \sum_{v \in \overline{\mathcal{W}}_{m-d}^{(d-h, d-k)}} c_v sgn(v) chV^2(\{v(-(\rho_1)|_{\mathfrak{h}_2} + \mu(\mathbf{a}, \mathbf{b}) + \rho^{A_{m-1}})\} - \rho^{A_{m-1}}). \end{aligned}$$

Once the character formula is written in this form, we can apply to it the argument given in Corollary 5.3, thus, with notation as in Section 5, we obtain

Corollary 8.6. *If $k \leq p$, $h \leq q$, $k + h \leq d$, $\mathbf{a} \in \mathbb{P}_k^*$, $\mathbf{b} \in \mathbb{P}_h^*$, and $\lambda_0 = -(\rho_1)|_{\mathfrak{h}_2} + \mu(\mathbf{a}, \mathbf{b}) + \rho^{A_{m-1}}$, then*

$$ch(L^2(\lambda_0)) = \sum_{w \in W_{\lambda_0}^A} (-1)^{\ell_{\lambda_0}(w)} V^2(\{w(\lambda_0)\} - \rho^{A_{m-1}}).$$

9. ON THE KAC-WAKIMOTO CONJECTURE

As a final application of our denominator formulas we verify Kac-Wakimoto conjecture in a remarkable special case. Let us recall briefly this conjecture. Let V be a finite dimensional irreducible highest weight module with highest weight Λ . Recall that the *atypicality* of Λ , denoted by $atp(\Lambda)$ is the maximal number of linearly independent mutually orthogonal isotropic roots which are orthogonal to Λ . The atypicality $atp(V)$ is defined as the atypicality of $\Lambda + \rho$. The ρ -shift makes this definition independent of the chosen set of positive roots. Also recall that the supercharacter of V is defined as $schV = \sum_{\lambda \in \mathfrak{h}^*} sdim(V_\lambda) e^\lambda$. In [14] the following conjecture is stated.

Conjecture. *There exists $b \in \mathbb{Q}$ such that*

$$(9.1) \quad \check{R}e^\rho schV = b \check{\mathcal{F}}_W \left(\frac{e^{\rho + \Lambda}}{\prod_{i=1}^{atp(V)} (1 - e^{-\beta_i})} \right),$$

where Λ is the highest weight of V and the set of simple roots contains mutually orthogonal isotropic roots $\beta_1, \dots, \beta_{atp(V)}$ satisfying $(\rho + \Lambda, \beta_i) = 0$.

We will refer to the latter condition as the KW condition. We prove this conjecture when V is the natural representation of Lie superalgebras of type $gl(m, n)$, $D(m, n)$, $D(n, m)$ with $m \geq n$ and $B(m, n)$, $B(n, m)$ with $m > n$. Recall that, in these cases, we have $atp(V) = \min(m - 1, n)$. Also note that $atp(V) = m$ for $B(m, m)$, and in this case KW condition does not hold for any choice of the set of simple roots, hence we exclude $B(m, m)$ from our consideration.

We fix the standard triangular decomposition of \mathfrak{g}_0 and consider the sets of simple roots compatible with this triangular decomposition. We embed our root system into the lattice spanned by $\{\epsilon_i\}_{i=1}^m \cup \{\delta_j\}_{j=1}^n$ (with $m \geq n$); for example, for $D(n, m)$, $m > n$, $2\epsilon_i$ is an even root. We choose the inner product $(\epsilon_i, \epsilon_j) = \delta_{ij} = -(\delta_i, \delta_j)$.

9.1. Formula for a special root system. Consider a set of simple roots Π which contains the set

$$\{\epsilon_1 - \epsilon_2, \dots, \epsilon_{m-1} - \epsilon_m, \epsilon_m - \delta_1, \delta_1 - \delta_2, \dots, \delta_{n-1} - \delta_n\}.$$

Then the corresponding total order on $\{\epsilon_i\}_{i=1}^m \cup \{\delta_j\}_{j=1}^n$ is given by $\epsilon_1 > \epsilon_2 > \dots > \epsilon_m > \delta_1 > \dots > \delta_n$. For $i = 1, \dots, n$ set

$$\gamma_i := \epsilon_{m+1-i} - \delta_i.$$

Consider the natural representation $V = V(\epsilon_1)$.

