

A SHORT PROOF OF THE MULTILINEAR KAKEYA INEQUALITY

LARRY GUTH

ABSTRACT. We give a short proof of a slightly weaker version of the multilinear Kakeya inequality proven by Bennett, Carbery, and Tao.

The multilinear Kakeya inequality is a geometric estimate about the overlap pattern of cylindrical tubes in \mathbb{R}^n pointing in different directions. This estimate was first proven by Bennett, Carbery, and Tao in [BCT]. Recently it has had some striking applications in harmonic analysis. Here is a short list of some applications. In [BCT], it was applied to prove a multilinear restriction estimate. In [BG], Bourgain and the author used this multilinear restriction estimate to make some progress on the original restriction problem, posed by Stein in [S]. In [B], Bourgain used it to prove new estimates for eigenfunctions of the Laplacian on flat tori. Most recently, in [BD], Bourgain and Demeter used the multilinear restriction estimate to prove the l^2 decoupling conjecture. As a corollary of their main result, they proved essentially sharp Strichartz estimates for the Schrodinger equation on flat tori.

The goal of this paper is to give a short proof of the multilinear Kakeya inequality. The original proof of [BCT] used monotonicity properties of heat flow and it is morally based on multiscale analysis. Later there was a proof in [G] using the polynomial method. The proof we give here is based on multiscale analysis. I think that the underlying idea is the same as in [BCT], but the argument is organized in a more concise way.

Here is the statement of the multilinear Kakeya inequality. Suppose that $l_{j,a}$ are lines in \mathbb{R}^n , where $j = 1, \dots, n$, and where $a = 1, \dots, N_j$. We write $T_{j,a}$ for the characteristic function of the 1-neighborhood of $l_{j,a}$.

Theorem 1. *Suppose that $l_{j,a}$ are lines in \mathbb{R}^n , and that each line $l_{j,a}$ makes an angle of at most $(10n)^{-1}$ with the x_j -axis.*

Let Q_S denote any cube of side length S . Then for any $\epsilon > 0$ and any $S \geq 1$, the following integral inequality holds:

$$(1) \quad \int_{Q_S} \prod_{j=1}^n \left(\sum_{a=1}^{N_j} T_{j,a} \right)^{\frac{1}{n-1}} \leq C_\epsilon S^\epsilon \prod_{j=1}^n N_j^{\frac{1}{n-1}}.$$

Theorem 1 is slightly weaker than the estimates proven in [BCT] or [G]. The strongest form of the estimate was proven in [G]: equation 1 holds without the factor S^ϵ . However, Theorem 1 is exactly what [BCT] use to prove the multilinear restriction theorem, Theorem 1.16 in [BCT]. (See Proposition 2.1 and the following paragraph in [BCT].)

There are some papers in the harmonic analysis literature that use a geometric setup similar to the one that we use here, such as [BB] and [BHT]. There are also some related ideas in recent work by Cs3rnyei and Jones [CJ] in geometric measure theory, connected with the tangent spaces

of nullsets and the sets of non-differentiability of Lipschitz functions. I heard Marianna Csörnyei give a talk about their work, and their approach helped suggest the argument in this paper.

1. THE PROOF OF THE MULTILINEAR KAKEYA INEQUALITY

In this section, we prove Theorem 1.

1.1. Reduction to nearly axis parallel tubes. The first observation of [BCT] is that it suffices to prove this type of estimate when the angle $(10n)^{-1}$ is replaced by an extremely small angle δ . More precisely, Theorem 1 follows from

Theorem 2. *For every $\epsilon > 0$, there is some $\delta > 0$ so that the following holds.*

Suppose that $l_{j,a}$ are lines in \mathbb{R}^n , and that each line $l_{j,a}$ makes an angle of at most δ with the x_j -axis. Then for any $S \geq 1$ and any cube Q_S of side length S , the following integral inequality holds:

$$(2) \quad \int_{Q_S} \prod_{j=1}^n \left(\sum_{a=1}^{N_j} T_{j,a} \right)^{\frac{1}{n-1}} \lesssim_{\epsilon} S^{\epsilon} \prod_{j=1}^n N_j^{\frac{1}{n-1}}.$$

We explain how Theorem 2 implies Theorem 1. Let e_j be the unit vector in the x_j direction, and let $S_j \subset S^{n-1}$ be a spherical cap around the point e_j of radius $(10n)^{-1}$. By the hypotheses of Theorem 1, every line in $l_{j,a}$ has direction in the cap S_j .

Given $\epsilon > 0$, we choose $\delta > 0$ as in Theorem 2.

