

Geometric constraints in dual F-theory and heterotic string compactifications

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ABSTRACT: We systematically analyze a broad class of dual heterotic and F-theory models that give four-dimensional supergravity theories, and compare the geometric constraints on the two sides of the duality. Specifically, we give a complete classification of models where the heterotic theory is compactified on a smooth Calabi-Yau threefold that is elliptically fibered with a single section and carries smooth irreducible vector bundles, and the dual F-theory model has a corresponding threefold base that has the form of a \mathbb{P}^1 bundle. We formulate simple conditions for the geometry on the F-theory side to support an elliptically fibered Calabi-Yau fourfold. We match these conditions with conditions for the existence of stable vector bundles on the heterotic side, and show that F-theory gives new insight into the conditions under which such bundles can be constructed. In particular, we find that many allowed F-theory models correspond to vector bundles on the heterotic side with exceptional structure groups, and determine a topological condition that is only satisfied for bundles of this type. We show that in many cases the F-theory geometry imposes a constraint on the extent to which the gauge group can be enhanced, corresponding to limits on the way in which the heterotic bundle can decompose. We explicitly construct all (4962) F-theory threefold bases for dual F-theory/heterotic constructions in the subset of models where the common twofold base surface is toric, and give both toric and non-toric examples of the general results.

KEYWORDS: F-Theory, Superstrings and Heterotic Strings, Superstring Vacua

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1 Introduction and overview

1.1 Introduction

Since the early days of string theory it has been known that a wide range of different physical theories in four and higher dimensions can be realized by compactifying ten-dimensional string theories (and their more recently discovered higher-dimensional relatives M-theory and F-theory) on different geometric spaces [1, 2]. Tremendous effort has been expended in exploring the range of theories that can be realized through such compactification. While for theories in higher dimensions with extended supersymmetry, the range of possible string models has a tractable scope, for theories in four dimensions with minimal supersymmetry known constructions seem to give rise to such a vast “landscape” [3–5] of possibilities that

it is difficult to systematically study the set of allowed models and the constraints that they impose on 4D physics. It is suspected, in fact, that the known constructions of 4D $\mathcal{N} = 1$ theories from string theory may represent only the tip of a much larger iceberg composed of compactifications described by more general mathematical objects including non-Kähler and non-geometric compactifications.

Nonetheless, it may be possible by analyzing specific string constructions to ascertain some global constraints and systematic features of the theories that arise from compactification of string theory. Recent work on globally classifying 6D string/F-theory compactifications and associated constraints on 6D supergravity theories [6–8] suggests that a systematic analysis is possible in six dimensions and may provide tools for a similar treatment of some aspects of the space of 4D compactifications [9]. In this paper we analyze how geometric constraints on two general classes of string compactifications to 4D are related, as a step towards a more systematic understanding of the space of 4D $\mathcal{N} = 1$ theories that can arise from string theory.

Compactifications of heterotic string theory and F-theory provide two corners of the string landscape where 4-dimensional $\mathcal{N} = 1$ supersymmetric theories with chiral matter and exceptional gauge symmetries arise naturally. There is a tremendous literature on heterotic string compactifications; some recent work has sought to explore and enumerate the possible effective theories that can be obtained from compactification of the heterotic theory on a smooth Calabi-Yau (see [10–18] for some recent systematic studies). The duality between smooth heterotic compactifications and equivalent 4D F-theory constructions has also been broadly explored; see e.g. [19, 20]. The effective low-dimensional theories arising from compactifications of both heterotic string theory and F-theory are highly constrained by the background geometry of the compact dimensions. Indeed, it is an attractive possibility that these constraints might be strong enough to characterize which effective theories can arise (in any dimension) from heterotic or F-theory compactifications, or in the case of compactifications to 4 dimensions, used to characterize which string geometries could be relevant for string phenomenology and give rise to the low-energy physics we see in nature.

A major obstacle in any systematic attempt to classify the possible compactification geometries and effective theories for either the heterotic string or F-theory is the current limitation on our mathematical understanding of the relevant geometries. It is not known, for example, whether the number of distinct diffeomorphism classes of Calabi-Yau threefolds and fourfolds is even finite, much less how to characterize all the properties of the manifolds that determine the effective theories. There is, however, at least one class of backgrounds, involving dual heterotic and F-theory compactifications on elliptically fibered Calabi-Yau threefolds and fourfolds, where the number of topologically distinct string geometries is finite, and some systematic analysis is possible.

For those theories that have dual heterotic and F-theory constructions, the compactification geometries take the form [21, 22]

$$\text{Heterotic on } X_n, \pi_h : X_n \xrightarrow{\mathbb{E}} B_{n-1} \Leftrightarrow \text{F-theory on } Y_{n+1}, \pi_f : Y_{n+1} \xrightarrow{K3} B_{n-1} \quad (1.1)$$

where X_n is elliptically fibered over B_{n-1} and the K3-fibered manifold Y_{n+1} admits a more detailed description as an elliptically-fibered Calabi-Yau $(n + 1)$ -fold with section over a

base \mathcal{B}_n which is itself \mathbb{P}^1 fibered over B_{n-1} . The classification of such dual theories can be done at increasing levels of complexity by including successively more information about the geometry. At the coarsest level, the theories can be classified by the topological type of the base B_{n-1} . In dimensions eight and six there is a unique B_{n-1} associated with smooth heterotic/F-theory dual pairs (respectively a point and \mathbb{P}^1), but in four dimensions there are many distinct possible bases B_{n-1} . At the next level of detail, theories can be classified by the geometry of the F-theory base \mathcal{B}_n . For any given B_{n-1} there are in general many distinct topological types of \mathcal{B}_n that characterize allowed F-theory geometries. Theories with different bases \mathcal{B}_n (including those with different B_{n-1}) are connected on the F-theory side by tensionless string transitions and on the heterotic side by small instanton transitions [22–24]. We primarily focus in this paper on classifying theories at the level of \mathcal{B}_n . The choice of \mathcal{B}_n on the F-theory side fixes some of the topology of the dual heterotic bundles, but not all (specifically, it fixes some components of the second Chern classes of the bundles). For a given choice of \mathcal{B}_n , Higgsing/unHiggsing transitions in the effective theory, which correspond to deforming along/tuning moduli in the F-theory and heterotic bundle pictures, can modify the gauge group of the low-energy effective theory, and correspondingly modify the bundle structure group on the heterotic side. For $\mathcal{N} = 1$ theories in 4D, G -flux on the F-theory side lifts some moduli and can give disjoint sets of string vacua associated with compactifications on a given \mathcal{B}_n . For the most part, in this paper we concentrate on features that depend only on the geometry of \mathcal{B}_n and are independent of the moduli lifting and other issues associated with G -flux. While a further understanding of the consequences of G -flux is clearly desirable, a good understanding of the underlying geometric structure that we focus on in this paper seems to be an important first step in a systematic understanding of general 4D F-theory models.

In this work we focus on 4-dimensional effective theories arising from heterotic string theory on a smooth elliptically fibered Calabi-Yau threefold and F-theory on a dual $K3$ -fibered Calabi-Yau fourfold that admits a compatible elliptic fibration and has a smooth resolution. For both the Calabi-Yau threefold and fourfold geometries, we consider only geometries where the elliptic fibrations admit a (single) section. We assume that the gauge bundle in the heterotic theory is smooth, and that there are no heterotic 5-branes wrapping curves in the base B_2 , which would be associated with singular small instanton configurations.¹

Beginning on the heterotic side of the duality, it is known that the number of topological types of smooth elliptically fibered Calabi-Yau threefolds with section is finite [25] (see also [7], for a more constructive argument in the context of Weierstrass models). In a heterotic dimensional reduction, the 10-dimensional gauge field and the vacuum gauge field configuration over the Calabi-Yau threefold must be taken into account. These are described in the $E_8 \times E_8$ heterotic theory by adding to the Calabi-Yau geometry a pair of holomorphic vector bundles V_i ($i = 1, 2$) on X_3 with structure groups $H_i \subseteq E_8$. In the $SO(32)$ heterotic theory, only a single vector bundle is used. For fixed bundle topology

¹In some cases, the dual geometries we consider may include heterotic 5-branes wrapping the elliptic fiber of the CY threefold. See [19] for a discussion of the different roles that heterotic 5-branes can play.

(specified by $\text{rank}(V)$ and a total Chern class, $c(V)$), it is known that the moduli space of bundles² compatible with $\mathcal{N} = 1$ supersymmetry in 4D has only finitely many components³ [26, 27]. Although these proofs of general finiteness are at present not constructive, it seems possible to systematically construct at least one important class of dual models.

As we discuss further in section 3, when the geometries are smooth on both the heterotic and F-theory sides the base surface B_2 is restricted to be a *generalized del Pezzo* surface [28], of which there are a finite number of topologically distinct types. Over these bases a rational (\mathbb{P}^1) fibered threefold \mathcal{B}_3 must be built and here we restrict our attention to the case where \mathcal{B}_3 can be constructed as a \mathbb{P}^1 bundle over B_2 . We demonstrate here that there are a finite number of topologically distinct \mathbb{P}^1 bundles over any generalized del Pezzo surface such that \mathcal{B}_3 supports an elliptically fibered Calabi-Yau fourfold. Moreover, we show that this finite set of \mathcal{B}_3 's can be enumerated for any B_2 . The number of distinct branches of the moduli space of Weierstrass models over any specific \mathcal{B}_3 corresponding to distinct gauge group and matter contents is finite by a similar argument to that for base surfaces in (section 6.5 of) [7].

This class of dual heterotic/F-theory models thus represents a reasonable starting point with which we can get a first foothold into the problem of classifying and characterizing 4D $\mathcal{N} = 1$ string vacua and their effective theories, as well as understanding constraints on the effective theories arising from string geometry.⁴

The general structure just detailed is illustrated clearly in the simple case of 4D models where the base B_2 is toric. The powerful mathematical toolkit of toric geometry allows for simple and direct computations in this class of examples. While there are hundreds of generalized del Pezzo surfaces B_2 that can act as bases of smooth dual heterotic/F-theory Calabi-Yau threefolds and fourfolds, only 16 of these B_2 's are toric. The direct enumeration of all associated F-theory bases \mathcal{B}_3 (built as \mathbb{P}^1 bundles) is a straightforward calculation, which we carry out in this paper as an example of the general theoretical framework.

1.2 Overview of main results

For the convenience of the reader, we summarize here some of the main results of the paper that we believe have some novelty, and indicate where in the paper these results are described in more detail.

²More precisely, the moduli space of Mumford semi-stable sheaves on X_3 .

³Finiteness of the number of heterotic geometries here is established in two steps. First, the results of [26, 27] guarantee that for stable, hermitian bundles with fixed first and second Chern classes c_1, c_2 there are a finite number of possible values for the third Chern class $c_3(V)$ (note that in the case exceptional structure groups, c_3 is no longer a topological invariant). To argue that the number of heterotic geometries is finite we must further observe that $c_1 \equiv 0 \pmod{2}$, and the second Chern class is bounded as $0 \leq c_2(V) \leq c_2(TX_3)$ by heterotic anomaly cancelation (see eq. (5.2)).

⁴As this paper was being completed, the paper [29] appeared, in which magnetized brane models were considered over smooth elliptically fibered Calabi-Yau threefolds over del Pezzo bases, and the number of models in this class was shown to be finite.

1.2.1 Classification and enumeration of models

Classification of \mathbb{P}^1 -bundle bases \mathcal{B}_3 for F-theory models with smooth heterotic duals

As described at the conclusion of the previous section, we show in section 4.1 that there are a finite number of \mathbb{P}^1 bundles \mathcal{B}_3 over smooth bases B_2 for F-theory models with smooth heterotic duals on elliptically fibered Calabi-Yau threefolds with section. This follows from the fact that the number of generalized del Pezzo surfaces B_2 is finite, and the number of possible “twists” of the \mathbb{P}^1 bundle \mathcal{B}_3 over any B_2 is finite. We construct explicit bounds on the twist that reduce the classification of \mathcal{B}_3 ’s to a finite enumeration problem in section 4.1.3, and write a simple set of topological conditions that characterize allowed \mathcal{B}_3 ’s in section 4.1.5 and section 4.1.6. These results are quite general, and do not depend on toric geometry or any other specific conditions on the F-theory base geometry beyond the \mathbb{P}^1 bundle structure.

Enumeration of models with toric B_2 and smooth heterotic duals

As a concrete example of the general classification results, we explicitly construct all F-theory bases \mathcal{B}_3 that can be built as \mathbb{P}^1 -bundles over toric surfaces B_2 giving rise to smooth elliptically fibered fourfolds for F-theory compactifications with smooth heterotic duals. For the 16 toric B_2 ’s we find 4962 threefolds \mathcal{B}_3 , and classify the generic associated effective theories. These manifolds add to the dataset of Calabi-Yau fourfolds that have been systematically studied to date (see [30–36]). These results are described in section 9.

1.2.2 Topological constraints on symmetries and spectra

Matching geometric F-theory constraints and heterotic bundle constraints

We show that there is a close correspondence between the geometric constraints on F-theory models and conditions for the existence of smooth, slope-stable bundles in heterotic theories. This extends earlier work of Rajesh [37] and Berglund and Mayr [38]. The details of this correspondence are elaborated in section 6. Some of the most interesting aspects of this correspondence arise when a constraint is better understood on one side of the duality than the other. In particular, the next two items describe constraints on the gauge group and bundle structure that are currently understood most clearly from the F-theory perspective, while the last item below describes aspects of matter content that are clearest from the heterotic point of view.

Heterotic bundles and the base-point free condition

One of the most general methods known for explicitly constructing bundles suitable for heterotic compactification is the *spectral cover construction* [19, 39]. This construction is used to build bundles with structure group $SU(N)$ or $Sp(N)$. Irreducible bundles can only be constructed via a spectral cover when the second Chern class of the bundle satisfies a condition of base-point freedom. We find that for $SU(N)$ or $Sp(N)$ structure groups the base point freeness condition can be derived from the F-theory geometry *independent of the assumption of any particular method of bundle construction*. Thus, for these structure

groups base-point-freeness of (a part) of the second Chern class is required for all bundles in the moduli space.

In addition, we find that many F-theory models that should have smooth heterotic duals violate this base-point free condition. We show that these are all associated with bundles having exceptional or $SO(8)$ structure groups and thus do not violate the above constraints for $SU(N)$ and $Sp(N)$ bundles. More general methods such as the *cameral cover construction* [40–42] and other approaches to constructing general G -bundles described in [19] based on a theorem of Looijenga [43, 44] can in principle provide constructions of bundles with these more general structure groups. Our analysis gives a general classification of situations in which bundles with exceptional structure groups are expected to exist — though in many cases explicitly describing the properties of such bundles is an open problem in geometry. The property of base-point-freeness and its violation also has important consequences for the problem of vector bundle deformations and symmetry group breaking/enhancement. The base-point free condition is described in section 5.3. The corresponding F-theory condition and circumstances for its failure are described in section 6.4. Examples of cases where this condition is violated in F-theory are described in sections 7, 9.

Limitations on gauge enhancement

Geometric constraints on the F-theory side not only provide a minimal gauge group for the low-energy theory given a compactification topology, but can also limit the extent to which the gauge group can be enhanced over a given base geometry. For example, in many situations $SU(2)$ and $SU(3)$ gauge groups are constrained by F-theory geometry so that they cannot be in a broken phase of an $SU(5)$ gauge group. In these cases the restriction is associated with the structure of a codimension one singularity in the F-theory geometry. In other cases, codimension two singularities related to matter fields constrain enhancement — so that, for example, in some cases an E_6 cannot be enhanced to an E_7 . These limitations on gauge enhancement are described in section 4.2 and section 8 with examples given in section 7.

Chiral matter

The circumstances under which the low-energy theory has chiral matter are better understood on the heterotic side. We identify a class of situations in which chiral matter must arise due to the heterotic geometry, with implications for the dual F-theory model when G-flux is incorporated. On the heterotic side these correspond to bundles built via the spectral cover construction with structure group given by $SU(2n + 1)$, giving rise to 4D GUT theories with, for example, E_6 or $SU(5)$ symmetry [45]. In particular, in the dual F-theory geometries enumerated in this work, we find that many examples of theories with generic E_6 symmetry contain chiral matter. Examples of this type are described in section 7.8.

