Microscopic Realization of Two-Dimensional Bosonic Topological Insulators

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It is well known that a bosonic Mott insulator can be realized by condensing vortices of a boson condensate. Usually, a vortex becomes an antivortex (and vice versa) under time reversal symmetry, and the condensation of vortices results in a trivial Mott insulator. However, if each vortex or antivortex interacts with a spin trapped at its core, the time reversal transformation of the composite vortex operator will contain an extra minus sign. It turns out that such a composite vortex condensed state is a bosonic topological insulator (BTI) with gapless boundary excitations protected by $U(1) \rtimes Z_2^T$ symmetry. We point out that in BTI, an external π-flux monodromy defect carries a Kramers doublet. We propose lattice model Hamiltonians to realize the BTI phase, which might be implemented in cold atom systems or spin-1 solid state systems.

Introduction.—Quantum phases beyond Landau symmetry breaking theory [1–3], including long-range entangled [4] intrinsically topologically ordered states [5–7] and short-range entangled symmetry-protected topological (SPT) states [8–12], have attracted great interest in condensed matter physics recently. Intrinsic topologically ordered states, such as fractional quantum Hall states, can be characterized by their bulk fractionalized excitations. On the other hand, SPT phases do not have nontrivial bulk excitations and can be adiabatically connected to a trivial product state if symmetry is broken in the bulk. The 1D Haldane phase for spin-1 chain [8,9,13,14] and topological insulators [15–20] are nontrivial examples of SPT phases. Bosonic SPT phases in spacial dimension d are (partially) classified by the $(d+1)$th group cohomology of the symmetry group [11]. Some SPT phases can also be described by Chern-Simons theory (2D) [21–23] or topological terms of the nonlinear sigma model [24]. Typically, nontrivial SPT phases can be characterized by their edge excitations. In one dimension, the edge state of a SPT phase is degenerate and transforms as a projective representation of the symmetry group [25,26]. In two dimensions, the edge state of the SPT phase is either gapless or symmetry breaking [10,11]. The boundary theory of 3D SPT phases is more interesting, it can be either gapless [27] or topologically ordered if symmetry is unbroken [28,29]. It is worthwhile to mention that if the ground state is long-range entangled, symmetry can also act on the bulk topological excitations nontrivially, resulting in different symmetry enriched topological (SET) phases [30–35].

In analogy to fermionic topological insulators, bosonic SPT phases protected by $U(1) \rtimes Z_2^T$ symmetry (where $U(1) = \{U_\theta; U_\theta = e^{i\theta}\}$, $Z_2^T = \{I, T\}$ and $U_\theta T = TU_{-\theta}$) are called bosonic topological (Mott) insulators. One kind

of nontrivial 3D bosonic topological insulators has been discussed via quantum field theory approach [28,29] and projective construction [36]. In this Letter, we will discuss how to realize 2D bosonic topological insulators (BTI). Since $\mathcal{H}^3(U(1) \rtimes Z_2^T, U(1)) = \mathbb{Z}_2$, there are two Mott insulating phases, one is trivial and the other is the nontrivial BTI. Our construction of the SPT phases contains two steps. The first step is condensing the boson field to break the $U(1)$ symmetry. The second step is condensing the vortex field to restore the broken symmetry. In the usual case, the vortex is charge neutral and changes into antivortex under time reversal. The resultant state after vortex condensation is a trivial Mott insulator. However, if there exists another layer of spins living on the dual lattice such that each vortex core traps a spin [37,38] [see Fig. 1(b)], the sign of the composite vortex operator is reversed under time reversal owing to interaction. The resultant state is a
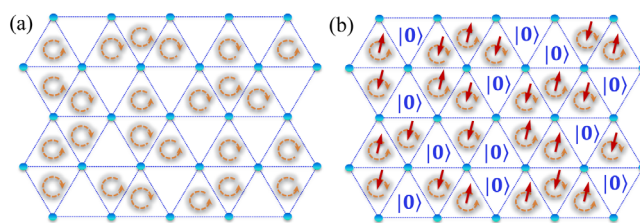