We subdivide the cap S_j into smaller caps $S_{j,\beta}$ of radius $\delta/10$. The number of caps $S_{j,\beta}$ is at most $\text{Poly}(\delta^{-1}) \lesssim_{\epsilon} 1$. We can break the left-hand side of Equation 2 into contributions from different caps $S_{j,\beta}$. We write $l_{j,a} \in S_{j,\beta}$ if the direction of $l_{j,a}$ lies in $S_{j,\beta}$. Since the number of caps is $\lesssim_{\epsilon} 1$, we get:

$$(3) \quad \int_{Q_S} \prod_{j=1}^n \left(\sum_{a=1}^{N_j} T_{j,a} \right)^{\frac{1}{n-1}} \lesssim_{\epsilon} \sum_{\beta_1, \dots, \beta_n} \int_{Q_S} \prod_{j=1}^n \left(\sum_{l_{j,a} \in S_{j,\beta_j}} T_{j,a} \right)^{\frac{1}{n-1}}.$$

We claim that each term on the right hand side of Equation 3 is controlled by Theorem 2.

If we choose β_j so that S_{j,β_j} contains e_j , then Theorem 2 directly applies. If not, we have to make a linear change of coordinates, mapping the center of S_{j,β_j} to e_j . The condition that the angle between $l_{j,a}$ and e_j is at most $(10n)^{-1}$ guarantees that this linear change of coordinates distorts lengths by at most a factor of 2 and distorts volumes by at most a factor of 2^n . In the new coordinates, the integral is controlled by Theorem 2.

1.2. The axis parallel case (Loomis-Whitney). We have just seen that the multilinear Kakeya inequality reduces to a nearly axis-parallel case. Next we consider the exactly axis-parallel case: the case that $l_{j,a}$ is parallel to the x_j -axis. In this axis-parallel case, the multilinear Kakeya inequality follows immediately from the Loomis-Whitney inequality, proven in [LW]. To state their result, we need a little notation.

Let $\pi_j : \mathbb{R}^n \rightarrow \mathbb{R}^{n-1}$ be the linear map that forgets the j^{th} coordinate:

$$\pi_j(x_1, \dots, x_n) = (x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_n).$$

Theorem 3. (*Loomis-Whitney*) *Suppose that $f_j : \mathbb{R}^{n-1} \rightarrow \mathbb{R}$ are (measurable) functions. Then the following integral inequality holds:*

$$\int_{\mathbb{R}^n} \prod_{j=1}^n f_j(\pi_j(x))^{\frac{1}{n-1}} \leq \prod_{j=1}^n \|f_j\|_{L^1(\mathbb{R}^{n-1})}^{\frac{1}{n-1}}.$$

If the line $l_{j,a}$ is parallel to the x_j -axis, then it can be defined by writing $\pi_j(x) = y_a$ for some $y_a \in \mathbb{R}^{n-1}$. Then the function $\sum_a T_{j,a}(x)$ is equal to $\sum_a \chi_{B(y_a,1)}(\pi_j(x))$. We apply the Loomis-Whitney inequality with $f_j = \sum_a \chi_{B(y_a,1)}$, which gives $\|f_j\|_{L^1(\mathbb{R}^{n-1})} = \omega_{n-1} N_j \sim N_j$. We get:

$$\int_{\mathbb{R}^n} \prod_{j=1}^n \left(\sum_{a=1}^{N_j} T_{j,a} \right)^{\frac{1}{n-1}} = \int_{\mathbb{R}^n} \prod_{j=1}^n f_j(\pi_j(x))^{\frac{1}{n-1}} \leq \prod_{j=1}^n \|f_j\|_{L^1(\mathbb{R}^{n-1})} \sim \prod_{j=1}^n N_j^{\frac{1}{n-1}}.$$

1.3. The multiscale argument. In this subsection, we will prove Theorem 2 - the multilinear Kakeya inequality for tubes that make a tiny angle with the coordinate axes. For any $\epsilon > 0$, we get to choose $\delta > 0$, and we know that each line $l_{j,a}$ makes an angle of at most δ with the x_j -axis. We have seen how to prove the inequality for lines that are exactly parallel to the axes. We just have to understand how to control the effect of a tiny tilt in the tubes.

The main idea of the argument is that instead of trying to directly perform an estimate at scale S , we work at a sequence of scales $\delta^{-1}, \delta^{-2}, \delta^{-3}$, etc. up to an arbitrary scale S . To get from each scale to the next scale, we use the Loomis-Whitney inequality.