1.3 Outline

This paper is organized as follows: we begin in section 2 with a brief review of the duality between heterotic string theory and F-theory in dimensions eight and six. We focus on the nature of heterotic/F-theory duality, the classification of models, and constraints

on the effective theory, illustrating features and tools that are helpful in analyzing four-dimensional compactifications in the remainder of the paper. In section 3 we summarize heterotic/F-theory duality in four dimensions and describe the range of constructions of interest. Section 4 gives a more detailed description of the geometric constraints on the F-theory side, and section 5 describes the constraints on both the Calabi-Yau threefold and bundle geometries on the heterotic side. In section 6 we compare the constraints on the two sides and show when they are equivalent and when one side of the duality provides new information about the geometry of the other side. Section 7 contains some examples. In section 8 we summarize the consequences of our study for heterotic bundle moduli spaces, and in section 9 we describe the results of the systematic enumeration of all smooth F-theory geometries with toric base B_2 and a smooth heterotic dual construction. Finally, a brief summary of this work and associated open questions are given in section 10. Some technical details are relegated to appendices.

2 Lessons from heterotic/F-theory duality in higher dimensions

2.1 Heterotic/F-theory duality in eight dimensions

Beginning with the initial formulation of F-theory in 8 dimensions [21, 46] (see [47] for a review), the duality of F-theory with the heterotic string has provided an important window through which both theories can be better understood. In 8 dimensions, F-theory compactified on an elliptically fibered $K3$ surface Y_2 , $\pi : Y_2 \rightarrow \mathbb{P}^1$, is dual in certain (separate) limits of its parameter space to the perturbative $E_8 \times E_8$ and $SO(32)$ heterotic string theories on T^2 . In the case of the 8D $E_8 \times E_8$ heterotic theory, this duality can be understood most explicitly in the weak coupling limit of the effective theory, which is realized by taking the volume of T^2 to be large in the heterotic theory. The heterotic T^2 volume modulus is mapped into a complex structure modulus of the $K3$ surface in F-theory. Geometrically, the $E_8 \times E_8$ limit corresponds to decomposing the $K3$ surface into a (singular) fiber product of two elliptically fibered dP_9 surfaces, glued together along an elliptic curve⁵ — the so-called “stable degeneration limit” [19, 21, 22].

In the 8D stable degeneration limit, all the features of the two theories, including the moduli parameterizing the vacua, can be matched exactly [22, 46, 48–51]. For example, the possible gauge groups arising from different configurations of the heterotic flat gauge bundles on T^2 (i.e. Wilson lines) can be matched to the symmetries arising from ADE degenerations of the elliptic fiber of $K3$ that produce different non-Abelian symmetries over points in the \mathbb{P}^1 base (corresponding to the positions of 7-branes in the language of Type IIB); these degenerations were classified mathematically by Kodaira [52, 53]. From the point of view of the classification of models and constraints, the 8D story is quite simple. In this case the base manifold B_0 is a point, and the F-theory base $\mathcal{B}_1 = \mathbb{P}^1$ is the unique \mathbb{P}^1 bundle over this point. Thus, there is a single moduli space of 8D models connected by “Higgsing” type transitions that reduce or increase the size of the gauge group by de-tuning

⁵For the limit which produces the $SO(32)$ heterotic theory, the $K3$ degenerates into a fiber-product of rational surfaces, see [48] for details.

or tuning moduli to modify the singularity structure of the elliptic fibration. On both the F-theory and heterotic sides, the only constraint is that the gauge algebra \mathcal{G} must have a root lattice that can be embedded into the unique signature $(2, 18)$ unimodular lattice $\Gamma^{2,18}$ ([54], reviewed in [55]).

In lower dimensions, heterotic/F-theory duality is understood by fibering the 8D duality over a nontrivial shared base manifold B_{n-1} . As in (1.1), a heterotic theory on an elliptically fibered Calabi-Yau n -fold X_n , $\pi_h : X_n \rightarrow B_{n-1}$ is dual to F-theory on a $K3$ -fibered $(n+1)$ -fold Y_{n+1} with the same base, $\pi_f : Y_{n+1} \rightarrow B_{n-1}$, in which the $K3$ fiber is in turn elliptically fibered as described above. The elliptic and $K3$ fibrations are taken to be compatible, and both are chosen to have sections. This duality has been studied primarily in the stable degeneration limit [19], though in this paper we describe aspects of the duality that are true more generally, independent of this limit.

2.2 Heterotic/F-theory duality in six dimensions

2.2.1 Dual 6D geometries

For dual heterotic/F-theory compactifications to six dimensions, the perturbative heterotic compactification space is a $K3$ surface that is elliptically fibered over the common base $B_1 = \mathbb{P}^1$, and the dual F-theory geometry is a Calabi-Yau threefold Y_3 that is elliptically fibered with section over a Hirzebruch surface, $\pi : Y_3 \rightarrow \mathbb{F}_n$, where the Hirzebruch surface $\mathcal{B}_2 = \mathbb{F}_n$ is itself a \mathbb{P}^1 bundle over B_1 . As in 8D, codimension one singularities in the elliptic fibration encode a gauge group in the F-theory picture, which in 6D can include non-simply laced groups when monodromy is present [56]. Codimension two singularities encode matter fields.

In principle, heterotic/F-theory duality can be extended beyond the set of smooth dual geometries by incorporating non-perturbative effects such as NS5-branes in the heterotic theory. In this case, the dual F-theory geometry Y_3 is an elliptic fibration over a more general 2-dimensional base \mathcal{B}_2 , which is a blow-up of a Hirzebruch surface [21, 22], corresponding to a more general \mathbb{P}^1 fibration over $B_1 = \mathbb{P}^1$. The base $\mathcal{B}_2 = \mathbb{P}^2$ can also be realized on the F-theory side, e.g. after a tensionless string transition from $\mathcal{B}_2 = \mathbb{F}_1$ [21, 24]. For each choice of \mathcal{B}_2 , there is a connected moduli space of elliptically fibered Calabi-Yau threefolds describing a set of 6D theories connected by Higgsing and unHiggsing transitions. The global space of 6D F-theory compactifications [7] consists of a finite family of such moduli spaces connected by tensionless string type transitions. The connectivity of the set of moduli spaces associated with distinct \mathcal{B}_2 's corresponds to the mathematical framework of *minimal surface theory* [57, 58], in which curves of self-intersection -1 are blown down until a minimal surface (in this case a Hirzebruch surface \mathbb{F}_n with $n \leq 12$, \mathbb{P}^2 , or the Enriques surface [59]) is reached. A systematic classification of F-theory bases \mathcal{B}_2 according to the intersection properties of effective divisors is given in [8]. A complete enumeration of allowed bases is in principle possible and has been carried out explicitly for toric \mathcal{B}_2 's [60], and the more general “semi-toric” class of \mathcal{B}_2 's that admit a single \mathbb{C}^* action [61]. The global description of the moduli space is much more complicated on the heterotic side, where multiple coincident small instantons must be analyzed systemati-

cally (section 2.2.4). We restrict attention in this paper to smooth heterotic/F-theory dual geometries where no small instantons arise, which in 6D limits us to $\mathcal{B}_2 = \mathbb{F}_n$.

2.2.2 Geometric conditions on vacua

Even in 6D, each side of the duality encodes some nontrivial information about the geometry of its dual theory. In early explorations of F-theory [21, 22, 56], the degrees of freedom and effective theory of heterotic compactifications were used to develop the “dictionary” of how the dual Calabi-Yau threefold geometry determines the gauge symmetries and matter spectra in the F-theory description. F-theory can in turn be used to enumerate possible heterotic backgrounds ($K3, V_1, V_2$) and to make useful statements about their properties.

One of the most significant aspects of this duality is the way that the dual theories realize the condition for $\mathcal{N} = 1$ (minimal) supersymmetry in 6 dimensions. On the F-theory side this appears as the condition that the total elliptically fibered compactification space is a Calabi-Yau threefold (more precisely, the manifold can be singular, in which case the resolved geometry is a Calabi-Yau threefold; the F-theory description can be thought of as a singular M-theory limit, as reviewed for example in [47]). On the heterotic side, this condition corresponds to the statement that the compactification manifold is a $K3$ surface and that the gauge bundles (V_1, V_2) with structure groups embedded into each E_8 factor satisfy the Hermitian Yang-Mills equations [1], that is, that they are slope-(poly)stable [62–64]. Furthermore, the first Chern class of the principal bundles must vanish,⁶ with $c_1(V_i) \equiv 0 \pmod{2}$.

The choice of a smooth heterotic/F-theory dual pair in 6D is determined by a single integer. In the heterotic theory this appears as the choice of a fixed second Chern class for the vector bundles $c_2(V_{1,2}) = 12 \pm n$. On the F-theory side, this corresponds to the choice of Hirzebruch surface \mathbb{F}_n for the two-fold base \mathcal{B}_2 . The constraint $n \leq 12$ (originally described in [21]) gives a simple example of the type of geometric constraint that we explore later in this paper for 4D compactifications. On the heterotic side this constraint follows from the slope-stability of the vector bundles and the heterotic anomaly cancellation condition which relates the second Chern class of the holomorphic tangent bundle of $K3$ with those of the gauge bundles $V_{1,2}$. On the F-theory side this constraint follows from the fact that for $n > 12$ the existence of an effective divisor in $\mathcal{B}_2 = \mathbb{F}_n$ with self-intersection $-n < -12$ yields a singularity in any elliptic fibration over \mathcal{B}_2 that cannot be resolved to yield a total space that is a Calabi-Yau manifold. Thus, in this case rather different geometric considerations on the two sides give the same analytic constraint on the structure of the allowed theories.

2.2.3 Moduli and the stable degeneration limit

It is worth briefly reviewing the moduli of heterotic/F-theory compactifications in 6D and the interpretation of these moduli in the two dual pictures. As described in [21, 22] and

⁶More precisely, since the generators of E_8 are traceless, all principal bundles \mathcal{V} that are sub-bundles of an E_8 bundle (i.e. that have structure group $H \subseteq E_8$) must have vanishing first Chern class. However, if the associated vector bundles $V, \wedge^2 V \dots$ (arising in the heterotic theory from the decomposition of the 248-dimensional adjoint of E_8) are reducible, then their first Chern classes can be non-zero and satisfy $c_1(V_i) \equiv 0 \pmod{2}$.

easily confirmed in the toric description (reviewed in section 3.3), a Calabi-Yau threefold Y_3 that is elliptically fibered over the Hirzebruch surface \mathbb{F}_n can be described in Weierstrass form as

$$y^2 = x^3 + f(z_1, z_2)x + g(z_1, z_2) \tag{2.1}$$

where (z_1, z_2) are coordinates on \mathbb{F}_n and

$$f(z_1, z_2) = \sum_{i=0}^I z_1^i f_{8+n(4-i)}(z_2) \quad I \leq 8 \text{ such that } 8 + n(4 - I) \geq 0 \tag{2.2}$$

$$g(z_1, z_2) = \sum_{j=0}^J z_1^j g_{12+n(6-j)}(z_2) \quad J \leq 12 \text{ such that } 12 + n(6 - J) \geq 0 \tag{2.3}$$

More abstractly, f, g are sections of the line bundles $\mathcal{O}(-4K), \mathcal{O}(-6K)$, where K is the canonical class of the base \mathcal{B}_2 ; these explicit expressions give a local coordinate description of generic sections of these line bundles.

As is now well understood [19, 21, 22], the heterotic/F-theory dictionary in 6D indicates that the “middle” polynomials (the coefficients of z_1^4 and z_1^6 in f and g , respectively in (2.2), (2.3)) correspond to the moduli of the heterotic $K3$ surface, while polynomials of low degree (coefficients of z_1^i with $i < 4$ in f and coefficients of z_1^j with $j < 6$ in g) parameterize one of the heterotic bundles V_1 , and polynomials of high degree ($i > 4, j > 6$) parameterize the other bundle V_2 . The bundle V_1 has structure group H_1 , which is embedded in E_8 , and the resulting gauge group is the commutant G_1 of $H_1 \subseteq E_8$. On the F-theory side this corresponds to a Calabi-Yau threefold with 7-branes wrapping the \mathbb{P}^1 base of the $K3$ -fibration, giving rise to symmetry G_1 encoded in the singularity structure of the elliptic fibration at the point $z_1 = 0$. Similarly, the second heterotic bundle V_2 has structure group H_2 with commutant G_2 , associated with the singularity structure of the F-theory elliptic fibration at $z_1 = \infty$. This correspondence can be made precise in the stable degeneration limit, in which $Y_3 \rightarrow Y_1 \cup_{K3} Y_2$ where $Y_{1,2}$ are (non-CY) dP_9 -fibered threefolds. In this limit, the infinitesimal deformation space, $\text{Def}(Y_i)$, of Y_i can be matched exactly to that of the bundles $\text{Def}(V_i)$ and the $K3$ surface, $\text{Def}(K3)$ (see [21, 22, 56, 65] and [66] for a modern treatment of this result in terms of limiting mixed Hodge structures). That is,

$$h^{2,1}(Y_i) = h^1(K3, \text{End}_0(V_i)) + 20 \tag{2.4}$$

The correspondence between the F-theory moduli in the Weierstrass model and moduli of the dual heterotic bundles is particularly transparent in the spectral cover construction (5.3), where the polynomials f_k, g_k play a dual role in parameterizing the spectral cover divisor on the heterotic side.

2.2.4 Constraints on bundles and gauge symmetry

For a fixed topology of \mathcal{B}_2 on the F-theory side, the moduli encoded in the functions f, g (2.2), (2.3) parameterize Weierstrass models for all elliptically (and $K3$) fibered threefolds Y_3 over the base \mathcal{B}_2 . Parts of this moduli space in principle give a complete encoding of each dual heterotic moduli space of sheaves with fixed total Chern class, denoted

$\mathcal{M}_\omega(\text{rank}, c_1, c_2)$, that are stable with respect to a chosen Kähler form⁷ ω on $K3$. In this context, nontrivial features of the heterotic and F-theory geometries can be exactly matched. Considering only F-theory on the elliptically fibered threefolds with the 13 Hirzebruch bases \mathbb{F}_n , $n = 0, \dots, 12$, it is possible to deduce a number of facts about the dual moduli spaces of sheaves $\mathcal{M}_\omega(\text{rank}, c_1, c_2)$. The first of these is that for $c_1 = 0$, fixed c_2 , and the structure group H_i of V_i fixed subject to $H_i \subseteq E_8$, \mathcal{M}_ω has only one component — corresponding to a connected deformation space of a dual Calabi-Yau threefold Y_3 as described above.⁸

The moduli space structure on the F-theory side is matched non-trivially not only to the irreducibility of $\mathcal{M}_\omega(\text{rank}, c_1, c_2)$ for *fixed rank*, but also provides information about deformations of V_i that change the rank. A change in the rank (and hence structure group H_i) of the bundles V_i corresponds to changing the gauge symmetry of the 6D effective theory ($G_i \subseteq E_8$). In the F-theory geometry, this gauge symmetry can be varied by changing the complex structure of Y_3 . By tuning the complex structure to special loci in moduli space, it is possible to augment the Kodaira singularity types of the elliptic fibration over divisors in the base, which enhances the gauge group symmetry of the 6D theory; this corresponds on the heterotic side to specializing V_i to bundles with smaller structure groups. In the reverse process, in cases where complex structure deformations exist to break a symmetry, there is a simple realization in the effective theory as a direct “Higgsing” of G by charged hypermultiplets. Complete “chains” of these breaking/enhancement patterns have been determined for the dual 6D theories and matched exactly to complex structure deformations of the corresponding threefolds Y_3 [56] (see also figure 1).

While the gauge symmetry of a 6D theory can often be made smaller or larger by Higgsing or un-Higgsing through de-tuning and tuning moduli in the F-theory picture, there are also constraints on both sides that can restrict the extent to which a gauge group can be broken or expanded. For $n \geq 3$, the threefolds $\pi : Y_3 \rightarrow \mathbb{F}_n$ are generically singular (though they admit a smooth resolution). This means that in these cases there is a 6-dimensional gauge symmetry that cannot be Higgsed away by giving vevs to the hypermultiplets associated to the complex structure moduli. In the F-theory geometry, this corresponds to the presence of a divisor of self-intersection $-n$ in \mathbb{F}_n , over which the elliptic fibration must become singular and has a Kodaira type associated with a nontrivial gauge group factor. On the heterotic side, this implies that for certain values of the second Chern class c_2 there is a maximum structure group H possible *for any bundle with that topology*, since $H \subseteq E_8$ encodes the unbroken symmetry G in the 6-dimensional heterotic

⁷It is important to note that while heterotic/F-theory duality is believed to hold for the full moduli space of the the two theories, the explicit “dictionary” between degrees of freedom is only well understood for suitably weakly coupled regions of parameter space. On the heterotic side, this corresponds to moduli spaces of stable sheaves that are stable for the appropriate “adiabatic” choice [19] of Kähler form on $K3$.