FIG. 1 (color online).   Two kinds of bosonic Mott insulators. The dots are bosons and the circles means vortex or antivortex. (a) Condensing the vortex or antivortex results in a trivial bose Mott insulator; (b) if $S = 1$ spins exist on the dual lattice, and creating a vortex or antivortex creation flips the spin momentum of the spin at its core, then after condensing the composite vortex or antivortex the resultant state is a bosonic topological Mott insulator. Here, $|0\rangle$ means $S_z|0\rangle = 0$, and up arrows and down arrows mean $|1\rangle$ and $|-1\rangle$, respectively.

nontrivial BTI [39], whose gapless edge states are symmetry protected. We illustrate that in the BTI a monodromy defect (i.e., a π flux) traps a Kramers doublet. Finally, we discuss Hamiltonian realization of the BTI in lattice models. These lattice Hamiltonians may shed light on further numerical studies and experimental realization of more nontrivial SPT phases.

Bosonic Mott insulators with $U(1) \rtimes Z_2^T$ *symmetry.—* Both the trivial and nontrivial bosonic Mott insulators can be constructed from Bose condensate. The idea is first condensing the bosons to break the $U(1)$ symmetry and then restoring the symmetry by condensing vortices.

Let us start with the trivial bosonic Mott insulator. Consider a familiar contact interacting boson model respecting $U(1) \rtimes Z_2^T$ symmetry:

$$\mathcal{L}_b = \frac{1}{2}(b^* \partial_\tau b + \text{H.c.}) + \frac{1}{2m} \partial_x b^* \partial_x b - \mu |b|^2 + \frac{g}{2} |b|^4, \tag{1}$$

when $\mu > 0, g > 0$, the bose field $b(x)$ condenses to a classical value $\sqrt{\rho_0} = \sqrt{(\mu/g)}$ and the $U(1)$ symmetry is spontaneously broken. We can write $b(x) = \sqrt{\rho_0 + \delta\rho} e^{i\theta(x)}$, where $\delta\rho$ and $\theta(x)$ are the density and phase fluctuations of the condensation, respectively. Integrating out the magnitude fluctuation $\delta\rho$, we obtain an XY model as the low energy effective Lagrangian [40]:

$$\mathcal{L}_{XY} = \frac{1}{2g}[(\partial_\tau \theta)^2 + v^2 (\nabla\theta)^2], \tag{2}$$

where $v = \sqrt{(\rho_0 g/m)}$ is the sound velocity. The boson density and current fluctuations are given by $J_0 \equiv \rho = \partial_\tau \theta$ and $J_i = \rho_0 \partial_i \theta$, respectively.

The XY model (2) is obviously gapless. There are two ways to open a gap. One way is to add a term $G \cos \theta$ to the model. This term explicitly breaks the $U(1)$ symmetry. The other way is to condense the vortex field. Condensing the vortex field will restore the $U(1)$ symmetry and the resultant state is a Mott insulating phase. We can introduce another Bose field $\tilde{b}$ to describe the vortex condensation. As derived in the Supplemental Material [41] the Bose field b and the vortex field $\tilde{b}$ can be described by two gauge fields a and $\tilde{a}$, respectively, where the boson current is given by $J^\mu = (i/2\pi)\varepsilon^{\mu\nu\lambda} \partial_\nu a_\lambda$ and the vortex current is given by $\tilde{J}^\mu = (i/2\pi)\varepsilon^{\mu\nu\lambda} \partial_\nu \tilde{a}_\lambda$. The two gauge fields have a mutual Chern-Simons coupling and the effective Lagrangian is

$$L_{CS} = -\frac{i}{4\pi} K_{IJ} \varepsilon^{\mu\nu\lambda} a_{I\mu} \partial_\nu a_{J\lambda} + L_{\text{Maxwell}}, \tag{3}$$

where $K = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $I, J = 1, 2$. Here have we changed the notation $a_{1\mu} = \tilde{a}_\mu$, $a_{2\mu} = a_\mu$ and the corresponding boson fields $b_1 = b$, $b_2 = \tilde{b}$. Under this notation, b_1 (the boson) carries the charge of the $a_{1\mu}$ gauge field and b_2 (the vortex) carries the charge of the $a_{2\mu}$ gauge field. The charges of both gauge fields a_1, a_2 are quantized since the boson number and vortex number are quantized. Therefore, the Lagrangian (3) is a well-defined effective field theory for the system.