Set

$$\Gamma := \begin{cases} \{\gamma_i\}_{i=1, \dots, n-1} & \text{if } m = n, \\ \{\gamma_i\}_{i=1, \dots, n} & \text{if } m > n. \end{cases}$$

Note that Γ is a maximal set of isotropic pairwise orthogonal roots which are orthogonal to $\rho + \epsilon_1$. We have excluded $B(m, m)$ since in this case the maximal set is $\{\gamma_i\}_{i=1}^{m-1} \cup \{\epsilon_1 + \delta_m\}$ and $\epsilon_1 + \delta_m$ is “bad” in the sense that this root does not lie in a set of simple roots obtained from Π by a series of odd reflections. Recall that $\llbracket \gamma_j \rrbracket = \sum_{i=1}^j \gamma_i$.

We are going to show that there is a constant j_V such that

$$(9.2) \quad \check{R}e^{\rho}schV = j_V \check{\mathcal{F}}_W \left(\frac{e^{\rho + \epsilon_1}}{\prod_{\gamma \in \Gamma} (1 - e^{-\llbracket \gamma \rrbracket})} \right).$$

Write $\check{R} = R_0/R_1$, $R_i = \prod_{\alpha \in \Delta_i^+} (1 - e^{-\alpha})$.

9.1.1. As \mathfrak{g}_0 -module, V is the sum of two simple modules with highest weights δ_1 and ϵ_1 respectively. By the Weyl character formula one has

$$(9.3) \quad R_0 e^{\rho_0} schV = \mathcal{F}_W (e^{\rho_0 + \epsilon_1} - e^{\rho_0 + \delta_1}).$$

9.1.2. Let \mathfrak{g}' be the simple subalgebra of \mathfrak{g} whose set of roots Δ' is the set of roots in Δ that are orthogonal to ϵ_1 . We have that \mathfrak{g}' is of type $gl(m - 1, n)$, $B(m - 1, n)$, $D(m - 1, n)$, $B(n, m - 1)$, $D(n, m - 1)$ respectively for \mathfrak{g} of type $gl(m, n)$, $B(m, n)$, $D(m, n)$, $B(n, m)$, $D(n, m)$. Observe that, if Σ is a set of simple roots for \mathfrak{g} such that $(\epsilon_1, \alpha) \geq 0$ for any $\alpha \in \Sigma$, then a set of simple roots for \mathfrak{g}' is given by $\Sigma' = \Sigma \setminus \{\alpha \in \Sigma \mid (\alpha, \epsilon_1) > 0\}$. In particular the set $\Pi' := \Pi \setminus \{\epsilon_1 - \epsilon_2\}$ is a set of simple roots for \mathfrak{g}' . We introduce $\rho', \rho'_0, R'_0, R'_1, \check{R}', W'$ for this root system in the standard way. Recall the denominator identity for \mathfrak{g}' in the form

$$\check{R}' e^{\rho'} = c \check{\mathcal{F}}_{W'} \left(\frac{e^{\rho'}}{\prod_{\gamma \in \Gamma} (1 - e^{-\llbracket \gamma \rrbracket})} \right),$$

with $c = \frac{atp(V)!}{C_{\delta'}}$, which can be rewritten as

$$R'_0 e^{\rho'_0} = c \mathcal{F}_{W'} \left(\frac{e^{\rho'_0} R'_1}{\prod_{\gamma \in \Gamma} (1 - e^{-\llbracket \gamma \rrbracket})} \right),$$

or

$$\mathcal{F}_{W'}(e^{\rho'_0}) = c \mathcal{F}_{W'} \left(\frac{e^{\rho'_0} R'_1}{\prod_{\gamma \in \Gamma} (1 - e^{-\llbracket \gamma \rrbracket})} \right).$$