⁸To avoid confusion it should be noted here that the rank appearing in the definition of the moduli space $\mathcal{M}_\omega(\text{rank}, c_1, c_2)$ refers to a fixed *fiber dimension* for a given vector bundle appearing in a representation of H_i determined by the decomposition of the adjoint of E_8 , *not* the rank of H_i as a Lie group. For example, although F_4 has rank 4 as a Lie group, one relevant moduli space for F_4 bundles appearing in heterotic theories would consist of rank 26 vector bundles. See section 5.2 for a general discussion of the relevant bundles, ranks and group representations.

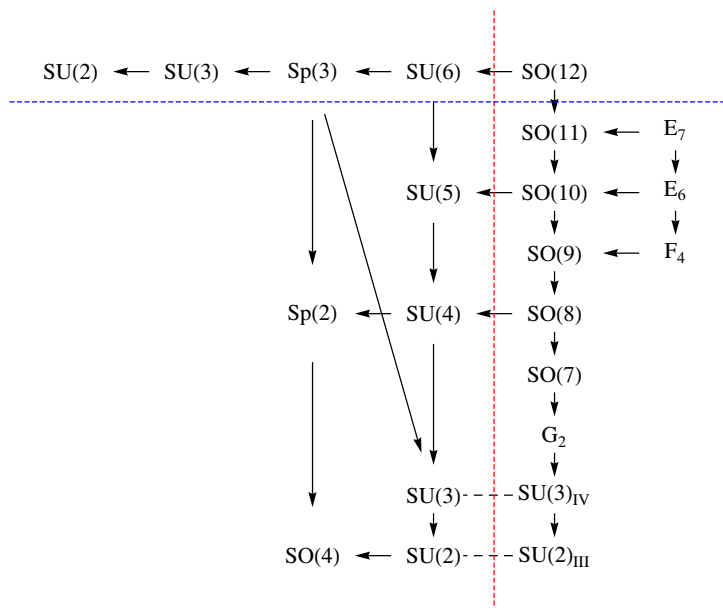


Figure 1. The possible Higgsing/Enhancement chains for smooth heterotic/F-theory dual pairs; modified from [56]. Figure depicts Higgsing possibilities based on heterotic bundles with structure group $H \subset E_8$, which match with dual F-theory models. F-theory gauge groups from Kodaira singularities with f, g having nonzero degrees of vanishing lie to the right of the vertical red dashed line, such gauge groups can be forced from the geometry (geometrically “non-Higgsable”) and cannot be unHiggsed to anything left of the line. The SU(3)’s and SU(2)’s connected near the bottom by horizontal dashed lines correspond to transitions between different Kodaira types in F-theory from type *IV, III* to type *I₃, I₂*. The top row above the horizontal blue dashed line corresponds to an alternative Higgsing sequence from E_8 to SU(3), SU(2) with non-standard commutants (e.g. $H = \text{SU}(3) \times G_2$ for upper SU(3)), generically associated with matter in the adjoint representation, which on the F-theory side involves wrapping on higher genus curves for 6D models. Note that in F-theory models that do not have heterotic duals, further unHiggsing (e.g. to $\text{SU}(N > 6)$) can occur. Note also that in the heterotic theory some Higgsing chains lead to product gauge groups, as discussed further in text.

theory. For example, if $c_2(V) = 4$, the maximal structure group of V is SU(2), for any such bundle on $K3$. It follows then that the moduli space of stable sheaves, $\mathcal{M}_\omega(r, 0, 4)$ contains no locally free sheaves (i.e. smooth bundles) for $r > 2$ (we will sometimes for brevity refer to such a moduli space as “empty”). This corresponds in F-theory to the fact that $\pi : Y_3 \rightarrow \mathbb{F}_8$ is singular with a generic, non-Higgsable E_7 symmetry [22].

A fact that is perhaps not generally well appreciated is that there are also cases in 6D where there are nontrivial constraints on the ways in which a gauge group can be enhanced by “unHiggsing” the generic model in a given component of the moduli space (i.e. possible up/rightward paths towards E_7 or SO(12) in figure 1). In particular, in any 6D F-theory construction where the low-energy theory has a generic (non-Higgsable) gauge group SU(3) arising from a Kodaira type *IV* singularity associated with a curve C of self-intersection -3 , the group cannot be enhanced to any larger SU(N). In F-theory this follows from the fact that the Weierstrass coefficients f, g must vanish to degrees 2, 2 on C , which is

incompatible with an A_n type singularity for $n > 2$. This condition corresponds on the heterotic side to a constraint on the extent to which the E_6 bundle over K3 with instanton number 9 can be deformed to a bundle with reduced structure group. In the case at hand, the enhancement $SU(3) \rightarrow G_2$ (i.e., the reduction of the E_6 bundle $V_{27} \rightarrow V_{26} \oplus \mathcal{O}$ in⁹ F_4) is possible, while the enhancement $SU(3) \rightarrow SU(4)$ via $V_{27} \rightarrow V_{10} + V_{16} \oplus \mathcal{O}$ in $SO(10)$ is not. Phrased differently, in terms of the Higgsing chains given in figure 1, if we begin with the lower right $SU(3)_{IV}$ for this case, it is possible to move upwards (un-Higgsing) and to the right along the right-hand path $G_2 \leftarrow SO(7) \leftarrow \dots$, but the group cannot be unHiggsed to the path $SU(4) \leftarrow SU(5) \leftarrow \dots$. While in 6D, those F-theory models with smooth heterotic duals have gauge groups that can always be enhanced to E_7 (or $SO(12)$) along *some* path in figure 1, the obstacles to gauge group enhancement can be stronger for more complicated 6D F-theory models (without smooth heterotic duals), and in four dimensions there are a number of constraints of this type. We explore the 4D constraints to gauge group enhancement in more detail in section 4.2.

The set of possible Higgsing/unHiggsing chains in dual pairs of 6D heterotic/F-theory models contains a number of other interesting features. As depicted in figure 1, a heterotic E_8 symmetry can be broken to $SU(3)$ in several distinct ways, depending on whether the commutant is $H = E_6$ or $H = G_2 \times SU(3)$, for example. In the latter case, explicit computation of the branching rules generically gives matter in the adjoint representation for the smaller groups along the Higgsing chain. In 6D F-theory models, $SU(N)$ matter in the adjoint representation is only possible when the gauge group arises on a curve of higher genus in the F-theory base B_2 . This is also true for $SU(N)$ models containing any representation other than the fundamental and k -fold antisymmetric tensor representation [67–69]. For F-theory models on \mathbb{F}_m dual to smooth heterotic models, the gauge group factors live on divisors of self-intersection $\pm m$. The divisors of self-intersection $-m$ with $m > 0$ are rigid and cannot support a higher genus curve. The divisors of self-intersection $+m$, however, can be taken with higher multiplicity, giving a higher genus Riemann surface. For example, in \mathbb{F}_2 there are irreducible curves of genus one that have twice the divisor class of the irreducible curve \tilde{S} of self-intersection $+2$. An $SU(3)$ gauge group with an adjoint results, corresponding to an $SU(3)$ factor in the dual heterotic model with the non-standard commutant. Note that for $m \neq 0$ this can only happen on one side, so only one gauge group can have adjoint matter representations and lie on the top line of figure 1, corresponding on the heterotic side to the fact that only bundles with $c_2(V) > 12$ can have a structure group such as $G_2 \times SU(3)$. Another interesting feature that can arise in this context is the appearance of product gauge groups as one of the factors $G_i \subset E_8$ in a heterotic model. On the F-theory side this corresponds again to a gauge group on a multiple of the divisor class with positive self-intersection, now given as a sum of two irreducible parts.

Constraints on heterotic bundles dual to F-theory models on $\mathcal{B}_2 = \mathbb{F}_n$ are particularly strong for $n = 9, \dots, 12$, in which cases the generic symmetry of Y_3 is E_8 (located on the patch containing $z_1 = 0$ in \mathbb{F}_n). This corresponds on the heterotic side to no structure group at all. That is, H is trivial and full E_8 symmetry is unbroken. In these cases, no smooth

⁹Here \mathcal{O} denotes the trivial line bundle over X_3 .

vector bundles exist on $K3$ with $c_2 = 1, 2, 3$. On the F-theory side this corresponds to the fact that the $-9, -10$ or -11 curve in \mathbb{F}_m must contain a point where the elliptic fibration is so singular that the point in the base must be blown up for the total space to have the structure of a Calabi-Yau threefold [8]. On the heterotic side this blowing up corresponds to the shrinking of an instanton to a point; in these cases there are *sheaves* (not locally free) with the desired topology. These cases go outside the smooth heterotic/F-theory dual paradigm that we focus on here; we encounter 4D analogues of these situations later but do not study them in detail. F-theory geometry and heterotic/F-theory duality thus lead to the inclusion of small instantons and sheafy degenerations in the heterotic picture, making it clear that the structure of the physical theory is in agreement with the mathematical notion of bundle/sheaf moduli spaces. Mathematically, any attempt to construct a moduli space of bundles alone results in a non-compact space. It is only by including sheaves (i.e., degenerations in the vector bundle) that a compact moduli space \mathcal{M}_ω arises. Physically, in the heterotic theory these sheaves¹⁰ correspond to point-like instantons (NS/M5 branes) on the K3 [21, 22], in the cases above with “instanton number” $c_2 = 1, 2, 3$. Thus for $0 < c_2 < 4$ we note that the moduli space \mathcal{M}_ω is non-empty but contains only sheaves and no smooth bundles. In principle, this line of development could be used to develop a full description of the 6D moduli space of non-perturbative vacua from the heterotic side of the duality which could be matched to the geometric F-theory description, though we do not pursue this further here.

2.2.5 Summary of 6D duality and relevance for 4D

To summarize this review of the 6D story, in 6 dimensions heterotic/F-theory duality encodes a deep and non-trivial correspondence between the moduli space of elliptically fibered Calabi-Yau threefolds (and their stable degeneration limits) and the moduli space of stable sheaves over $K3$. In six dimensions, essentially all of the information that can be inferred from the F-theory geometry about the heterotic bundle moduli space, including a) the irreducibility of $\mathcal{M}_\omega(r, c_1, c_2)$ and b) the existence of a “maximal” rank/structure group for a given c_2 , can be independently determined using known mathematics to study the heterotic geometry. Both of the facts, a) and b) were previously known in the mathematics literature in the study of moduli spaces of stable sheaves on $K3$ and Donaldson-Thomas invariants on $K3$ (see for example [72]).

In this paper, we ask many of the questions described above in the context of 4D heterotic/F-theory duality, which relates the moduli space of vector bundles (V_1, V_2) on an elliptically fibered Calabi-Yau threefold to the moduli of a $K3$ fibered Calabi-Yau four-fold. In this context, however, the information obtained is made more significant by the fact that far fewer mathematical techniques are known for determining the moduli space of stable sheaves on Calabi-Yau threefolds. Indeed, aside from a handful of examples with special topology (see e.g. [73–78]), no systematic tools exist for constructing the moduli spaces $\mathcal{M}_\omega(\text{rank}, c_1, c_2, c_3)$ or the corresponding Donaldson-Thomas Invariants [79].

In the following sections we use heterotic/F-theory duality to develop analogous statements to those made above for bundles on $K3$, and explore a number of new features,

¹⁰More precisely, skyscraper sheaves supported over points in the \mathbb{P}^1 base of $K3$ [70, 71].

unique to the 4D theory. These include deriving strong upper and *lower* bounds (based on $c_2(V)$) on the bundle structure group H , as well as constraints tying the matter spectrum of the theories to topology.

3 Heterotic/F-theory duality in four dimensions

We now briefly review the duality between heterotic string and F-theory compactifications for four-dimensional $\mathcal{N} = 1$ supergravity theories [19, 80]. We begin with a general abstract formulation of the duality in section 3.1, and then characterize the possible compactification geometries that are smooth on both sides of the duality in section 3.2. In section 3.3, we give a more detailed description of constructions that involve a toric base surface B_2 .

3.1 Geometry of heterotic/F-theory duality

We focus on the best understood class of dualities, in which the heterotic compactification is on a smooth Calabi-Yau threefold X_3 that is elliptically fibered with a single section over a base B_2 , the heterotic bundles are smooth and irreducible and there are no additional 5-branes wrapping curves in the base. In the dual F-theory compactification, we consider a threefold \mathcal{B}_3 that is a \mathbb{P}^1 bundle over B_2 . The F-theory compactification space \mathcal{B}_3 in turn acts as a base for an elliptically-fibered Calabi-Yau fourfold Y_4 . Following [19], we can construct the \mathbb{P}^1 bundle \mathcal{B}_3 as a projectivization of a sum of two line bundles

$$\mathcal{B}_3 = \mathbb{P}(\mathcal{O} \oplus \mathcal{L}), \tag{3.1}$$

where \mathcal{L} is a general line bundle on the base B_2 . On \mathcal{B}_3 we have the classes $R = c_1(\mathcal{O}(1)), T = c_1(\mathcal{L})$, where $\mathcal{O}(1)$ is a bundle that restricts to the usual $\mathcal{O}(1)$ on each \mathbb{P}^1 fiber. There are sections Σ_- and $\Sigma_+ = \Sigma_- + T$ of \mathcal{B}_3 that satisfy $\Sigma_- \cdot \Sigma_+ = 0$, corresponding to the relation $R(R + T) = 0$ in cohomology.

An F-theory model on \mathcal{B}_3 is dual to a heterotic model on X_3 . For the $E_8 \times E_8$ heterotic theory, the bundle decomposes as $V_1 \oplus V_2$, and the curvatures split as (see appendix A)

$$\frac{1}{30} \text{Tr } F_i^2 = \eta_i \wedge \omega_0 + \zeta_i, \quad i = 1, 2 \tag{3.2}$$

where η_i, ζ_i are (pullbacks of) 2-forms and 4-forms on B_2 and ω_0 is Poincaré dual to the section. The Bianchi identity gives $\eta_1 + \eta_2 = 12c_1(B_2)$. Heterotic/F-theory duality is possible when

$$\eta_{1,2} = 6c_1(B_2) \pm T, \quad (E_8 \times E_8). \tag{3.3}$$

This correspondence between η_i and T was identified by Friedman, Morgan, and Witten for bundles in the stable degeneration limit in [19]. It was shown more generally in [9] that this correspondence follows directly from the structure of axion-curvature squared terms in the dimensionally reduced supergravity action, independent of the stable degeneration limit or type of bundle construction. For the SO(32) heterotic string, the analysis [9] of the axion-curvature squared terms in the 4D supergravity action constrains the twisting T of the bundle on the F-theory side to satisfy

$$T = 2c_1(B_2), \quad (\text{SO}(32)) \tag{3.4}$$

for a dual $SO(32)$ heterotic compactification to exist. This generalizes the corresponding 6D case where the $SO(32)$ heterotic theory is dual to F-theory on \mathbb{F}_4 ($n = 4 \Rightarrow T = 4H = 2c_1(B_1 = \mathbb{P}^1)$).

Note that we assume that the elliptic fibration on the heterotic side has precisely one section. It is possible that fibrations with more than one independent section (i.e., with nontrivial Mordell-Weil group) — or with multisections — may admit some more general kind of F-theory dual. We leave this interesting question for further work. On the F-theory side, elliptic fibrations without a global section were explored in [81], and included into the moduli space of Weierstrass models in [82]. As described in [82], such models can be understood from Higgsing abelian $U(1)$ symmetries, which can in turn be understood from Higgsing nonabelian symmetries, all of which should have a clear parallel between the F-theory and heterotic descriptions, though we do not pursue this here.

For elliptic fibrations X_3 with one section and smooth total space, not all topological features of the heterotic bundle are determined by the dual F-theory geometry. Knowing the base B_2 and the twist T of the dual F-theory \mathbb{P}^1 bundle (i.e., knowing \mathcal{B}_3) allows for the identification of all components of $c_2(V_i)$ except for ζ_i in (3.2); these components together satisfy $\zeta_1 + \zeta_2 = 11c_1(B_2)^2 + c_2(B_2)$ (appendix A, [19]). The F-theory fourfold geometry also does not fix $c_3(V_i)$; for this we must consider in addition G -flux on Y_4 . These features each correspond to one undetermined parameter on the heterotic side. It is interesting that many of the consequences that can be derived from F-theory for the structure of heterotic bundles — discussed later in this paper — are largely independent of any possible freedom in these two parameters on the heterotic side.