To set up our multiscale argument, we use not just tubes of radius 1 but tubes of multiple scales. We define

$T_{j,a,W} :=$ the characteristic function of the W -neighborhood of $l_{j,a}$.

$$f_{j,W} := \sum_{a=1}^{N_j} T_{j,a,W}.$$

The key step in the argument is the following lemma relating one scale to a scale δ^{-1} times larger:

Lemma 4. *Suppose that $l_{j,a}$ are lines with angle at most δ from the x_j axis. Let $T_{j,a,W}$ and $f_{j,W}$ be as above.*

If $S \geq \delta^{-1}W$, and if Q_S is any cube of sidelength S , then

$$\int_{Q_S} \prod_{j=1}^n f_{j,W}^{\frac{1}{n-1}} \leq C_n \delta^n \int_{Q_S} \prod_{j=1}^n f_{j,\delta^{-1}W}^{\frac{1}{n-1}}.$$

Proof. We divide Q_S into subcubes Q of side length between $\frac{1}{20n}\delta^{-1}W$ and $\frac{1}{10n}\delta^{-1}W$. For each such cube Q , it suffices to prove that

$$\int_Q \prod_{j=1}^n f_{j,W}^{\frac{1}{n-1}} \leq C_n \delta^n \int_Q \prod_{j=1}^n f_{j,\delta^{-1}W}^{\frac{1}{n-1}}.$$

Because the side length of Q is $\leq (1/10n)\delta^{-1}W$, the intersection of $T_{j,a,W}$ with Q looks fairly similar to an axis-parallel tube. More precisely, for each j, a there is an axis-parallel tube $\tilde{T}_{j,a,2W}$, of radius $2W$, so that for $x \in Q$, $T_{j,a,W}(x) \leq \tilde{T}_{j,a,2W}(x)$. Therefore,

$$\int_Q \prod_{j=1}^n f_{j,W}^{\frac{1}{n-1}} = \int_Q \prod_{j=1}^n \left(\sum_a T_{j,a,W} \right)^{\frac{1}{n-1}} \leq \int_Q \prod_{j=1}^n \left(\sum_a \tilde{T}_{j,a,2W} \right)^{\frac{1}{n-1}}.$$

This last integral involves axis-parallel tubes and we can bound it using the Loomis-Whitney inequality, as in Subsection 1.2. Let $N_j(Q)$ be the number of tubes $T_{j,a,W}$ that intersect Q . By Loomis-Whitney, we get

$$\int_Q \prod_{j=1}^n \left(\sum_a \tilde{T}_{j,a,2W} \right)^{\frac{1}{n-1}} \leq C_n W^n \prod_{j=1}^n N_j(Q)^{\frac{1}{n-1}}.$$

Since the sidelength of Q is at most $(1/10n)\delta^{-1}W$, the diameter of Q is at most $(1/10)\delta^{-1}W$. If $T_{j,a,W}$ intersects Q , then $T_{j,a,\delta^{-1}W}$ is identically 1 on Q . Therefore,

$$C_n W^n \prod_{j=1}^n N_j(Q)^{\frac{1}{n-1}} \leq C_n W^n \oint_Q \prod_{j=1}^n \left(\sum_a T_{j,a,\delta^{-1}W} \right)^{\frac{1}{n-1}} = C_n \delta^n \int_Q \prod_{j=1}^n \left(\sum_a T_{j,a,\delta^{-1}W} \right)^{\frac{1}{n-1}}.$$

This finishes the proof of the lemma. \square

Using Lemma 4, we can now prove Theorem 2. Suppose first that S is a power $S = \delta^{-M}$, for an integer $M \geq 0$. Using Lemma 4 repeatedly we get:

$$\begin{aligned} \int_{Q_S} \prod_{j=1}^n \left(\sum_a T_{j,a} \right)^{\frac{1}{n-1}} &= \int_{Q_S} \prod_{j=1}^n f_{j,1}^{\frac{1}{n-1}} \leq C_n \delta^n \int_{Q_S} \prod_{j=1}^n f_{j,\delta^{-1}}^{\frac{1}{n-1}} \leq \\ &\leq C_n^2 \delta^{2n} \int_{Q_S} \prod_{j=1}^n f_{j,\delta^{-2}}^{\frac{1}{n-1}} \leq \dots \leq \\ &\leq C_n^M \delta^{Mn} \int_{Q_S} \prod_{j=1}^n f_{j,\delta^{-M}}^{\frac{1}{n-1}} = C_n^M \oint_{Q_S} \prod_{j=1}^n f_{j,\delta^{-M}}^{\frac{1}{n-1}}. \end{aligned}$$