9.1.3. We now prove

$$(9.4) \quad \mathcal{F}_W(e^{\rho_0 + \epsilon_1} - e^{\rho_0 + \delta_1}) = j_V \mathcal{F}_W \left(\frac{e^{\rho_0 + \epsilon_1} R_1}{\prod_{\gamma \in \Gamma} (1 - e^{-\llbracket \gamma \rrbracket})} \right),$$

where $j_V = \frac{1}{c}$ except for the case $D(m, m)$ with $\delta_i \in \Delta$ and in the latter case $j_V = \frac{1}{2c}$. Note that, by (9.3), formula (9.2) is equivalent to formula (9.4).

Set

$$A = \mathcal{F}_W \left(\frac{e^{\rho_0 + \epsilon_1} R_1}{\prod_{\gamma \in \Gamma} (1 - e^{-\llbracket \gamma \rrbracket})} \right).$$

Since $\frac{R_1}{R'_1}$ and $e^{\rho_0 - \rho'_0}$ are W' -invariant, one has

$$\begin{aligned} A &= \mathcal{F}_{W/W'} \left(\frac{R_1}{R'_1} e^{\epsilon_1 + \rho_0 - \rho'_0} \cdot \mathcal{F}_{W'} \left(\frac{R'_1 e^{\rho'_0}}{\prod_{\gamma \in \Gamma} (1 - e^{-\llbracket \gamma \rrbracket})} \right) \right) \\ &= \mathcal{F}_{W/W'} \left(\frac{R_1}{R'_1} e^{\epsilon_1 + \rho_0 - \rho'_0} \cdot \frac{1}{c} \mathcal{F}_{W'}(e^{\rho'_0}) \right) \\ &= \frac{1}{c} \mathcal{F}_W(e^{\rho_0 + \epsilon_1} \prod_{\beta \in \Delta_1^+ \setminus \Delta'} (1 - e^{-\beta})) \end{aligned}$$

so

$$A = \frac{1}{c} \sum_{J \subset \Delta_1^+ \setminus \Delta'} (-1)^{|J|} \mathcal{F}_W(e^{\lambda_J}), \quad \text{where } \lambda_J = \rho_0 + \epsilon_1 - \sum_{\beta \in J} \beta.$$

Now formula (9.4) is equivalent to the formula

$$(9.5) \quad A = a \mathcal{F}_W(e^{\lambda_\emptyset} - e^{\lambda_{\{\epsilon_1 - \delta_1\}}}),$$

where $a = 1$ except for the case $D(m, m)$ with $\delta_i \in \Delta$ and $a = 2$ in the latter case.

Recall that $\mathcal{F}_W(e^\lambda) = 0$ if λ is not regular. Let us find J such that λ_J is regular. Write $\Delta_0 = \Delta_\epsilon \amalg \Delta_\delta$ with Δ_ϵ in the span of $\{\epsilon_i\}_{i=1}^m$ and $\rho_0 = \rho_\epsilon + \rho_\delta$, where ρ_ϵ (resp., ρ_δ) is the standard “ ρ ” for Δ_ϵ (resp., for Δ_δ). For any $\beta \in \Delta_1^+ \setminus \Delta'$ one has $(\beta, \epsilon_1) = 1$, $(\beta, \epsilon_i) = 0$ for $i > 1$. Thus $\lambda_J = \rho_\epsilon + (1 - |J|)\epsilon_1 + \nu_J$, where ν_J lies in the span of $\{\delta_j\}_{j=1}^n$. The regularity of λ_J is equivalent to the regularity of $\rho_\epsilon + (1 - |J|)\epsilon_1$ and of ν_J .

For gl_m the element $\rho_\epsilon + (1 - |J|)\epsilon_1$ is regular if and only if $|J| = 0, 1$ or $|J| > m$, which is impossible since $|\Delta_1^+ \setminus \Delta'| = n \leq m$. For $|J| = 1$ one has $\nu_J = \rho_\delta + \delta_i$, which is regular only for $i = 1$. This gives (9.5).