A central piece of information on which we focus in our analysis is the generic gauge group in the low-energy 4D supergravity theory corresponding to a given heterotic/F-theory dual pair. On the F-theory side, the threefold base \mathcal{B}_3 defined by B_2 and T supports an elliptic fibration that may have singularities along certain divisors. The Kodaira classification of such singularities indicates the presence of nonabelian gauge group factors in the 4D supergravity theory [21, 22]. There can only be a smooth heterotic dual when the only nonabelian gauge group factors are associated with the divisors Σ_-, Σ_+ . For any given base \mathcal{B}_3 , there is therefore a minimal gauge group $G = G_1 \times G_2$, corresponding to singularity structures present over Σ_-, Σ_+ in a completely generic elliptic fibration over \mathcal{B}_3 . When such a gauge group is present, it implies that the largest possible structure group for a bundle on the heterotic side over B_2 with the topological data $\eta_{1,2}$ fixed by (3.3) is $H = H_1 \times H_2$ where the commutant of H_i in E_8 is G_i . The singularity structure of the F-theory elliptic fibration is determined in terms of a Weierstrass model $Y^2 = X^3 + fX + g$ over \mathcal{B}_3 , parameterized by f and g , which are as before sections of line bundles $\mathcal{O}(-4K), \mathcal{O}(-6K)$ where K is the canonical class of \mathcal{B}_3 . The minimal group factors G_i are determined by the minimal degrees of vanishing of f, g as listed in table 1, and in some cases distinguished by monodromy around the singular divisor [22, 56, 83]. Note that the Kodaira singularity dictates the physical gauge algebra only; different theories may have gauge groups that differ by a discrete factor that does not affect the algebra. In much of this paper we are somewhat cavalier about the distinction between gauge algebra and gauge group, but the reader should keep in mind that in most cases the only structure fixed by the local

deg f	deg g	\mathcal{G}_i	\mathcal{H}_i
1	2	\mathfrak{su}_2	\mathfrak{e}_7
2	2	$\mathfrak{su}_3, \mathfrak{su}_2$	$\mathfrak{e}_6, \mathfrak{e}_7$
2	3	$\mathfrak{so}_8, \mathfrak{so}_7, \mathfrak{g}_2$	$\mathfrak{so}_8, \mathfrak{so}_9, \mathfrak{f}_4$
3	4	$\mathfrak{e}_6, \mathfrak{f}_4$	$\mathfrak{su}_3, \mathfrak{g}_2$
3	5	\mathfrak{e}_7	\mathfrak{su}_2
4	5	\mathfrak{e}_8	trivial

Table 1. The gauge algebra summands associated with group factors G_i arising in 4D supergravity theory from divisors on which Weierstrass parameters f, g vanish to various degrees, and the associated structure group factors for dual heterotic bundles.

singularity structure of the F-theory geometry is the gauge algebra.

The only gauge factors listed in table 1 are those that can be forced to arise from the structure of the F-theory base threefold, independent of the choice of elliptic fibration, in cases with a smooth heterotic dual. This is analogous to the generic gauge groups for 6D F-theory models over Hirzebruch surfaces \mathbb{F}_m [21], with for example a generic gauge group of E_6 over the curve of self-intersection -6 in \mathbb{F}_6 . For more general F-theory models (i.e., those without smooth heterotic duals) there can be more complicated minimal gauge groups, some involving multiple gauge group factors — in analogy to the the general set of structures arising in maximally Higgsed 6D F-theory constructions, which can contain “non-Higgsable” matter [8]. In general, the gauge group in a particular model may be larger than the minimum group dictated by the structure of the base. For example, over some F-theory base threefolds it is possible to tune the Weierstrass coefficients f, g to have an A_4 singularity corresponding to a gauge factor $SU(5)$ over certain divisors, though this group does not arise as an automatic consequence of the geometry of any base corresponding to a singularity that arises in the generic elliptic fibration over that base. While a wide variety of models with different gauge groups can be tuned over each base, we are focused here on the minimal gauge group for each base.

In a number of places we will need to know the precise gauge algebra associated with given degrees of vanishing of f, g over the divisors Σ_{\pm} , including the effects of monodromy. As described in [56, 83], this can be determined by performing an expansion $f = f_0 + f_1z + f_2z^2 + \dots, g = g_0 + g_1z + g_2z^2 + \dots$ around the divisor D of interest, where z is an algebraic coordinate that vanishes on D . Since the precise conditions that determine the monodromy are expressed differently in various places in the literature, we collect here a succinct summary of the possible situations. When $\text{deg } f = 2, \text{deg } g = 2$, the gauge algebra is \mathfrak{su}_3 when g_2 is a perfect square, and \mathfrak{su}_2 otherwise. Similarly, when $\text{deg } f = 3, \text{deg } g = 4$ the algebra is \mathfrak{e}_6 when g_4 is a perfect square, and \mathfrak{f}_4 otherwise. The case $\text{deg } f = 2, \text{deg } g = 3$ is somewhat more complicated; in this case, the algebra depends on the factorization properties of the cubic $X^3 + f_2X + g_3$. If this cubic can be algebraically factorized into a product of three terms to the form $(X - a)(X - b)(X + (a + b))$ then the gauge algebra is \mathfrak{so}_8 , if it factorizes into the form $(X - a)(X^2 + aX + b)$ the algebra is \mathfrak{so}_7 ,

and if it does not factorize algebraically, the algebra is \mathfrak{g}_2 . We use these conditions in the analysis in several places in the remainder of the paper.

It is also worth noting that in the cases $T = 2c_1(B_2)$ that admit an $SO(32)$ heterotic dual, the twist T — and hence the base \mathcal{B}_3 on the F-theory side — is fixed uniquely for any B_2 . The corresponding minimal gauge group is always \mathfrak{so}_8 , matching with the expectation from the heterotic side. This is shown in section 6.3

3.2 Possible base surfaces for smooth heterotic/F-theory duals

There are only a limited class of bases B_2 over which the elliptic fibration geometry is a smooth Calabi-Yau on the heterotic side. The set of complex base surfaces over which an elliptically fibered Calabi-Yau threefold exists can be classified according to the intersection structure of effective divisors on the base [8]. When the base contains a curve of self-intersection -3 or below, the total space of the elliptic fibration becomes singular, and in general the heterotic theory acquires an enhanced gauge group. Note that there are some special cases where an F-theory construction on a \mathbb{P}^1 bundle over a base surface with -3 curves can apparently exist without an extra nonabelian gauge group, though there is still no smooth heterotic dual in such cases as the dual Calabi-Yau geometry would be singular; an example of such a model is given in section 7.3. Our restriction to models with smooth heterotic duals means that we limit our analysis here to bases B_2 that only have effective curves of self-intersection -2 or above.

The set of bases B_2 that contain no curves of self-intersection -3 or below consist of the *del Pezzo* surfaces dP_n , given by \mathbb{P}^2 blown up at $n \leq 9$ points, the bases $\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$ and \mathbb{F}_2 (which is a limit of \mathbb{F}_0 with a -2 curve), and the broader class of *generalized del Pezzo* surfaces, which are in general described as limits of del Pezzo surfaces containing curves of self-intersection -2 [28]. For each $n, 1 < n < 9$, there are generalized del Pezzo surfaces corresponding to limits of dP_n with a set of -2 curves having an intersection structure corresponding to any proper subgraph of the extended Dynkin diagram \hat{E}_n [84]. For $n = 9$, the classification is slightly more complicated. There are 279 rational elliptic surfaces with different combinations of -2 curves, corresponding to generalized del Pezzo surfaces with $n = 9$; these surfaces are classified in [85, 86]. Over each of the possible base surfaces B_2 there are a wide range of possible twists T giving different geometries on the F-theory side. Each such geometry will correspond to a different class of bundles on the heterotic side on the Calabi-Yau describing an elliptic fibration over B_2 . In principle, all possible F-theory bases \mathcal{B}_3 with a smooth heterotic dual can be classified by determining all allowed twists T for each del Pezzo and generalized del Pezzo (and for $\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$).

3.3 Heterotic/F-theory duals with toric base surfaces

A particularly simple class of bases B_2 can be described using toric geometry. In [60], the complete set of toric bases for elliptic threefold fibrations was enumerated. Here we are only interested in those cases where all effective curves have self-intersection -2 or above, which restricts us to only 16 possible bases: the del Pezzo surfaces dP_n with $0 \leq n \leq 3$, the surfaces \mathbb{F}_0 and \mathbb{F}_2 , and 11 other toric generalized del Pezzo surfaces with various combinations of curves of self-intersection -2 . These 16 bases are listed in table 4.

Following standard methods in toric geometry [87, 88], we characterize the base B_2 by the toric fan, consisting of vectors $v_0, \dots, v_{k+1} \in N_2 = \mathbb{Z}^2$. This describes a base with $h_{1,1}(B_2) = k$. We can choose a basis in which $v_0 = (1, 0)$ and $v_{k+1} = (0, 1)$. The coordinates of the remaining vectors defining B_2 can be written as $v_i = (x_i, y_i)$ in this basis. Any toric \mathbb{P}^1 bundle over B_2 can then be described in terms of a 3D toric fan

$$w_0 = (1, 0, 0) \tag{3.5}$$

$$w_i = (x_i, y_i, t_i), \quad 1 \leq i \leq k \tag{3.6}$$

$$w_{k+1} = (0, 1, 0) \tag{3.7}$$

$$w_{k+2, k+3} = s_{\pm} = (0, 0, \mp 1), \tag{3.8}$$

where s_{\pm} correspond to the divisors Σ_{\pm} , and $w_a, 0 \leq a \leq k+1$, are 3D rays that project to v_a , with the third component t_i parameterizing the twist T that defines the \mathbb{P}^1 bundle. The vanishing of the third component of w_0, w_{k+1} is a coordinate choice on $N_3 = \mathbb{Z}^3$ used to eliminate two redundant degrees of freedom in the twist T . For convenience we will use indices $i \in \{1, \dots, k\}$, $a \in \{0, \dots, k+1\}$, and $\alpha \in \{0, \dots, k+3\}$, and write $r = k+4$ for the total number of rays generating the 3D toric fan.

The toric language gives a simple description of the monomials available in the Weierstrass description of the F-theory model. If the set of 1D rays describing the toric threefold \mathcal{B}_3 are $w_{\alpha}, \alpha = 0, \dots, r-1$, then the monomials in f are in one-to-one correspondence with the elements $m = (a, b, c) \in M = N^*$ of the dual lattice whose inner product with all of the w_{α} is not less than -4 ,

$$\mathcal{F} = \{m \in N_3^* : \langle m, w_{\alpha} \rangle \geq -4, \forall \alpha\}. \tag{3.9}$$

Similarly, the monomials in g are associated with

$$\mathcal{G} = \{m \in N_3^* : \langle m, w_{\alpha} \rangle \geq -6, \forall \alpha\}. \tag{3.10}$$

As a simple example, for the case $\mathcal{B}_3 = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, which is the trivial \mathbb{P}^1 bundle over $\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$, the rays w_{α} are the basis vectors $(\pm 1, 0, 0), (0, \pm 1, 0), (0, 0, \pm 1)$, and the monomials in f, g are the triplets $(a, b, c) \in \mathbb{Z}^3$ with $|a|, |b|, |c| \leq 4, 6$.

For a toric F-theory base, we can compute the (anti)canonical class of B_2 directly from the toric description. There are two equivalence relations on the set of divisors D_a associated with the rays v_a , giving

$$D_0 \sim \sum_{i=1}^k -x_i D_i \tag{3.11}$$

$$D_{k+1} = \sum_{i=1}^k -y_i D_i. \tag{3.12}$$

We have then

$$-K_2 = c_1(B_2) = \sum_{a=0}^{k+1} D_a = \sum_i (1 - x_i - y_i) D_i. \tag{3.13}$$

Similarly, we have

$$-K_3 = 2\Sigma_- - K_2 + T, \tag{3.14}$$

where $T = \sum t_i D_i$. As discussed below for more general F-theory geometries with a heterotic dual, the formula (3.14) also follows straightforwardly from the definition of \mathcal{B}_3 given in (3.1) and the adjunction formula [19].

4 F-theory constraints

In this section we describe the geometric constraints on \mathbb{P}^1 -bundle threefold bases in the class of models with smooth heterotic duals. In section 4.1 we describe constraints on the threefold geometry, and in section 4.2 we describe further constraints on the gauge group of the corresponding 4D supergravity theory and the extent to which it can be enhanced through “unHiggsing” by moving on the Calabi-Yau moduli space.

4.1 Constraints on threefold base geometry

The basic conditions on an F-theory threefold base geometry \mathcal{B}_3 are that there are no codimension one or codimension two loci with singularities worse than the ϵ_8 singularity in the Kodaira classification. These conditions can be described in terms of constraints on the base geometry B_2 and twist T describing \mathcal{B}_3 as a \mathbb{P}^1 bundle over B_2 . We begin (section 4.1.1) with a brief overview of the general F-theory constraints, which are easy to make explicit in the toric context (section 4.1.2). In section 4.1.3, we use the general conditions to derive a set of local constraints on the twist T associated with specific divisors in the base. The toric description of these constraints is given in section 4.1.4. In section 4.1.5, we derive a simple set of necessary conditions associated with the divisors Σ_{\pm} . Combined with the constraints on T , this gives a set of conditions that are necessary, but not sufficient, for the existence of a good F-theory compactification geometry. In section 4.1.6, we include more general, nonlocal conditions associated with curves in \mathcal{B}_3 , which give a set of sufficient conditions for an acceptable F-theory model, subject to issues from codimension three singularities and G-flux that we do not address here (see section 7.8 for some relevant aspects of G-flux for 4D F-theory compactifications).

In the analysis in this section we repeatedly use a basic result from algebraic geometry, which states that if an effective divisor A on a surface S has a negative intersection $A \cdot D < 0$ with an irreducible effective divisor D having negative self-intersection $D \cdot D < 0$, then A contains D as a component, meaning that $A = D + X$ with X effective. This means in particular that any section $s \in \mathcal{O}(A)$ must vanish on D . This result was used in [8] to identify the “non-Higgsable clusters” that classify the intersection structure of base surfaces \mathcal{B}_2 for 6D F-theory compactifications. More generally, an effective divisor may contain a number of rigid divisors D_i with multiplicity γ_i by repeated applications of the preceding rule

$$A = \sum_i \gamma_i D_i + X, \quad (X \text{ effective}). \tag{4.1}$$

When such a decomposition is carried out over the rational numbers $\gamma_i \in \mathbb{Q}$, it is called the *Zariski decomposition* of A . While in higher dimensions the Zariski decomposition can

be more subtle, for surfaces the computation of the terms $\gamma_i D_i$, known as the *base locus* of A is straightforward. For example, if D is a curve of self-intersection $D \cdot D = -2$, and $A \cdot D = -4$, then $A = 2D + X$ with X effective, $X \cdot D = 0$.

In a number of places in this section we focus on curves in \mathcal{B}_3 of the form $C = \Sigma_{\pm} \cap D$, where D is a divisor on \mathcal{B}_3 pulled-back from a corresponding divisor in the base surface B_2 . We will generally use C for the curve in \mathcal{B}_3 , while D can refer either to the divisor in \mathcal{B}_3 or in B_2 , depending on context.

4.1.1 General constraints from F-theory geometry

We begin with a general statement of the F-theory constraints that hold for any geometry. For a good F-theory model to exist on a base \mathcal{B}_3 there must be a Calabi-Yau fourfold that is elliptically fibered over \mathcal{B}_3 [21, 22, 46]. As described in section 3.1, when the Weierstrass coefficients f, g and the discriminant $\Delta = 4f^3 + 27g^2$ vanish on a divisor in \mathcal{B}_3 , the corresponding 4D supergravity theory gets a nonabelian gauge group contribution depending upon the Kodaira type of the corresponding singularity in the elliptic fibration. When the vanishing degrees of (f, g, Δ) reach or exceed $(4, 6, 12)$ on a divisor, the fibration becomes too singular to admit a Calabi-Yau resolution. Thus, a constraint on \mathcal{B}_3 is that $-nK_3$ must admit a section of vanishing degree $< n$ for $n = 4$ or 6 on any irreducible effective divisor D . Similarly, f, g cannot vanish to orders $4, 6$ on any curve, or the curve would need to be blown up, giving a different base structure, for a Calabi-Yau resolution of the singular elliptic fibration to exist. This provides a strong set of constraints on bases \mathcal{B}_3 that admit good F-theory models. The constraint on codimension three loci (points) on the base is less clear; if the degrees of f, g reach $8, 12$ on a point then the point must be blown up for a good Calabi-Yau resolution. On the other hand, if the degrees of vanishing reach $4, 6$ on a codimension three locus but do not exceed $8, 12$ then the model may be problematic yet cannot be blown up directly [89]. We focus in this paper on the constraints associated with codimension one and two loci, associated with gauge groups and matter content in the low-energy theory. We include therefore in our analysis models with codimension 3 singularities, leaving the resolution of the status of these models to future work. Codimension three singularities are discussed further in section 7.8.