If we gauge the $U(1)$ symmetry and probe the system through such an external gauge field A_μ, a coupling term should be added to the Lagrangian (3): $\mathcal{L}_{\text{prob}} = (i/2\pi)q_I \varepsilon^{\mu\nu\lambda} A_\mu \partial_\nu a_{I\lambda}$, where the charge vector is given by $q = (0, 1)^T$ [42]. The response to the probe field A_μ is the Hall conductance $\sigma_H = (1/2\pi)q^T K^{-1} q$. It is easy to see that the above Mott insulator has zero Hall conductance (it is the same with the nontrivial bosonic topological insulator that will be discussed below).

From the theory of the quantum Hall effect [43], the boundary of (3) is given by $\mathcal{L}_{\text{bdry}} = -(i/4\pi)K_{IJ}\partial_\tau \phi_I \partial_x \phi_J + V_{IJ}\partial_x \phi_I \partial_x \phi_J$, where $\partial_i \phi_I = a_{Ii}$, and the boson density operator ρ_I and creation operator b_I can be expressed in forms of the ϕ_I field as

$$\rho_I = \partial_x \phi_I, \quad b_I \sim e^{-i\phi_I}. \tag{4}$$

Before further discussion, let us see how the boson fields change under the symmetry group. Recalling that the boson field b_1 carries $U(1)$ symmetry charge and the vortex field b_2 is neutral, and the vortex becomes an antivortex under time reversal, they vary in the following way under $U(1) \rtimes Z_2^T$ symmetry: $U_\theta b_1 U_\theta^{-1} = e^{-i\theta} b_1, T b_1 T^{-1} = b_1$ [44], $U_\theta b_2 U_\theta^{-1} = b_2, T b_2 T^{-1} = b_2^\dagger$. From Eq. (4), the above relations can also be written as

$$\begin{aligned} U_\theta \phi_1 U_\theta^{-1} = \phi_1 + \theta, &\qquad T\phi_1 T^{-1} = -\phi_1, \\ U_\theta \phi_2 U_\theta^{-1} = \phi_2, &\qquad T\phi_2 T^{-1} = \phi_2, \end{aligned} \tag{5}$$

The bulk of the doubled Chern-Simons theory is gapped. What is interesting is the edge spectrum. Since the field ϕ_I satisfies the following Kac-Moody algebra,

$$[\partial_x \phi_I, \partial_y \phi_J] = 2\pi i K_{IJ}^{-1} \partial_x \delta(x - y), \tag{6}$$

the edge excitations may be gapless. As pointed out in Ref. [21], a perturbing term $G \cos \phi_2$ with $G < 0$ locates the ϕ_2 field to its classical value $\phi_2 = 0$ and gaps out the boundary without breaking any symmetry [see Eq. (5)]. That is to say, it is a trivial SPT phase.

In a Mott insulator, the boson number per site is an integer. It can be simply realized in a lattice Hamiltonian—the Bose Hubbard model

$$H = -\sum_{\langle ij \rangle} t_{ij} b_i^\dagger b_j + \sum_i [U(b_i^\dagger b_i - 1)^2 + \mu b_i^\dagger b_i]. \tag{7}$$

If U is larger than a critical value U_c, the boson will be localized and the boson number per site approaches to 1, which forms a (trivial) Mott insulator.

Since the bose Hubbard model only realizes the trivial Bose Mott insulator, it is mystical how to construct the nontrivial one. It turns out that it can be realized as long as the vortex of the Bose condensate varies nontrivially under time reversal [21]. To this end, we couple the Bose current to a second layer of $S = 1$ spins on the dual lattice [see Fig. 1(b), the spins are neutral under $U(1)$ symmetry].

Since a vortex or antivortex carries a nontrivial Bose current, we assume that under certain interactions creating a vortex or antivortex will increase or decrease the spin angular momentum of the $S = 1$ spin at the vortex or antivortex core. Thus, we obtain a composite vortex b'_2 as a combination of b_2 and S^-,

$$b'_2 = b_2 S^-.$$

where $S^\pm = S^x \pm iS^y$.