For $B(m, n)$, $m > n$ the root system Δ_ϵ (resp., Δ_δ) is of type B_m (resp., C_n). The element $\rho_\epsilon + (1 - |J|)\epsilon_1$ is regular if and only if $|J| = 0, 1$ or $|J| \geq 2m$, which is impossible since $|\Delta_1^+ \setminus \Delta'| = 2n < 2m$. For $|J| = 1$ one has $\nu_J = \rho_\delta \pm \delta_i$, which is regular only for $\nu_J = \rho_\delta + \delta_1$; this gives (9.5).

For $B(n, m)$, $m > n$ the root system Δ_ϵ (resp., Δ_δ) is of type C_m (resp., B_n). The element $\rho_\epsilon + (1 - |J|)\epsilon_1$ is regular if and only if $|J| = 0, 1$ or $|J| \geq 2m + 1$, which is impossible because $|\Delta_1^+ \setminus \Delta'| = 2n + 1 < 2m + 1$. Consider the case $|J| = 1$. Then $\nu_J = \rho_\delta$ if $J = \{\epsilon_1\}$ or $\nu_J = \rho_\delta \pm \delta_i$ for $J = \{\epsilon_1 \mp \delta_i\}$. Therefore λ_J is regular for $J = \{\epsilon_1\}, \{\epsilon_1 - \delta_1\}, \{\epsilon_1 + \delta_n\}$. One has $\lambda_{\{\epsilon_1\}} = s_{\delta_n}(\lambda_{\{\epsilon_1 + \delta_n\}})$ so $\mathcal{F}_W(e^{\lambda_{\{\epsilon_1\}}} + e^{\lambda_{\{\epsilon_1 + \delta_n\}}}) = 0$. This establishes (9.5).

For $D(n, m)$ the root system Δ_ϵ (resp., Δ_δ) is of type C_m (resp., D_n). The element $\rho_\epsilon + (1 - |J|)\epsilon_1$ is regular if and only if $|J| = 0, 1$ or $|J| \geq 2m + 1$, which is impossible since $|\Delta_1^+ \setminus \Delta'| = 2n < 2m + 1$. For $|J| = 1$ one has $\nu_J = \rho_\delta \pm \delta_i$ and this element is regular only if $\nu_J = \rho_\delta + \delta_1$; this gives (9.5).

For $D(m, n)$, $m > n$ the root system Δ_ϵ (resp., Δ_δ) is of type D_m (resp., C_n). The element $\rho_\epsilon + (1 - |J|)\epsilon_1$ is regular if and only if $|J| = 0, 1$ or $|J| \geq 2m - 1$, which is impossible since $|\Delta_1^+ \setminus \Delta'| = 2n < 2m - 1$. For $|J| = 1$ one has $\nu_J = \rho_\delta \pm \delta_i$, which is regular only for $\nu_J = \rho_\delta + \delta_1$; this gives (9.5).

Consider the case $D(m, m)$ with Δ_ϵ (resp., Δ_δ) of type D_m (resp., C_n). The element $\rho_\epsilon + (1 - |J|)\epsilon_1$ is regular if and only if $|J| = 0, 1$ or $|J| \geq 2m - 1$. As above, for $|J| = 1$ the element ν_J is regular only if $J = \{\epsilon_1 - \delta_1\}$, $\nu_J = \rho_\delta + \delta_1$. If $|J| = 2m$, then $J = \Delta_1^+ \setminus \Delta' = \{\epsilon_1 \pm \delta_i\}_{i=1}^m$ so $\lambda_J = \rho_0 + (1 - 2m)\epsilon_1 = s_{\epsilon_1} s_{\epsilon_m}(\rho_0 + \epsilon_1)$. If $|J| = 2m - 1$, then $J = (\Delta_1^+ \setminus \Delta') \setminus \{\beta\}$, where $\beta = \epsilon_1 \pm \delta_i$ and so $\nu_J = \rho_\delta \mp \delta_i$ which is regular only if $\nu_J = \rho_\delta + \delta_1$ and $\lambda_J = \rho_0 + (2 - 2m)\epsilon_1 + \delta_1 = s_{\epsilon_1} s_{\epsilon_m}(\rho_0 + \delta_1)$. Since $s_{\epsilon_1} s_{\epsilon_m} \in W$ we obtain