4.1.2 Constraints for toric bases

The F-theory constraints described above are particularly simple to describe for toric F-theory bases \mathcal{B}_3 using the explicit description of the Weierstrass monomials as elements of the dual lattice, as described in the previous section.¹¹ The degrees of vanishing of f, g on the divisor D_{α} associated with the ray w_{α} are given by

$$\deg_{D_{\alpha}} f = \min_{m \in \mathcal{F}} \langle m, w_{\alpha} \rangle + 4, \quad \deg_{D_{\alpha}} g = \min_{m \in \mathcal{G}} \langle m, w_{\alpha} \rangle + 6. \quad (4.2)$$

These are easily computed for any given base and divisor. For a good F-theory base, these degrees cannot both reach or exceed $4, 6$ on any divisor D_{α} . When the degrees are nonzero,

¹¹Related constraints in the toric language of “tops” were described in [91].

they indicate the presence of a generic gauge algebra factor on D_α , according to table 1. The degrees of vanishing of f, g on the toric curve $D_\alpha \cap D_\beta$ are given by

$$\deg_{D_\alpha \cap D_\beta} f = \min_{m \in \mathcal{F}} \langle m, w_\alpha + w_\beta \rangle + 8, \quad \deg_{D_\alpha \cap D_\beta} g = \min_{m \in \mathcal{G}} \langle m, w_\alpha + w_\beta \rangle + 12. \quad (4.3)$$

Again, these degrees cannot both reach or exceed 4, 6 on any curve or the F-theory base must be blown up along that curve to give a new base. When these degrees reach or exceed 1, 2 along a given curve they generally indicate the presence of $\mathcal{N} = 2$ matter transforming under gauge groups carried on the divisors D_α, D_β , in analogy to the 6D situation. In some toric 4D cases, however, there are no such nonabelian gauge groups. An example of such a situation is described explicitly in section 7.2, with \mathcal{B}_3 a P^1 bundle over $B_2 = \mathbb{F}_1$. Such codimension two singularities may simply represent cusps where the discriminant becomes singular, as occurs in 6D compactifications (see e.g. [90]), or may in some cases represent matter charged under U(1) gauge factors.

4.1.3 F-theory bounds on twists

We now specialize to the class of F-theory geometries that have smooth heterotic duals as described in section 3. In this case, the relation (3.14) derived above in the toric context holds more generally from the adjunction formula applied to \mathcal{B}_3 defined in (3.1)

$$-K_3 = 2\Sigma_- - K_2 + T. \quad (4.4)$$

Writing the Weierstrass functions f, g locally in a region around the locus Σ_- defined by the coordinate $z = 0$, where Σ_+ is at $z = \infty$, we have

$$f = f_0 + f_1 z + f_2 z^2 + \cdots + f_8 z^8 \quad (4.5)$$

$$g = g_0 + g_1 z + g_2 z^2 + \cdots + g_{12} z^{12}. \quad (4.6)$$

The term $f_k z^k$ vanishes to order k on Σ_- and to order $8 - k$ on Σ_+ , and f_k is a section of $\mathcal{O}(-4K_3 - k\Sigma_- - (8 - k)\Sigma_+)$. Similarly, $g_k z^k$ vanishes to order k on Σ_- and to order $12 - k$ on Σ_+ , and g_k is a section of $\mathcal{O}(-6K_3 - k\Sigma_- - (12 - k)\Sigma_+)$. Thus, f_k and g_k are sections of $\mathcal{O}(-4K_2 - (4 - k)T)$ and $\mathcal{O}(-6K_2 - (6 - k)T)$ respectively. We can use this fact to determine constraints on the possible twists T compatible with any particular base B_2 .

A set of necessary conditions on T can be determined by imposing the condition that f, g should not vanish to orders 4, 6 on any curve C in \mathcal{B}_3 that is of the form $\Sigma_- \cap D$, where D is associated with an irreducible effective divisor on B_2 . We consider the various possibilities depending upon the self-intersection of D in B_2 , using the fact that $(K_2 + D) \cdot D = -2$ when D is a rational curve. We focus on divisors D with non-positive self-intersection $D \cdot D \leq 0$, since determining conditions on these divisors is sufficient to bound the total number of twists T over any base B_2 . Note that, as shown in [8], any higher genus divisor of negative self-intersection in the base B_2 gives a singular elliptic fibration that cannot be resolved to a Calabi-Yau, so it is sufficient to restrict attention to rational curves D .

We begin with the case $D \cdot D = -2$, where $K_2 \cdot D = 0$. Consider then the intersection

$$(-nK_2 - (n - k)T) \cdot D = -(n - k)T \cdot D, \quad k < n. \quad (4.7)$$

When $T \cdot D > 0$, this intersection is negative, and $(-nK_2 - (n - k)T)$ contains D as a component for $n = 4, 6$ when $0 \leq k < n$. This means that the sections f_k, g_k must vanish on D . If $T \cdot D \geq 2$, then f_k, g_k must vanish on D at least to order $4 - k, 6 - k$ respectively, which would mean that f, g vanish to degrees 4, 6 on C . Thus, the twist must satisfy $T \cdot D < 2$ for any rational curve D in the base having self-intersection $D \cdot D = -2$. A similar argument for Σ_+ shows that $T \cdot D > -2$, so $|T \cdot D| \leq 1$ for any -2 curve in B_2 .

Now, consider the case $D \cdot D = -1$. In this case, $-K_2 \cdot D = 1$, so we have $(-nK_2 - (n - k)T) \cdot D = n - (n - k)T \cdot D$. This is less than or equal to $-(n - k)$ for all $k < n = 4, 6$ when $T \cdot D > 6$, which would force f, g to vanish on $D \cap \Sigma_-$ to degrees 4, 6, with a similar constraint with the opposite sign for Σ_+ , so we have a bound in this case of $|T \cdot D| \leq 6$. (For example, if $T \cdot D = 6$, then $(-6K_2 - T) \cdot D = 0$, so g_5 need not vanish on D , though all other f_k, g_k vanish on D to degree $n - k$.)

Similar reasoning shows that analogous constraints hold for curves of self-intersection 0 and for curves of more negative self-intersection; the complete set of constraints for the twist over any rational curve of (non-positive) self-intersection $-n$ is

$$n = 0 : \quad |T \cdot D| \leq 12 \tag{4.8}$$

$$n = 1 : \quad |T \cdot D| \leq 6 \tag{4.9}$$

$$6 \geq n \geq 2 : \quad |T \cdot D| \leq 1. \tag{4.10}$$

$$n \geq 7 : \quad T \cdot D = 0. \tag{4.11}$$

We only need the results for $n \leq 2$ in this paper.

These bounds provide strong constraints on the twists that are allowed for a \mathbb{P}^1 bundle over any base B_2 . As promised in section 1, these reduce the problem of identifying all smooth F-theory bases \mathcal{B}_3 with smooth heterotic dual geometries to a finite enumeration problem, since the curves of negative or 0 self-intersection in the base B_2 form a connected set. We summarize the results of a complete enumeration of all twists over toric bases with smooth heterotic duals in section 9.

4.1.4 Toric bounds on twists

The bounds on twists can be seen explicitly in the toric context. We can identify the bounds on the twist t_i over the divisor D_i associated with a given base ray v_i by considering the local geometry of the \mathbb{P}^1 bundle over the sequence of rays v_{i-1}, v_i, v_{i+1} . (If $i = 0$ or $i = k + 1$, we replace $i - 1$ or $i + 1$ with $k + 1$ or 0 respectively in the obvious fashion to respect the cyclic ordering of rays). We can choose a basis for N_2 so that $v_{i-1} = (1, 0)$ and $v_i = (0, -1)$ (note that this is a different choice of basis than that used in (3.5)–(3.8)). Associated with the three 2D rays v_{i-1}, v_i, v_{i+1} there are twists t_{i-1}, t_i, t_{i+1} , associated with the extension of the corresponding 3D rays w_{i-1}, w_i, w_{i+1} in the third dimension (3.6). In the 3D toric lattice N_3 we can perform a linear transformation taking

$$w_{i-1} = (1, 0, t_{i-1}) \rightarrow \tilde{w}_{i-1} = (1, 0, 0), \quad w_i = (0, -1, t_i) \rightarrow \tilde{w}_i = (0, -1, 0). \tag{4.12}$$

Since we are assuming that the base B_2 is smooth, the third ray has the form $w_{i+1} = (-1, -n, t_{i+1})$, where the integer $-n$ is the self-intersection of the divisor D_i [87]. The

linear transformation (4.12) takes

$$w_{i+1} \rightarrow \tilde{w}_{i+1} = (-1, -n, \tilde{t}_i), \quad \text{where } \tilde{t}_i = t_{i-1} + t_{i+1} - nt_i. \quad (4.13)$$

The parameter $\tilde{t}_i = T \cdot D_i$ determines the nontrivial part of the twist around the ray v_i , and can be constrained geometrically depending upon n , to reproduce the conditions (4.8)–(4.11). As in the general situation described in the previous subsection, the strongest constraint on the twist component \tilde{t}_i comes from the condition that f, g do not have degrees 4, 6 on the curves associated with $w_i + s_{\pm}$. Let us assume that $\tilde{t}_i \geq 0$. Then we have the following constraints on the monomials associated with $m = (a, b, c) \in N_3^*$ for f, g with $B = 4, 6$

$$\langle m, \tilde{w}_{i-1} \rangle \geq -B \quad \rightarrow \quad a \geq -B \quad (4.14)$$

$$\langle m, \tilde{w}_i \rangle \geq -B \quad \rightarrow \quad b \leq B \quad (4.15)$$

$$\langle m, s_- \rangle \geq -B \quad \rightarrow \quad c \geq -B \quad (4.16)$$

$$\exists \langle m, \tilde{w}_i + s_- \rangle < -B \quad \rightarrow \quad b - c > B \quad (4.17)$$

$$\langle m, \tilde{w}_{i+1} \rangle \geq -B \quad \rightarrow \quad -a - nb + \tilde{t}_i c \geq -B. \quad (4.18)$$

All of these inequalities except (4.17) must be satisfied for all monomials in \mathcal{F}, \mathcal{G} . Inequality (4.17) on the other hand, need only be satisfied by at least one point m , to avoid having a 4, 6 singularity on $w_i + s_-$. But this means that there must be at least one integral point satisfying all of these inequalities. The second through fourth inequalities define a simple triangle in the b - c plane within which any solutions must lie (figure 2). The final constraint imposes a condition that restricts the solutions within this triangle. This constraint is weakest when a is maximally negative, so if there are any solutions in b, c for any a they will also be acceptable for $a = -B$. Therefore, we need only ask whether there can exist any solutions of the form $(-B, b, c)$ to inequalities (4.15)–(4.18). For $n = 0$, (4.18) becomes $\tilde{t}_i c \geq -2B$, and since $c < 0$ from (4.17) and (4.15), we need $\tilde{t}_i \leq 2B$ for a solution of all the inequalities to exist. This is weakest for $B = 6$, so we have the constraint $\tilde{t}_i \leq 12$ when $n = 0$. For $n = 1$, (4.18) becomes $b \leq \tilde{t}_i c + 2B$, which combined with (4.17) becomes $c + B < \tilde{t}_i c + 2B$, so $(\tilde{t}_i - 1)(-c) < B$ and since $-c > 0$ we need $\tilde{t}_i \leq 6$. Finally, for $n = 2$, (4.18) becomes $2b \leq \tilde{t}_i c + 2B$, which combined with (4.17) becomes $2c < \tilde{t}_i c$, and since $c < 0$ this implies $\tilde{t}_i \leq 1$. To summarize, we have reproduced the constraints (4.8)–(4.11) in the toric context

$$n = 0 : \quad |\tilde{t}_i| \leq 12 \quad (4.19)$$

$$n = 1 : \quad |\tilde{t}_i| \leq 6 \quad (4.20)$$

$$n \geq 2 : \quad |\tilde{t}_i| \leq 1. \quad (4.21)$$

The constraints on coefficients in g ($B = 6$) for $n = 2, \tilde{t}_i = 1$ are shown in figure 2.

4.1.5 General constraints on geometry from f, g on divisors

In section 4.1.3 we used curves of the form $C = \Sigma_{\pm} \cap D$ in the base \mathcal{B}_3 to determine bounds on the individual components $T \cdot D$ of the twist T parameterizing the \mathbb{P}^1 bundle over B_2 . We now describe more general constraints on T associated directly with the divisors Σ_{\pm} .

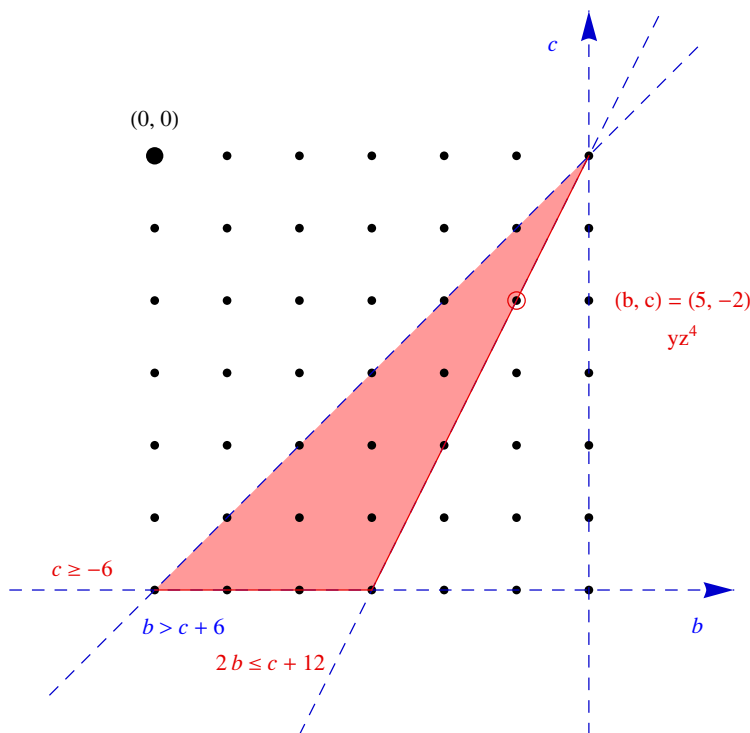


Figure 2. Constraints imposed on dual monomials in the Weierstrass function g parameterized by a, b, c in the toric description of a twist $\tilde{t}_i = 1$ over a curve of self-intersection $-n = -2$; the depicted constraints on b, c correspond to weakest conditions, which hold at $a = -B = -6$. For a smooth F-theory geometry, at least one monomial in the shaded region (not including the boundary at $b > c + 6$, or the corresponding region for $f(B = 4)$ must be nonzero. The circled point $(5, -2)$ corresponds to the monomial yz^4 in coordinates where $z = 0$ corresponds to Σ_- and $y = 0$ corresponds to D_i . This point is relevant in ruling out gauge algebra factors $\mathfrak{e}_7, \mathfrak{e}_8$ on Σ_- under these conditions (section 6.4).

Note that the divisors D on the base cannot give further constraints on T since f, g cannot vanish on D to higher degree than they do on the corresponding curve in B_2 . This follows from the fact that f_4, g_6 are sections of $-4K_2, -6K_2$ respectively. Thus, we need only consider constraints associated with the degrees of vanishing of f, g on the divisors Σ_{\pm} .