Notice that the spin is charge neutral, $U_\theta S^- U_\theta^{-1} = S^-$, and reverses its sign under time reversal $TS^-T^{-1} = -(S^-)^\dagger = -S^+$, so, for the composite vortex, we have $U_\theta(b'_2)U_\theta^{-1} = b'_2, T(b'_2)T^{-1} = -(b'_2)^\dagger$. When the new vortex field condenses, the $U(1)$ symmetry of the bulk in the boson layer is restored and a gap is opened. Repeating the previous argument, we can formulate the effective theory as a doubled Chern-Simons theory (3). However, what is different here is that the boundary remains gapless if the $U(1) \rtimes Z_2^T$ symmetry is unbroken. To see this, we rewrite the new vortex as $b'_2 = e^{-i\phi'_2}$ at the boundary, ϕ'_2 varies in the following way under the symmetry group:

$$U_\theta \phi'_2 U_\theta^{-1} = \phi'_2, \qquad T\phi'_2 T^{-1} = \phi'_2 + \pi. \qquad (8)$$

ϕ_1 varies in the same way as given in (5). ϕ_1 and ϕ'_2 still satisfy Kac-Moody algebra $[\partial_x\phi_1, \partial_y\phi'_2] = 2\pi i\partial_x\delta(x - y)$. To gap out the boundary, either $U(1)$ or time reversal symmetry Z_2^T should be broken explicitly or spontaneously. For instance, the perturbation term $G\cos(2\phi'_2)$ is invariant under the symmetry group and can gap out the boundary, but there are two ground states $\phi'_2 = 0$ and $\phi'_2 = \pi$. Time reversal $T\phi'_2 T^{-1} = \phi'_2 + \pi$ transforms one ground state into the other, so the Z_2^T symmetry is spontaneously broken.

Noticing that $T(b'_2)T^{-1} = -(b'_2)^\dagger$, careful readers may ask why time reversal symmetry is not broken by condensing $b'_2 = b_2 S^-$. At first glance, it seems that T is broken since $\langle b'_2 \rangle$ is not invariant under T. However, since b'_2 carries the charge of the gauge field a_2, it is not gauge invariant. In other words, the Lagrangian (3) is invariant under the following gauge transformation

$$b'_2 \to b'_2 e^{i\varphi}, \qquad a_2 \to a_2 - \partial\varphi.$$

The sign change of b'_2 under time reversal can be compensated for by a gauge transformation $b'_2 \to -b'_2$. When averaging all the gauge fluctuations, the vortex condensate has a zero expectation value $\langle b'_2 \rangle = 0$ [45]. In this way, we conclude that time reversal symmetry is not broken in the ground state even when b'_2 is condensed.

The BTI may also be realized in superfluids or superconductors (in the molecule limit) with spin orbital coupling. Supposing that the boson carries a soft spin-1 momentum which is initially staying at the $|0\rangle$ state, and the vortex or antivortex flips the spin momentum at its core to $|1\rangle/|-1\rangle$ because of spin-orbital coupling, then a nontrivial BTI is obtained if the vortex or antivortex condense.

Symmetry-protected invariants and possible lattice model realization of the BTI.—The nontriviality of the BTI constructed above can also be illustrated by its symmetry-protected invariant [46]. The invariant of the BTI is that a π flux of the $U(1)$ symmetry [47] carries a Kramers doublet. We will show how this is true.

