POISSON TRACES FOR SYMMETRIC POWERS OF SYMPLECTIC VARIETIES

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Abstract. We compute the space of Poisson traces on symmetric powers of affine symplectic varieties. In the case of symplectic vector spaces, we also consider the quotient by the diagonal translation action, which includes the quotient singularities $T^*\mathbb{C}^{n-1}/S_n$ associated to the type A Weyl group S_n and its reflection representation $\mathbb{C}^{n-1}$. We also compute the full structure of the natural $\mathcal{D}$-module, previously defined by the authors, whose solution space over algebraic distributions identifies with the space of Poisson traces. As a consequence, we deduce bounds on the numbers of finite-dimensional irreducible representations and prime ideals of quantizations of these varieties. Finally, motivated by these results, we pose conjectures on symplectic resolutions, and give related examples of the natural $\mathcal{D}$-module. In an appendix, the second author computes the Poisson traces and associated $\mathcal{D}$-module for the quotients $T^*\mathbb{C}^n/D_n$ associated to type D Weyl groups. In a second appendix, the same author provides a direct proof of one of the main theorems.

Contents

1. Introduction

1.1. **Poisson traces for S^nY and type A Weyl group quotient singularities.** Given a Poisson algebra A over $\mathbb{C}$, a *Poisson trace* is a functional $A \to \mathbb{C}$ which annihilates $\{A, A\}$. These may also be viewed as functionals on the associated Poisson variety $\mathsf{Spec}\, A$ which are invariant under Hamiltonian flows. The space of such traces is the dual, $\mathsf{HP}_0(A)^*$, to the *zeroth Poisson homology*, $\mathsf{HP}_0(A) := A/\{A, A\}$, which also coincides with the zeroth Lie homology.

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Given an affine variety $Y = \mathsf{Spec}\, A$, we use the notation $\mathcal{O}_Y := A$. Let $S^n Y := Y^n/S_n = \mathsf{Spec}\,\mathsf{Sym}^n A$ be the n-th symmetric power of Y. By a symplectic variety, we always mean a smooth symplectic variety. Let the symbol & denote the tensor product in the symmetric algebra.

Our first main result is:

Theorem 1.1.1. Let Y be an affine symplectic variety. Then, there is a canonical isomorphism of graded algebras,

$$\mathsf{Sym}(\mathsf{HP}_0(\mathcal{O}_Y)^*[t]) \xrightarrow{\sim} \bigoplus_{n\geq 0} \mathsf{HP}_0(\mathcal{O}_{S^n Y})^*, \tag{1.1.2}$$

$$\phi \cdot t^{m-1} \mapsto \Big((f_1 \& \cdots \& f_m) \mapsto \phi(f_1 \cdots f_m)\Big),$$

wherein the grading is given by $|\mathsf{HP}_0(\mathcal{O}_{S^n Y})^*| = n$ (on both sides of the isomorphism), and $|t| = 1$.

This will be proved in §2. It is well known that, if Y is connected, then $\mathsf{HP}_0(\mathcal{O}_Y) \cong H^{\dim Y}(Y)$, the top cohomology of Y, via the isomorphism $[f] \mapsto f \cdot \mathrm{vol}_Y$, where vol_Y is the canonical volume form (i.e., the $\frac{1}{2}\dim Y$-th exterior power of the symplectic form). We can write the above more explicitly using the coefficients $a_n(i)$ which give the number of i-multipartitions of n (i.e., collections of i ordered partitions whose sum of sizes is n), i.e.,

$$\prod_{m\geq 1} \frac{1}{(1-t^m)^i} = \sum_{n\geq 0} a_n(i) \cdot t^n. \tag{1.1.3}$$

Corollary 1.1.4. If Y is connected, then $\dim \mathsf{HP}_0(\mathcal{O}_{S^n Y}) = a_n(\dim H^{\dim Y}(Y))$.

We may derive a relationship with Hochschild homology of a quantization, as follows. Given an associative algebra B, recall that the zeroth Hochschild homology is $\mathsf{HH}_0(B) := B/[B,B]$. Recall also that a (formal) deformation quantization of a Poisson algebra $\mathcal{O}_X$ is an associative algebra $(A_\hbar, \star)$ over $\mathbb{C}[\![\hbar]\!]$ which is isomorphic to $\mathcal{O}_X[\![\hbar]\!] := \{\sum_{m\geq 0} f_m \cdot \hbar^m \mid f_m \in \mathcal{O}_X\}$ as a $\mathbb{C}[\![\hbar]\!]$-module, and such that, for $a, b \in \mathcal{O}_X$, the deformed multiplication satisfies $a \star b = ab + O(\hbar)$ and $a \star b - b \star a = \hbar\{a,b\} + O(\hbar^2)$. Here $f = g + O(\hbar^i)$ means that $f - g \in \hbar^i A_\hbar$. Note that $A_\hbar$ and $A_\hbar[\hbar^{-1}] \supset A_\hbar$ are filtered by powers of $\hbar$: $F^i A_\hbar[\hbar^{-1}] = \hbar^i A_\hbar$. Hence, $\mathsf{HH}_0(A_\hbar[\hbar^{-1}])$ inherits the filtration.

Corollary 1.1.5. Let Y be an affine symplectic variety, and $A_\hbar$ be any deformation quantization of $\mathcal{O}_Y$ (so $\mathsf{Sym}^n A_\hbar$ is a deformation quantization of $\mathcal{O}_{S^n Y}$). Then, the natural surjection $\mathsf{HP}_0(\mathcal{O}_{S^n Y})(\!(\hbar)\!) \twoheadrightarrow \mathsf{gr}\,\mathsf{HH}_0(\mathsf{Sym}^n A_\hbar[\hbar^{-1}])$ is an isomorphism.

The above corollary uses the computation of $\mathsf{HH}_0(\mathsf{Sym}^n A_\hbar[\hbar^{-1}])$ from [EO06]. For arbitrary quantizations of $\mathsf{Sym}^n \mathcal{O}_Y$ (not necessarily of the form $\mathsf{Sym}^n A_\hbar$), we can still deduce

Corollary 1.1.6. Let Y be a connected affine symplectic variety and $B_\hbar$ be an arbitrary deformation quantization of $\mathcal{O}_{S^n Y}$. Then, $\dim_{\mathbb{C}(\!(\hbar)\!)} \mathsf{HH}_0(B_\hbar[\hbar^{-1}]) \leq a_n(\dim H^{\dim Y}(Y))$.

Example 1.1.7. In the case that Y is a connected smooth surface and $H^1(Y) = 0$, if $A_\hbar$ is any quantization of $\mathcal{O}_Y$, then [EO06, §6] constructs the universal formal deformation of $A_\hbar[\hbar^{-1}]^{\otimes n} \rtimes S_n$, called $A_\hbar[\hbar^{-1}](n,c,k)$, over $\mathbb{C}(\!(\hbar)\!)[\![H^2(Y) \oplus \mathbb{C}]\!]$, i.e., $\mathbb{C}(\!(\hbar)\!)$-valued functions on the formal neighborhood of zero in $H^2(Y) \oplus \mathbb{C}$. Here, $c := (c_1, \ldots, c_{\dim H^2(Y)})$ and k denote bases for linear functions on $H^2(Y)$ and $\mathbb{C}$, respectively, so we write $\mathbb{C}(\!(\hbar)\!)[\![H^2(Y) \oplus \mathbb{C}]\!] = \mathbb{C}(\!(\hbar)\!)[\![c,k]\!]$. The deformation $A_\hbar[\hbar^{-1}](n,c,k)$ is topologically free over $\mathbb{C}(\!(\hbar)\!)[\![c,k]\!]$, i.e., isomorphic to $A(\!(\hbar)\!)[\![c,k]\!]$ as a $\mathbb{C}(\hbar))[\![c,k]\!]$-module.

Suppose in addition that there exists an integral form of this algebra, i.e., a $\mathbb{C}[\![\hbar]\!][\![c,k]\!]$-subalgebra $A_\hbar(n,c,k) \subseteq A_\hbar[\hbar^{-1}](n,c,k)$ satisfying

(a) $A_\hbar(n,c,k)$ is topologically free over $\mathbb{C}[\![\hbar]\!][\![c,k]\!]$, and
(b) $A_\hbar(n,c,k)[\hbar^{-1}] \cong A_\hbar[\hbar^{-1}](n,c,k)$ as algebras over $\mathbb{C}(\!(\hbar)\!)[\![c,k]\!]$.

Such an $A_\hbar(n,c,k)$ exists in all examples we know. Then, the subalgebra $eA_\hbar(n,\hbar c,\hbar k)e$, where $e = \frac{1}{n!}\sum_{\sigma\in S_n}\sigma \in \mathbb{C}[S_n]$ is the symmetrizer, is a quantization of $\mathcal{O}_{S^nY}[\![c,k]\!]$, and the above corollary applies to show that $\mathsf{HH}_0(eA_\hbar(n,\hbar c,\hbar k)[\hbar^{-1}]e)$ is generated by at most $a_n(\dim H^2(Y))$ elements. In particular, if one specializes at any values of c and k, then one obtains a deformation quantization of A and the dimension of the resulting zeroth Hochschild homology as a vector space over $\mathbb{C}(\!(\hbar)\!)$ is at most $a_n(\dim H^2(Y))$.

Note that this is essentially a global version of the Cherednik algebra associated to S_n: when one replaces Y by $\mathbb{C}^2$, one can recover the Cherednik algebra associated to the Weyl group S_n from the above (more precisely, one recovers the usual Cherednik algebra tensored by $A_\hbar$, since the Cherednik algebra itself involves deforming $A_\hbar^{\otimes n-1} \rtimes S_n$, corresponding to the reflection representation of S_n). One can conjecture that, parallel to Corollary 1.1.14, in fact $\mathsf{HH}_*(eA_\hbar[\hbar^{-1}](n,c,k)e) \cong \mathsf{HH}_*(\mathsf{Sym}^n A_\hbar[\hbar^{-1}])[\![c,k]\!] \cong H^{2n-*}(\mathrm{Hilb}^n Y)(\!(\hbar)\!)[\![c,k]\!]$, where $\mathrm{Hilb}^n Y$ is the Hilbert scheme of the surface Y; see also §1.3 below. The second isomorphism here follows by comparing [EO06, Corollary 3.3] with the known cohomology of the Hilbert scheme, because $A_\hbar[\hbar^{-1}]$ is an infinite-dimensional simple algebra with trivial center.

Given an affine Poisson variety X such that $\mathcal{O}_X$ is nonnegatively graded and equipped with a Poisson bracket of degree $-d$, one defines a filtered quantization to be a filtered algebra B over $\mathbb{C}$ such that $\mathsf{gr}\, B = \mathcal{O}_X$, $[B_{\leq i}, B_{\leq j}] \subseteq B_{\leq i+j-d}$, and for $a \in B_{\leq i}$ and $b \in B_{\leq j}$, $\{\mathsf{gr}_i\, a, \mathsf{gr}_j\, b\} = \mathsf{gr}_{i+j-d}[a,b]$.

In the case $X = V$ is a symplectic vector space, the standard quantization is given as follows: Write $V = U \oplus U^*$ where U and U^* are complementary Lagrangians. Let $x_1,\ldots,x_n$ be a basis of $U^* \subseteq \mathcal{O}_U$ and $\partial_1,\ldots,\partial_n \in U$ be the dual basis. Then, the standard quantization is the ring of differential operators $\mathcal{D}_U$ filtered by the Bernstein filtration,[1] where $(\mathcal{D}_U)_{\leq k}$ is spanned by elements of the form $x_{i_1}\cdots x_{i_j}\partial_{i_{j+1}}\cdots\partial_{i_\ell}$, for $\ell \leq k$. In other words, this is the filtration generated by $|x_i| = |\frac{\partial}{\partial x_i}| = 1$. Then, $\mathsf{gr}\,\mathcal{D}_U \cong \mathcal{O}_V$ (with $d=2$). (The algebra $\mathcal{D}_U$ is also known as the Weyl algebra of V.) Similarly, one could consider the deformation quantization $\mathcal{D}_{U,\hbar}$, which as a $\mathbb{C}[\![\hbar]\!]$-module is isomorphic to $\mathcal{D}_U[\![\hbar]\!]$, but with the commutation relations multiplied by $\hbar$: namely, $\mathcal{D}_{U,\hbar}$ is generated by x_i and p_i with relations $[p_i, x_j] = \hbar\delta_{ij}$ (one can think of p_i as $\hbar\frac{\partial}{\partial x_i}$). In the case $X = V/G$, then $\mathcal{D}_U^G$ and $\mathcal{D}_{U,\hbar}^G$ are filtered and deformation quantizations of $\mathcal{O}_V^G = \mathcal{O}_{V/G}$.

Similarly to the preceding theorem, we can consider quantizations of $X = S^{n+1}V$, for V a symplectic vector space. In this case, we have a decomposition $S^{n+1}V = V^n/S_{n+1} \times V$, where the second factor is the diagonally embedded V, and the S_{n+1} action on V^n is by the identification $V^n \cong (\mathbb{C}^n \otimes V)$, where $\mathbb{C}^n$ is the reflection representation and V is a trivial representation. So, $\mathsf{HP}_0(\mathcal{O}_{S^{n+1}V}) = 0$, since $\mathsf{HP}_0(\mathcal{O}_V) = 0$. On the other hand:

Theorem 1.1.8. $\mathsf{HP}_0(\mathcal{O}_{V^n/S_{n+1}})^* \cong \mathbb{C}$, spanned by the augmentation map $\mathcal{O}_{V^n} \to \mathbb{C}$.

As we will see, Theorem 1.1.1 reduces, in a sense, to the above theorem, using $\mathcal{D}$-modules to localize the problem. An elementary proof of the above theorem, that does not require anything in the main body of the paper, is provided in Appendix B.

Remark 1.1.9. In [RS10], building on seminal work of Mathieu [Mat95], the second author computes more generally some of the structure of $\mathsf{HP}_0(\mathcal{O}_{V^n/S_{n+1}}, \mathcal{O}_{V^n}) := \mathcal{O}_{V^n}/\{\mathcal{O}_{V^n}^{S_{n+1}}, \mathcal{O}_{V^n}\}$. This is an S_{n+1}-representation whose invariants are $\mathsf{HP}_0(\mathcal{O}_{V^n/S_{n+1}})$. In [RS10], the argument used here is

[1]Since our groups G will be of the form $G < \mathsf{GL}(U) < \mathsf{Sp}(V)$, one could alternatively use the order filtration and change the grading on $\mathcal{O}_V$ accordingly; this would have the effect of halving the degrees appearing in HP_0 and HH_0.

generalized to show, among other things, that the isotypic part of $\mathsf{HP}_0(\mathcal{O}_{V^n/S_{n+1}}, \mathcal{O}_{V^n})$ corresponding to Young diagrams with at most $\dim V + 1$ boxes below the top row coincide with the same isotypic part of the subspace of the free Poisson algebra on n variables $z_1, \ldots, z_n$ which has degree one in each variable, with S_{n+1} action by the reflection representation, and with grading given by twice the number of pairs of brackets $\{-,-\}$ which appear. Then, Theorem 1.1.8 above follows from the fact that the only multilinear Poisson polynomial in $z_1, \ldots, z_n$ which is symmetric in all the variables is the product $z_1 \cdots z_n$. The result of *op. cit.* also implies that the reflection representation $\mathfrak{h}$ of S_{n+1} does not occur, and that the isotypic component of $\wedge^2\mathfrak{h}$ occurs with multiplicity $\lfloor \frac{n}{2} \rfloor$. In terms of affine symplectic varieties Y, these results translate into information about the structure of $\mathsf{HP}_0(\mathcal{O}_{\mathsf{Sym}^n Y}, \mathcal{O}_{Y^n})$ as an S_n-representation; see Remark 1.2.3.

In fact, in §2, we will deduce the above two theorems from a more general result (Theorem 1.2.1) on the $\mathcal{D}$-module $M(X)$ from [ES10a], which is essentially the quotient of $\mathcal{D}_{S^nY}$ by the right ideal generated by Hamiltonian vector fields. See §1.2 for the statements.

Corollary 1.1.10. Let $U \subseteq V$ be a Lagrangian subspace. Then, the natural surjection $\mathsf{HP}_0(\mathcal{O}_{V^n}^{S_{n+1}}) \twoheadrightarrow \mathsf{gr}\,\mathsf{HH}_0(\mathcal{D}_{U^n}^{S_{n+1}})$ is an isomorphism, and both are isomorphic to $\mathbb{C}$.

Remark 1.1.11. As a consequence, if B is any filtered quantization of $\mathcal{O}_{V^n}^{S_{n+1}}$, then, $\dim \mathsf{HH}_0(B) \leq 1$, and hence B admits at most one finite-dimensional irreducible representation. However, when $\dim V > 2$, we do not know if filtered quantizations not isomorphic to $\mathsf{Weyl}(V^n)^{S_{n+1}}$ exist, and for the latter the zeroth Hochschild homology was already computed in [AFLS00] (we discuss the case $\dim V = 2$ below).

In the case $V = \mathbb{C}^2$, then $V^n = \mathfrak{h} \oplus \mathfrak{h}^*$, where $\mathfrak{h} \cong \mathbb{C}^n$ is the reflection representation of S_{n+1}, viewed as a type A_n Weyl group. In this case, the theorem specializes to

Corollary 1.1.12. Let $\mathfrak{h} \cong \mathbb{C}^n$ be the reflection representation of the Weyl group S_{n+1} of type A_n. Then

$$\mathsf{HP}_0(\mathcal{O}_{(\mathfrak{h}\oplus\mathfrak{h}^*)/S_{n+1}}) \cong \mathbb{C} \cong \mathsf{HH}_0(\mathcal{D}(\mathfrak{h})^{S_{n+1}}). \tag{1.1.13}$$

This corollary was verified by computer by Justin Sinz for small values of n; the cases $n \leq 2$ and $n = 3$ are also proved in [AF09] and [But09].