As described in section 4.1.3, the components f_k, g_k of the discriminant locus that vanish to degree k on Σ_- are associated with sections of the line bundles $\mathcal{O}(-4K_2 - (4 - k)T)$ and $\mathcal{O}(-6K_2 - (6 - k)T)$ respectively. In particular, g_5 is a section of $\mathcal{O}(-6K_2 - T)$. If $-6K_2 - T$ is effective, then this line bundle admits a section, so generically g_5 is nonzero and g does not vanish to degree 6 on Σ_- . On the other hand, if $-6K_2 - T$ is not effective then there are no sections and $g_5 = 0$. Furthermore, if $-6K_2 - T$ is not effective then $-6K_2 - nt$ cannot be effective for $n \geq 2$, since

$$n(-6K_2 - T) = -6K_2 - nt + (n - 1)(-6K_2) \tag{4.22}$$

and $-6K_2$ is effective, as is any positive combination of effective divisors. Similarly, if

$-6K_2 - T$ is not effective then $-4K_2 - mt$ cannot be effective for $m \geq 1$ since

$$m(-6K_2 - T) = -4K_2 - mt + (6m - 4)(-K_2), \tag{4.23}$$

where again the last term is effective. This shows that $-6K_2 - T$ is effective if and only if f, g do not vanish to degrees 4, 6 on Σ_- . A parallel argument shows that $-6K_2 + T$ is effective if and only if f, g do not vanish to degrees 4, 6 on Σ_+ .

From these considerations we can distinguish several possible configurations of allowed geometries on the F-theory side

A) One possibility is that

$$T = -6K_2. \tag{4.24}$$

In this case all coefficients of f_k, g_k vanish up to but not including degrees (4, 5).

B) If $T \neq -6K_2$ then g_5 is not a constant and must vanish on some curves D in B_2 . To avoid having f, g both vanish to degrees 4, 6 on the associated curve $C = \Sigma_- \cap D$ in \mathcal{B}_3 at least one other coefficient $f_{k \leq 3}, g_{k \leq 4}$ must be nonvanishing. But by a parallel argument to the above this means that

$$-4K_2 - T \text{ is effective.} \tag{4.25}$$

C) If there is any curve D in the base with self-intersection $D \cdot D = -2$, where the associated component of the twist is $T \cdot D = +1$, then by the analysis of section 4.1.3 we know that f_3, g_5 both vanish on D . This means that one of $f_{k \leq 2}, g_{k \leq 4}$ must be nonvanishing. Again, a parallel argument to the above means that in this case

$$\exists D : D \cdot D = -2, T \cdot D = +1 \Rightarrow -3K_2 - T \text{ is effective.} \tag{4.26}$$

Any good F-theory base geometry must satisfy these conditions and must fit into one of the 3 categories (A-C). A similar set of conditions hold for Σ_+ where the sign is changed for T on all equations.

4.1.6 General constraints on geometry from f, g on curves

The conditions (A-C) derived in section 4.1.5, along with the local twist conditions (4.8)–(4.11), give a set of *necessary* conditions that must be satisfied for any F-theory compactifications on a space \mathcal{B}_3 that is a \mathbb{P}^1 bundle over a base B_2 without divisors of self-intersection below -2. These conditions are not, however, sufficient. While the conditions derived in the preceding subsection are both necessary and sufficient for f, g to be well-behaved on divisors, the local constraints on twist components $T \cdot D$ are not sufficient to guarantee that f, g are well-behaved on all curves $C = \Sigma_- \cap D$ with D a divisor in B_2 . In general, “non-local” effects from other divisors can limit the range of allowed twists more stringently than the local conditions (4.8)–(4.11). Note, however, that there are no further constraints associated with curves formed from the intersection of two divisors D_i, D_j since the vanishing on such curves cannot be greater than at the corresponding points in B_2 .

As an example of a further constraint following from the interaction between twists on different divisors, consider a base B_2 that contains two divisors D_1, D_2 with $D_1 \cdot D_1 = D_2 \cdot D_2 = -2, D_1 \cdot D_2 = 1$. While the bound (4.10) seems to allow $T \cdot D_1 = T \cdot D_2 = 1$, if this were to hold then f, g would vanish to degrees 4, 6 on $D_1 \cap \Sigma_-, D_2 \cap \Sigma_-$. This can be seen by considering (as usual, for $n = 4, 6$ and $0 \leq k < n$) $Q = (-nK_2 - (n - k)T)$, which satisfies

$$Q \cdot D_1 = -(n - k)T \cdot D_1 = -(n - k) < 0 \tag{4.27}$$

$$Q \cdot D_2 = -(n - k)T \cdot D_2 = -(n - k) < 0. \tag{4.28}$$

We then have a decomposition (4.1) of Q of the form

$$Q = \gamma D_1 + \eta D_2 + X, \tag{4.29}$$

with $X \cdot D_1 \geq 0, X \cdot D_2 \geq 0$, so $(\eta - 2\gamma) \leq -(n - k), (\gamma - 2\eta) \leq -(n - k)$, from which it follows that $\eta \geq n - k, \gamma \geq n - k$.

A general statement of the sufficient conditions on T for f, g to be well-behaved on all curves $D \cap \Sigma_{\pm}$ (and all divisors Σ_-) is that for each divisor D in B_2 , there is at least one n, k ($n = 4$ or $6, k < n$) so that $Q_{n,k} = -nK_2 - (n - k)T$ has a decomposition of the form (4.1) with $\gamma_D < k$. This condition is automatically satisfied for any base B_2 without -2 curves, since for a -1 curve D as long as $-6K_2 - T$ is effective, $Q_{6,5} \cdot D = 6 - T \cdot D \geq 0$ for $T \cdot D \leq 6$, so $Q_{6,5}$ is effective and has sections that do not vanish on any -1 curves D in the base.

This shows that while the conditions described in section 4.1.5 and (4.8)–(4.11) are sufficient for a model to have acceptable f, g on all divisors and curves when the base is del Pezzo (or \mathbb{P}^2 or \mathbb{F}_0) with no -2 curves, when the base has -2 curves the more general conditions stated above must be included to give a set of sufficient conditions.

In formulating these conditions, note again that we have not considered potential problems with codimension 3 singularities or G-flux, which may make a geometry unsuitable for F-theory compactification even when the sufficient conditions discussed here are satisfied. These other issues are deferred to future work. In specific, however, note that any curve on which f, g vanish to degrees 4, 5 but are not constants will generically have points where f, g vanish to degrees 4, 6, which are of codimension 3 in the full base \mathcal{B}_3 .

4.2 Constraints on gauge enhancement

The constraints described so far limit the possible geometries that can be used for F-theory compactification, and for any given geometry impose a minimum gauge group that cannot be broken without changing the F-theory base B_2 . In some situations, the vanishing conditions on f, g also impose constraints that limit the extent to which gauge group factors in the effective supergravity theory can be enhanced by “unHiggsing” matter fields to form larger gauge groups. These constraints give nontrivial limitations on bundle structure in the dual heterotic picture. We consider two specific types of such constraints. In the first type, generic $SU(2)$ and $SU(3)$ gauge groups are constrained from being enhanced to $SU(N)$ with $N > 3$; in the second type, constraints are associated with codimension two loci in the F-theory picture. In both cases, the restriction on enhancement is related to the absence of sufficient matter to represent a Higgsed phase of a theory with higher symmetry.

Note that while the bounds considered here on gauge enhancement are a consequence of the geometry, it is possible that in some cases G-flux may decrease the size of the gauge group, as discussed further in later sections. This would not affect the upper bounds described in this section.

4.2.1 Constraints on SU(2) and SU(3) enhancement

One unusual feature of the gauge groups SU(2) and SU(3) is that they can be realized in two separate ways in F-theory, associated with two different Kodaira singularities. There is a standard A_{N-1} realization of SU(N) where f and g do not vanish on a given divisor but Δ vanishes to order N (an I_N singularity in the Kodaira classification). Another realization of SU(2) arises when f, g, Δ vanish to degrees 1, 2, 3 and SU(2) or SU(3) can be realized when f, g, Δ vanish to degrees 2, 2, 4 (Type III and IV singularities in the Kodaira classification). While type III and IV singularities are in one sense simply special limit points on the loci of type I_2 and I_3 singularities, their physical properties are rather different. When \mathfrak{su}_2 or \mathfrak{su}_3 gauge algebras are forced to exist on a divisor D by type III or type IV singularities in the F-theory geometry, there is no way to tune the Weierstrass moduli to realize any SU(N) gauge group with $N > 3$ on D since this can only happen from a type I_N singularity where f, g do not vanish on D . This means that for many theories with generic SU(2) or SU(3) gauge symmetries there is in principle no branch of the theory with enhanced SU(N) gauge symmetry (particularly no SU(5) gauge symmetry).

As a concrete example of where this constraint is relevant, we begin by considering the 6D case of F-theory compactification on \mathbb{F}_3 . In this case, the base $\mathcal{B}_2 = \mathbb{F}_3$ contains a divisor Σ with self-intersection $\Sigma \cdot \Sigma = -3$ on which f, g, Δ necessarily vanish to degrees 2, 2, 4. The resulting SU(3) gauge group in the corresponding 6D supergravity theory *cannot* be enhanced to SU(4) or any higher SU(N) by tuning the Weierstrass moduli to get an A_{N-1} singularity. This is clear in the low-energy 6D theory, as there is no matter charged under the SU(3), such as would arise under Higgsing from a larger gauge group. As we discuss in section 7, this corresponds in the heterotic dual picture to a constraint on how the associated bundle over K3 can be decomposed.

A similar constraint occurs for a wide range of 4D F-theory compactifications. In many 4D compactifications the structure of the base \mathcal{B}_3 is such that f, g, Δ are forced to vanish to degree 1, 2, 3 or 2, 2, 4, giving rise to a non-Higgsable SU(2) or SU(3) gauge group. In such cases these gauge groups cannot be enhanced to higher SU(N) with $N > 3$ anywhere in the moduli space, though the SU(2) gauge factors can generally be enhanced to SU(3) by tuning moduli. We describe some specific examples where these kinds of constraints appear in section 7.

4.2.2 Constraints from codimension two loci

In another class of situations, the extent to which a gauge group factor can be enhanced is limited by the degrees of vanishing of f, g, Δ on a locus of codimension 2. This occurs when the enhancement of the factor G on a divisor D to a given Kodaira singularity type automatically raises the degrees of vanishing of f, g, Δ on a codimension two locus to 4, 6, 12 or beyond.

In six dimensions, constraints of this type only appear for F-theory constructions without smooth heterotic duals. A simple example is when the base \mathcal{B}_2 contains two effective irreducible divisors C_1, C_2 with self and mutual intersections $C_1 \cdot C_1 = -2, C_2 \cdot C_2 = -3, C_1 \cdot C_2 = 1$. In such a situation, the 6D theory has a non-Higgsable gauge group with Lie algebra summands $\mathfrak{su}_2 \oplus \mathfrak{g}_2$, and there is matter charged under both groups, as described in [8]. While naively the G_2 factor can be tuned to an E_6 or F_4 by increasing the degrees of vanishing of f and g on C_2 to 3, 4, doing this raises the degrees of vanishing of f, g to 4, 6 on the intersection point $C_1 \cdot C_2$, leading to a point in the base that must be blown up. Writing the Weierstrass coefficients explicitly, in a coordinate system where z, w vanish on C_1, C_2 respectively we have $f = azw^2 + \mathcal{O}(z, w)^4, g = bz^2w^3 + \mathcal{O}(z, w)^6$, from which the above conclusions follow directly.

While this kind of enhancement constraint only arises for 6D F-theory models that do not have heterotic duals, in 4D the issue is much more general. One key class of examples, which we discuss further in section 6, are 4D compactifications of F-theory that have heterotic duals violating the base-point free condition (see section 5.3). In such situations an F_4 symmetry can have an obstruction to enhancement to an E_6, E_7 or E_8 on the F-theory side as such an enhancement would lead to a 4, 6 vanishing of f, g on a codimension two locus.

5 Heterotic constraints

As noted in section 1, the compactification of the $E_8 \times E_8$ heterotic string gives rise to a number of consistency constraints, linking the topological data of the Calabi-Yau threefold with that (i.e., Chern classes) of the vector bundles V_1, V_2 . We will be interested here in these bounds on topology, as well as in the conditions for supersymmetric $\mathcal{N} = 1$ heterotic vacua.

We consider a pair (V_1, V_2) of vector bundles on a Calabi-Yau threefold with structure groups $H_i \subseteq E_8, i = 1, 2$, which break each E_8 factor to the commutant G_i of H_i in E_8 . The bundles must satisfy the topological constraints

$$c_1(V_1) \equiv c_1(V_2) \equiv 0 \pmod{2} \tag{5.1}$$

$$\text{ch}_2(TX_3) - \text{ch}_2(V_1) - \text{ch}_2(V_2) + [W]_{\text{eff}} = 0 \tag{5.2}$$

The first of these conditions is equivalent to the vanishing of the second Steifel-Whitney class of the bundles V_i , a necessary condition for the existence of spinors; in the case of irreducible principal bundles, this reduces to the condition $c_1 = 0$ (see the discussion in section 2.2.2). Henceforth in this paper we focus attention on irreducible bundles and take $c_1 = 0$. The second constraint on the second Chern characters of the bundles is the familiar 10D anomaly cancellation condition we have already encountered in section 3.1 in the context of heterotic/F-theory duality. The last term in (5.2) is a non-perturbative contribution arising from NS5-branes (equivalently M5 branes, in heterotic M-theory), where $[W]_{\text{eff}}$ denotes the total class of effective curves wrapped by 5-branes. In this work, we will not include 5-branes wrapping curves in the base B_2 (i.e., degenerations of the bundle corresponding to sheaves supported over curves) and as a result, any possible term $[W]_{\text{eff}} \neq 0$ will not affect (3.3), the definition of η given in section 3.1. In some cases,

non-perturbative effects in the form of 5-branes wrapping the elliptic fibers may be present (see [19] for a discussion of such heterotic 5-branes and G-flux).

For a supersymmetric vacuum, the vanishing of the 10-dimensional gaugino variation requires each bundle V_i to satisfy the well-known “Hermitian-Yang-Mills” equations [1]

$$F_{ab} = F_{\bar{a}\bar{b}} = 0, \quad g^{a\bar{b}}F_{a\bar{b}} = 0 \tag{5.3}$$

The first half of these conditions, namely the vanishing of $F^{2,0} = F^{0,2} = 0$, is by definition the condition that the vector bundle is holomorphic (i.e., that its transition functions are holomorphic functions over the base X_3). The consequences of the condition $g^{a\bar{b}}F_{a\bar{b}} = 0$ are not so easy to state, however; solving this partial differential equation has historically posed a significant challenge to the construction of supersymmetric heterotic vacua, since the background Ricci-flat Calabi-Yau metric $g^{a\bar{b}}$ and the field strength F associated to the bundle V are not known analytically except in very special cases.¹² Thanks, however, to the powerful Donaldson-Uhlenbeck-Yau theorem [62–64], it is possible to translate this problem in differential geometry into one in algebraic geometry. According to the DUY theorem, a holomorphic bundle V admits a connection A that solves (5.3) if and only if V is *slope poly-stable*. A bundle V is defined to be *slope stable* with respect to a given Kähler form $\omega \in H^{1,1}(X_3)$ if for all sub-sheaves $\mathcal{F} \subset V$, with $0 < rk(\mathcal{F}) < rk(V)$,

$$\mu(\mathcal{F}) < \mu(V), \tag{5.4}$$

where for any sheaf,

$$\mu(\mathcal{F}) = \frac{1}{rk(\mathcal{F})} \int_{X_3} c_1(\mathcal{F}) \wedge \omega \wedge \omega. \tag{5.5}$$

A bundle is called semi-stable¹³ if $\mu(\mathcal{F}) \leq \mu(V)$ for all sub-sheaves, and “poly-stable” if $V = \bigoplus_i V_i$ with V_i stable and $\mu(V) = \mu(V_i) \forall i$. Regardless of the structure group H of V , vector bundles describing a good heterotic vacuum must be holomorphic, slope poly-stable and satisfy

$$\mu(V) = 0 \tag{5.6}$$

for the physical Kähler form ω .

It is poly-stable bundles that we must consider in the context of $\mathcal{N} = 1$ 4D heterotic Calabi-Yau vacua, and although the study of such bundles and their moduli spaces is a rich and ongoing subject in algebraic geometry, at present very little is known in general about how to fully classify and enumerate the moduli space of stable bundles (sheaves) on Calabi-Yau threefolds.¹⁴ One of our goals in this work is to try to use heterotic/F-theory duality to understand as much as possible about which stable bundles can exist on Calabi-Yau threefolds and what properties characterize the associated heterotic effective theories.

¹²For recent progress in solving these equations via numeric approximations, see for example [92–94].

¹³Note that all poly-stable bundles are automatically semi-stable, but the converse does not hold.

¹⁴See [78, 95, 96] for some recent results in the math/physics literature on bundle moduli spaces on Calabi-Yau threefolds in examples with special topology.