In continuous field theory, the flux is dispersed in space and, consequently, a π flux (noted as $|\pi\rangle$) is distinct from $-\pi$ flux (noted as $|-\pi\rangle$). Under time reversal, they transform into each other

$$T|\pi\rangle \propto |-\pi\rangle, \qquad T|-\pi\rangle \propto |\pi\rangle.$$

To construct a time reversal invariant π flux centered at x, we have to consider a superposition of states $|\pi\rangle$ and $|-\pi\rangle$. On the other hand, $|-\pi\rangle$ and $|\pi\rangle$ are related by a vortex creation or annihilation operator,

$$b'_2|\pi\rangle = \eta|-\pi\rangle, \qquad b'_2|-\pi\rangle = \eta|\pi\rangle, \qquad (9)$$

where η is a constant. Notice that $Tb'^\dagger_2 T^{-1} = -b'_2$, we have $Tb'^\dagger_2 T^{-1}T|-\pi\rangle = \eta T|\pi\rangle = -b'_2 T|-\pi\rangle$ and, similarly, $Tb'_2 T^{-1}T|\pi\rangle = \eta T|-\pi\rangle = -b'^\dagger_2 T|\pi\rangle$. Comparing with (9), we can choose a proper gauge such that

$$T|\pi\rangle = |-\pi\rangle,$$
$$T|-\pi\rangle = -|\pi\rangle. \qquad (10)$$

This means that when acting on the two-dimensional space $|\pi\rangle$ and $|-\pi\rangle$, time reversal T behaves as $\hat{T} = i\sigma_y K$, satisfying $\hat{T}^2 = -1$. Since $\hat{T}$ is irreducible, $|\pi\rangle$ and $|-\pi\rangle$ (generally $|(2n + 1)\pi\rangle$ and $|-(2n + 1)\pi\rangle$) form a Kramers' doublet. As a contrast, in the trivial Mott insulating phase, the time-reversal invariant π flux is generally nondegenerate.

In a lattice model, we don't need to distinguish π flux and $-\pi$ flux. In this case, a π flux is still a doublet since the degeneracy can never be removed unless time reversal symmetry is broken. This is the symmetry-protected invariant of the topological Mott phase and can be used to distinguish from the trivial one.

We have proposed a mechanism to obtain the nontrivial Bose insulating state. Now we propose a possible interaction that may stabilize this state. The key point is to flip a spin when creating a vortex or antivortex. However, the vortex or antivortex creation operator is usually nonlocal and it is difficult to give its explicit form. Noticing that a vortex or antivortex carries Bose current, we can couple the Bose current with the spins. In addition to the Hamiltonian (7), a possible $U(1) \rtimes Z_2^T$ invariant interaction is [see Fig. 2(a)]

$$H_{\text{int}} = -\sum_i gS_i^z(J_{i_1 i_2} + J_{i_2 i_3} + J_{i_3 i_1}), \qquad (11)$$

where $g > 0$ is the interaction strength and $J_{ij} = (-ib_i^\dagger b_j + \text{H.c.})$ is the Bose current operator. Here we
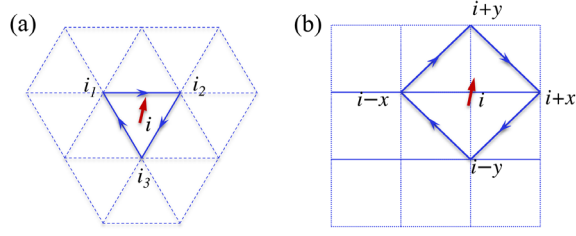
FIG. 2 (color online).   Interactions that stabilizes nontrivial SPT states. The Bose current interact with the spin such that the spin feel an effective "Zeeman field" (coupling to S_z) induced by the vortex current. (a) BTI on a double layer system. The spinless bosons live on triangular lattice and the spins (at boson vortex cores) locate on the dual lattice. (b) BTI of spin-1 bosons on square lattice. The vortex core of condensate on A sublattice locates on B sublattice, and vice versa.

consider a triangular lattice for the boson, and the spins live on the dual honeycomb lattice. The spins are initially staying in the eigenstate of S^z, $S^z|0\rangle = 0$ (Notice that the spin layer has time reversal symmetry but no spin rotational symmetry. The spins may have the following interaction $\sum_{\langle i,j\rangle} J\mathbf{S}_i \cdot \mathbf{S}_j + \sum_i D(S_i^z)^2$ [48] plus some perturbations to remove accidental symmetries). If $g > D > J$, the spin at the vortex (or antivortex) core will be flipped from $|0\rangle$ to $|1\rangle$ (or $|-1\rangle$), in agreement with our expectation that $b_2'^{\dagger} = b_2^{\dagger}S^+$, $b_2' = b_2 S^-$.