More generally, we can extend the corollary to the case of spherical rational Cherednik algebras associated to S_{n+1}. Recall (see, e.g., [EG02]) that these are certain filtered algebras B of the form $e\widetilde{B}e$, where $\widetilde{B}$ is a filtered algebra such that $\mathsf{gr}\,\widetilde{B} \cong \mathcal{O}_{\mathbb{C}^{2n}} \rtimes S_{n+1}$, and $e = \frac{1}{(n+1)!}\sum_{\sigma\in S_{n+1}} \sigma$ is the symmetrizer.

Corollary 1.1.14. Let B an arbitrary noncommutative spherical rational Cherednik algebra deforming $\mathcal{O}_{\mathbb{C}^{2n}}^{S_{n+1}}$. Then, $\dim \mathsf{HH}_0(B) = 1$.

In particular, this also gives another proof of the result from [BEG04] that B can have at most one irreducible finite-dimensional representation.

Remark 1.1.15. If B admits any other filtered quantizations aside from the Cherednik algebras, then for these one concludes at least that $\dim \mathsf{HH}_0(B) \leq 1$ and B admits at most one finite-dimensional irreducible representation. However, we do not know if there exist any quantizations other than the Cherednik algebras; cf. the comments in §1.3 below.

1.1.1. *Prime ideals of quantizations.* Returning to the case of S^nY where Y is affine symplectic, we remark that there can never be any finite-dimensional representations of quantizations of S^nY when $\dim Y > 0$ and Y is connected, since S^nY has no zero-dimensional symplectic leaves (i.e., subvarieties closed under the flow of Hamiltonian vector fields $\xi_f := \{f, -\}$). In more detail, recall that a primitive ideal of an associative algebra A is the kernel of an irreducible representation. If A is a filtered quantization of an affine Poisson variety X, then a primitive ideal J is the kernel of a finite-dimensional representation if and only if the support of $\mathsf{gr}\, J$ is zero-dimensional. However, it is well known (and easy to check) that the support of $\mathsf{gr}\, J$ must be closed under Hamiltonian flow on X. Since $S^n Y$ has no zero-dimensional symplectic leaves, it follows that A cannot have any finite-dimensional irreducible representations.

However, we can still make a nontrivial statement about more general primitive ideals of quantizations of $\mathcal{O}_{S^n Y}$. In fact, we can consider more generally prime ideals: recall that a (two-sided) ideal $J \subseteq A$ is prime if $R = A/J$ is a prime ring, i.e., $aRb = 0$ if and only if either a or b is zero. All primitive ideals are prime.

Using the method of I. Losev's appendix to [ES10a], we may then deduce

Corollary 1.1.16. Let Y be connected affine symplectic and let $A_\hbar$ or B be a deformation or filtered quantization of $\mathcal{O}_{S^n Y}$, respectively. For each $i \leq n$, the number of prime ideals of $A_\hbar[\hbar^{-1}]$ or B (over $\mathbb{C}((\hbar))$ or $\mathbb{C}$, respectively) whose support has codimension $i \dim Y$ in $S^n Y$ is at most $p_{n,i}$, which is given by the generating function

$$\sum_{n,i \geq 0} p_{n,i} s^i t^n = \prod_{m \geq 0} \frac{1}{1 - s^m t^{m+1}}, \tag{1.1.17}$$

i.e., the number of partitions of n with $n - i$ parts. There are no prime ideals whose support has codimension not a multiple of $\dim Y$.

In particular, the bound on the number of prime ideals is independent of Y.

Similarly, in the case of V^n / S_{n+1}, we may deduce

Corollary 1.1.18. Let V be a symplectic vector space, and $A_\hbar$ or B be a deformation or filtered quantization of $\mathcal{O}_{V^n}^{S_{n+1}}$. Then, for each $i \leq n$, the number of prime ideals of $A_\hbar[\hbar^{-1}]$ or B whose support has codimension $i \dim V$ in V^n / S_{n+1} is at most $p_{n+1,i}$.

1.1.2. *Poisson deformations and zero-dimensional symplectic leaves.* Assume that A is a graded algebra which is Poisson with Poisson bracket of degree $-d$. Then, recall that a filtered Poisson deformation is a Poisson algebra B whose Poisson bracket satisfies $\{B_{\leq i}, B_{\leq j}\} \subseteq B_{\leq i+j-d}$, and such that $\mathsf{gr}\, B = A$ as a Poisson algebra. Similarly, if A is an arbitrary Poisson algebra, one can consider Poisson algebras $(A_\hbar, \star, \{-,-\}_\star)$ over $\mathbb{C}[\![\hbar]\!]$, which are isomorphic to $A[\![\hbar]\!]$ as $\mathbb{C}[\![\hbar]\!]$-modules, and satisfy $a \star b = ab + O(\hbar)$ and $\{a, b\}_\star = \hbar\{a, b\} + O(\hbar^2)$ for all $a, b \in A$. Let us call these *formal Poisson deformations*. The two deformations above are analogous to filtered and deformation quantizations, respectively (there is a slight discrepancy with the use of the term "deformation" which refers to a formal parameter in the quantization case but not in the Poisson case).

In the filtered case, one has a surjection $\mathsf{HP}_0(A) \twoheadrightarrow \mathsf{gr}\, \mathsf{HP}_0(B)$, and in the formal case, one has $\mathsf{HP}_0(A)((\hbar)) \twoheadrightarrow \mathsf{gr}\, \mathsf{HP}_0(B_\hbar[\hbar^{-1}])$.

Finally, recall that a zero-dimensional symplectic leaf of a Poisson variety X is a point $x \in X$ at which all Hamiltonian vector fields vanish. Equivalently, x is a point at which the evaluation map $\mathrm{ev}_x : \mathcal{O}_X \to \mathbb{C}$ is a Poisson trace. Note that the evaluation maps at distinct points of X are linearly independent.

Therefore, as before, we deduce

Corollary 1.1.19. Let Y be a connected affine symplectic variety. If $B_\hbar$ is a formal Poisson deformation of $S^n Y$, then $\dim_{\mathbb{C}((\hbar))} \mathsf{HP}_0(B_\hbar[\hbar^{-1}]) \leq a_n(\dim H^{\dim Y}(Y))$.

Now consider the linear case. By [GK04, Proposition 1.16], the second Poisson cohomology $\mathsf{HP}^2(\mathcal{O}_{V^n}^{S_{n+1}})$ is zero unless $\dim V = 2$, in which case it is one-dimensional. So, there are only nontrivial (filtered or deformation) Poisson deformations when $\dim V = 2$. Moreover, by [GK04, Theorem 1.18], there is a universal Poisson deformation of $\mathcal{O}_{\mathbb{C}^{2n}}^{S_{n+1}}$, given by the family of commutative spherical rational Cherednik algebras. Using results of [EG02], we may conclude

Corollary 1.1.20. Let V be a symplectic vector space and B be a nontrivial filtered Poisson deformation of $\mathcal{O}_{V^n}^{S_{n+1}}$. Then $V \cong \mathbb{C}^2$, B is a commutative spherical rational Cherednik algebra, $\dim \mathsf{HP}_0(B) = 1$, and $\operatorname{\mathsf{Spec}} B$ has at most one zero-dimensional symplectic leaf.

1.2. **A canonical $\mathcal{D}$-module on S^nY for Y symplectic.** Here we will explain and generalize Theorem 1.1.1 using $\mathcal{D}$-modules.

We first recall the basic construction from [ES10a] for Poisson varieties. Let X be an affine Poisson variety, i.e., $X = \operatorname{\mathsf{Spec}} A$ where A is a Poisson algebra over $\mathbb{C}$ which is finitely generated as an algebra over $\mathbb{C}$. Let $i : X \hookrightarrow V$ be an embedding of X into a smooth affine variety V. Then, [ES10a] defined the right $\mathcal{D}_V$-module $M(X, i)$ on V as the quotient of the ring $\mathcal{D}(V)$ of differential operators on V with polynomial coefficients by the right ideal generated by functions vanishing on X and vector fields which, on X, are parallel to X and restrict to Hamiltonian vector fields. As explained there, this does not depend on the choice of embedding $i : X \hookrightarrow V$, in the sense that, given two embeddings $i_1 : X \hookrightarrow V_1$ and $i_2 : X \hookrightarrow V_2$, the resulting $\mathcal{D}_V$-modules $M(X, i_1)$ and $M(X, i_2)$ are images of each other (up to isomorphism) under Kashiwara's equivalence of categories of $\mathcal{D}_V$-modules on V_1 and V_2 supported on X. We may thus refer to the module as $M(X)$ when not using the embedding. (Note that one can also define $M(X)$ without using an embedding at all, as a quotient of the canonical right $\mathcal{D}$-module $\mathcal{D}(X)$ by the left action of Hamiltonian vector fields: see [ES10a].)

The motivation for the definition of $M(X)$ is the formula $\pi_0(M(X)) \cong \mathsf{HP}_0(\mathcal{O}_X)$, where $\pi : X \to \mathrm{pt}$ is the projection to the point, and π_0 is the underived direct image.

On the other hand, since the definition of the $\mathcal{D}$-module $M(X)$ is local, as explained in [ES10a], it makes sense to define $M(X)$ even in the case that X is not affine.

We now present a theorem giving the structure of $M(X)$ when $X = S^nY$, for Y a symplectic variety that need not be affine. Let $\Delta_i : Y \hookrightarrow S^iY$ be the diagonal embedding, and for $\sum_{j=1}^k r_j i_j = n$, let $q : (S^{i_1}Y)^{r_1} \times \cdots \times (S^{i_k}Y)^{r_k} \twoheadrightarrow S^nY$ be the obvious projection.

Theorem 1.2.1.

(1.2.2)

$$M(S^nY) \cong \bigoplus_{r_1 \cdot i_1 + \cdots + r_k \cdot i_k = n, 1 \le i_1 < \cdots < i_k, r_j \ge 1 \, \forall j} q_*\big((\Delta_{i_1})_*(\Omega_Y)^{\boxtimes r_1} \boxtimes \cdots \boxtimes (\Delta_{i_k})_*(\Omega_Y)^{\boxtimes r_k}\big)^{S_{r_1} \times \cdots \times S_{r_k}}.$$

Remark 1.2.3. Let Y be connected and V be a symplectic vector space of dimension equal to the dimension of Y. The results of [RS10] on $\mathsf{HP}_0(\mathcal{O}_{V^{n-1}}^{S_n}, \mathcal{O}_{V^{n-1}})$ (cf. Remark 1.1.9) imply information on the structure of the S_n-equivariant $\mathcal{D}$-module $M_\phi(Y^n)$, where $\phi : Y^n \twoheadrightarrow Y^n/S_n = S^nY$, and $M_\phi(Y^n)$ is defined as the quotient of D_{Y^n} by the S^n-invariant Hamiltonian vector fields (more generally, for $\psi : X \to Z$, $M_\psi(X)$ is locally the quotient of $\mathcal{D}_X$ by the action of Hamiltonian vector fields associated to Hamiltonians pulled back from Z: see [ES10a]). The computation of $M_\phi(Y^n)$ reduces to a study of the diagonal $Y \hookrightarrow Y^n$ as before, and there one obtains the S_n-module $\Omega_Y \otimes \mathsf{HP}_0(\mathcal{O}_{V^{n-1}}^{S_n}, \mathcal{O}_{V^{n-1}})$. Then, the results of *op. cit.* mentioned in Remark 1.1.9 say that the reflection representation $\mathfrak{h}$ of S_n does not occur here, and the representation $\wedge^2\mathfrak{h}$ occurs there tensored by $\lfloor \frac{n-1}{2} \rfloor$ copies of Ω_Y. Using this and the theorem, one can obtain analogues of Theorem 1.1.1 giving information on the structure of $\mathsf{HP}_0(\mathcal{O}_{S^nY}, \mathcal{O}_{Y^n})$.

This implies the following "derived" generalization of Theorem 1.1.1. For any affine Poisson variety X, let $\mathsf{HP}_i^{DR}(X) := L^i\pi_*(M(X))$ be the i-th derived pushforward of $M(X)$ to a point, where $\pi : X \to \mathrm{pt}$ is the projection. This is called the i-th *Poisson-de Rham homology* of X and was defined in [ES10a]. Note that $\mathsf{HP}_0^{DR}(X) = \mathsf{HP}_0(\mathcal{O}_X)$. Moreover, when X is symplectic and connected, $\mathsf{HP}^{DR}_i(X) \cong H^{\dim X - i}(X) \cong \mathsf{HP}_i(\mathcal{O}_X)$, where the latter is the usual i-th Poisson homology (which differs from HP^{DR}_i for general affine X when $i > 0$).

Corollary 1.2.4. Let Y be affine symplectic and connected. Then, as bigraded algebras (in de Rham degree and in symmetric power plus t degrees, i.e., $|\mathsf{HP}^{DR}_i(S^n Y)| = (i, n)$ and $|t| = (0, 1)$),

$$\bigoplus_{n \geq 0} \mathsf{HP}^{DR}_\bullet(S^n Y)^* \cong \mathsf{Sym}(\mathsf{HP}^{DR}_\bullet(Y)^*[t]) = \mathsf{Sym}(H^{\dim Y - \bullet}(Y)^*[t]). \tag{1.2.5}$$

Next, continuing to assume that Y is affine symplectic and connected, let $A_\hbar$ be a deformation quantization of $\mathcal{O}_Y$. Then, we deduce the following generalization of Corollary 1.1.5:

Corollary 1.2.6. Taking the $\mathbb{C}((\hbar))$-linear dual, we obtain an isomorphism of bigraded algebras over $\mathbb{C}((\hbar))$ (with $|\mathsf{HH}_i(\mathsf{Sym}^n A_\hbar)[\hbar^{-1}]^*| = (i, n) = |\mathsf{HP}^{DR}_i(S^n Y)((\hbar))^*|$):

$$\bigoplus_{n \geq 0} \mathsf{HH}_\bullet(\mathsf{Sym}^n A_\hbar)[\hbar^{-1}]^* \cong \bigoplus_{n \geq 0} \mathsf{HP}^{DR}_\bullet(S^n Y)((\hbar))^*. \tag{1.2.7}$$

1.3. **Conjectures on symplectic resolutions.** In this subsection, we explain some conjectures related to symplectic resolutions motivated by the preceding results and also [ES09]. The material of this subsection will not be needed elsewhere in this paper.

For an irreducible (affine) Poisson variety X, we say that a morphism $\widetilde{X} \to X$ is a symplectic resolution if $\widetilde{X}$ is symplectic and the morphism is proper, birational, and Poisson (the latter condition means that its pullback is a morphism of sheaves of Poisson algebras).

When Y is a connected affine symplectic surface, $S^n Y$ admits a symplectic resolution $\operatorname{Hilb}^n Y \twoheadrightarrow S^n Y$, and we can deduce from Corollary 1.2.4 and the known description of the cohomology of $\operatorname{Hilb}^n Y$ that $\mathsf{HP}^{DR}_\bullet(S^n Y) \cong H^{2n-\bullet}(\operatorname{Hilb}^n Y)$. This suggests

Conjecture 1.3.1. Let X be an irreducible affine Poisson variety with a symplectic resolution $\rho : \widetilde{X} \twoheadrightarrow X$. Then:

(a) $\mathsf{HP}_0(\mathcal{O}_X) \cong H^{\dim \widetilde{X}}(\widetilde{X})$.
(b) $\mathsf{HP}^{DR}_\bullet(X) \cong H^{\dim \widetilde{X} - \bullet}(\widetilde{X})$.
(c) $M(X) \cong \rho_* \Omega_{\widetilde{X}}$.

In part (c), ρ_* refers to the derived pushforward. Clearly, $(c) \Rightarrow (b) \Rightarrow (a)$. Also, we remark that part (c) makes sense even when X is not affine, so it is reasonable to conjecture that the affine assumption is not needed (and this also would imply the generalization of (b) to nonaffine X, if we extend the definition of $\mathsf{HP}^{DR}_\bullet(X)$ by taking the appropriate derived pushforward of $M(X)$ to a point).

Note that (c) would imply that $M(X)$ is semisimple holonomic with regular singularities, by the decomposition theorem [BBD82, Théorème 6.2.5] (although the holonomicity already follows from [ES10a, Theorem 3.1] once one notices that X necessarily has finitely many symplectic leaves; however, as pointed out in [ES10a, Example 4.11], the latter condition does not imply that the singularities are regular, and in fact neither does it imply that $M(X)$ is semisimple). Similarly, it would follow immediately from the conjecture that $\rho_* \Omega_{\widetilde{X}}$ is a $\mathcal{D}$-module rather than a complex, although this already follows from the fact, [Kal06, Lemma 2.11], that ρ is a semismall morphism.

We can prove the conjecture in three cases:

(A) If $\widetilde{X} = \operatorname{Hilb}^n Y$ and $X = S^n Y$, part (c) follows from Theorem 1.2.1 together with the standard computation of $\rho_* \Omega_{\operatorname{Hilb}^n Y}$ (see [GS93, Theorem 3]).
(B) If $\widetilde{X} = T^*(G/B)$ is the Springer resolution of the nilpotent cone $X \subseteq \operatorname{Lie} G$, for G a semisimple connected complex Lie group and $B < G$ a Borel, or more generally the restriction of this to the resolution of a Slodowy slice of X (a transverse slice at a point $e \in X$ to its coadjoint orbit), part (c) follows from the main result of [ES10b].

(C) If $\widetilde{X} = \mathrm{Hilb}^n(\widetilde{\mathbb{C}^2/G}) \overset{\rho}{\twoheadrightarrow} \mathsf{Sym}^n(\mathbb{C}^2/G) = X$, where $G < \mathsf{SL}_2(\mathbb{C})$ is a finite subgroup, i.e., X is a symmetric power of a Kleinian singularity, and $\widetilde{\mathbb{C}^2/G}$ is the minimal resolution of the Kleinian singularity. Then, the argument is similar to that of (i), using the computation of $\mathsf{HP}_0(\mathcal{O}_X)$ from [ES09, Theorem 1.1.14]: see §1.3.2 below.

Remark 1.3.2. We stress that, for all parts (a)–(c) of the conjecture, we only conjecture an abstract isomorphism (which is confirmed in the cases mentioned above), not a canonical isomorphism; i.e., in the cases of (a) and (b), we conjecture only an equality of dimensions. It would be desirable to refine the conjecture to give a more precise relationship between the two conjecturally isomorphic objects.