We know that $f_{j,\delta^{-M}} \leq N_j$, and so we get

$$\int_{Q_S} \prod_{j=1}^n \left(\sum_a T_{j,a} \right)^{\frac{1}{n-1}} \leq C_n^M \prod_{j=1}^n N_j^{\frac{1}{n-1}}.$$

Since $S = \delta^{-M}$, we see that $M = \frac{\log S}{\log \delta^{-1}}$. Therefore,

$$C_n^M = S^{\frac{C_n}{\log \delta^{-1}}}.$$

Now we choose $\delta > 0$ sufficiently small so that $\frac{C_n}{\log \delta^{-1}} \leq \epsilon$. For $S = \delta^{-M}$, we have now proven that

$$\int_{Q_S} \prod_{j=1}^n \left(\sum_a T_{j,a} \right)^{\frac{1}{n-1}} \leq S^\epsilon \prod_{j=1}^n N_j^{\frac{1}{n-1}}.$$

Finally, for an arbitrary $S \geq 1$, we can find an integer $M \geq 0$ and cover Q_S with $C(\delta)$ cubes of side length δ^{-M} . Therefore, for any $S \geq 1$, we see

$$\int_{Q_S} \prod_{j=1}^n \left(\sum_a T_{j,a} \right)^{\frac{1}{n-1}} \leq C_\epsilon S^\epsilon \prod_{j=1}^n N_j^{\frac{1}{n-1}}.$$

This finishes the proof of Theorem 2 and hence the proof of Theorem 1.

2. SOME SMALL GENERALIZATIONS

In this section, we mention two minor generalizations of Theorem 1 that can be useful in applications.

One minor generalization is to add weights.

Corollary 5. *Suppose that $l_{j,a}$ are lines in \mathbb{R}^n , and that each line $l_{j,a}$ makes an angle of at most $(10n)^{-1}$ with the x_j -axis. Suppose that $w_{j,a} \geq 0$ are numbers. Let $T_{j,a}$ be the characteristic function of the 1-neighborhood of $l_{j,a}$. Define*

$$f_j := \sum_a w_{j,a} T_{j,a}.$$

Let Q_S denote any cube of side length S . Then for any $\epsilon > 0$ and any $S \geq 1$, the following integral inequality holds:

$$(4) \quad \int_{Q_S} \prod_{j=1}^n f_j^{\frac{1}{n-1}} \leq C_\epsilon S^\epsilon \prod_{j=1}^n \left(\sum_a w_{j,a} \right)^{\frac{1}{n-1}}.$$

Proof. If the weights $w_{j,a}$ are positive integers, the result follows from Theorem 1 by including each tube multiple times. (The tubes $T_{j,a}$ in Theorem 1 do not need to be distinct.) Then by scaling the theorem holds for rational weights and by continuity for real weights. \square

In Theorem 1, we assumed that $l_{j,a}$ makes an angle less than $(10n)^{-1}$ with the x_j -axis. This was simple to state, but it is not the most general condition we could make about the angles of the lines $l_{j,a}$. Here is a more general setup that can be useful in applications.

Corollary 6. *Suppose that $S_j \subset S^{n-1}$. Suppose that $l_{j,a}$ are lines in \mathbb{R}^n and that the direction of $l_{j,a}$ lies in S_j . Suppose that for any vectors $v_j \in S_j$,*

$$|v_1 \wedge \dots \wedge v_n| \geq \nu.$$

Let $T_{j,a}$ be the characteristic function of the 1-neighborhood of $l_{j,a}$.

Let Q_S denote any cube of side length S . Then for any $\epsilon > 0$ and any $S \geq 1$, the following integral inequality holds:

$$(5) \quad \int_{Q_S} \prod_{j=1}^n \left(\sum_{a=1}^{N_j} T_{j,a} \right)^{\frac{1}{n-1}} \leq C(\epsilon) \text{Poly}(\nu^{-1}) S^\epsilon \prod_{j=1}^n N_j^{\frac{1}{n-1}}.$$