Realization of BTI in a single layer of spin-1 bosonic system.—In the above discussion we proposed to realize the BTI in a double layer model. In the following we show that it can also be realized in a single layer of spin-1 bosons, which is more likely to be implemented in cold atom systems [49,50]. A spin-1 boson can be considered as three species of bosons b_α ($\alpha = 1, 0, -1$). The $U(1)$ symmetry is defined by $U(1) = \{e^{i\hat{N}\theta}; \theta \in [0, 2\pi)\}$, where $\hat{N} = \sum_i \hat{N}_i$, $\hat{N}_i = \sum_\alpha b_{\alpha,i}^{\dagger}b_{\alpha,i}$. As usual, time reversal operator T will reverse the spin and is defined as $T = e^{i\sum_i S_i^y \pi}K$, where $S_i^y = (1/\sqrt{2})[-i(b_{1,i}^{\dagger}b_{0,i} + b_{0,i}^{\dagger}b_{-1,i}) +$ H.c.] (S_i^x, S_i^z are defined similarly) and K means complex conjugation. When acting on the boson operators, we obtain $U_\theta b_\alpha U_\theta^{\dagger} = b_\alpha e^{i\theta}$ ($U_\theta \equiv e^{i\hat{N}\theta}$), $Tb_0T^{-1} = -b_0$, $Tb_1T^{-1} = b_{-1}$, $Tb_{-1}T^{-1} = b_1$, and $TS_i^m T^{-1} = -S_i^m$, where $m = x, y, z$.

We consider the square lattice, which contains two sublattices labeled as A and B. Suppose the boson can only hop within each sublattice, namely, we only consider the next-nearest neighbor hopping. The idea to realize the topological Mott insulator is the same as what we discussed in the double layer model. We first condense the boson and break the $U(1)$ symmetry. For instance, we can condense the b_0 boson by tuning the interactions [see Eq. (12)]. The second step is to restore the broken $U(1)$ symmetry by condensing the vortex field. A vortex becomes an antivortex under time reversal. The resultant state is a (trivial) Mott insulating phase, which can be realized by the following simple Hamiltonian,

$$H = -\sum_{\langle\langle ij\rangle\rangle,\alpha,\beta} t_{ij}^{\alpha\beta} b_{\alpha,i}^{\dagger}b_{\beta,j} + \sum_i [U(\hat{N}_i - 1)^2 + \mu\hat{N}_i + D(S_i^z)^2],$$

(12)

where $\langle\langle\rangle\rangle$ means the next-nearest neighbor and $U > 0, D > 0$. Since $(S^z)^2 = b_1^{\dagger}b_1 + b_{-1}^{\dagger}b_{-1}$, the D term changes the chemical potential of b_1 and b_{-1} bosons.

To obtain the nontrivial Mott insulator, it is required that under time reversal the vortex reverses its sign and becomes antivortex. One way out is to redefine the vortex operator as $\tilde{b}' = \tilde{b}S^-$. Similar to the previous discussion, we couple the Bose current (carried by the vortex or antivortex) to the spin degrees of freedom (at the vortex or antivortex core).

Notice that we only considered next-nearest neighbor hopping, so the two condensates on two different sublattices can be considered independent of each other. The vortex core of the condensate on the A sublattice locates on the B sublattice, and vice versa [see Fig. 2(b)]. The two condensates (on two different sublattices) couple to each other via the following interaction,

$$H_{\text{int}} = -\sum_i gS_i^z(J_{i+y,i+x} + J_{i+x,i-y} + J_{i-y,i-x} + J_{i-x,i+y}),$$

(13)

where $J_{ij} = \sum_{\alpha,\beta}(-ib_{\alpha,i}^{\dagger}b_{\beta,j} + \text{H.c.})$ is the current operator of the condensate, and $g > D$. Under the above interaction, the vortex or antivortex tends to increase or decrease the spin momentum at its core and as a consequence the vortex (antivortex) will change its sign under time reversal. According to our previous discussion of Chern-Simons effective field theory, this interaction will possibly stabilize the nontrivial SPT phase.