At least for part (a), we can do this: (1.3.7) below should imply that, for suitable deformation quantizations $B_\hbar$ of X, one has a *canonical* isomorphism $\mathsf{HH}_\bullet(B_\hbar[\hbar^{-1}]) \cong H^{\dim X - \bullet}(\widetilde{X})((\hbar))$. Since there is a canonical surjection $\mathsf{HP}_0(\mathcal{O}_X)((\hbar)) \twoheadrightarrow \mathsf{gr}\,\mathsf{HH}_0(B_\hbar[\hbar^{-1}])$, this suggests that, in the formal version with $\hbar$, there may rather be a filtration on the right-hand side of (a) whose associated graded vector space is the left-hand side, i.e., that there is a canonical isomorphism $\mathsf{HP}_0(\mathcal{O}_X)((\hbar)) \overset{\sim}{\to} \mathsf{gr}\, H^{\dim X}(\widetilde{X})((\hbar))$.

Moreover, if X has a contracting $\mathbb{C}^\times$ action, we can eliminate the $\hbar$ using this grading, and should obtain a canonical isomorphism $\mathsf{HP}_0(\mathcal{O}_X) \overset{\sim}{\to} \mathsf{gr}\, H^{\dim X}(\widetilde{X})$. This holds in all cases we have checked (e.g., cases (B) and (C); note that there is no $\mathbb{C}^\times$ action in case (A) in general).

For parts (b) and (c) of the conjecture, it would be desirable to have a similar statement. However, we know of no direct relationship between the Poisson-de Rham homology of X and the Hochschild homology of a quantization (there is a spectral sequence from ordinary Poisson homology of $\mathcal{O}_X$ to this Hochschild homology, but ordinary Poisson homology only coincides with Poisson-de Rham homology in degree zero). Perhaps this problem could be alleviated using the universal formal deformation $\mathcal{X}$ of $\widetilde{X}$ of [KV02, Theorem 1.1] discussed below, which is generically affine symplectic, and which maps to the formal deformation $\mathsf{Spec}\,\Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ of X in a way which is generically an isomorphism, since for affine symplectic varieties the Poisson-de Rham and ordinary Poisson homology coincide.

Next, we can pose a conjecture on the Hochschild homology of quantizations. To motivate this, note that, in the case of (A) above, if $A_\hbar$ is a deformation quantization of $\mathcal{O}_Y$, then by Corollary 1.2.6, $\mathsf{HP}^{DR}_\bullet(S^nY)((\hbar)) \cong \mathsf{HH}_\bullet(\mathsf{Sym}^n A_\hbar[\hbar^{-1}])$. We would like to generalize this to the case of general symplectic resolutions.

We will be particularly interested in quantizations obtainable by quantizing the symplectic resolution in the sense of [BK04]. Namely, according to [BK04, Definition 1.3], a quantization of $\widetilde{X}$ is a sheaf $\mathcal{B}_\hbar$ of associative flat $\mathbb{C}[\![\hbar]\!]$-algebras on X equipped with an isomorphism $\mathcal{B}_\hbar/\hbar\mathcal{B}_\hbar \cong \mathcal{O}_{\widetilde{X}}$. We will additionally require that the induced Poisson structure on $\mathcal{O}_{\widetilde{X}}$ is the one coming from the symplectic form. By [BK04, Theorem 1.8], there is a semiuniversal family of such quantizations, parameterized by $\hbar H^2(\widetilde{X})[\![\hbar]\!]$. (Moreover, it seems reasonable to ask if these produce all quantizations of X, or if there is a semiuniversal family of all quantizations in which these map to a dense subset.)

Conjecture 1.3.3. Let X be an irreducible affine Poisson variety which admits a symplectic resolution.

(i) For every deformation quantization $A_\hbar$ of $\mathcal{O}_X$, the canonical surjection is an isomorphism

$$\mathsf{HP}_0(\mathcal{O}_X)((\hbar)) \overset{\sim}{\to} \mathsf{gr}\,\mathsf{HH}_0(A_\hbar[\hbar^{-1}]). \tag{1.3.4}$$

(ii) There is a countable collection of $\hbar$-homogeneous hypersurfaces in $\hbar H^2(\widetilde{X})[\![\hbar]\!]$ such that, for $A_\hbar$ obtained as the global sections of a quantization in the family $\hbar H^2(\widetilde{X})[\![\hbar]\!]$ outside of this collection, one has (abstractly)

$$\mathsf{HP}^{DR}_{\bullet}(X)(\!(\hbar)\!) \cong \mathsf{gr}\,\mathsf{HH}_{\bullet}(A_{\hbar}[\hbar^{-1}]). \tag{1.3.5}$$

Here, by an $\hbar$-homogeneous hypersurface in $\hbar H^2(\widetilde{X})[\![\hbar]\!]$, we mean by definition a subvariety of the form $Z \times \hbar^m H^2(\widetilde{X})[\![\hbar]\!]$, where $Z \subseteq \bigoplus_{j=1}^{m-1} \hbar^j H^2(\widetilde{X})$ is cut out by an equation which is homogeneous in $\hbar$ of some degree.

Remark 1.3.6. As in Remark 1.3.2 above, it would be better if in (ii) one could construct a canonical map from the LHS to the RHS which is conjecturally an isomorphism, but we are not sure how to do this.

Moreover, given a semiuniversal quantization of X, one can ask if (1.3.5) still holds for this family. Note that (ii) implies (i) (for quantizations considered in (ii)), since $\dim_{\mathbb{C}(\!(\hbar)\!)} \mathsf{HH}_0(A_{\hbar}[\hbar^{-1}])$ is upper semicontinuous and bounded above by $\dim \mathsf{HP}_0(\mathcal{O}_X)$ (and $\mathsf{HP}_0(\mathcal{O}_X) = \mathsf{HP}_0^{DR}(X)$, unlike in higher degrees).

Also, note that the genericity assumption of (ii) above is needed: already in the case $X = \mathbb{C}^2/(\mathbb{Z}/2)$, there exist quantizations for which (1.3.5) does not hold (this follows from [FSSÁ03, Theorem 2.1]; see also [ES10b, Remark 1.14]). Indeed, only in degree zero does one obtain a (natural) surjection from $\mathsf{HP}^{DR}_{\bullet}(X)(\!(\hbar)\!)$ to $\mathsf{gr}\,\mathsf{HH}_{\bullet}(A_{\hbar}[\hbar^{-1}])$.

In cases (B) and (C) above, we can prove this conjecture, at least when (i) is restricted to quantizations coming from the symplectic resolution. In case (B), one should be able to check that the algebras $A_{\hbar}$ appearing in the conjecture are the Rees algebras of the quantum W-algebras deforming $\mathcal{O}_X$. For these algebras, parts (i) and (ii) of the conjecture follow from [ES10b, Theorem 1.10.(ii)] and [ES10b, Theorem 1.13].

In case (C), the algebras $A_{\hbar}$ appearing in the conjecture should be the Rees algebras of the spherical symplectic reflection algebras [EG02] deforming $\mathcal{O}_{\mathsf{Sym}^n(\mathbb{C}^2/G)} = \mathcal{O}_{\mathbb{C}^{2n}}^{G^n \rtimes S_n}$. For these algebras, part (i) of the conjecture is a consequence of [ES09, Corollary 1.3.2]. Part (ii) follows by comparing the explicit description of $M(X)$ given in §1.3.2 below (for the LHS) with the description of $\mathsf{HH}_{\bullet}(\mathcal{D}_{\mathbb{C}^n}^{G^n \rtimes S_n})$ from [AFLS00], as well as the fact from [EG02, Theorem 1.8] that this coincides with $\mathsf{HH}_{\bullet}(A)$ for generic spherical symplectic reflection algebras A quantizing $\mathcal{O}_{\mathbb{C}^{2n}}^{G^n \rtimes S_n} = \mathcal{O}_{\mathsf{Sym}^n(\mathbb{C}^2/G)}$.

If true, the conjecture would yield a necessary criterion for existence of symplectic resolutions (where in (ii) we take a semiuniversal family of quantizations). This condition does *not* appear to be sufficient, however: already in the case that $X = \mathsf{Sym}^n V$ for V a symplectic vector space of dimension ≥ 4, our main theorem implies that (1.3.5) holds for the quantization $\mathsf{Sym}^n \mathsf{Weyl}(V)$. We are not sure if there exist other quantizations: for the quasiclassical analogue, there exist no nontrivial Poisson deformations as discussed after Corollary 1.1.19. On the other hand, X does not admit a symplectic resolution by [Ver00] (since G is not generated by symplectic reflections, i.e., elements $g \in G$ such that $g - \mathrm{Id}$ has rank two; in fact, G has no symplectic reflections, which is why $\mathsf{HP}^2(\mathcal{O}_X) = 0$).

Finally, we remark that Conjecture 1.3.1 almost implies Conjecture 1.3.3 (at least if we restrict part (i) to quantizations coming from the resolution). First of all, by [KV02, Theorem 1.1], there is a universal formal deformation $\mathcal{X}$ of $\widetilde{X}$ in the category of symplectic schemes, which lies over the formal completion $\widehat{H}^2(\widetilde{X})$ of $H^2(\widetilde{X})$ at the origin. By [BK04, Theorem 1.8, Lemma 6.4], $\mathcal{X}$ also admits a canonical quantization over $\widehat{H}^2(\widetilde{X})$, so that the quantization $\mathcal{B}_{\hbar}$ corresponding to a formal power series $P \in \hbar H^2(\widetilde{X})[\![\hbar]\!]$ is the pullback of the canonical quantization $\mathcal{B}_{\mathcal{X}}$ of $\mathcal{X}$ by the formal point $p \in \widehat{H}^2(\widetilde{X})$ corresponding to P. Now, according to [Kal08, Lemma 2.5], for generic p, the fiber of $\mathcal{X}$ over p is affine. For such p, it should follow that

$$\mathsf{HH}_*(\Gamma(\mathcal{X}_p, p^*\mathcal{B}_{\mathcal{X}}[\hbar^{-1}])) \cong H^{\dim X - *}(\mathcal{X}_p)(\!(\hbar)\!) \cong H^{\dim X - *}(\widetilde{X})(\!(\hbar)\!), \tag{1.3.7}$$

adapting the usual identification of Hochschild homology of quantizations of an affine symplectic variety with the de Rham cohomology of the variety for the first isomorphism, and applying topological triviality of the family of deformations for the second isomorphism. This would yield Conjecture 1.3.3.(ii). Then, to deduce part (i), we apply part (ii) together with the fact that $\dim \mathsf{HP}_0(\mathcal{O}_X) \geq \dim_{\mathbb{C}((\hbar))} \mathsf{HH}_0(A_\hbar[\hbar^{-1}])$ for all quantizations, along with upper semicontinuity of $\dim \mathsf{HH}_0(\Gamma(\mathcal{X}_p, p^*\mathcal{B}_\mathcal{X}[\hbar^{-1}]))$ in p. Thus, Conjecture 1.3.1 should also imply Conjecture 1.3.3, at least in (ii) if we ask only for an abstract isomorphism of $\mathbb{C}((\hbar))$-vector spaces which preserves the homological grading ($\bullet$).

1.3.1. *The case of linear quotient singularities.* In the case when $X = V/G$ is a linear quotient singularity with $G < \mathsf{Sp}(V)$, the main result of [AFLS00] computes $\dim \mathsf{HH}_{2i}(\mathsf{Weyl}(V)^G)$: this is the number of conjugacy classes of $g \in G$ such that $\dim \ker(g - \mathrm{Id}) = 2i$. Here, $\mathsf{Weyl}(V)$ is the Weyl algebra and $\mathsf{Weyl}(V)^G$ is therefore a filtered quantization of $\mathcal{O}_V^G$. This would imply the first part of the following conjecture:

Conjecture 1.3.8. Suppose that $G < \mathsf{Sp}(V)$ is finite and V/G admits a symplectic resolution. Then

(i) The canonical surjection is an isomorphism $\mathsf{HP}_0(\mathcal{O}_V^G) \overset{\sim}{\twoheadrightarrow} \mathsf{gr}\,\mathsf{HH}_0(\mathsf{Weyl}(V)^G)$. In particular, $\dim \mathsf{HP}_0(\mathcal{O}_V^G)$ is the number of conjugacy classes of elements $g \in G$ such that $g - \mathrm{Id}$ is invertible.

(ii) For all $i \geq 0$, abstractly, $\mathsf{HP}_{2i}^{DR}(V/G) \cong \mathsf{HH}_{2i}(\mathsf{Weyl}(V)^G)$, i.e., $\dim \mathsf{HP}_{2i}^{DR}(V/G)$ is the number of conjugacy classes of $g \in G$ such that $\ker(g-\mathrm{Id})$ has dimension $2i$ (and $\mathsf{HP}_{2i+1}^{DR}(V/G) = 0$).

Conversely, part (i) of the above conjecture would imply Conjecture 1.3.3 for noncommutative spherical symplectic reflection algebras deforming $\mathcal{O}_V^G$: this follows from reasoning similar to the proof of Corollary 1.1.14. Namely, by [EG02, Theorem 1.8], for generic such algebras, $\dim \mathsf{HH}_0(A)$ coincides with $\dim \mathsf{HH}_0(\mathsf{Weyl}(V)^G)$. Hence, by upper semicontinuity of $\dim \mathsf{HH}_0$ in the family, $\dim \mathsf{HH}_0(A)$ is at least $\dim \mathsf{HH}_0(\mathsf{Weyl}(V)^G)$ for all noncommutative spherical symplectic reflection algebras deforming $\mathcal{O}_V^G$. Then, Conjecture 1.3.8 would say that this coincides also with $\dim \mathsf{HP}_0(\mathcal{O}_V^G)$, which is an upper bound for $\dim \mathsf{HH}_0(A)$ for all quantizations A. Hence, $\dim \mathsf{HH}_0(A)$ would be constant in the family, implying Conjecture 1.3.3.(i) for such algebras.

In case (C) above, i.e., for G a wreath product of a finite subgroup of $\mathsf{SL}_2(\mathbb{C})$ with S_n for some $n \geq 1$, we can prove the above conjecture: it follows from our proof of Conjecture 1.3.1 below (or alternatively, it follows from Conjecture 1.3.3, since in this case the family of quantizations obtained from the resolution of singularities is exactly the noncommutative spherical symplectic reflection algebras). Note that, in this case, statement (i) was a conjecture by J. Alev of [But08, Remark 40] (he possibly conjectured it for some other groups G as well elsewhere); this conjecture was first proved in [ES09], apart from the cases $n = 2$ and $n = 3$ where it was proved in [AF09] and [But08], respectively.

1.3.2. *Proof of Conjecture 1.3.1 in the case $X = \mathsf{Sym}^n(\mathbb{C}^2/G)$.* Let $X = \mathsf{Sym}^n(\mathbb{C}^2/G)$ where $G < \mathsf{SL}_2(\mathbb{C})$ is a finite group. By [ES10a, Corollary 4.16], $M(X)$ is a direct sum of IC D-modules of the symplectic leaves with some multiplicities. These leaves are indexed by tuples $(r, r_1, \ldots, r_k)$ of nonnegative integers, such that $r + \sum_{j=1}^k j \cdot r_j = n$. This symplectic leaf, $X_{(r,r_1,\ldots,r_k)}$, has closure given by the image of

$$\begin{aligned} (1.3.9)\quad & \{0\} \times \mathsf{Sym}^{r_1}(\mathbb{C}^2/G) \times \cdots \times \mathsf{Sym}^{r_k}(\mathbb{C}^2/G) \\ & \hookrightarrow \{0\}^r \times \mathsf{Sym}^{r_1}((\mathbb{C}^2/G)^1) \times \mathsf{Sym}^{r_2}((\mathbb{C}^2/G)^2) \times \cdots \times \mathsf{Sym}^{r_k}((\mathbb{C}^2/G)^k) \to \mathsf{Sym}^n(\mathbb{C}^2/G) = X. \end{aligned}$$

The multiplicity of $IC(\overline{X_{(r,r_1,\ldots,r_k)}})$, by *op. cit.*, is $\dim \mathsf{HP}_0(\mathcal{O}_{Z_{(r,r_1,\ldots,r_k)}})$, where $Z_{(r,r_1,\ldots,r_k)} = \mathsf{Sym}^r(\mathbb{C}^2/G) \times \prod_{j=2}^k (\mathbb{C}^{2(j-1)}/S_j)^{r_j}$. By Theorem 1.1.8 and [ES09, Theorem 1.1.14], this multiplicity is equal to the number of $\dim \mathsf{HP}_0(\mathbb{C}^2/G)$-multipartitions of r. By, e.g., [AL98], $\dim \mathsf{HP}_0(\mathbb{C}^2/G)$ is the number of isomorphism classes of nontrivial representations of G, which is well known to be the number of irreducible components of the fiber $\pi^{-1}(0)$ of the resolution of Kleinian singularities, $\pi : \widetilde{\mathbb{C}^2/G} \to \mathbb{C}^2/G$.

Then, it remains to show that the above is the same as $\rho_* \Omega_{\widetilde{X}}$. We can argue similarly to the aforementioned result, [GS93, Theorem 3] (which dealt with the case where the fibers of ρ were irreducible). Namely, since the above map is semismall (this is well known in this case, and is also more generally true for all symplectic resolutions by the aforementioned [Kal06, Lemma 2.11]), $\rho_* \Omega_{\widetilde{X}}$ decomposes as a direct sum of intermediate extensions of local systems (i.e., $\mathcal{O}$-coherent $\mathcal{D}$-modules) on each symplectic leaf of X. Moreover, the local systems occurring on each symplectic leaf are the top cohomology of the fibers of ρ restricted to that leaf. By restricting to a formal neighborhood of a symplectic leaf, using the explicit description of the symplectic leaves above, the computation reduces to the case of the point $\{0\} \in \mathsf{Sym}^{n'}(\mathbb{C}^2/G)$ for all $n' \leq n$. In this case, we evidently get a direct sum of delta-function $\mathcal{D}$-modules, with multiplicity given by the number of irreducible components of $\rho^{-1}(0)$ of dimension n'. This is equal to the number of m-multipartitions of n', where m is the number of irreducible components of the zero fiber of $\widetilde{\mathbb{C}^2/G} \to \mathbb{C}^2/G$. This is, however, the same multiplicity as for $M(X)$, as mentioned above. We conclude that $\rho_* \Omega_{\widetilde{X}} \cong M(X)$, as desired.