$$A = 2\mathcal{F}_W(e^{\rho_0 + \epsilon_1} - e^{\rho_0 + \delta_1}).$$

This establishes (9.5) for this case. Having established (9.5) in all cases, we have proven (9.2).

9.1.4. Let us deduce from (9.2) Kac-Wakimoto formula (9.1) for the natural representation. If $\tilde{\Pi}$ is a set of simple roots for \mathfrak{g} and $V(\Lambda)$ is an irreducible finite dimensional \mathfrak{g} -module of highest weight Λ , recall that we say that the pair $(\tilde{\Pi}, V(\Lambda))$ satisfies the KW condition if there are $\{\beta_1, \dots, \beta_{\text{atp}(V)}\} \subseteq \tilde{\Pi}$ with $(\beta_i, \beta_j) = 0$ for all i, j and $(\Lambda + \rho_{\tilde{\Pi}}, \beta_i) = 0$ for all i .

Assume that $(\tilde{\Pi}, V)$ satisfies the KW condition, where, as above, V is the natural representation of \mathfrak{g} . Since KW condition is obviously invariant under the action of the Weyl group we can consider the total order on $\{\epsilon_i\}_{i=1}^m \cup \{\delta_i\}_{i=1}^n$ corresponding to $\tilde{\Pi}$. We can also assume that $\epsilon_1 > \epsilon_2 > \dots > \epsilon_m$ and $\delta_1 > \delta_2 > \dots > \delta_n$ with $\epsilon_m, \delta_n \geq 0$ for $\mathfrak{g} \neq gl(m, n)$ (this means that $\tilde{\Pi}$ induces the standard triangular decomposition on \mathfrak{g}_0).

Let Λ be the highest weight of the standard representation. Clearly, Λ is the maximal element in $\{\epsilon_i\}_{i=1}^m \cup \{\delta_i\}_{i=1}^n$ so $\Lambda \in \{\epsilon_1, \delta_1\}$. For $gl(m, m)$, $D(m, m)$ we may (and will) assume that $\epsilon_1 > \delta_1$ (since we can switch $\{\epsilon_i\}$ and $\{\delta_i\}$ in subsection 9.1). Let us show that $\Lambda = \epsilon_1$ for $m > n$. Indeed, for $m > n$ one has $\text{atp}(V) = n$ and so $(\beta_i, \delta_1) \neq 0$ for some i . If $\Lambda = \delta_1$, then $(\Lambda + \rho, \beta_i) = (\delta_1, \beta_i) \neq 0$ which contradicts to the definition of β_i . Hence $\Lambda = \epsilon_1 > \delta_1$.

Let $\alpha \in \tilde{\Pi}$ be a root satisfying $(\alpha, \epsilon_1) = 1$. Notice that $(\rho, \alpha) \geq 0$ so $(\rho + \epsilon_1, \alpha) \neq 0$ and thus $\alpha \notin \{\beta_i\}_{i=1}^{\text{atp}(V)}$. The set $\tilde{\Pi}' = \tilde{\Pi} \setminus \{\alpha\}$ is a set of simple roots for \mathfrak{g}' . Since $\text{def } \mathfrak{g}' = \text{atp}(V)$, the set $\{e_i \widehat{d}_i \mid \beta_i = e_i - d_i\}$ is an arc diagram \tilde{X} for $\tilde{\Pi}'$. Similarly, let X be the arc diagram for Π' having the elements of Γ as arcs. Denominator

identity for \mathfrak{g}' gives, letting $\tilde{\rho} = \rho_{\tilde{\Pi}}$ and $\tilde{\rho}' = \rho_{\tilde{\Pi}'}$,

$$(9.6) \quad atp(V)! \cdot \check{\mathcal{F}}_{W'}\left(\frac{e^{\rho'}}{\prod_{\gamma \in \Gamma}(1 - e^{-\llbracket \gamma \rrbracket})}\right) = \check{\mathcal{F}}_{W'}\left(\frac{e^{\tilde{\rho}'}}{\prod_{i=1}^{atp(V)}(1 - e^{-\beta_i})}\right).$$