Similar ideas can also be implemented in solid state systems. In the Supplemental Material [41] we realize the BTI in a pure spin-1 systems in the limit $U \to +\infty$ where the "charge" degrees of freedom are frozen. The $U(1)$ symmetry of the system is defined by $U(1) = \{e^{i(3S_z^2 - 2I)\theta}\}$, which preserves spin nematic momentum instead of magnetic momentum, and the time reversal symmetry is defined as usual. The BTI phase realized this way corresponds to a topologically nontrivial spin nematic phase with gapless edge modes.

Conclusion and discussion.—We constructed Bosonic topological insulators protected by $U(1)\rtimes Z_2^T$ symmetry. We first let the boson field condense and break the $U(1)$ symmetry. To gap out the Goldstone modes and restore the symmetry, we condense the vortex or antivortex field. If the vortex or antivortex creation operator does not reverse its sign under time reversal T, we obtain a trivial SPT phase. On the other hand, if the vortex or antivortex creation operator flips the spin momentum at the vortex or antivortex core such that it reverses its sign under T, we obtain a nontrivial SPT phase—the bosonic topological insulator after condensing the composite vortex. This mechanism might be realized in interacting superfluids or

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superconductors with spin-orbital coupling. Based on the above mechanism, we construct lattice models to realize BTI. Although our constructed Hamiltonians are still toy models, we mark an important step toward experimental realization of interacting bosonic SPT phases. For the purpose of detecting nontrivial BTI phase, we show that a π flux in the bulk of BTI carries a Kramers doublet.

Our method can be used to realize other SPT phases protected by different symmetries, and might shed some light on experimental realizations. There are also some questions remaining open, for instance, the models we constructed have not been studied numerically, and we didn't discuss how to identify the nontrivial SPT phases experimentally. These issues will be addressed in our further work.