1.4. **Examples of nontrivial local systems in $M(X)$.** Note that, in all of the examples of affine Poisson varieties X studied thus far in this paper, $M(X)$ is a direct sum of intermediate extensions of trivial local systems on the symplectic leaves of X. *Here and below, "local system" refers to an $\mathcal{O}$-coherent $\mathcal{D}$-module on a smooth variety.* Note that these were all examples of the form $X = U/G$ with U affine symplectic and G a finite group of symplectic automorphisms of U. In this subsection, which will not be required in the remainder of the paper, we construct other examples of this form such that nontrivial local systems do appear in $M(X)$. This fulfills the promise of [ES10a, footnote 6].

In fact, by [ES10a, Corollary 4.16], $M(X)$ is always semisimple if $X = V/G$ for V a symplectic vector space and $G < \mathsf{Sp}(V)$ finite. Also, by [ES10a, Theorem 4.21], whenever $X = U/G$, U is a symplectic variety (not necessarily a vector space or even affine), and G is a finite group of symplectic automorphisms of U, then $M(X)$ is always a direct sum of intermediate extensions of one-dimensional local systems on symplectic leaves of X; these local systems all have monodromy valued in ± 1. Moreover, there is a simple necessary (but not sufficient) criterion for the local systems to be nontrivial: roughly, the action of G on normal bundles to preimages of symplectic leaves must contain quaternionic representations. More precisely, let $X_0 \subseteq X$ be a symplectic leaf, and fix $x \in X_0$ with preimage $u \in U$. Then, there can only be a nontrivial local system appearing in $M(X)|_{X_0}$ if the $\mathrm{Stab}_G(u)$-representation $(T_u U)^\perp$ contains a quaternionic irreducible representation. In particular, this implies the aforementioned result (which we also explain directly in §2.1.1 below) that $M(S^n Y)$ is a direct sum of intermediate extensions of trivial local systems, when Y is a symplectic variety. This is because the $\mathrm{Stab}_G(u)$-representations $(T_u U)^\perp$ are all products of (reducible) representations $\mathbb{C}^{2m}$ of S_{m+1} associated to type A_m Weyl groups, and in particular all irreducible summands are of real, not quaternionic, type.

Now, let $X = U/G$ where U is a symplectic variety and G is a finite group of symplectic automorphisms. Let $x \in X_0$ and $u \in U$ be as above. Let us describe the local system $M(X)|_{X_0}$ more explicitly. As observed in [ES10a, §4], this local system has fiber $\mathsf{HP}_0(\mathcal{O}_{(T_u U)^\perp}^{\mathrm{Stab}_G(u)})$. The monodromy is given by the composition $\pi_1(X_0) \to \mathsf{Sp}_{\mathrm{Stab}_G(U)}((T_u U)^\perp) \to \mathrm{Aut}(\mathsf{HP}_0(\mathcal{O}_{(T_u U)^\perp})),$

where $\mathsf{Sp}_{\mathrm{Stab}_G(u)}((T_uU)^\perp)$ denotes the group of automorphisms of the symplectic vector space $(T_uU)^\perp$ preserving the G-action. The first map is given by the Hamiltonian flow along X_0, as explained in *op. cit.*. Moreover, as explained in *op. cit.*, since $M(X)$ is locally constant along Hamiltonian vector fields, the first map factors through $\mathsf{Sp}_{\mathrm{Stab}_G(u)}((T_uU)^\perp)/\mathsf{Sp}_{\mathrm{Stab}_G(u)}((T_uU)^\perp)^\circ \cong \prod_{Q\in R_q((T_uU)^\perp)} \mathbb{Z}/2$, where $R_q((T_uU)^\perp)$ denotes the set of isomorphism classes of quaternionic representations of $\mathrm{Stab}_G(u)$ occurring in $(T_uU)^\perp$.

We therefore have to consider the two resulting maps: (a) $\pi_1(X_0) \to \prod_{Q\in R_q((T_uU)^\perp)} \mathbb{Z}/2$, and (b) $\prod_{Q\in R_q((T_uU)^\perp)} \mathbb{Z}/2 \to \mathrm{Aut}(\mathsf{HP}_0(\mathcal{O}_{(T_uU)^\perp}))$. We first consider (a):

Claim 1.4.1. For any symplectic vector space V, finite subgroup $G < \mathsf{Sp}(V)$ such that $V^G = \{0\}$, symplectic variety Y, and homomorphism $\pi_1(Y) \to \prod_{Q\in R_q(U)} \mathbb{Z}/2$, one can construct a symplectic variety U with an action of G such that

(i) $U^G \cong Y$;
(ii) For $u \in U^G$, $(T_uU)^\perp \cong V$ as symplectic G-representations;
(iii) The map $\pi_1(U^G) \to \prod_{Q\in R_q((T_uU)^\perp)} \mathbb{Z}/2$ coincides with the given map under (i) and (ii).

Using the claim, it will remain only to exhibit a pair (V, G) such that the map (b) is nonzero. Let us explain such an example. We begin by describing the map (b) more explicitly. It is easy to see that each generator $1_Q \in \mathbb{Z}/2$ corresponding to $Q \in R_q((T_uU)^\perp)$ maps to $(-\mathrm{Id})^{\mu_Q}$, where μ_Q is the operator $f \mapsto |f|_Q$, assigning to functions their parity of degree in any fixed summand of $(T_uU)^\perp$ isomorphic to Q (this parity of degree is independent of the choice of summand). In particular, if $(T_uU)^\perp$ is itself an irreducible quaternionic representation, μ_Q is the parity of the polynomial degree.

More generally, if $(T_uU)^\perp$ is a direct sum of distinct irreducible quaternionic representations, then the image of $(1, \ldots, 1) \in \prod_{Q\in R_q((T_uU)^\perp)} \mathbb{Z}/2$ in $\mathrm{Aut}(\mathsf{HP}_0(\mathcal{O}_{(T_uU)^\perp}))$ is $(-\mathrm{Id})^{\deg}$, where deg is the polynomial degree. In particular, this is nontrivial in the case that $\mathsf{HP}_0(\mathcal{O}_{(T_uU)^\perp}^{\mathrm{Stab}_G(u)})$ is nontrivial in odd degrees.

An example of a pair (V, G) of a symplectic vector space V and a finite subgroup $G < \mathsf{Sp}(V)$ such that $\mathsf{HP}_0(\mathcal{O}_V^G)$ is nontrivial in odd degrees was exhibited in the appendix to [EGP$^+$]: there $V = V_1 \oplus V_2 \oplus V_3$ with V_i irreducible quaternionic representations of G, with $\dim V_1 = \dim V_2 = \dim V_3 = 2^m$ for $m \geq 2$. In particular, the smallest dimension of such V found there is 12. We can apply this to the claim with $Y = \mathbb{C}^\times \times \mathbb{C}$, where $\mathbb{C}^\times$ is the punctured complex plane, together with the map $\pi_1(Y) \cong \mathbb{Z} \ni 1 \mapsto (1, \ldots, 1)$. The resulting X has dimension 14 (for the case $m = 2$), and $M(X)|_{X_0}$ is nontrivial.

Remark 1.4.2. The same analysis as above can be applied more generally to the $\mathcal{D}$-module $M_\phi(U)$, where $\phi : U \to U/G$ is the quotient map (cf. Remark 1.2.3). The only difference is that the fiber $\mathsf{HP}_0(\mathcal{O}_{(T_uU)^\perp}^{\mathrm{Stab}_G(u)})$ is replaced by the $\mathrm{Stab}_G(u)$-representation $\mathsf{HP}_0(\mathcal{O}_{(T_uU)^\perp}^{\mathrm{Stab}_G(u)}, \mathcal{O}_{(T_uU)^\perp})$. Then, one produces examples of nontrivial local systems in $M_\phi(U)$ from any triple (V, G, Y) as in the claim such that $H_1(Y, \mathbb{Z}/2) \neq 0$ with V an *irreducible* quaternionic representation of G, since in this case, $\mathsf{HP}_0(\mathcal{O}_V^G, \mathcal{O}_V)$ is already nontrivial in degree one, where it is V itself. (Here, by nontrivial, we mean that they are nontrivial even considered as ordinary local systems, not merely as G-equivariant local systems.) For example, one can take $Y = \mathbb{C}^\times \times \mathbb{C}$, $V = \mathbb{C}^2$, and $G < \mathsf{SL}_2(\mathbb{C})$ any finite nonabelian subgroup. Then, $M_\phi(U)$ is nontrivial, and $\dim U = 4$. (Note that this is the minimum possible dimension of a symplectic variety U such that, for some finite group of automorphisms G, $M_\phi(U)$ can restrict to a nontrivial local system on some some locally closed subvariety, which we may assume is the locus $\{u \in U : \mathrm{Stab}(u) = K\}$ for some subgroup $K < G$.)

Finally, we can generalize the above argument to obtain information about the G-isotypic components of $M_\phi(U)$. In particular, if V is a direct sum of distinct irreducible quaternionic representations of G, and one constructs the associated G-variety U as above, then any irreducible representation of G that occurs in odd degree in $\mathsf{HP}_0(\mathcal{O}_V^G, \mathcal{O}_V)$ also occurs tensored by a nontrivial local system in $M_\phi(U)|_{U^G}$.

For example, in the case $V = \mathbb{C}^2$ and $G < \mathsf{SL}_2(\mathbb{C})$ is nonabelian, such irreducible representations of G are exactly the ones occurring in the odd tensor powers of V, since these are the ones where $-\operatorname{Id} \in G$ acts by multiplication by -1. Moreover, for such irreducible representations of G, the isotypic part of $M_\phi(U)|_{U^G}$ occurs without a summand of the trivial local system on U^G.

Proof of Claim 1.4.1. We recall first the description given in *op. cit.*: for each $Q \in R_q((T_u U)^\perp)$, let $((T_u U)^\perp_Q \subseteq (T_u U)^\perp$ be the isotypic component of Q. Note that

$$\mathsf{Sp}_{\operatorname{Stab}_G(u)}((T_u U)^\perp_Q) \cong O(\operatorname{Hom}_{\operatorname{Stab}_G(u)}(Q, (T_u U)^\perp)),$$

the orthogonal group acting on the associated vector space $\operatorname{Hom}_{\operatorname{Stab}_G(u)}(Q, (T_u U)^\perp))$, i.e., the multiplicity space of $(T_u U)^\perp_Q$. As explained in *op. cit.*, the composition $\pi_1(X) \to \prod_{Q \in R_q((T_u U)^\perp)} \mathbb{Z}/2 \twoheadrightarrow \mathbb{Z}/2$ with the projection to the factor Q is nothing but application of the first Stiefel-Whitney class $w_1(\operatorname{Hom}_{\operatorname{Stab}_G(u)}(Q, (T_u U)^\perp))$ of this orthogonal vector bundle.

Now, let $\widetilde{Y}$ be the cover of Y corresponding to the kernel of the map $\pi_1(Y) \to H_1(Y, \mathbb{Z}/2)$. Set $U := (\widetilde{Y} \times V)/H_1(Y, \mathbb{Z}/2)$, where $H_1(Y, \mathbb{Z}/2)$ acts as follows. First, it acts by the defining action on the factor of $\widetilde{Y}$. Next, for each $Q \in R_q(V)$, fix an isomorphism $V_Q \cong Q^{m_Q}$. Then, let each $\gamma \in H_1(Y, \mathbb{Z}/2)$ act on $V_Q \cong Q^{m_Q}$ by $\pm \operatorname{Id} \oplus \operatorname{Id}^{m_Q - 1}$, where the sign is the image of γ under the composite map $\pi_1(Y) \to \prod_{Q' \in R_q(V)} \mathbb{Z}/2 \twoheadrightarrow \mathbb{Z}/2 \cong \{\pm 1\}$ corresponding to Q. Taking the direct sum, we obtain an action of $H_1(Y, \mathbb{Z}/2)$ on V, and taking the product with the defining action on $\widetilde{Y}$, we obtain an action of $H_1(Y, \mathbb{Z}/2)$ on $\widetilde{Y} \times V$.

It follows from the construction that $X_0 := Y$ is a symplectic leaf of $X := U/G$: since $V^G = \{0\}$, $X_0 = U^G$. Moreover, for $x \in X_0$ and $u \in U$ mapping to x, $(T_u U)^\perp = U$. It is straightforward to check that the resulting map (b) is the given one. ☐

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2. Proofs

2.1. Proofs of theorems and corollaries from §1.2.

2.1.1. *Proof of Theorem 1.2.1.* The symplectic leaves of $X := S^n Y$ are exactly the quotients $\{z \in Y^n \mid \operatorname{Stab}_{S_n}(z) = G\}/N_{S_n}G$, where $G < S_n$ has the form $G = S_{i_1}^{r_1} \times \cdots \times S_{i_k}^{r_k}$, and $N_{S_n}G < S_n$ is its normalizer. Here, we may choose $i_1 < i_2 < \cdots < i_k$. By [ES10a, Theorem 3.1] and its proof, $M(X)$ is a holonomic $\mathcal{D}$-module whose composition factors are intermediate extensions of local systems (i.e., $\mathcal{O}$-coherent $\mathcal{D}$-modules) on the symplectic leaves. Let $G = S_{i_1}^{r_1} \times \cdots \times S_{i_k}^{r_k}$ be a fixed subgroup of S_n as above (with $r_1 i_1 + \cdots + r_k i_k = n$). Let $(X^G)^\circ \subseteq X$ be the corresponding symplectic leaf, and let X^G denote its closure. Set $U := X \setminus (X^G \setminus (X^G)^\circ)$. One has an obvious surjection $M(U) \twoheadrightarrow \Omega_{(X^G)^\circ}$ sending 1 to the volume form. As a result, the intermediate extension of $\Omega_{(X^G)^\circ}$ is a composition factor of $M(X)$. To deduce the desired result, therefore, it suffices to show that these are all the composition factors, occurring with multiplicity one, and that $M(X)$ is semisimple.

In order to prove that $M(X)$ is semisimple, we prove a more general result: let $\phi : Y^n \twoheadrightarrow X$ be the defining surjection, and consider the $\mathcal{D}$-module $M_\phi(Y^n)$ defined in [ES10a]: this is the quotient of D_{Y^n} by the right ideal generated by Hamiltonian vector fields of the form $\xi_{\phi^* f}$ for $f \in \mathcal{O}_X = \mathcal{O}_Y^{S_n}$. According to [ES10a, Theorem 3.1], $M_\phi(Y^n)$ is a holonomic $\mathcal{D}$-module on Y^n, and moreover $(\pi_* M_\phi(Y^n))^{S_n} \cong M(X)$. Thus, it suffices to show that $M_\phi(Y^n)$ is semisimple.

To prove this, we recall again from [ES10a, Theorem 3.1] and its proof that the singular support of $M_\phi(Y^n)$ in $T^*(Y^n)$ is contained in the locus of pairs (z, v) with $z \in Y^n, v \in T^*_z Y^n$, such that $v \cdot \xi_{\phi^* f}|_v = 0$ for all $f \in \mathcal{O}_X$. This is the union of the conormal bundles of the inverse images of symplectic leaves on X. Specifically, the closure of the inverse image of each symplectic leaf is of the form $Y^{r_1+\cdots+r_k} \subseteq (Y^{i_1})^{r_1} \times \cdots \times (Y^{i_k})^{r_k} = Y^n$. Hence, the composition factors of $M_\phi(Y^n)$ are S_n-equivariant local systems on these smooth, closed subvarieties.

We claim that Ext^1 between any two such $\mathcal{D}$-modules supported on distinct diagonals is trivial. Since the singular supports of these $\mathcal{D}$-modules are the conormal bundles of the given smooth symplectic subvarieties, the claim follows from the more general

Lemma 2.1.1. Suppose that Z is a smooth variety, and $Z_1, Z_2 \subseteq Z$ as well as $Z_1 \cap Z_2$ are smooth closed subvarieties, all of pure dimension. Let $\mathcal{L}_1, \mathcal{L}_2$ be local systems on Z_1 and Z_2, respectively, and let $i_1 : Z_1 \to Z$ and $i_2 : Z_2 \to Z$ be the inclusions. Then,

$$\mathrm{Ext}^j((i_1)_* \mathcal{L}_1, (i_2)_* \mathcal{L}_2) = 0, \text{ for } j < (\dim Z_1 - \dim Z_1 \cap Z_2) + (\dim Z_2 - \dim Z_1 \cap Z_2). \tag{2.1.2}$$

Namely, the result follows from the lemma since, in our case, Z_1, Z_2, and $Z_1 \cap Z_2$ are all even dimensional and $Z_1 \neq Z_2$.

Proof of Lemma 2.1.1. By adjunction, the LHS of (2.1.2) identifies with

$$\mathrm{Ext}^j(i_2^*(i_1)_* \mathcal{L}_1, \mathcal{L}_2). \tag{2.1.3}$$

Next, let $i_{12,k} : Z_1 \cap Z_2 \to Z_k$ be the inclusions for $k \in \{1, 2\}$. Then, applying proper base change for the closed embedding $i_{12,2}$, we can rewrite (2.1.3) as

$$\mathrm{Ext}^j((i_{12,2})_* i_{12,1}^* \mathcal{L}_1, \mathcal{L}_2). \tag{2.1.4}$$

Since $i_{12,2}$ is a closed embedding, $(i_{12,2})_* = (i_{12,2})_!$. Applying adjunction, we obtain

$$\mathrm{Ext}^j(i_{12,1}^* \mathcal{L}_1, i_{12,2}^! \mathcal{L}_2). \tag{2.1.5}$$

Now, $i_{12,1}^* \mathcal{L}_1$ is a local system shifted by $-(\dim Z_1 - \dim Z_1 \cap Z_2)$, and $i_{12,2}^! \mathcal{L}_2$ is a local system shifted by $\dim Z_2 - \dim Z_1 \cap Z_2$. So, the above vanishes when $j < (\dim Z_1 - \dim Z_1 \cap Z_2) + (\dim Z_2 - \dim Z_1 \cap Z_2)$ (or when $j > (\dim Z_1 - \dim Z_1 \cap Z_2) + (\dim Z_2 - \dim Z_1 \cap Z_2) + \dim Z_1 \cap Z_2$). ☐

Remark 2.1.6. In fact, the above lemma is needed for the omitted proof of [ES10a, Theorem 4.21]. So, even though we could have deduced semisimplicity from that theorem, the above argument cannot be avoided.