Let us show that $\rho - \rho' = \tilde{\rho} - \tilde{\rho}'$. Indeed, observe that $\rho - \rho', \tilde{\rho} - \tilde{\rho}'$ are orthogonal to Δ' . Since ϵ_1 is maximal the inequality $(\alpha, \epsilon_1) > 0$ for $\alpha \in \Delta$ forces $\alpha \in \Delta^+ \cap \tilde{\Delta}^+$. Therefore

$$\begin{aligned} 2(\rho - \rho', \epsilon_1) &= \sum_{\alpha \in \Delta_0^+ \setminus \Delta'} (\alpha, \epsilon_1) - \sum_{\alpha \in \Delta_1^+ \setminus \Delta'} (\alpha, \epsilon_1) = \sum_{\alpha \in \tilde{\Delta}_0^+ \setminus \Delta'} (\alpha, \epsilon_1) - \sum_{\alpha \in \tilde{\Delta}_1^+ \setminus \Delta'} (\alpha, \epsilon_1) \\ &= 2(\tilde{\rho} - \tilde{\rho}', \epsilon_1). \end{aligned}$$

Set $\xi = (\rho - \rho') - (\tilde{\rho} - \tilde{\rho}')$. We conclude that ξ is orthogonal to Δ' and ϵ_1 . For $\mathfrak{g} \neq gl(m, n)$ this means that $\rho - \rho' = \tilde{\rho} - \tilde{\rho}'$.

For $gl(m, n)$ we obtain that ξ is proportional to $\sum_{i=2}^m \epsilon_i - \sum_{j=1}^n \delta_j$. The roots $\pm(\epsilon_1 - \epsilon_2)$ are the only roots in $\Delta \setminus \Delta'$ which are not orthogonal to ϵ_2 . Since $\Delta_0^+ = \tilde{\Delta}_0^+$, one has

$$2(\tilde{\rho} - \tilde{\rho}', \epsilon_2) = (\epsilon_1 - \epsilon_2, \epsilon_2) = 2(\rho - \rho', \epsilon_2).$$

Hence $(\xi, \epsilon_2) = 0$ so $\xi = 0$ as required.

Substituting (9.6) in (9.2), we obtain,

$$\begin{aligned} \check{R}e^{\rho} schV &= j_V \check{\mathcal{F}}_W\left(\frac{e^{\rho + \epsilon_1}}{\prod_{\gamma \in \Gamma}(1 - e^{-\llbracket \gamma \rrbracket})}\right) = j_V \mathcal{F}_{W/W'}\left(e^{\rho - \rho' + \epsilon_1} \mathcal{F}_{W'}\left(\frac{e^{\rho'}}{\prod_{\gamma \in \Gamma}(1 - e^{-\llbracket \gamma \rrbracket})}\right)\right) \\ &= \frac{j_V}{atp(V)!} \mathcal{F}_{W/W'}\left(e^{\rho - \rho' + \epsilon_1} \mathcal{F}_{W'}\left(\frac{e^{\tilde{\rho}'}}{\prod_{i=1}^{atp(V)}(1 - e^{-\beta_i})}\right)\right) = \\ &= \frac{j_V}{atp(V)!} \mathcal{F}_W\left(\frac{e^{\tilde{\rho} + \epsilon_1}}{\prod_{i=1}^{atp(V)}(1 - e^{-\beta_i})}\right). \end{aligned}$$

This proves (9.1) with $b = \frac{j_V}{atp(V)!}$. Using (1.13), one can check that b has an expression depending only on $atp(V)$.

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