[1] L. D. Landau, Phys. Z. Sowjetunion **11**, 26 (1937).
[2] V. L. Ginzburg and L. D. Landau, Zh. Eksp. Teor. Fiz. **20**, 1064 (1950).
[3] L. D. Landau and E. M. Lifschitz, *Statistical Physics-Course of Theoretical Physics* (Pergamon, London, 1958), Vol. 5.
[4] X. Chen, Z.-C. Gu, and X.-G. Wen, Phys. Rev. B **82**, 155138 (2010).
[5] X.-G. Wen, Phys. Rev. B **40**, 7387 (1989).
[6] X.-G. Wen and Q. Niu, Phys. Rev. B **41**, 9377 (1990).
[7] X.-G. Wen, Int. J. Mod. Phys. B **04**, 239 (1990).
[8] Z.-C. Gu and X.-G. Wen, Phys. Rev. B **80**, 155131 (2009).
[9] F. Pollmann, E. Berg, A. M. Turner, and M. Oshikawa, Phys. Rev. B **85**, 075125 (2012).
[10] X. Chen, Z.-X. Liu, and X.-G. Wen, Phys. Rev. B **84**, 235141 (2011).
[11] X. Chen, Z.-C. Gu, Z.-X. Liu, and X.-G. Wen, Phys. Rev. B **87**, 155114 (2013).
[12] X. Chen, Z.-C. Gu, Z.-X. Liu, and X.-G. Wen, Science **338**, 1604 (2012).
[13] F. D. M. Haldane, Phys. Lett. A **93**, 464 (1983).
[14] I. Affleck, T. Kennedy, E. H. Lieb, and H. Tasaki, Commun. Math. Phys. **115**, 477 (1988).
[15] C. L. Kane and E. J. Mele, Phys. Rev. Lett. **95**, 226801 (2005).
[16] B. A. Bernevig and S.-C. Zhang, Phys. Rev. Lett. **96**, 106802 (2006).
[17] C. L. Kane and E. J. Mele, Phys. Rev. Lett. **95**, 146802 (2005).
[18] J. E. Moore and L. Balents, Phys. Rev. B **75**, 121306 (2007).
[19] L. Fu, C. L. Kane, and E. J. Mele, Phys. Rev. Lett. **98**, 106803 (2007).
[20] X.-L. Qi, T. L. Hughes, and S.-C. Zhang, Phys. Rev. B **78**, 195424 (2008).
[21] Y.-M. Lu and A. Vishwanath, Phys. Rev. B **86**, 125119 (2012).
[22] Z.-X. Liu and X.-G. Wen, Phys. Rev. Lett. **110**, 067205 (2013).
[23] M. Cheng and Z.-C. Gu, Phys. Rev. Lett. **112**, 141602 (2014).
[24] Z. Bi, A. Rasmussen, and C. Xu, arXiv:1309.0515.
[25] X. Chen, Z.-C. Gu, and X.-G. Wen, Phys. Rev. B **83**, 035107 (2011).
[26] X. Chen, Z.-C. Gu, and X.-G. Wen, Phys. Rev. B **84**, 235128 (2011).
[27] P. Ye and J. Wang, Phys. Rev. B **88**, 235109 (2013).
[28] A. Vishwanath and T. Senthil, Phys. Rev. X **3**, 011016 (2013).
[29] M. A. Metlitski, C. L. Kane, and M. P. A. Fisher, Phys. Rev. B **88**, 035131 (2013).
[30] X.-G. Wen, Phys. Rev. B **65**, 165113 (2002).
[31] S.-P. Kou, M. Levin, and X.-G. Wen, Phys. Rev. B **78**, 155134 (2008).
[32] S.-P. Kou and X.-G. Wen, Phys. Rev. B **80**, 224406 (2009).
[33] A. Mesaros and Y. Ran, Phys. Rev. B **87**, 155115 (2013).
[34] L.-Y. Hung and Y. Wan, Phys. Rev. B **87**, 195103 (2013).
[35] Y.-M. Lu and A. Vishwanath, arXiv:1302.2634.
[36] P. Ye and X.-G. Wen, Phys. Rev. B **89**, 045127 (2014).
[37] Z. Y. Weng, D. N. Sheng, Y. C. Chen, and C. S. Ting, Phys. Rev. B **55**, 3894 (1997).
[38] P. Ye, C.-S. Tian, X.-L. Qi, and Z.-Y. Weng, Phys. Rev. Lett. **106**, 147002 (2011).
[39] C. Xu and T. Senthil, Phys. Rev. B **87**, 174412 (2013).
[40] X.-G. Wen, *Quantum Field Theory of Many-Body Systems—From the Origin of Sound to an Origin of Light and Electrons* (Oxford Univ. Press, Oxford, 2004).
[41] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.113.267206 for derivation of effective double Chern-Simons Lagrangian and realization of BTI in pure spin systems.
[42] Noticing that $J_{2\mu} = (1/2\pi)\varepsilon_{\mu\nu\lambda}\partial_\nu a_{2\lambda} = (1/2\pi)\varepsilon_{\mu\nu\lambda}\partial_\nu a_\lambda$ is the boson current, it is natural that the charge vector is equal to $(0,1)^T$. This is consistent with the fact that the b_1 boson [described by particle vector $l = (1,0)^T$] carries $U(1)$ symmetry charge, $Q = l^T K^{-1} q = 1$.
[43] X.-G. Wen, Phys. Rev. B **43**, 11025 (1991).
[44] Notice that an arbitrary phase factor $Tb_1T^{-1} = e^{-i2\phi}b_1$ arises if we redefine the boson operator $b_1 \rightarrow b_1 e^{i\phi}$. So this phase factor is not important and can be fixed to 1.
[45] S. Elitzur, Phys. Rev. D **12**, 3978 (1975).
[46] X.-G. Wen, Phys. Rev. B **89**, 035147 (2014).
[47] The physical excitations of the system only contain integer times of 2π fluxes (namely, the vortices), so the π flux can be seen as external perturbation, the so-called monodromy defect.
[48] The Heisenberg interaction may explain that the π-flux monodromy defects, which can be considered as ends of a string, carry spin-1/2 Kramers doublets.
[49] F. Zhou and M. Snoek, Ann. Phys. (Berlin) **308**, 692 (2003).
[50] X.-J. Liu, Z.-X. Liu, and M. Cheng, Phys. Rev. Lett. **110**, 076401 (2013).