Proof. This estimate follows from Theorem 2 by essentially the same argument as in subsection 1.1. We again cover S_j by caps $S_{j,\beta}$ of a small radius ρ . As long as $\rho \leq \frac{1}{100n}\nu$, we can guarantee that $|v_1 \wedge \dots \wedge v_n| \geq \nu/2$ for all $v_j \in S_{j,\beta}$. We pick a sequence of caps $S_{1,\beta_1}, \dots, S_{n,\beta_n}$. We change coordinates so that the center of the cap S_{j,β_j} is mapped to the coordinate unit vector e_j . The distortion of lengths and volumes caused by the coordinate change is $\text{Poly}(\nu^{-1})$. We apply Theorem 2 in the new coordinates. If $\rho = \rho(\epsilon)$ is small enough, the image of $S_{j,\beta}$ is contained in a cap of radius $\delta = \delta(\epsilon)$ as in Theorem 2 – and this gives the desired estimate with error factor $C_\epsilon \text{Poly}(\nu^{-1}) S^\epsilon$. Finally, we have to sum over $C(\epsilon) \text{Poly}(\nu^{-1})$ different choices of $S_{1,\beta_1} \dots S_{n,\beta_n}$. \square

This is not the sharpest known estimate when it comes to the dependence on ν . See [BCT] or [G] for sharper estimates. However, in the applications I know, this estimate is sufficient.

3. ON LIPSCHITZ CURVES

The proof of Theorem 2 applies not just to straight lines, but also to Lipschitz curves with Lipschitz constant at most δ .

Suppose that $g_{j,a} : \mathbb{R} \rightarrow \mathbb{R}^{n-1}$ is a Lipschitz function with Lipschitz constant at most δ . We let $\gamma_{j,a}$ be the graph given by $(x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_n) = g_{j,a}(x_j)$. The curves $\gamma_{j,a}$ will play the role of the lines $l_{j,a}$ – lines are the special case that the functions $g_{j,a}$ are affine. We let $T_{j,a}$ be the characteristic function of the 1-neighborhood of $\gamma_{j,a}$. With this setup, Theorem 2 generalizes to Lipschitz curves with the same proof.

Theorem 7. *For every $\epsilon > 0$, there is some $\delta > 0$ so that the following holds.*

Suppose that $\gamma_{j,a}$ are as above: Lipschitz curves in \mathbb{R}^n with an angle of at most δ with the x_j -axis. Then for any $S \geq 1$ and any cube Q_S of side length S , the following integral inequality holds:

$$(6) \quad \int_{Q_S} \prod_{j=1}^n \left(\sum_{a=1}^{N_j} T_{j,a} \right)^{\frac{1}{n-1}} \lesssim_\epsilon S^\epsilon \prod_{j=1}^n N_j^{\frac{1}{n-1}}.$$

This multilinear estimate for Lipschitz curves has a similar flavor to some estimates of Csörnyei and Jones described in [CJ].

REFERENCES

- [B] J. Bourgain, Moment inequalities for trigonometric polynomials with spectrum in curved hypersurfaces. Israel J. Math. 193 (2013), no. 1, 441-458.
- [BB] J. Bennett and N. Bez, Some nonlinear Brascamp-Lieb inequalities and applications to harmonic analysis. J. Funct. Anal. 259 (2010), no. 10, 2520-2556.
- [BHT] I. Bejenaru, S. Herr, and D. Tataru, A convolution estimate for two-dimensional hypersurfaces. Rev. Mat. Iberoam. 26 (2010), no. 2, 707-728.
- [BD] J. Bourgain and C. Demeter, The proof of the l^2 decoupling conjecture, arXiv:1403.5335.
- [BG] J. Bourgain, L. Guth, Bounds on oscillatory integral operators based on multilinear estimates. Geom. Funct. Anal. 21 (2011), no. 6, 1239-1295.

- [BCT] J. Bennett, A. Carbery, and T. Tao, On the multilinear restriction and Keakeya conjectures. *Acta Math.* 196 (2006), no. 2, 261-302.
- [CJ] M. Csörnyei and P. Jones, Product Formulas for Measures and Applications to Analysis and Geometry. URL: www.math.sunysb.edu/Videos/dfest/PDFs/38-Jones.pdf
- [G] L. Guth, The endpoint case of the Bennett-Carbery-Tao multilinear Keakeya conjecture. *Acta Math.* 205 (2010), no. 2, 263-286.
- [LW] L. Loomis and H. Whitney, An inequality related to the isoperimetric inequality. *Bull. Amer. Math. Soc* 55, (1949). 961-962.
- [S] E. Stein, Some problems in harmonic analysis. *Harmonic analysis in Euclidean spaces (Proc. Sympos. Pure Math., Williams Coll., Williamstown, Mass., 1978), Part 1*, pp. 3-20, *Proc. Sympos. Pure Math., XXXV, Part*, Amer. Math. Soc., Providence, R.I., 1979.