5.1 The Bogomolov bound

There are a number of constraints that slope-stability places on the topology of a holomorphic vector bundle. One of the most important of these is the so-called “Bogomolov bound” (see [97] for a review), which states that if a rank N bundle V is slope (poly-) stable with respect to a choice of Kähler form $\omega = t^k \omega_k$ (with $k = 1, \dots, h^{1,1}(X_3)$) on the CY 3-fold X_3 , then

$$\int_{X_3} (2Nc_2(V) - (N - 1)c_1^2(V)) \wedge \omega \geq 0 \tag{5.7}$$

For simplicity, let us consider first the case of vector bundles with $c_1(V) = 0$, in which case the Bogomolov bound reduces to $\int_{X_3} c_2(V) \wedge \omega \geq 0$.

Thus far our discussion of the consistency conditions on heterotic vacua has been general. We restrict our attention now to those threefolds that can give rise to F-theory duals, namely smooth, elliptically fibered Calabi-Yau threefolds, $\pi : X_3 \rightarrow B_2$ (with section). More specifically, we restrict our consideration to the case in which there is a single section (i.e., the Mordell-Weil group of sections is trivial and $h^{1,1}(X_3) = 1 + h^{1,1}(B_2)$) and the manifold can be put in Weierstrass form as

$$\hat{Y}^2 = \hat{X}^3 + f(u)\hat{X}\hat{Z}^4 + g(u)\hat{Z}^6 \tag{5.8}$$

where $\{\hat{X}, \hat{Y}, \hat{Z}\}$ are coordinates on the elliptic fiber (described as a degree six hypersurface in \mathbb{P}_{231}) and $\{u\}$ are coordinates on the base B_2 .

Let us consider the consequences of the Bogomolov bound for a bundle over an X_3 defined as above. Recalling the geometric identities in appendix A, as in (A.9) we can expand the second Chern class of the bundle as

$$c_2(V) = \pi^*(\eta) \wedge \omega_0 + \pi^*(\zeta) \tag{5.9}$$

where $\pi^*(\eta)$ and $\pi^*(\zeta)$ are pullbacks of, respectively, $\{1, 1\}$ and $\{2, 2\}$ forms on the base B_2 . (Note that in other sections we often use the notation e.g. η both for the form on B_2 and for the pullback form — technically $\pi^*(\eta)$ — on B_3 . Which form is used should be apparent in any given equation from context.) Expanding the Kähler form ω in the explicit basis of appendix section A, we have

$$\omega = a\omega_0 + \pi^*(\omega^{base}) \tag{5.10}$$

where a is a constant and $\omega_{base} = b^\alpha \omega_\alpha$ ($\alpha = 1, \dots, h^{1,1}(B_2)$) is an ample divisor on B_2 . Without loss of generality, we can scale the Kähler form to set $a = 1$

$$\omega = \omega_0 + M\pi^*(\omega^{base}) \tag{5.11}$$

for some constant M . Substituting this form for the Kähler moduli into the Bogomolov bound, the constraint becomes

$$\begin{aligned} & \int_X (\pi^*(\eta) \wedge \omega_0 + \pi^*(\zeta)) \wedge (\omega_0 + M\pi^*(\omega_{base})) \\ & = M\omega_0 \wedge \pi^*(\eta \wedge \omega_{base}) + (\omega_0 \wedge \pi^*(\zeta) + \pi^*(\eta) \wedge \omega_0 \wedge \omega_0) \geq 0. \end{aligned} \tag{5.12}$$

Moreover, we recall that by the triple intersection numbers in (A.5),

$$\int_{X_3} \omega_0 \wedge \pi^*(\eta \wedge \omega_{base}) = \eta \cdot \omega_{base}, \tag{5.13}$$

and $\pi^*(\eta) \wedge \omega_0 \wedge \omega_0 = \eta \cdot K_{B_2}$. Thus, the Bogomolov bound becomes the condition

$$\int_{X_3} c_2(V) \wedge \omega = M\eta \cdot \omega_{base} + \zeta + \eta \cdot K_{B_2} \geq 0 \tag{5.14}$$

(with ζ viewed now as the coefficient of the $\{2,2\}$ -form on B_2). Note that in (5.13) and (5.14) while the l.h.s. is computed by integrating over the threefold X_3 , the r.h.s. is computed in terms of intersection products on the base B_2 .

In general, to extract consistency conditions on η from the Bogomolov bound, the condition in (5.14) must be examined in a case-by-case manner. That is, given a choice of ζ and a Kähler form in (5.11), it is possible to derive consistency conditions on η associated to the underlying bundle being slope-stable (though once again it is important to recall that the Bogomolov bound is *necessary* but not sufficient for the stability of V).

There is one limit, however, that is of particular interest in heterotic/F-theory duality. In order to take the stable degeneration limit of section 3, it is necessary that we evaluate this expression with not just any Kähler form, but one chosen in the appropriate “Adiabatic limit” [19] in which the volume of the elliptic fiber is small compared to that of the base (and the volume of X_3 , as given in (A.6), is large). This limit is achieved by taking $M \gg 1$ in (5.14). For M sufficiently large, it is clear that the dominant constraint from the Bogomolov bound in (5.14) is that η has positive intersection with the Kähler form of the base B_2 . In the adiabatic limit, it is impossible for the Bogomolov bound to be satisfied unless

$$\eta \cdot \omega_{base} \geq 0 \tag{5.15}$$

If this is taken to hold for any ample Kähler form ω_{base} of the base then, by definition, this is simply the condition that η is an effective divisor in B_2 .

5.2 Matter spectra in heterotic theories

As described above, the presence of a vector bundle V_i with structure group H_i on the Calabi-Yau threefold breaks E_8 to G_i , the commutant of H_i inside of E_8 . All matter in the low-energy effective theory arises under dimensional reduction from components of the 10-dimensional gauge field, that is from the decomposition of the 248-dimensional representation of E_8 under the direct product $G_i \times H_i$ (see [1] for a standard review):

$$248 \rightarrow (\text{Ad}(G), 1) + \bigoplus_A (R_A, r_A) \tag{5.16}$$

where $\text{Ad}(G)$ represents the adjoint representation of G and $\{(R_A, r_A)\}$ denotes a set of representations of $G \times H$. For example, the presence of a bundle with $H = \text{SU}(3)$ over X_3 breaks one E_8 factor down to $G = E_6$ and the possible states in the theory are determined by the decomposition

$$248 \rightarrow (\mathbf{78}, \mathbf{1}) + (\mathbf{27}, \mathbf{3}) + (\overline{\mathbf{27}}, \overline{\mathbf{3}}) + (\mathbf{1}, \mathbf{8}) \tag{5.17}$$

Field	Multiplicity
27	$h^1(X, V)$
$\overline{\mathbf{27}}$	$h^1(X, V^\vee)$
1	$h^1(X, \text{End}_0(V))$

Table 2. Matter multiplicities for the E_6 example.

That is, in addition to the adjoint-valued E_6 gauge boson, the low energy theory can contain charged matter in the **27** and $\overline{\mathbf{27}}$ representations, as well as E_6 -singlet fields.

While the decomposition above is sufficient to determine the *type* of matter in the 4-dimensional E_6 theory, to find the *multiplicity* of these massless scalar fields it is necessary to count the number of bundle-valued 1-forms on the Calabi-Yau manifold. More precisely, under dimensional reduction, the zero-modes of the 4-dimensional theory are determined by the dimensions of vector bundle-valued cohomology groups (in the representation r_A) over the Calabi-Yau threefold, such as

$$H^1(X_3, V), H^1(X_3, \wedge^2 V), H^1(X, \text{End}_0(V)), \text{ etc.} \dots \tag{5.18}$$

For the E_6 example above this leads to the bundle-valued cohomology groups shown in table 2 where V is the rank 3 vector bundle valued in the fundamental of $SU(3)$ (and hence V^\vee is associated to the $\overline{\mathbf{3}}$ and $\text{End}_0(V)$ to the **8**).

The chiral index¹⁵ of the $\mathcal{N} = 1$ theory is determined by the Atiyah-Singer index theorem [98] as the alternating sum

$$\text{Ind}(V) = h^0(X, V) - h^1(X, V) + h^2(X, V) - h^3(X, V) \ . \tag{5.19}$$

But by slope stability of the bundle V and the condition $c_1(V) = 0$,

$$H^0(X_3, V) = H^0(X_3, V^\vee) = 0 \tag{5.20}$$

(as well as the induced representations; i.e., $H^0(X, \wedge^n V) = 0$, with $n < rk(V)$, etc). Finally, it should be recalled that by Serre duality [98], $h^m(X, V) = h^{3-m}(X, V^\vee)$ and as a result, the chiral index can be expressed simply as the difference,

$$\text{Ind}(V) = -h^1(X, V) + h^1(X, V^\vee) \ . \tag{5.21}$$

In the case that $c_1(V) = 0$ this is further given by $\text{Ind}(V) = \text{Ch}_3(V) = \frac{1}{2}c_3(V)$, the third Chern character. Finally, it should be noted that the index given in (5.21) is written in terms of the vector bundle associated to the fundamental representation of the underlying principal H -bundle. The chiral asymmetry in all other representations (i.e., for the induced vector bundles, $V^\vee, \wedge^m V, S^p V, \dots$) is fact determined by the index of the fundamental representation given above (see [99] for further details). In heterotic theories, the exact

¹⁵Physically, the chiral index in a heterotic compactification counts the number of generations minus the number of anti-generations of chiral particles. For example, the number of **27** multiplets minus the number of $\overline{\mathbf{27}}$'s in the E_6 theory given above, or the number of families in a heterotic Standard Model.

massless matter spectrum is frequently easier to compute than in F-theory, and in this work we use the simple structure of heterotic matter to extract useful information about the spectrum and chiral index arising from F-theory compactifications on Calabi-Yau fourfolds.

For heterotic theories defined over elliptically fibered Calabi-Yau threefolds, the bundle-valued cohomology groups defined above have a simple decomposition in terms of the base/fiber geometry. Using the techniques of Leray spectral sequences, there exists a decomposition

$$H^m(X, V) = \bigoplus_{p+r=m} H^r(B_2, R^p \pi_*(V)) \tag{5.22}$$

where $R^p \pi_*(V)$ is the p -th derived push forward [98] of V . On any open set \mathcal{U} , $R^p \pi_*(V)$ can locally be represented on B_2 by the pre-sheaf

$$\mathcal{U} \rightarrow H^p(\pi^{-1}(\mathcal{U}), V) \tag{5.23}$$

This formalism allows for a very precise notion of “localized” matter in the heterotic theory (supported over loci in the base B_2) which (in the case of simply-laced G) can be matched exactly to the localized matter associated to 7-brane intersections in the dual F-theory geometry. We explore this localized matter and the chiral index further in section 7.8 and in the context of the spectral cover construction of vector bundles below.

5.3 The spectral cover construction

Thus far our discussion of heterotic compactifications and constraints has been completely general, and throughout this work we attempt as far as possible to keep our study of the holomorphic vector bundles (V_1, V_2) independent of any particular method of bundle construction. It will, however, be useful in certain examples to appeal to one method of constructing vector bundles on elliptically fibered manifolds in which heterotic/F-theory duality is particularly well understood [19, 42]. This is the well-known “spectral cover” construction [39, 41].

5.3.1 Spectral covers

The spectral cover construction can be used to build rank N bundles, $V \rightarrow X_3$, with structure group $SU(N)$ or $Sp(2N)$. Moreover, for some bundles with these structure groups, which are slope-stable in the adiabatic region described in the previous section, the Fourier-Mukai transform¹⁶ [40] provides a 1–1 map (in fact a full functor on the category of coherent sheaves) from V to a pair $(\mathcal{S}, L_{\mathcal{S}})$ where \mathcal{S} is a divisor in X_3 that is an N -fold cover of the base B_2 and $L_{\mathcal{S}}$ is a rank-1 sheaf on \mathcal{S} . The class of \mathcal{S} is given by

$$[\mathcal{S}] = N[\sigma] + \pi^*(\eta) \tag{5.24}$$

where σ is the zero section of $\pi : X_3 \rightarrow B_2$ and η is defined as in (5.9). As in (5.8), let $\hat{X}, \hat{Y}, \hat{Z}$ be the coordinates of the elliptic fiber (where $\hat{Z} = 0$ defines the section σ). Then

¹⁶The precise conditions for stability and consistency of spectral cover bundles will be discussed further in the next section.

in the case that the structure group of V is $SU(N)$, the *spectral cover* \mathcal{S} can be represented as the zero set of the polynomial

$$s = a_0 \hat{Z}^N + a_2 \hat{X} \hat{Z}^{N-2} + a_3 \hat{Y} \hat{Z}^{N-3} + \dots \tag{5.25}$$

ending in $a_N \hat{X}^{\frac{N}{2}}$ for N even and $a_N \hat{X}^{\frac{N-3}{2}} \hat{Y}$ for N odd [19] (see [19] also for the analogous construction for $Sp(2N)$ structure group). The coefficients a_j are sections of line bundles over the base B_2

$$a_j \in H^0(B_2, K_{B_2}^{\otimes j} \otimes \mathcal{O}(\eta)) = H^0(B_2, \mathcal{O}(\eta + jK_2)), \tag{5.26}$$

which can locally be described as polynomial functions with appropriate degrees. Note that in the duality to F-theory, the coefficients a_j play a dual role as coefficients f_k, g_k in the F-theory Weierstrass model, providing a direct map between the moduli on the two sides of the duality. In order for the spectral cover to be an actual algebraic surface in X_3 it is necessary that \mathcal{S} be an effective class in $H_4(X_3, \mathbb{Z})$. It is straightforward to show [19, 39, 100] that this is true if and only if η is an effective class in B_2 . This can be seen by noting that η must be effective for a_0 to be nonvanishing, and since $-K_2$ is effective no other coefficient a_j can be nonvanishing if η is not effective. In view of the Bogomolov condition in the previous section, it is clear that spectral cover bundles are built to be slope-stable in the adiabatic region of Kähler moduli space.¹⁷

There is a further condition that must be imposed in order for the spectral cover bundle V to be slope stable. By construction, *irreducible* spectral covers are stable in the adiabatic region given above [19, 39]. However, the condition that the cover is irreducible places another condition on η . It can be argued that \mathcal{S} is irreducible¹⁸ if η is base-point free (*i.e.*, has no base locus in a decomposition of the type (4.1)) and $\eta - Nc_1(B_2)$ is effective (see [102] for example). The condition of base point free-ness will be explored in further detail for surfaces in section 6.4, but for now it should simply be noted that this condition guarantees that there exist irreducible curves in the class $[\eta]$.

To fully determine the bundle V and its topology after Fourier-Mukai transform, it is necessary to specify not only the class η in (5.24), but also the rank-1 sheaf $L_{\mathcal{S}}$ above. The condition that $c_1(V) = 0$ fixes the first Chern class of $L_{\mathcal{S}}$ to be [19]

$$c_1(L_{\mathcal{S}}) = N \left(\frac{1}{2} + \lambda \right) \sigma + \left(\frac{1}{2} - \lambda \right) \pi_S^* \eta + \left(\frac{1}{2} + N\lambda \right) \pi_S^* c_1(B_2) \tag{5.27}$$

¹⁷More precisely, it is known [100] that for a spectral cover bundle there exists some value M_0 such that for $M \gg M_0$ in (5.11), the bundle associated to that spectral data is slope stable for the given region of Kähler moduli space. For Kähler twofolds the proof of stability is constructive and yields an explicit value of M_0 defining the stable region of Kähler moduli space. For Calabi-Yau threefolds however, the arguments are not constructive and we are restricted to considering the limit $M \gg 1$ [19, 100].