It remains to prove that the intermediate extensions $\Omega_{(X^G)^\circ}$ are all of the composition factors of $M(X)$, and that they occur with multiplicity one. Then, the irreducible composition factors of $M_\phi(Y^n)$ are all supported on distinct diagonal subvarieties of Y^n, so the above argument implies that $M_\phi(Y^n)$, and hence $M(X)$, are semisimple. Since the composition factors are exactly the claimed direct summands of $M(X)$, the theorem also follows.

So, we prove that the intermediate extensions of $\Omega_{(X^G)^\circ}$, i.e., the IC D-modules of $(X^G)^\circ$, are all of the composition factors of $M(X)$, and that they occur with multiplicity one. It suffices to consider the formal neighborhood of a point of $(X^G)^\circ$. Then, the computation reduces to the case that $G = S_n$ and $X^G = (X^G)^\circ = Y \subseteq S^n Y$, and moreover, we may reduce to the case that $Y = V$ is a symplectic vector space, and consider the formal neighborhood of zero,

$\widehat{\mathcal{O}_V} = \mathbb{C}[\![x_1, \ldots, x_d, y_1, \ldots, y_d]\!]$. Let δ_V be the delta-function $\mathcal{D}$-module of the diagonal $V \subseteq S^n V$. Since V is now a symplectic vector space [ES10a, Corollary 4.16] implies that $M(S^n V)$ is semisimple, and a direct sum of IC D-modules of the symplectic leaves with some multiplicities (in fact, *op. cit.* implies that the multiplicity of δ_V is $\dim \mathsf{HP}_0(\mathcal{O}_{V^{n-1}}^{S_n})$, which would reduce us to Theorem 1.1.8, but we will instead deduce that theorem from the present one). It suffices to prove

$$\operatorname{Hom}_{D_{S^n V}}(M(S^n V), \delta_V) \cong \mathbb{C}. \tag{2.1.7}$$

This may be restated and proved without the use of $\mathcal{D}$-modules:

Lemma 2.1.8. The space of symmetric polydifferential operators $\psi : (\mathcal{O}_V)^{\otimes(n-1)} \to \mathcal{O}_V$ invariant under Hamiltonian flow is one-dimensional, and spanned by the multiplication map.

Note that, actually, we only need to show that there are no S_n-invariant operators, with the S_n action given by viewing the polydifferential operators in the lemma as distributions on n functions; the lemma is a slightly more general result, requiring only S_{n-1}-invariance.

We remark that this lemma is tantamount to Theorem 1.1.8, i.e., one can directly show that the above space of polydifferential operators is identified with $\mathsf{HP}_0(\mathcal{O}_{V^{n-1}/S_n}, \mathcal{O}_V)$ (at least if we require that the operators be S_n-invariant). For details, see [RS10, §4].

We further remark that the space mentioned in the lemma can alternatively be viewed as the space of C^∞ Hamiltonian-invariant distributions on $S^n V$ supported on the diagonal, since finite-dimensionality guarantees that $\operatorname{Hom}_{D_{S^n V}}(M(S^n V), \delta_V)$ is the same when considered in the C^∞ context. Then, a polydifferential operator ψ of degree $n-1$ becomes a distribution Ψ on V^n by the prescription $\Psi(f_1, \ldots, f_n) = \int \psi(f_1, \ldots, f_{n-1}) f_n$. They are supported on the diagonal since they depend only on (finitely many) partial derivatives of $f_1 \times \cdots \times f_n$ evaluated at the diagonal.

Proof. It suffices to pass to the formal completion and consider polydifferential operators on $\widehat{\mathcal{O}_V}$. Such polydifferential operators are determined by their value on elements $f^{\otimes(n-1)}$ for $f \in \widehat{\mathcal{O}_V}$, since they are symmetric and hence determined by their restriction to $\mathsf{Sym}^{n-1} \widehat{\mathcal{O}_V}$. Furthermore, we can assume that $f'(0) \neq 0$, since the complement of this locus in the pro-vector space $\widehat{\mathcal{O}_V}$ has codimension equal to $\dim V \geq 2$.

Next, by the formal Darboux theorem, by applying a formal symplectomorphism of V, we may assume $f = x_1$. Since all formal symplectomorphisms are obtained by integrating Hamiltonian vector fields, it suffices to consider the value $\psi(x_1^{\otimes(n-1)})$. This value must be a function that depends only on x_1, since these are the only functions invariant under all symplectomorphisms fixing x_1. By linearity and invariance under conjugation by rescaling x_1 (and applying the inverse scaling to y_1), we deduce that $\psi(x_1^{\otimes(n-1)}) = \lambda \cdot x_1^{n-1}$ for some $\lambda \in \mathbb{C}$. Thus, on $x_1^{\otimes(n-1)}$, ψ coincides with λ times the multiplication operator, $f_1 \otimes \cdots \otimes f_{n-1} \mapsto \lambda f_1 \cdots f_{n-1}$. The latter operator is evidently symmetric and invariant under Hamiltonian flow. On the other hand, we have argued that a symmetric operator invariant under Hamiltonian flow is uniquely determined by its value on $x_1^{\otimes(n-1)}$. So ψ is equal to λ times the multiplication operator, as desired. □

2.1.2. *Proofs of Corollaries 1.2.4 and 1.2.6.*

Proof of Corollary 1.2.4. By definition, for all (affine) Poisson varieties X, $\mathsf{HP}_\bullet^{DR}(X) = \pi_* M(X)$, where π_* is the *derived* pushforward of the $\mathcal{D}$-module $M(X)$ to a point. This identifies the first term in (1.2.5) with the LHS of (1.2.2).

Next, since Y is symplectic, by [ES10a, Example 2.6], $M(Y) = \Omega_Y$. Therefore, $\pi_*((\Delta_i)_*(\Omega_Y)^{\boxtimes r}) \cong (\mathsf{HP}_\bullet^{DR}(Y))^{\otimes r}$, with the canonical S_r action given by permutation of components. This identifies the second term in (1.2.5) with the RHS of (1.2.2).

It remains to consider the third term in (1.2.5). Here, we use again the fact that $M(Y) = \Omega_Y$, together with the standard fact that, since Y is smooth and connected, $H_\bullet(\pi_* \Omega_Y) \cong H^{\dim Y - \bullet}(Y)$. ☐

Proof of Corollary 1.2.6. It suffices to show that the third term in (1.2.5) is identified with the first term in (1.2.7). By results of Nest-Tsygan [NT95], one has an isomorphism

$$H^{\dim Y - \bullet}(Y)(\!(\hbar)\!) \cong \mathsf{HP}_\bullet(Y)(\!(\hbar)\!) \xrightarrow{\sim} \mathsf{HH}_\bullet(A_\hbar[\hbar^{-1}]), \tag{2.1.9}$$

where $\mathsf{HP}_\bullet(Y)$ is the usual Poisson homology (which is well known to be isomorphic to $H^{\dim Y - \bullet}(Y)$ when Y is symplectic, since the Poisson homology complex identifies with the de Rham complex). Thus, it suffices to show that

$$\bigoplus_{n \geq 0} \mathsf{HH}_\bullet(\mathsf{Sym}^n A_\hbar)[\hbar^{-1}]^* \cong \mathsf{Sym}(\mathsf{HH}_\bullet(A_\hbar[\hbar^{-1}])^*[t]), \tag{2.1.10}$$

again taking the $\mathbb{C}(\!(\hbar)\!)$-linear dual. Since $A_\hbar$ is an infinite-dimensional, simple algebra with trivial center, this follows from [EO06, Corollary 3.3]. Since we will need this again later, we state it below. ☐

We used here and will continue to use the following result from [EO06], which we state somewhat more explicitly than is in *op. cit.* (we omit the proof of the more explicit formula, as we do not essentially need it):

Theorem 2.1.11. [EO06, Corollary 3.3] Let A be an infinite-dimensional simple algebra over a field of characteristic zero with trivial center. Then, the coalgebra $\mathcal{H}_\bullet(A) := \bigoplus_{n \geq 0} \mathsf{HH}_\bullet(\mathsf{Sym}^n A)$ is a polynomial coalgebra,

$$\Psi : \mathcal{H}_\bullet(A) \xrightarrow{\sim} \mathsf{Sym}(\mathsf{HH}_\bullet(A)[t]), \tag{2.1.12}$$

where the isomorphism is the unique coalgebra map which is graded with respect to $|\mathsf{HH}_\bullet(A)| = |t| = 1$ such that, for every n, composition with the projection to $t^{n-1}\mathsf{HH}_\bullet(A)$ restricts on $\mathsf{HH}_\bullet(\mathsf{Sym}^n A)$ to a map

$$\mathsf{HH}_\bullet(\mathsf{Sym}^n A) \to \mathsf{Sym}(\mathsf{HH}_\bullet(A)[t]) \twoheadrightarrow t^{n-1}\mathsf{HH}_\bullet(A) \tag{2.1.13}$$

of the form, in Hochschild degree zero,

$$[a_1 \& \cdots \& a_n]) \mapsto \frac{1}{n!} \sum_{\sigma \in S_n} t^{n-1}[a_{\sigma(1)} \cdots a_{\sigma(n)}], \tag{2.1.14}$$

and similarly is the natural multiplication map on Hochschild m-chains for all $m \geq 0$,

$$c_0 \otimes \cdots \otimes c_m \mapsto c_0' \otimes \cdots \otimes c_m',$$

where $c_i \mapsto c_i'$ is the map (2.1.14).

2.2. Proofs of theorems and corollaries from §1.1.

2.2.1. *Proofs of Theorems 1.1.1 and 1.1.8.* Theorem 1.1.1 already follows from the corollary 1.2.4 of Theorem 1.2.1, so it remains only to prove Theorem 1.1.8.

Proof of Theorem 1.1.8. As in the introduction, write $S^n V \xrightarrow{\sim} (V \times V^{n-1}/S_n)$ where the map to the first factor is given by averaging. We deduce that $M(S^n V) \cong \Omega_V \boxtimes M(V^{n-1}/S_n)$. Recall from [ES10a, Theorem 4.13] that, for any symplectic vector space U and finite subgroup $G < \mathsf{Sp}(U)$, the space $\mathsf{HP}_0(\mathcal{O}_U^G)$ naturally identifies with the multiplicity space of the delta-function $\mathcal{D}$-module of the origin in $M(U/G)$, which is semisimple.[2] Hence, it also identifies with the multiplicity space

[2]For general affine Poisson varieties X and $x \in X$, the space $\mathrm{Hom}_{D_X}(M(X), \delta_x)$ identifies with a subspace of $\mathsf{HP}_0(\mathcal{O}_X)^*$; when $\mathcal{O}_X$ is nonnegatively graded with the ideal of x as the augmentation ideal, this is an equality.

of the delta-function $\mathcal{D}$-module of the diagonal $V \subset S^n V$ in $M(S^n V)$. This multiplicity space is one-dimensional (in fact, the main step of the proof of Theorem 1.2.1 was to show this). $\square$

In the appendix, we will give a different, elementary proof of Theorem 1.1.8. A proof without using $M(S^n V)$, requiring only the Darboux theorem, can also be obtained from Lemma 2.1.8 following the comments after the statement of the lemma.

2.2.2. *Proofs of Corollaries 1.1.4–1.1.6.*

Proof of Corollary 1.1.4. This is immediate by expanding the LHS of (1.1.2), since $\mathsf{HP}_0(\mathcal{O}_Y) \cong H^{\dim Y}(Y)$: namely, the subspace of the LHS spanned by terms of the form $f_1 t^{r_1} \& \cdots \& f_m t^{r_m}$ for fixed $r_1 \leq r_2 \leq \cdots \leq r_m$ has a basis by requiring that the f_i lie in a fixed basis of $\mathsf{HP}_0(\mathcal{O}_Y)$; then the number of these is the number of $\dim \mathsf{HP}_0(\mathcal{O}_Y)$-multipartitions $(\lambda_1, \ldots, \lambda_{\dim \mathsf{HP}_0(\mathcal{O}_Y)})$ of n, i.e., $|\lambda_1| + \cdots + |\lambda_{\dim \mathsf{HP}_0(\mathcal{O}_Y)}| = n$, such that there are a total of m parts appearing in all the partitions, of lengths $r_1 + 1, \ldots, r_m + 1$. (In particular, $(r_1 + 1) + \cdots + (r_m + 1) = n$.) $\square$

Proof of Corollary 1.1.5. This is a direct consequence of Corollary 1.2.6 (or we can prove it in the same manner, using only Theorem 1.1.1 rather than Theorem 1.2.1). $\square$

Proof of Corollary 1.1.6. This is an immediate consequence of Corollary 1.1.4, using the canonical surjection $\mathsf{HP}_0(\mathcal{O}_{S^n Y})(\!(\hbar)\!) \twoheadrightarrow \mathsf{HH}_0(B_\hbar[\hbar^{-1}])$. $\square$

2.2.3. *Proofs of Corollaries 1.1.10–1.1.14.*

Proof of Corollary 1.1.10. By Theorem 1.1.8, $\mathsf{HP}_0(\mathcal{O}_{V^n}^{S_{n+1}}) \cong \mathbb{C}$; it suffices to show that $\mathsf{HH}_0(\mathcal{D}_{U^n}^{S_{n+1}}) \cong \mathbb{C}$. This is a consequence of [AFLS00]: the dimension of $\mathsf{HH}_0(\mathcal{D}_{U^n}^{S_{n+1}})$ is equal to the number of conjugacy classes of elements in S_{n+1} which act without eigenvalue one on U^n; there is exactly one such conjugacy class, namely the conjugacy class of the $(n+1)$-cycle. $\square$

Proof of Corollary 1.1.12. This is Corollary 1.1.10 in the case that $\dim V = 2$. $\square$

Proof of Corollary 1.1.14. By upper semicontinuity of $\dim \mathsf{HH}_0(B)$ in the family of filtered quantizations B, it suffices to show that, for generic spherical rational Cherednik algebras B deforming $\mathcal{O}_{\mathbb{C}^{2n}}^{S_{n+1}}$, $\dim \mathsf{HH}_0(B) = 1$. By [EG02, Theorem 1.8], for generic B, $\mathsf{HH}_\bullet(B) \cong \mathsf{HH}_\bullet(\mathcal{D}_{\mathbb{C}^n}^{S_{n+1}})$. Thus, the result follows from [AFLS00], as already explained in the proof of Corollary 1.1.10 (or one can simply refer to that corollary or Corollary 1.1.12). $\square$

2.2.4. *Proofs of Corollaries 1.1.16 and 1.1.18.*

Proof of Corollary 1.1.16. Losev's [ES10a, Appendix A] implies the following result. Let X be an affine Poisson variety with finitely many (locally closed) symplectic leaves $X_1, \ldots, X_k$. Let $B_\hbar$ or B be a deformation or filtered quantization of $\mathcal{O}_X$ (the latter only in the case that $\mathcal{O}_X$ is nonnegatively graded). For each symplectic leaf X_i let $x_i \in X_i$ be a point. Let $\hat{\mathcal{O}}_{X,x_i}$ be the formal completion of $\mathcal{O}_X$ at x_i. Now, write $\hat{X}_{x_i} := \mathsf{Spf}\, \hat{\mathcal{O}}_{X,x_i}$ for the formal neighborhood of x_i in X, where Spf refers to the "formal" spectrum of prime ideals in $\mathcal{O}_{X,x_i}$ which are closed under the $\mathfrak{m}_{x_i}$-adic topology, and $\mathfrak{m}_{x_i}$ is the maximal ideal associated to $x_i \in X$. According to [Kal06, Proposition 3.3], there is an isomorphism $\hat{X}_{x_i} \cong \hat{(X_i)}_{x_i} \hat{\times} \hat{Z}_i$, for some "slice" subvariety $\hat{Z}_i \subseteq \hat{X}_{x_i}$. That is, $\hat{\mathcal{O}}_{X,x_i} \cong \hat{\mathcal{O}}_{X_i,x_i} \hat{\otimes} \mathcal{O}_{\hat{Z}_i}$, where $\mathcal{O}_{\hat{Z}_i}$ is a quotient of $\mathcal{O}_{X,x_i}$ by a complete ideal, and $\hat{\otimes}$ denotes the completed tensor product.

We will need to consider the space $\mathsf{HP}_0(\mathcal{O}_{\hat{Z}_i}) = \mathcal{O}_{\hat{Z}_i} / \{\mathcal{O}_{\hat{Z}_i}, \mathcal{O}_{\hat{Z}_i}\}$. (Note that, as pointed out in [ES10a, Proposition 3.10], $\{\mathcal{O}_{\hat{Z}_i}, \mathcal{O}_{\hat{Z}_i}\}$ is a closed subspace of $\mathcal{O}_{\hat{Z}_i}$ in the adic topology, and $\mathsf{HP}_0(\mathcal{O}_{\hat{Z}_i}) = \lim_{n \to \infty} \mathsf{HP}_0(\mathcal{O}_{\hat{Z}_i} / \mathfrak{m}_{x_i}^n)$.)

The following is a direct consequence of [ES10a, Appendix A], using [ES10a, Proof of Corollary 3.13]:

Theorem 2.2.1. [ES10a, Appendix A] Every prime ideal of $B_\hbar[\hbar^{-1}]$ over $\mathbb{C}((\hbar))$ is supported on $\overline{X_i}$ for some i. For each i, the number of such ideals is at most $\dim_{\mathbb{C}((\hbar))} \mathsf{HP}_0(\mathcal{O}_{\hat{Z}_i})((\hbar))$.

Now, the closures of the symplectic leaves of S^nY are exactly the images of all possible compositions $Y^m \to (S^{i_1}Y)^{r_1} \times \cdots \times (S^{i_k}Y)^{r_k} \to S^nY$, where $m = r_1 + \cdots + r_k$, $n = r_1 i_1 + \cdots + r_k i_k$, the first map is the product of the m diagonal embeddings $\Delta_{i_1}^{r_1} \times \cdots \times \Delta_{i_k}^{r_k}$, and the second map is the obvious projection. At a point x_i of the locally closed symplectic leaf X_i with this closure, the slice $\hat{Z}_i$ such that $(\widehat{S^nY})_{x_i} \cong \hat{X}_{x_i} \hat{\times} \hat{Z}_i$ can be taken to be isomorphic to the formal neighborhood of the origin in $(V^{i_1-1}/S_{i_1})^{r_1} \times \cdots \times (V^{i_k-1}/S_{i_k})^{r_k}$, where V is a symplectic vector space of dimension equal to $\dim Y$. Namely, if $\phi : Y^n \to S^nY$ is the projection, we can consider a preimage $\widetilde{x}_i \in S^nY$ of Y^n and look at the completed conormal fiber of $\phi^{-1}(X_i)$ at $\widetilde{x}_i$, then project back down to S^nY, to get $\hat{Z}_i$.

Therefore, $\mathsf{HP}_0(\mathcal{O}_{\hat{Z}_i}) \cong \mathsf{HP}_0(V^{i_1-1}/S_{i-1})^{\otimes r_1} \otimes \cdots \otimes \mathsf{HP}_0(V^{i_k-1}/S_{i-1})^{\otimes r_k} \cong \mathbb{C}$. So, there is at most one prime ideal supported on $\overline{X_i}$. Note that $\operatorname{codim}_{S^nY}(\overline{X_i}) = (n-m)\dim Y$, where $m = r_1 + \cdots + r_k$ as above.

Thus, the number of prime ideals with support of codimension $(n-m)\dim Y$ is at most the number of partitions of n with m parts. This immediately implies the statement. □

Proof of Corollary 1.1.18. Note that $V^{n+1}/S_{n+1} \cong V \times V^n/S_{n+1}$, with the projection $V^{n+1}/S_{n+1} \to V$ given by averaging the $n+1$ elements of V in the ordered $(n+1)$-tuple, and the map $V^{n+1}/S_{n+1} \to V^n/S_{n+1}$ given by subtracting the average from each element of the $(n+1)$-tuple. Therefore, the symplectic leaves of V^{n+1}/S_{n+1} are all of the form $V \times X_i$ where X_i is a symplectic leaf of V^n/S_{n+1}, and this establishes a bijection between the symplectic leaves of V^{n+1}/S_{n+1} and those of V^n/S_{n+1}. The corollary then follows from Corollary 1.1.16. □

2.2.5. *Proofs of Corollaries 1.1.19–1.1.20.*

Proof of Corollary 1.1.19. This follows from Corollary 1.1.4 and the surjection $\mathsf{HP}_0(\mathcal{O}_{S^nY})((\hbar)) \twoheadrightarrow \operatorname{gr} \mathsf{HP}_0(B_\hbar[\hbar^{-1}])$. □

Proof of Corollary 1.1.20. By the comments before the corollary, it suffices to show that $\dim \mathsf{HP}_0(B) = 1$ when B is a commutative spherical Cherednik algebra which is a filtered Poisson deformation of $\mathcal{O}_{\mathbb{C}^{2n}}^{S_{n+1}}$. The fact that $\dim \mathsf{HP}_0(B) \leq 1$ follows from the surjection $\mathsf{HP}_0(\mathcal{O}_{V^n}^{S_{n+1}}) \twoheadrightarrow \operatorname{gr} \mathsf{HP}_0(B)$ and Theorem 1.1.8. For the opposite inequality, by upper semicontinuity of $\dim \mathsf{HP}_0(B)$, it suffices to show that, for generic commutative spherical Cherednik algebras B deforming $\mathcal{O}_{\mathbb{C}^{2n}}^{S_{n+1}}$, one has $\mathsf{HP}_0(B) \neq 0$. This result follows because, by [EG02, Corollary 1.14], generic spherical rational Cherednik algebras B deforming $\mathcal{O}_{\mathbb{C}^{2n}}^{S_{n+1}}$ are of the form $B = \mathcal{O}_X$ for X smooth and symplectic, with one-dimensional top cohomology, i.e., $\mathsf{HP}_0(B) \cong \mathbb{C}$. (Alternatively, without using that $\operatorname{\mathsf{Spec}} B$ is smooth for generic B, one could take a deformation quantization $B_\hbar$ of B such that $B_\hbar[\hbar^{-1}]$ is isomorphic to a noncommutative spherical Cherednik algebra over $\mathbb{C}((\hbar))$ deforming $\mathcal{O}_{\mathbb{C}^{2n}}^{S_{n+1}}((\hbar))$, so $\dim \mathsf{HH}_0(B_\hbar[\hbar^{-1}]) = 1$ by [EG02, Theorem 1.8] (or by Corollary 1.1.14, which uses *op. cit.*). Then, one concludes using the canonical surjection $\mathsf{HP}_0(B)((\hbar)) \to \operatorname{gr} \mathsf{HH}_0(B_\hbar[\hbar^{-1}])$.) □

Appendix A. Type D Weyl groups, by T. Schedler

In this appendix, we compute $\mathsf{HP}_0(\mathcal{O}_{\mathbb{C}^{2n}}^{D_n})$, where $D_n < \mathsf{GL}(\mathbb{C}^n) < \mathsf{Sp}(\mathbb{C}^{2n})$ is the type D_n Weyl subgroup. Recall that $D_n = S_n \ltimes (\mathbb{Z}/2)^{n-1}$, and we let $\mathbb{C}^n$ be its reflection representation, where S_n acts by permuting components, and $(\mathbb{Z}/2)^{n-1}$ acts by diagonal matrices whose diagonal entries are ± 1 which have determinant one (i.e., an even number of -1 entries).

Note that D_n is an index-two subgroup of $B_n = C_n = S_n \ltimes (\mathbb{Z}/2)^n$. Also, $\bigoplus_{n\geq 0} \mathsf{HP}_0(\mathcal{O}_{\mathbb{C}^{2n}}^{B_n})^*$ is a bigraded algebra, graded by the symmetric power degree, n, and the weight degree (degree of polynomials in $\mathcal{O}_{\mathbb{C}^{2n}}$ for all n). Recall from [ES09]:

Theorem A.0.2. [ES09][3] There is an isomorphism of bigraded algebras

$$\bigoplus_{n\geq 0} \mathsf{HP}_0(\mathcal{O}_{\mathbb{C}^{2n}}^{B_n})^* \cong \mathbb{C}[s_1, s_2, \ldots], \tag{A.0.3}$$

where s_i has symmetric power degree i and weight $4(1-i)$.[4]

Here, the algebra structure on the LHS arises from the symmetrization map: precisely, given $\phi \in (\mathsf{Sym}^m \mathcal{O}_{\mathbb{C}^2}^{\mathbb{Z}/2})^* = (\mathcal{O}_{\mathbb{C}^{2m}}^{B_m})^*$ and $\psi \in (\mathsf{Sym}^n \mathcal{O}_{\mathbb{C}^2}^{\mathbb{Z}/2})^* = (\mathcal{O}_{\mathbb{C}^{2n}}^{B_n})^*$, then $\phi \cdot \psi \in (\mathsf{Sym}^{m+n} \mathcal{O}_{\mathbb{C}^2}^{\mathbb{Z}/2})^*$ is defined by $\phi \cdot \psi = \phi \boxtimes \psi$, viewing $\mathsf{Sym}^{m+n} \mathcal{O}_{\mathbb{C}^2}^{\mathbb{Z}/2}$ as the subspace of $T^{m+n}\mathcal{O}_{\mathbb{C}^2}^{\mathbb{Z}/2}$ of symmetric tensors.

We now compute $\mathsf{HP}_0(\mathcal{O}_{\mathbb{C}^{2n}}^{D_n})$. Let us view s_i as the coordinate functions on the infinite-dimensional space $\mathbb{C}[\![x^2]\!]$ (we will explain why in the proof), so that, for all $f \in \mathbb{C}[\![x^2]\!]$,

$$f = s_1(f) + s_2(f)x^2 + s_3(f)x^4 + \cdots .$$

We need to define certain vector fields ξ_k on $\mathbb{C}[\![x^2]\!]$ for $k \geq 1$. First, let $Q(z)$ be the Taylor series of $\sqrt{1+z}$, i.e., $Q(z) = 1 + \frac{z}{2} - \frac{z^2}{8} + \frac{z^3}{16} - \cdots$. Then, we define

$$\xi_k := \mathcal{V}\Big(\frac{d}{dx}\Big(x^{2k-1} \cdot Q\big(\frac{1}{s_{2k}} \sum_{i\geq 2k+1} s_i x^{2(i-2k)}\big)\Big)\Big), \tag{A.0.4}$$

where

$$\mathcal{V}\big(\sum_{i\geq 0} f_i x^{2i}\big) := \sum_{i\geq 0} f_i \partial_{s_{i+1}}, \quad f_i \in \mathbb{C}[s_1, s_2, \ldots].$$

Explicitly, the first few terms of ξ_k can be written out as

$$\xi_k = \big((2k-1)\partial_{s_k} + \frac{2k+1}{2}\frac{s_{2k+1}}{s_{2k}}\partial_{s_{k+1}} + \cdots\big), \tag{A.0.5}$$

where here $\cdots$ means terms that are multiples of s_{2k+j} for $j \geq 2$.

Theorem A.0.6. The sum $\bigoplus \mathsf{HP}_0(\mathcal{O}_{\mathbb{C}^{2n}}^{D_n})^*$ is naturally a bigraded subalgebra of $\bigoplus_{n\geq 0} \mathsf{HP}_0(\mathcal{O}_{\mathbb{C}^{2n}}^{B_n})^*$. In terms of (A.0.3), it is the subalgebra of elements f such that, for all $k \geq 1$,

$$\xi_k(f)|_{s_1=\cdots=s_{2k-1}=0, s_{2k}\neq 0} = 0. \tag{A.0.7}$$

Remark A.0.8. [5] It is interesting to try to integrate the above vector fields, in order to interpret solutions $f \in \bigoplus_{n\geq 0} \mathsf{HP}_0(\mathcal{O}_{\mathbb{C}^{2n}}^{D_n})^*$ as functions on $\mathbb{C}[\![x^2]\!]$ invariant under a certain flow. We can interpret this flow as follows: A curve $h(t)$ in $\mathbb{C}[\![x^2]\!]$ is invariant if and only if $h_t = -(\sqrt{h})_x$, i.e., setting $u := 2\sqrt{h}$, we should have

$$u_x + uu_t = 0.$$

This equation is the well known inviscid Burgers equation (with t and x swapped). Then, the solutions should look like $u = f(t - ux)$ for some function f.

At $t = 0$, we obtain $u(0,x) = f(-u(x,0)x)$. So, in the case that $u(x,0) \in x\mathbb{C}[\![x^2]\!]^\times$, i.e., $h(0) \in \mathbb{C}^\times \cdot x^2 + x^4\mathbb{C}[\![x^2]\!]$, this implies that $f = \sqrt{g}$ where g has linear behavior near 0. This implies

[3]Note that [ES09, Theorem 1.1.3] is for the much more general situation of symmetric powers of isolated surface singularities in $\mathbb{C}^3$ with a contracting $\mathbb{C}^*$-action, but we only need the case of the surface $\mathbb{C}^2/(\mathbb{Z}/2)$.

[4]We assign s_i *nonpositive* weight because it lies in the dual space to $\mathsf{HP}_0(\mathcal{O}_{\mathbb{C}^{2n}}^{B_{i+1}})$, which is assigned nonnegative weight.

[5]Thanks to P. Etingof for pointing out this observation.

that $u^2 = g(t - ux)$, and letting G be an inverse of g, we can write $G(u^2) + ux - t = 0$, which can now be solved for u. For example, if $g(z) = -z$, then we obtain $u(x,t) = \frac{x+\sqrt{x^2-4t}}{2}$.

However, it is not clear whether one can use this to simplify the description of the algebra $\bigoplus_{n\geq 0} \mathsf{HP}_0(\mathcal{O}_{\mathbb{C}^{2n}}^{D_n})^*$.

Corollary A.0.9. For $n \geq 7$, $\mathsf{gr}\,\mathsf{HH}_0(\mathcal{D}_{\mathbb{C}^n}^{D_n})^*$ is a proper subspace of $\mathsf{HP}_0(\mathcal{O}_{\mathbb{C}^{2n}}^{D_n})^*$. For $n \leq 6$, the natural inclusion is an equality, $\mathsf{gr}\,\mathsf{HH}_0(\mathcal{D}_{\mathbb{C}^n}^{D_n})^* = \mathsf{HP}_0(\mathcal{O}_{\mathbb{C}^{2n}}^{D_n})^*$.

Proof. Clearly it suffices to show that $\dim \mathsf{HH}_0(\mathcal{D}_{\mathbb{C}^n}^{D_n}) = \dim \mathsf{HP}_0(\mathcal{O}_{\mathbb{C}^{2n}}^{D_n})$ if and only if $n \leq 6$. By the main result of [AFLS00], for an arbitrary symplectic vector space V, $G < \mathsf{Sp}(V)$, and Lagrangian $U \subseteq V$, the dimension of $\mathsf{HH}_0(\mathcal{D}_U^G)$ is equal to the number of conjugacy classes of elements $g \in G$ such that $g - Id$ is invertible (acting on V). (However, this says nothing about the *filtration* on $\mathsf{HH}_0(\mathcal{D}_U^G)$, which we deduce in this corollary.) In the case at hand with $G = D_n$, the dimension of $\mathsf{HH}_0(\mathcal{D}_{\mathbb{C}^n}^{D_n})$ therefore equals the number of partitions of n with an even number of parts.

Note that solutions of (A.0.7), in particular, include all multiples of s_1^2. One can inductively prove that, for $n > 10$, there are more of the latter type of partitions than there are of the former. Alternatively, more linearly independent solutions of (A.0.7) are given, for every monomial g in $s_2, s_3, \ldots, s_{k+1}$, by $s_2^k s_{k+1} \cdot g - s_1 \xi_1(s_2^k s_{k+1} \cdot g)$ (this is a polynomial, and not merely a Laurent polynomial, because of the restriction on g). One can inductively prove that the number of these plus the number of monomial multiples of s_1^2 exceed the number of even partitions of n for $n > 8$ and $n = 7$; then it remains only to consider the case $n = 8$, where one can find an additional solution not spanned by these (as reported in Figure 1); it lies in weight -20. The fact that the isomorphism stated in the corollary holds for $n \leq 6$ is a consequence of a straightforward explicit computation, or see Figure 1. ☐

The above theorem, along with Theorem 1.1.8 (for the type A_n cases) and the results of [ES09] (which imply the $B_n = C_n$ cases) complete the computation of Poisson traces for varieties $(\mathfrak{h}\oplus\mathfrak{h}^*)/W$ for W one of the classical series ($A, B = C$, and D) of finite Weyl groups and $\mathfrak{h}$ its reflection representation. Little is known about the exceptional cases: only the case G_2 was computed in [AF09]. We also remark that, if we consider also the finite Coxeter groups, the additional rank ≤ 3 cases ($I_2(m)$ and H_3) are computed in [EGP+]. In all of these cases, one has $\mathsf{HP}_0(\mathcal{O}_{\mathfrak{h}\oplus\mathfrak{h}^*}^W) \cong \mathsf{gr}\,\mathsf{HH}_0(\mathcal{D}_{\mathfrak{h}}^W)$.

A.1. **Filtered quantizations and Poisson deformations.** Here we explain the analogous corollaries to those in the main body of the paper, now for type D_n rather than type A_n Weyl groups. For all n, let d_n be the dimension of $\mathsf{HP}_0(\mathcal{O}_{\mathbb{C}^{2n}}^{D_n})$, as follows from the theorem (for $n \leq 34$, this can also be obtained by evaluating the polynomials in Figures 1 and 2 at $t = 1$). The next corollary is an analogue of Corollary 1.1.10, and is proved in the same manner:

Corollary A.1.1. Let B be a filtered quantization of $\mathcal{O}_{\mathbb{C}^{2n}}^{D_n}$. Then, $\dim \mathsf{HH}_0(B) \leq d_n$, and the number of irreducible finite-dimensional representations of B is at most d_n.

In particular, this includes the noncommutative spherical Cherednik algebras deforming $\mathcal{O}_{\mathbb{C}^{2n}}^{D_n}$. (Note that we cannot obtain an equality in this case since $\dim \mathsf{HP}_0(\mathcal{O}_{\mathbb{C}^{2n}}^{D_n}) > \dim \mathsf{HH}_0(\mathcal{D}_{\mathbb{C}^n}^{D_n}) =$ the dimension of $\mathsf{HH}_0(B)$ for generic noncommutative spherical Cherednik algebras deforming $\mathcal{O}_{\mathbb{C}^{2n}}^{D_n}$. This is partly a reflection of the fact that $\mathbb{C}^{2n}/D_n$ does not admit a symplectic resolution; see §1.3 of the main text.)

The next corollary is an analogue of Corollary 1.1.20, proved in the same manner:

Corollary A.1.2. Let B be a filtered Poisson deformation of $\mathcal{O}_{\mathbb{C}^{2n}}^{D_n}$. Then, $\dim \mathsf{HP}_0(B) \leq d_n$, and the number of zero-dimensional symplectic leaves of $\mathsf{Spec}\, B$ is at most d_n.

In particular, this includes the commutative spherical Cherednik algebras deforming $\mathcal{O}_{\mathbb{C}^{2n}}^{D_n}$. (For the same reason as before, we cannot obtain an equality in this case.)

One can also formulate an analogue of Corollary 1.1.18 (which can also be proved in the same manner; see also the proof of Theorem A.2.1 in §A.5). Recall the definition of $p_{n,i}$ from there. Let $p'_{n,i}$ be the number of $(n-i)$-multipartitions of n such that every cell has an even number of elements, e.g., $(2,2,4)$ is allowed, but not $(1,2,3,4)$.

Corollary A.1.3. Let B be an arbitrary filtered quantization of $\mathcal{O}_{\mathbb{C}^{2n}}^{D_n}$. Then, the number of prime ideals of B whose support has codimension $2i$ in V/D_n is at most

$$p'_{n,i} + \sum_{j=0}^{i} d_j p_{n-j,i-j}. \tag{A.1.4}$$

A.2. **The $\mathcal{D}$-module $M(\mathbb{C}^{2n}/D_n)$.** Similarly to the case of symmetric powers of symplectic varieties in §1.2, we may deduce the structure of $M(X)$ for $X = \mathbb{C}^{2n}/D_n$.

When U is a vector space, let $\delta_{0\in U}$ denote the δ-function D_U-module at the origin. Let $q : \mathbb{C}^{2n} \twoheadrightarrow \mathbb{C}^{2n}/D_n$ be the quotient map. Let $\Delta_i : \mathbb{C}^2 \hookrightarrow (\mathbb{C}^2)^i$ denote the diagonal embedding. Also, define the modified embedding $\Delta'_i : \mathbb{C}^2 \hookrightarrow (\mathbb{C}^2)^i$ by $\Delta'_i(x,y) = ((-x,-y),(x,y),\ldots,(x,y))$, i.e., the composition of $-\mathrm{Id}\times \mathrm{Id}^{i-1}$ with Δ_i.

Theorem A.2.1.

$$M(X) \cong \bigoplus_{\substack{r+r_1\cdot i_1+\cdots+r_k\cdot i_k=n\\ r_j\geq 1\,\forall j, 1\leq i_1<\cdots<i_k, r\geq 0}} \mathsf{HP}_0(\mathcal{O}_{\mathbb{C}^{2r}}^{D_r}) \otimes q_*\big(\delta_{0\in\mathbb{C}^{2r}} \boxtimes \big((\Delta_{i_1})_*(\Omega_{\mathbb{C}^2})^{\boxtimes r_1} \boxtimes \cdots \boxtimes (\Delta_{i_k})_*(\Omega_{\mathbb{C}^2})^{\boxtimes r_k}\big)\big)^{\mathrm{Stab}}$$
$$\oplus \bigoplus_{\substack{r_1\cdot i_1+\cdots+r_k\cdot i_k=n\\ r_j\geq 1\,\forall j, 2\leq i_1<\cdots<i_k, 2|i_j,\forall j}} q_*\big(\big((\Delta'_{i_1})_*(\Omega_{\mathbb{C}^2})\boxtimes(\Delta_{i_1})_*(\Omega_{\mathbb{C}^2})^{\boxtimes r_1-1}\big)\boxtimes(\Delta_{i_2})_*(\Omega_{\mathbb{C}^2})^{\boxtimes r_2}\boxtimes\cdots\boxtimes(\Delta_{i_k})_*(\Omega_{\mathbb{C}^2})^{\boxtimes r_k}\big)\big)^{\mathrm{Stab}}. \tag{A.2.2}$$

Here, the superscript of Stab refers to the subgroup of D_n which preserves the support of the $\mathcal{D}_{\mathbb{C}^{2n}}$-module we are pushing forward by q: for example, in the first big direct sum, this will be the subgroup for each summand preserving the locus $\{(0,0)\}^r \times \Delta_{i_1}(\mathbb{C}^2)^{r_1} \times \cdots \times \Delta_{i_k}(\mathbb{C}^2)^{r_k}$. This group is explicitly

$$D_n \cap \left(B_r \times \prod_{j=1}^{k}((\pm S_{i_j})^{r_j} \rtimes S_{r_j})\right),$$

where here $\pm S_i \cong S_i \times \mathbb{Z}/2$ is the group generated by permutation matrices and $-\mathrm{Id}$. One can express the second stabilizer in a similar way, and it is isomorphic to $\prod_{j=1}^{k}((\pm S_{i_j})^{r_j} \rtimes S_{r_j})$ (with the case $j=1$ of the product acting in a modified way so as to preserve the locus $\Delta'_{i_1}(\mathbb{C}^2)\times\Delta_{i_1}(\mathbb{C}^2)^{r_1-1}$ rather than $\Delta_{i_1}(\mathbb{C}^2)^{r_1}$).

A.3. **Explicit computational results.** Using programs [Sch11] written in Magma [BCP97], we explicitly solved (A.0.7) for $n \leq 34$ (and double-checked, for $n \leq 7$ and low enough degrees for $n \in \{8,9\}$, that the result matches a direct computation of HP_0 without using Theorem A.0.6). The result is given in Figures 1 and 2.

A.4. **Proof of Theorem A.0.6.** Set $A := \mathcal{O}_{\mathbb{C}^{2n}}^{D_n}$. (When we need n to vary later on, we will also denote A by $A^{(n)}$.) Fix a basis $x_1,\ldots,x_n,y_1,\ldots,y_n$ of $(\mathbb{C}^{2n})^*$ such that $\{x_i,y_j\} = \delta_{ij}$ and $\{x_i,x_j\} = 0 = \{y_i,y_j\}$ for all i,j. Decompose $A = A_+ \oplus A_-$ as the eigenspaces of the diagonal

n	$h(\mathsf{HP}_0(\mathcal{O}_{\mathbb{C}^{2n}}^{D_n}); t^{\frac{1}{4}})$
4	t^2+t+1
5	t^2+t+1
6	$2t^3+2t^2+t+1$
7	$2t^4+2t^3+2t^2+t+1$
8	$2t^5+4t^4+3t^3+2t^2+t+1$
9	$2t^6+4t^5+4t^4+3t^3+2t^2+t+1$
10	$t^7+6t^6+6t^5+5t^4+3t^3+2t^2+t+1$
11	$t^8+6t^7+8t^6+6t^5+5t^4+3t^3+2t^2+t+1$
12	$t^9+8t^8+10t^7+10t^6+7t^5+5t^4+3t^3+2t^2+t+1$
13	$t^{10}+7t^9+13t^8+12t^7+10t^6+7t^5+5t^4+3t^3+2t^2+t+1$
14	$t^{11}+8t^{10}+16t^9+17t^8+14t^7+11t^6+7t^5+5t^4+3t^3+2t^2+t+1$
15	$t^{12}+6t^{11}+19t^{10}+21t^9+19t^8+14t^7+11t^6+7t^5+5t^4+3t^3+2t^2+t+1$
16	$t^{13}+7t^{12}+22t^{11}+28t^{10}+25t^9+21t^8+15t^7+11t^6+7t^5+5t^4+3t^3+2t^2+t+1$
17	$t^{14}+7t^{13}+25t^{12}+33t^{11}+33t^{10}+27t^9+21t^8+15t^7+11t^6+7t^5+5t^4+3t^3+2t^2+t+1$
18	$t^{15}+8t^{14}+27t^{13}+43t^{12}+42t^{11}+37t^{10}+29t^9+22t^8+15t^7+11t^6+7t^5+5t^4+3t^3+2t^2+t+1$
19	$t^{16}+8t^{15}+29t^{14}+49t^{13}+54t^{12}+47t^{11}+39t^{10}+29t^9+22t^8+15t^7+11t^6+7t^5+5t^4+3t^3+2t^2+t+1$

FIGURE 1. Poisson traces on type D_n singularities for $n \leq 19$

matrix $T := \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} \in \mathsf{GL}(\mathbb{C}^n) \subseteq \mathsf{Sp}(\mathbb{C}^{2n})$, in the basis of the x_i (or equivalently, any element of $B_n \setminus D_n$). Then, $A_+ = \mathcal{O}_{\mathbb{C}^{2n}}^{B_n}$ is the ring of polynomials which are symmetric under the action of S_n simultaneously on x_i and y_i and for which every monomial has an even sum of degrees in the index-i variables x_i and y_i, for all i. Similarly, A_- is the space of symmetric polynomials such that every monomial has *odd* total degree in x_i and y_i, for all i. Note the formula

$$A_- = \sum_{z_i \in \{x_i, y_i\}} z_1 z_2 \cdots z_n A_+. \tag{A.4.1}$$

We would like to compute $A/\{A,A\} = A_+/(\{A_+,A_+\} + \{A_-,A_-\}) \oplus A_-/\{A_+,A_-\}$.

Lemma A.4.2. $A_- = \{A_+, A_-\}$.

Proof. Let $\mathrm{symm}(f) = \frac{1}{n!}\sum_{\sigma \in S_n} \sigma(f)$ be the symmetrization map. We need to show that, for all monomials $x_1^{a_1}y_1^{b_1}\cdots x_n^{a_n}y_n^{b_n}$ such that a_i+b_i is odd for all i, the symmetrization $\mathrm{symm}(x_1^{a_1}y_1^{b_1}\cdots x_n^{a_n}y_n^{b_n})$ is a sum of Poisson brackets.

To do so, we consider a filtration on A (which we will not label by integers) given by an ordering on monomials. First, take the ordering on monomials in $\mathbb{C}[x,y]$ of the form $x^a y^b > x^{a'} y^{b'}$ if either $a+b > a'+b'$ or $a+b = a'+b'$ and $a > a'$. Extend this to symmetrizations of monomials in $\mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_n]$: assuming that $x^{a_1}y^{b_1} \geq x^{a_2}y^{b_2} \geq \cdots \geq x_n^{a_n}y_n^{b_n}$ and similarly $x^{a'_1}y^{b'_1} \geq x^{a'_2}y^{b'_2} \geq \cdots \geq x_n^{a'_n}y_n^{b'_n}$, then we say that $\mathrm{symm}(x_1^{a_1}y_1^{b_1}\cdots x_n^{a_n}y_n^{b_n}) \geq \mathrm{symm}(x_1^{a'_1}y_1^{b'_1}\cdots x_n^{a'_n}y_n^{b'_n})$ if, for some i, $a_j = a'_j$ and $b_j = b'_j$ for all $j < i$ and $x^{a_i}y^{b_i} > x^{a'_i}y^{b'_i}$.

n	$h(\mathsf{HP}_0(\mathcal{O}_{\mathbb{C}^{2n}}^{D_n}); t^{\frac{1}{4}})$
20	$t^{17}+9t^{16}+30t^{15}+60t^{14}+67t^{13}+63t^{12}+51t^{11}+41t^{10}+30t^9+22t^8+15t^7+11t^6+7t^5+5t^4+3t^3+2t^2+t+1$
21	$t^{18}+9t^{17}+30t^{16}+68t^{15}+83t^{14}+78t^{13}+68t^{12}+53t^{11}+41t^{10}+30t^9+22t^8+15t^7+11t^6+7t^5+5t^4+3t^3+2t^2+t+1$
22	$t^{19}+10t^{18}+33t^{17}+80t^{16}+101t^{15}+101t^{14}+87t^{13}+72t^{12}+55t^{11}+42t^{10}+30t^9+22t^8+15t^7+11t^6+7t^5+5t^4+3t^3+2t^2+t+1$
23	$t^{20}+10t^{19}+37t^{18}+87t^{17}+122t^{16}+124t^{15}+112t^{14}+92t^{13}+74t^{12}+55t^{11}+42t^{10}+30t^9+22t^8+15t^7+11t^6+7t^5+5t^4+3t^3+2t^2+t+1$
24	$t^{21}+11t^{20}+40t^{19}+100t^{18}+145t^{17}+156t^{16}+142t^{15}+121t^{14}+96t^{13}+76t^{12}+56t^{11}+42t^{10}+30t^9+22t^8+15t^7+11t^6+7t^5+5t^4+3t^3+2t^2+t+1$
25	$t^{22}+11t^{21}+44t^{20}+105t^{19}+171t^{18}+188t^{17}+179t^{16}+153t^{15}+126t^{14}+98t^{13}+76t^{12}+56t^{11}+42t^{10}+30t^9+22t^8+15t^7+11t^6+7t^5+5t^4+3t^3+2t^2+t+1$
26	$t^{23}+12t^{22}+48t^{21}+115t^{20}+199t^{19}+232t^{18}+222t^{17}+197t^{16}+162t^{15}+130t^{14}+100t^{13}+77t^{12}+56t^{11}+42t^{10}+30t^9+22t^8+15t^7+11t^6+7t^5+5t^4+3t^3+2t^2+t+1$
27	$t^{24}+12t^{23}+52t^{22}+120t^{21}+230t^{20}+275t^{19}+276t^{18}+245t^{17}+208t^{16}+167t^{15}+132t^{14}+100t^{13}+77t^{12}+56t^{11}+42t^{10}+30t^9+22t^8+15t^7+11t^6+7t^5+5t^4+3t^3+2t^2+t+1$
28	$t^{25}+13t^{24}+56t^{23}+136t^{22}+262t^{21}+333t^{20}+337t^{19}+310t^{18}+263t^{17}+217t^{16}+171t^{15}+134t^{14}+101t^{13}+77t^{12}+56t^{11}+42t^{10}+30t^9+22t^8+15t^7+11t^6+7t^5+5t^4+3t^3+2t^2+t+1$
29	$t^{26}+13t^{25}+61t^{24}+150t^{23}+296t^{22}+390t^{21}+411t^{20}+381t^{19}+333t^{18}+274t^{17}+222t^{16}+173t^{15}+134t^{14}+101t^{13}+77t^{12}+56t^{11}+42t^{10}+30t^9+22t^8+15t^7+11t^6+7t^5+5t^4+3t^3+2t^2+t+1$
30	$t^{27}+14t^{26}+65t^{25}+169t^{24}+330t^{23}+462t^{22}+497t^{21}+474t^{20}+415t^{19}+351t^{18}+283t^{17}+226t^{16}+175t^{15}+135t^{14}+101t^{13}+77t^{12}+56t^{11}+42t^{10}+30t^9+22t^8+15t^7+11t^6+7t^5+5t^4+3t^3+2t^2+t+1$
31	$t^{28}+14t^{27}+70t^{26}+185t^{25}+365t^{24}+534t^{23}+597t^{22}+576t^{21}+518t^{20}+438t^{19}+362t^{18}+288t^{17}+228t^{16}+175t^{15}+135t^{14}+101t^{13}+77t^{12}+56t^{11}+42t^{10}+30t^9+22t^8+15t^7+11t^6+7t^5+5t^4+3t^3+2t^2+t+1$
32	$t^{29}+15t^{28}+75t^{27}+206t^{26}+399t^{25}+624t^{24}+711t^{23}+706t^{22}+639t^{21}+552t^{20}+456t^{19}+371t^{18}+292t^{17}+230t^{16}+176t^{15}+135t^{14}+101t^{13}+77t^{12}+56t^{11}+42t^{10}+30t^9+22t^8+15t^7+11t^6+7t^5+5t^4+3t^3+2t^2+t+1$
33	$t^{30}+15t^{29}+80t^{28}+225t^{27}+432t^{26}+710t^{25}+845t^{24}+849t^{23}+786t^{22}+683t^{21}+575t^{20}+467t^{19}+376t^{18}+294t^{17}+230t^{16}+176t^{15}+135t^{14}+101t^{13}+77t^{12}+56t^{11}+42t^{10}+30t^9+22t^8+15t^7+11t^6+7t^5+5t^4+3t^3+2t^2+t+1$
34	$t^{31}+16t^{30}+85t^{29}+249t^{28}+480t^{27}+816t^{26}+994t^{25}+1028t^{24}+959t^{23}+849t^{22}+717t^{21}+593t^{20}+476t^{19}+380t^{18}+296t^{17}+231t^{16}+176t^{15}+135t^{14}+101t^{13}+77t^{12}+56t^{11}+42t^{10}+30t^9+22t^8+15t^7+11t^6+7t^5+5t^4+3t^3+2t^2+t+1$

FIGURE 2. Poisson traces on type D_n singularities for $20 \le n \le 34$

For each degree $m \geq 0$, we will consider the induced filtration on the vector space $(A_-)_m$ in total degree m, given by the union

$$(\text{A.4.3}) \quad (A_-)_m = \bigcup_{\substack{a_1+\cdots+a_n+b_1+\cdots+b_n=m \\ a_i+b_i \text{ is odd}, \forall i}} \operatorname{Span}\{\operatorname{symm}(x_1^{a_1'} y_1^{b_1'} \cdots x_n^{a_n'} y_n^{b_n'}) \text{ s.t. } \operatorname{symm}(x_1^{a_1'} y_1^{b_1'} \cdots x_n^{a_n'} y_n^{b_n'}) > \operatorname{symm}(x_1^{a_1} y_1^{b_1} \cdots x_n^{a_n} y_n^{b_n})\},$$

i.e., the map $(A_-)_m \to \mathsf{gr}(A_-)_m$ with respect to this filtration takes the *lowest symmetrized monomial* with respect to the ordering on monomials, which has nonzero coefficient.

It suffices to show that $\mathsf{gr}\{A_-, A_+\}_m = \mathsf{gr}(A_-)_m$ for all $m \geq 0$. That is, for each monomial $x_1^{a_1} y_1^{b_1} \cdots x_n^{a_n} y_n^{b_n}$ of total degree m and with $a_i + b_i$ odd for all i, we need to show that

$\mathrm{symm}(x_1^{a_1}y_1^{b_1}\cdots x_n^{a_n}y_n^{b_n})+\text{higher terms}\in\{A_-,A_+\}_m$, for some linear combination of greater symmetrizations of monomials of total degree m.

So, assume $x^{a_i}y^{b_i}\geq x^{a_{i+1}}y^{b_{i+1}}$ for all $1\leq i\leq n-1$. We compute that

$$\text{(A.4.4)}\quad \{\mathrm{symm}(x_1^{a_1+1}y_1^{b_1}),\mathrm{symm}(y_1\cdot x_2^{a_2}y_2^{b_2}\cdots x_n^{a_n}y_2^{b_n})\} = \frac{1+c}{n}\cdot(a_1+1)\mathrm{symm}(x_1^{a_1}y_1^{b_1}\cdots x_n^{a_n}y_n^{b_n})+\text{higher terms},$$

where $c=|\{i\in\{2,3,\ldots,n\}\mid(a_i,b_i)=(0,1)\}|\geq 0$. ☐

We conclude that

$$\mathsf{HP}_0(A)=\mathsf{HP}_0(\mathcal{O}_{\mathbb{C}^{2n}}^{B_n})/\{A_-,A_-\}. \tag{A.4.5}$$

Next, recall that, for every Poisson algebra P which is Poisson generated by elements $p_1,\ldots,p_k$, $\{P,P\}=\{\langle p_1,\ldots,p_k\rangle,P\}$. This is a result of the Jacobi identity and the identity $\{ab,c\}+\{bc,a\}+\{ca,b\}=0$. Then, note that $A_+\subset A$ contains the copy of $\mathfrak{sl}_2$ spanned by $\sum_i x_i^2$, $\sum_i y_i^2$, and $\sum_i x_iy_i$. Here and below all sums over i will range from 1 to $n+1$ (not only 1 to n) unless otherwise specified.

As a result of this and (A.4.1), A is Poisson generated by A_+ and the single element $y_1y_2\cdots y_n$. Hence, we deduce that

$$\mathsf{HP}_0(A)=\mathsf{HP}_0(\mathcal{O}_{\mathfrak{h}\oplus\mathfrak{h}^*}^{B_n})/\{y_1y_2\cdots y_n,A_-\}. \tag{A.4.6}$$

As a result, there is a natural inclusion of dual spaces $\mathsf{HP}_0(A)^*\subseteq\mathsf{HP}_0(\mathcal{O}_{\mathbb{C}^{2n}}^{B_n})^*$. For convenience, when we allow n to vary, let $A^{(n)}:=\mathcal{O}_{\mathbb{C}^{2n}}^{D_n}=A_+^{(n)}\oplus A_-^{(n)}$ be the above decomposition (note that $A_+^{(n)}=\mathcal{O}_{\mathbb{C}^{2n}}^{B_n}$). In these terms, we showed that $\mathsf{HP}_0(A^{(n)})^*\subseteq\mathsf{HP}_0(A_+^{(n)})^*$.

Claim A.4.7. The image of the inclusion

$$\bigoplus_n\mathsf{HP}_0(A^{(n)})^*\subseteq\bigoplus_n\mathsf{HP}_0(A_+^{(n)})^* \tag{A.4.8}$$

is a bigraded subalgebra.

Proof. We have to show that, if $f\in\mathsf{HP}_0(A_+^{(m)})^*\subseteq(A_+^{(m)})^*$ and $g\in\mathsf{HP}_0(A_+^{(n)})^*\subseteq(A_+^{(n)})^*$ satisfy

$$f(\{y_1\cdots y_m,A_-^{(m)}\})=0,\quad g(\{y_1\cdots y_n,A_-^{(n)}\})=0, \tag{A.4.9}$$

i.e., $f\in\mathsf{HP}_0(A^{(m)})^*$ and $g\in\mathsf{HP}_0(A^{(n)})^*$, then

$$(f\cdot g)(\{y_1\cdots y_{m+n},A_-^{(m+n)}\})=0. \tag{A.4.10}$$

This follows immediately from the Leibniz rule and the fact that, as subspaces of $\mathcal{O}_{\mathbb{C}^{2(m+n)}}=T^{m+n}\mathcal{O}_{\mathbb{C}^2}$, $A_-^{(m+n)}\subseteq A_-^{(m)}\otimes A_-^{(n)}$. ☐

We now explicitly describe the subalgebra $\bigoplus_n\mathsf{HP}_0(A^{(n)})^*\subseteq\bigoplus_n\mathsf{HP}_0(A_+^{(n)})^*=\mathbb{C}[s_i]_{i\geq 1}$. This depends on the choice of the s_i, each of which is canonical up to scaling. We will make use of the construction of [ES09], as we recall in the proof.

Let us recall the definition of the functions s_i from [ES09, §4]. It is convenient to view s_i as a degree-i function on $\mathbb{C}[x^2,xy,y^2]$, i.e., $s_i(f):=s_i(f^{\otimes i})$. Since they are homogeneous, the s_i extend to continuous functions on the completion $\mathbb{C}[\![x^2,xy,y^2]\!]$. It is proved in *op. cit.* that every f in the completion $\mathbb{C}[\![x^2,xy,y^2]\!]$ of $\mathcal{O}_{\mathbb{C}^2}^{\mathbb{Z}/2}$ with nonvanishing second derivative ("nondegenerate") is equivalent up to continuous Poisson automorphisms (i.e., even symplectomorphisms of the formal disc) to a unique element of the form

$$f\sim y^2+s_1+s_2x^2+s_3x^4+\cdots. \tag{A.4.11}$$

Then, $s_i(f)$ is defined as the above coordinate in this normal form. This extends uniquely to the entire pro-vector space $\mathbb{C}[\![x^2, xy, y^2]\!]$ (no longer requiring the nonvanishing second derivative condition) since the degenerate elements form a codimension-two subspace. These s_i restrict to functions on $\mathbb{C}[x^2, xy, y^2]$ and have degree i and weight $4-4i$, i.e., s_i is a degree-i polynomial and $s_i(f(\lambda x, \lambda y)) = \lambda^{4i-4} s_i(f(x,y))$.

We need to consider the value of the functions s_n on brackets of the form $\{y_1 y_2 \cdots y_n, g\}$ for $g \in A_-^{(n)}$. Since $A_-^{(n)} = \mathsf{Sym}^n A_-^{(1)} = \langle x, y \rangle \mathbb{C}[x^2, xy, y^2]$, it follows that $A_-^{(n)}$ is spanned by elements of the form $g = f^{\otimes n}$ for $f \in A_-^{(1)}$. Then, we notice that

$$\{y_1 y_2 \cdots y_n, f^{\otimes n}\} = \operatorname{symm}(n\{y, f\} \otimes (yf)^{\otimes(n-1)}). \tag{A.4.12}$$

Thus, the subalgebra $\bigoplus_n \mathsf{HP}_0(A^{(n)})^* \subseteq \mathbb{C}[s_i]_{i \geq 1}$ consists of those polynomials F such that, working over $\mathbb{C}[\varepsilon]/(\varepsilon^2)$,

$$F(yf + \varepsilon\{y, f\}) - F(yf) = 0, \quad \forall f \in A_-^{(1)}. \tag{A.4.13}$$

Next, we write yf in normal form up to even formal symplectomorphisms. We claim that the result is of the form $(y+h)(y-h)$, where $h \in x\mathbb{C}[\![x^2]\!]$. Indeed, since $yf(0) = 0$, the result must lie in $y^2 + x^2\mathbb{C}[\![x^2]\!]$, and since $\mathbb{C}[\![x^2]\!]$ admits square roots in $\mathbb{C}[\![x]\!]$, the result must be of the form $y^2 - h^2$ for some $h \in x\mathbb{C}[\![x]\!]$.

The next step is to write $yf + \varepsilon\{y, f\}$ in normal form up to even formal symplectomorphisms. First let φ be the aforementioned symplectomorphism satisfying $\varphi(yf) = (y+h)(y-h)$. Then, up to choice of h, φ takes y to $u(y+h)$ and f to $u^{-1}(y-h)$, for some even unit $u \in \mathbb{C}[\![x^2, xy, y^2]\!]$. Therefore,

$$\begin{aligned} \varphi(\{y, f\}) &= \{u(y+h), u^{-1}(y-h)\} \\ &= 2\{h, y\} + u^{-1}(y+h)\{u, y-h\} - u(y-h)\{u^{-1}, y+h\} \\ &= 2\{h, y\} + u^{-1}(y+h)\{u, y-h\} + u^{-1}(y-h)\{u, y+h\} \\ &= 2\{h, y\} + 2y\{\log(u), y\} - 2h\{\log(u), h\} \\ &= 2\{h, y\} + \{\log(u), y^2 - h^2\}. \end{aligned} \tag{A.4.14}$$

Hence, $\varphi(yf + \varepsilon\{y, f\}) = (y^2 - h^2) + 2\varepsilon\{h, y\} + \varepsilon\{\log(u), y^2 - h^2\}$. Further, we may apply the symplectomorphism $e^{-\varepsilon \operatorname{ad}(\log u)}$ and we obtain $(y^2 - h^2) + 2\varepsilon\{h, y\}$. Therefore, for $F \in \mathsf{HP}_0(A^{(n)})^*$, (A.4.13) becomes

$$F(y^2 - h^2 + 2\varepsilon\{h, y\}) - F(y^2 - h^2) = 0, \forall h \in x\mathbb{C}[\![x^2]\!]. \tag{A.4.15}$$

Next, consider the s_i as coordinate functions on the infinite-dimensional affine space $y^2 + \mathbb{C}[\![x^2]\!]$, or just $\mathbb{C}[\![x^2]\!]$ by deleting the y^2. Then, the above equation says that F is annihilated by a particular (discontinuous) vector field up to sign, which is supported on $x^2\mathbb{C}[\![x^2]\!]$, and at the point $g = -h^2$ is the vector $2h' = 2(\sqrt{-g})'$. For all $k \geq 0$, the square-root function is a regular multivalued function on the locus $\mathbb{C}^\times \cdot x^{2k} + x^{2k+2}\mathbb{C}[\![x^2]\!]$, defined by

$$\sum_{i \geq k} s_{i+1} x^{2i} \mapsto \pm\sqrt{s_{k+1}} x^k Q\Big(\frac{1}{s_{k+1}} \sum_{i \geq k+1} s_{i+1} x^{2(i-k)}\Big), \tag{A.4.16}$$

where Q is the Taylor series for $\sqrt{1+x}$. That is, it says that

$$\xi_k F|_{s_1 = s_2 = \cdots = s_{2k-1} = 0, s_{2k} \neq 0} = 0, \quad \forall k \geq 1, \tag{A.4.17}$$

with ξ_k as in the statement of the theorem.

A.5. **Proof of Theorem A.2.1.** Let $X := \mathbb{C}^{2n}/D_n$. By [ES10a, Theorem 3.1] and its proof, it suffices to analyze $M(X)$ in the formal neighborhood of each symplectic leaf. These leaves are the image under $\mathbb{C}^{2n} \twoheadrightarrow X$ of two types of partitions:

(i) Partitions $\{1,\ldots,n\} := I \sqcup J_1 \sqcup \cdots \sqcup J_\ell$, with the leaf an open subset of the locus where $x_i = 0 = y_i$ for all $i \in I$, and $x_i = x_j$ and $y_i = y_j$ for all $i,j \in I_k$, for all $1 \leq k \leq \ell$;

(ii) Partitions $\{1,\ldots,n\} := J_1 \sqcup \cdots \sqcup J_\ell$ with each $|J_i|$ even and $1,2 \in J_1$; the leaf is an open subset of the locus where $x_i = x_j$ and $y_i = y_j$ for all $i,j \in I_k$, for all $1 \leq k \leq \ell$, except that $x_1 = -x_2$ and $y_1 = -y_2$.

Namely, the leaves are the complement in such loci of properly contained such loci, i.e., those corresponding to partitions obtained by joining some cells of the given partition (and if I joins with any other cell J_i, the new label must remain I). The computation therefore reduces to the cases $\{1,\ldots,n\} = I$ or $\{1,\ldots,n\} = J_1$. In the former case, the local system is just a multiple of the delta-function local system at zero, whose multiplicity must be $\dim \mathsf{HP}_0(\mathcal{O}_X)$, and in the latter case, the problem reduces to the computation of Theorem 1.2.1.

Appendix B. Direct proof of Theorem 1.1.8, by T. Schedler

We need to show that $\mathsf{HP}_0(\mathcal{O}_V^{S_n}, \mathcal{O}_V^{S_n}) = \mathbb{C}$. In fact, we will show the stronger result that $\mathsf{HP}_0(\mathcal{O}_V^{S_n}, \mathcal{O}_V^{S_{n-1}}) = \mathbb{C}$ (note that this is also what Lemma 2.1.8 proves). It will be convenient to remember that $\{\mathcal{O}_V^{S_n}, \mathcal{O}_V\}^{S_{n-1}} = \{\mathcal{O}_V^{S_n}, \mathcal{O}_V^{S_{n-1}}\}$.

In degree zero, we clearly get $\mathsf{HP}_0(\mathcal{O}_V^{S_n}, \mathcal{O}_V^{S_{n-1}})_0 = \mathbb{C}$. So it suffices to show that the positively-graded part vanishes, i.e., $\mathsf{HP}_0(\mathcal{O}_V^{S_n}, \mathcal{O}_V^{S_{n-1}})_{>0} = 0$.

It will be helpful to explicitly write V in terms of coordinates. Let $V = \mathfrak{h} \oplus \mathfrak{h}^*$ where $\mathfrak{h} \cong \mathbb{C}^{n-1}$ is the reflection representation. We can consider $\mathfrak{h} \subseteq \mathbb{C}^n$ to be the subset where all coordinates sum to zero. Hence we can write $\mathcal{O}_V = \mathbb{C}[x_1,\ldots,x_n,y_1,\ldots,y_n]/(x_1+\cdots+x_n, y_1+\cdots+y_n)$. Moreover, we can choose coordinates so that the permutation action of S_n on $x_1,\ldots,x_n$ and on $y_1,\ldots,y_n$ is by the usual action (by simultaneous permutations of indices using the same permutation), and the Poisson bracket is given by

$$\{x_i, y_j\} = \delta_{ij} - \frac{1}{n}.$$

(Note that the $-\frac{1}{n}$ is required here because, for instance, $x_1 + \cdots + x_n = 0$.)

Consider the sub-Lie algebra of $\mathcal{O}_V^{S_n}$ spanned by $\sum_{i=1}^n x_i^2$, $\sum_{i=1}^n y_i^2$, and $h := \sum_{i=1}^n x_i y_i$. This is isomorphic to $\mathfrak{sl}_2$, and we will simply call it $\mathfrak{sl}_2$. Moreover, the action of S_n commutes with the action of $\mathfrak{sl}_2$.

Since the adjoint action of $\mathfrak{sl}_2$ preserves degree on $\mathcal{O}_V$, in each degree we obtain a semisimple representation. Hence, $\{\mathcal{O}_V^{S_n}, \mathcal{O}_V\}$ contains the sum of all nontrivial $\mathfrak{sl}_2$-representations in $\mathcal{O}_V$, $\{\mathfrak{sl}_2, \mathcal{O}_V\}$. Because the action of S_n commutes with that of $\mathfrak{sl}_2$, also $\{\mathcal{O}_V^{S_n}, \mathcal{O}_V^{S_{n-1}}\}$ contains the S_{n-1}-invariants of the sum of nontrivial $\mathfrak{sl}_2$-representations, $\{\mathfrak{sl}_2, \mathcal{O}_V\}^{S_{n-1}}$.

Next, for any two finite-dimensional $\mathfrak{sl}_2$-representations W and W', if $w' \in W'$ is a highest (or lowest) weight vector for $h \in \mathfrak{sl}_2$, it is easy to see that $W \otimes w'$ generates $W \otimes W'$ as a $\mathfrak{sl}_2$-representation (e.g., one can assume W is irreducible, and then show that all tensor products $w_1 \otimes w_2$ of h-weight vectors are generated, by induction on the weights). Since y_n^k is a highest weight vector for the representation $\mathbb{C}[y_n]_k$ (the subscript denotes degree k), it follows that, for all $1 \leq j < n$,

$$(\mathbb{C}[x_1,\ldots,x_j,y_1,\ldots,y_j] \cdot \mathbb{C}[y_n]) + \{\mathfrak{sl}_2, \mathcal{O}_V\} \supseteq \mathbb{C}[x_1,\ldots,x_j,y_1,\ldots,y_j] \cdot \mathbb{C}[x_n,y_n]. \tag{B.0.1}$$

Let $\mathrm{symm} : \mathcal{O}_V \to \mathcal{O}_V^{S_n}$ be the symmetrization map, and $\mathrm{symm}' : \mathcal{O}_V \to \mathcal{O}_V^{S_{n-1}}$ be the symmetrization for the subgroup $S_{n-1} \subset S_n$.

In view of the above, it suffices to prove that, for all $1 \leq j < n$ and all $a_1, \ldots, a_j, b_1, \ldots, b_j \geq 0$ and $m \geq 0$ such that the a_i, b_i, and m are not all zero,

$$\text{symm}'\big(x_1^{a_1} y_1^{b_1} \cdots x_j^{a_j} y_j^{b_j} \cdot y_n^m\big) \in \{\mathcal{O}_V^{S_n}, \mathcal{O}_V^{S_{n-1}}\}. \tag{B.0.2}$$

It suffices to fix the total degree $d = \sum_i (a_i + b_i) + m \geq 1$, which we fix from now on. Note that, once we prove (B.0.2) for all a_i, b_i, m such that the total degree is d, it follows from (B.0.1) that

$$\text{symm}'\mathbb{C}[x_1, y_1, \ldots, x_j, y_j, x_n, y_n]_d \subseteq \{\mathcal{O}_V^{S_n}, \mathcal{O}_V^{S_{n-1}}\}, \tag{B.0.3}$$

for the same value of j (and the same degree d). Here the subscript of d denotes the part of total degree d. Now, we will prove (B.0.2), for fixed total degree d, by a double induction: we induct on j, and for each value of j, we also perform a reverse induction on m (beginning with $m = d$, the total degree).

For the base case(s) of the (double) induction, for any value of j, it is enough to show that $\text{symm}'(y_n^d)$ is in $\{\mathcal{O}_V^{S_n}, \mathcal{O}_V^{S_{n-1}}\}$. This follows because y_n^d generates a nontrivial representation of $\mathfrak{sl}_2$ in $\mathcal{O}_V$, since $d \geq 1$.

The inductive step follows from the computation

$$\begin{aligned} &\text{symm}'(x_1^{a_1} y_1^{b_1} \cdots x_j^{a_j} y_j^{b_j} y_n^{a_n}) \\ &\quad = -\frac{n^2}{(n-j)(a_1+1)(a_n+1)} \text{symm}'\Big(\{\text{symm}(x_1^{a_1+1} y_1^{b_1}), x_2^{a_2} y_2^{b_2} \cdots x_j^{a_j} y_j^{b_j} y_n^{a_n+1}\} + \text{h.o.t.}\Big), \end{aligned} \tag{B.0.4}$$

where "h.o.t."="higher order terms" refers to a linear combination of monomials with fewer indices appearing (in this case, the only variables which occur in this part of the sum will be $x_2, \ldots, x_j, y_2, \ldots, y_j, x_n$, and y_n), or where the exponent of y_n appearing is greater (in this case, it will be $y_n^{a_n+1}$). These are already in $\{\mathcal{O}_V^{S_n}, \mathcal{O}_V^{S_{n-1}}\}$ by hypothesis. This completes the induction.

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