¹⁸Note that if \mathcal{S} is irreducible as an algebraic curve in X_3 , the associated vector bundle under Fourier-Mukai transform will be indecomposable (*i.e.* not a direct sum $V_1 \oplus V_2 \oplus \dots$). However, the converse does not hold. Some reducible spectral covers can still correspond to indecomposable vector bundles. See [71, 101] for examples.

where $\pi_{\mathcal{S}} : \mathcal{S} \rightarrow B_2$, the bundle V has structure group $H = \text{SU}(N)$, and the parameter λ is either integer or half-integer depending on N :

$$\lambda = \begin{cases} m + \frac{1}{2}, & \text{if } N \text{ is odd} \\ m, & \text{if } N \text{ is even} \end{cases} \quad (5.28)$$

where $m \in \mathbb{Z}$. This condition arises from the fact that $c_1(L_{\mathcal{S}})$ must be an integral class in $H^{1,1}(\mathcal{S}, \mathbb{Z})$. When N is even it is clear that this integrality condition imposes

$$\eta \equiv c_1(B_2) \pmod{2} \quad (5.29)$$

where “mod 2” indicates that η and $c_1(B_2)$ differ only by an even element of $H^2(B_2, \mathbb{Z})$. The relation (5.27) holds when the cohomology of \mathcal{S} is spanned by the class σ and the pullback of the cohomology in the base. While this is expected to be true generically, there can be situations in which \mathcal{S} has a larger Picard group (i.e. more independent divisor classes). Examples where this increase in $h^{1,1}(\mathcal{S})$ may occur include Noether-Lefschetz loci in the complex structure of \mathcal{S} and degenerate (singular) spectral covers (see [103] for some generalizations of (5.27) for such examples). In these more general situations, the constraint (5.29) may not hold. We will see the need for such interesting possibilities (and their F-theory duals) in later sections.

Finally, with this data in hand it is possible to extract the full topology of V , including the chiral index, $\text{Ind}(V) = -h^1(X_3, V) + h^1(X_3, V^\vee)$. The Chern classes of a spectral cover bundle V , specified by η and the integers N and λ , are [19, 39, 45, 103]

$$c_1(V) = 0 \quad (5.30)$$

$$c_2(V) = \eta \wedge \sigma - \frac{N^3 - N}{24} c_1(B_2)^2 + \frac{N}{2} \left(\lambda^2 - \frac{1}{4} \right) \eta \wedge (\eta - N c_1(B_2)) \quad (5.31)$$

$$c_3(V) = 2\lambda\sigma \wedge \eta \wedge (\eta - N c_1(B_2)) = 2\lambda\eta \cdot (\eta - N c_1(B_2)) \quad (5.32)$$

Note that since $c_1(V) = 0$, $\text{Ind}(V) = ch_3(V) = \frac{1}{2}c_3(V)$.

The essential heterotic constraints on a bundle constructed via spectral covers can be simply encapsulated by the $\{1, 1\}$ form η . For a bundle V_i , if η_i and $\eta_i - N c_1(B_2)$ are effective, η_i is base-point free, and an $L_{\mathcal{S}}$ is chosen subject to (5.27), (5.28) and (5.29), a spectral cover bundle is guaranteed to exist and to be slope stable (for a region in Kähler moduli space in which heterotic/F-theory duality is well understood). In section 6 we explore the way that some of these same constraints appear in the dual fourfold geometry, giving information about when these constraints must be true based on topological data independent of a specific method of bundle construction.

5.3.2 Localized matter and spectral covers

In $\text{SU}(N)$ spectral covers, at least some of the zero-modes of the theory have a simple realization in terms of the geometry of the spectral cover $\mathcal{S} \subset X_3$. By the Leray spectral sequence arguments outlined above, it can be shown that the matter in the theory determined by $H^1(X, V)$ is localized at the intersections $\mathcal{S} \cap \sigma$ in B_2 (see section (6.2) of [19]

for a review). For example, for an $SU(2)$ spectral cover of the form

$$a_0 \hat{Z}^2 + a_2 \hat{X} = 0 \tag{5.33}$$

\mathcal{S} intersects the zero-section σ at the zero-locus of the section $a_2(u) \in H^0(B_2, K_2^{\otimes 2} \otimes \mathcal{O}(\eta))$ over B_2 . This localized matter appears as the **56** multiplets of the 4-dimensional E_7 theory. The exact multiplicity of these fields can be found by a Leray calculation to determine the exact zero-mode spectrum along the curve $a_2 = 0$ in B_2 .¹⁹

More generally, given an $SU(N)$ spectral cover of the form shown in (5.25), the localized matter counted by $H^1(X, V)$ (i.e., the matter valued in the fundamental representation of V) will be controlled by the zeros of $a_N(u) \in H^0(B_2, K_2^{\otimes N} \otimes \mathcal{O}(\eta))$. Note that this matter need not be chiral, and this is of course not the full matter spectrum of the low-energy 4D theory — for that other representations appearing in the decomposition of the **248** of E_8 in (5.16), such as $H^1(X, \wedge^2 V)$, $H^1(X, \text{End}_0(V))$ etc., must be considered. For these, we must consider not just the N -sheeted spectral cover associated to the fundamental representation of the $SU(N)$ bundle, but other curves associated to other induced vector bundles (such as $\wedge^k V, S^p V$, etc) as well. At present, it is not known how to construct all such associated spectral covers in full generality (see [20, 104] for some progress in this direction), though in the case of some of these representations there will likewise be a notion of localized matter [20].

Finally, it is worth noting that although the presence of “matter curves” in the class $[a_N = 0]$ in B_2 indicates the presence of charged matter in the 4D theory, the exact matter spectrum with multiplicities cannot be determined without fully specifying *all* the data of the bundle, including detailed properties of a particular V (not just its topology) and of course, the third Chern class (5.21). With this data in hand, the restriction of V to both the fiber and base implicit in (5.22) and (5.23) can be explicitly calculated. From the perspective of F-theory, we expect this further data to be necessary, since it is known that the exact (chiral) matter spectrum depends crucially not just on the fourfold geometry, but also a choice of G -flux. We will return to the issue of chiral matter and heterotic/F-theory duality again in section 7.8.

5.3.3 Limitations of spectral covers

Despite the fact that it constitutes one of the most studied and best understood corners of the dual heterotic/F-theory landscape, the spectral cover construction is far from general and care must be taken in generalizing results derived in this context to the full vector bundle moduli space or the generic dual fourfold geometries. As pointed out above, the spectral cover construction is valid only for special structure groups (i.e., $H = SU(N)$ or $Sp(2N)$) and perhaps more importantly, its applicability is limited even in these settings. That is, not all consistent $SU(N)$ or $Sp(2N)$ bundles arising in heterotic compactifications can be represented by well-behaved spectral covers.

¹⁹Note that in the dual F-theory geometry matter curves appear in the shared base B_2 in exactly the same way. In the notation of (4.5)–(4.6), the dual Weierstrass model to (5.33) is $y^2 = x^3 + (a_2 z^3 + \dots)x + (a_0 z^5 + \dots)$ and on the vanishing locus $a_2 = 0$ there is an enhancement of $E_7 \rightarrow E_8$ as expected.

Since this has an impact on the heterotic/F-theory comparisons undertaken in this work, it is worth briefly reviewing some of these constraints here. The Fourier-Mukai transform is a well-behaved functor on the moduli space of sheaves, subject to the following conditions: that the restriction $V|_{E_p}$ of the bundle to each elliptic fiber E_p is

1. Semistable
2. Regular (i.e., that the restricted rank N bundle on the elliptic curve has an automorphism group of the minimum possible dimension: $\dim(\text{Aut}(V|_{E_p})) = N$)

For the first of these, the semi-stability of V is defined with respect to the restricted slope in (5.5). For indecomposable stable bundles V this will be true for generic fibers when the Kähler class is chosen to be

$$\omega = \omega_0 + M\omega_\alpha \quad , \quad M \gg 1 \tag{5.34}$$

where, as above, for threefolds the stability proof is based on $M > M_0$ for some unknown M_0 and is not constructive. The condition above is sufficient to guarantee that a stable region for V exists, compatible with the large volume, weakly-coupled limit that we require for heterotic/F-theory dual pairs. It is important to note, however, that this limitation of moduli space is certainly not necessary for the consistency of heterotic theories [105, 106]. This point is explored further in appendix B.

The second condition of “regularity” appearing above was introduced by Atiyah in his classification of semi-stable sheaves on elliptic curves [107]. In particle, V is called “regular” if when it is restricted to every elliptic fiber it decomposes into a poly-stable sum of line bundles (i.e., by the divisor-line bundle correspondence, a set of points on E_i summing to zero) rather than a non-trivial extension (for example, Atiyah’s I_2 bundle of the form $0 \rightarrow \mathcal{O}_{E_i} \rightarrow I_2 \rightarrow \mathcal{O}_{E_i} \rightarrow 0$). Unlike the previous one, this condition cannot be stated as a simple global restriction on the heterotic moduli space and is harder to characterize for generic bundles. Indeed, it is significant that many good heterotic bundles will fail the regularity criterion. For instance, it is known that most bundles described via the monad construction [108, 109] are not regular [110]. In fact, there are indications that this criteria can sometimes be consistently violated in the context of perturbative heterotic/F-theory dual geometry (including in such well-known examples [71] as the heterotic “Standard Embedding” in which $V = TX_3$). See [101] for some recent results on degenerate spectral covers and dual F-theory geometry. The regularity condition will not be explored here in detail, but it may be relevant in explaining some of the unsolved questions regarding geometric constraints arising in dual heterotic/F-theory pairs, including exotic G-flux (see section 7.9) and the role of quantization conditions like (5.28) and (5.29).

6 Equivalence of constraints

We now consider the relation between constraints on F-theory geometry and the constraints on bundle constructions on the heterotic side.

6.1 Effective condition on η_i

We begin by showing that the F-theory condition that f, g do not vanish to degrees 4, 6 on either Σ_- or Σ_+ is equivalent to the heterotic condition that η_1 and η_2 are effective. This follows directly from the analysis of section 4.1.5, where it was shown that $-6K_2 \pm T$ must be effective in any F-theory geometry where f, g are well-behaved on Σ_{\pm} . With the identification

$$\eta_{\pm} = \eta_{1,2} = -6K_2 \pm T \tag{6.1}$$

it follows that the condition that η_{\pm} are effective is a necessary condition for the existence of a good F-theory geometry. This is thus a necessary condition for the existence of a good heterotic dual. This matches with what is known of the heterotic theory, where this bound is necessary in order to satisfy the Bogomolov bound in the adiabatic limit of the stable degeneration limit. The fact that this bound is necessary for any good F-theory geometry shows that this bound on the heterotic side must be more general and applies to any bundle construction, irrespective of the stable degeneration limit.

In fact, from the F-theory side the constraint is significantly stronger. The constraint on the F-theory side states that either $\eta_- = 0$, or $-4K_2 - T = \eta_- - 2c_1$ must be effective. This constraint, and the analogous constraint for η_+ must be necessary conditions on the heterotic side for the existence of any smooth bundle with the specified components of c_2 .

6.2 Effective constraint and gauge groups

In [38] the effective constraint on η for a bundle to exist on the heterotic side was generalized to situations where the 4D gauge group can be seen to be restricted in specific ways on the F-theory side. These constraints can be readily attained by a generalization of the analysis in the previous section. For example, if we consider a divisor Σ_- that carries a gauge group no larger than E_7 , then $-4K_2 - T$ must be effective for f to have a term of degree ≤ 3 , so $\eta = 6c_1 - T \geq 2c_1$, where by $A \geq B$ we mean that $A - B$ is effective. This corresponds on the heterotic side to a constraint on η for bundles with structure group $H = \text{SU}(2)$, so we conclude that $\eta \geq 2c_1$ is a necessary condition for the existence of a bundle with structure algebra \mathfrak{su}_2 (or greater). This matches with the result found in [38]. Similarly, bundles with structure \mathfrak{su}_3 or \mathfrak{g}_2 correspond to gauge algebras $\mathcal{G} = \mathfrak{e}_6, \mathfrak{f}_4$, which have g_4 or a lower term in g or f vanishing, so $-6K_2 - 2t = 2\eta - 6c_1 = 2(\eta - 3c_1)$ is effective. The results for these and the other minimal gauge group types are shown in table 3, again in agreement with [38]. Note that these conditions for bundles with structure group $\text{SU}(2), \text{SU}(3)$ precisely agree with the condition that $\eta - Nc_1(B_2)$ is effective that was needed in section 5.3, showing that this constraint is more general and independent of bundle construction. In the case where G is trivial and the structure group on the heterotic side is E_8 , it is only necessary that $-6K_2 - 5t = 5(\eta - 24c_1/5)$ be effective (corresponding to a nontrivial g_1). For $\text{SU}(N)$ groups with $N > 3$, on the other hand, f_0 and g_0 must be nonvanishing so $-6K_2 - 6t = 6(\eta - 5c_1)$ must be effective and $\eta \geq 5c_1$.

\mathcal{G}	\mathcal{H}	bound
\mathfrak{e}_8	1	$\eta = 0$
\mathfrak{e}_7	\mathfrak{su}_2	$\eta \geq 2c_1$
$\mathfrak{e}_6, \mathfrak{f}_4$	$\mathfrak{su}_3, \mathfrak{g}_2$	$\eta \geq 3c_1$
$\mathfrak{so}_8, \mathfrak{g}_2$	$\mathfrak{so}_8, \mathfrak{f}_4$	$\eta \geq 4c_1$
\mathfrak{su}_3	\mathfrak{e}_6	$\eta \geq \frac{9}{2}c_1$
\mathfrak{su}_2	\mathfrak{e}_7	$\eta \geq \frac{14}{3}c_1$
1	\mathfrak{e}_8	$\eta \geq \frac{24}{5}c_1$

Table 3. Constraints on η for certain structure groups H of heterotic bundles, identified from F-theory bounds on η for a given 4D gauge group G ; $\eta \geq ac_1$ means that $\eta - ac_1(B_2)$ is an effective divisor on B_2 . For example, a heterotic bundle with structure group $SU(2)$ is only possible when $\eta - 2c_1$ is effective. These results match those found in [38].

6.3 F-theory constraints and $SO(32)$ models

As discussed in section 3, for any given base B_2 there is a unique twist $T = 2c_1(B_2)$ so that F-theory on the resulting \mathcal{B}_3 is dual to the $SO(32)$ heterotic theory on a generic elliptically fibered Calabi-Yau threefold over B_2 . In this case the F-theory conditions are that f_2 is a section of $\mathcal{O}(-4K_2 - 2T) = \mathcal{O}_{\mathcal{B}_3}$ and g_3 is a section of $\mathcal{O}(-6K_2 - 3T) = \mathcal{O}_{\mathcal{B}_3}$. Both of these are therefore simply complex numbers so that the cubic $x^3 + f_2x + g_3$ has three complex roots and the resulting gauge algebra is \mathfrak{so}_8 . This matches the dual heterotic theory where a generic choice of bundle will break the full $SO(32)$ (really $Spin(32)/Z_2$) down to $SO(8)$, just as in the 6D case where the F-theory dual of heterotic on K3 is F-theory on an elliptically fibered Calabi-Yau threefold over the base F_4 .

6.4 Base-point free condition

We now consider the conditions that the heterotic base-point free condition imposes on F-theory geometry. On the heterotic side, the constraint is that η_i is base-point free on B_2 . As discussed in section 4.1, on a surface this simply means that there does not exist any effective divisor (curve) D of negative self-intersection such that $\eta_i \cdot D < 0$. In the F-theory picture this means that

$$(-6K_2 \pm T) \cdot D \geq 0 \tag{6.2}$$

for all effective divisors D in B_2 . Let us examine the consequences of this condition for curves satisfying $D \cdot D = -n$ where $n = 1, 2$.

For a curve D with $n = 1$, we have $-K_2 \cdot D = 1$, since $(K_2 + D) \cdot D = 2g - 2 = -2$ where D is a rational curve, so (6.2) becomes

$$6 \pm T \cdot D \geq 0. \tag{6.3}$$

This is automatically satisfied from (4.9). Thus, the base-point free condition imposes no additional conditions for F-theory bases associated with twists over curves of self-intersection -1 . In particular, this means that any F-theory base \mathcal{B}_3 formed as a \mathbb{P}^1 bundle

over a del Pezzo base B_2 automatically gives rise to η_i that satisfy the base-point free condition on the heterotic side, since B_2 contains no curves of self-intersection lower than -1 .

Now consider the case of a curve D with self-intersection $-n = -2$. In this case, we have $-K_2 \cdot D = 0$, so (6.2) becomes

$$\pm T \cdot D \geq 0. \tag{6.4}$$

This relation is only satisfied with both signs when $T \cdot D = 0$. Thus, the base-point free condition will be violated whenever the twist over a -2 curve has $T \cdot D = \pm 1$. From (4.10), these are the only nonzero possibilities.

We can now analyze the consequences for the gauge group on the divisors Σ_{\pm} when the base-point free condition is violated. This corresponds to the condition (C) analyzed in section 4.1.5. Assume that D is a curve of self-intersection -2 with an associated twist having $T \cdot D = +1$. In this situation, $(-nK_2 - (n - k)T) \cdot D = -(n - k)$. As described in section 4.1.3, this means that g_5, g_4, f_3, f_2 vanish on D to degree at least 1, g_3, g_2, f_1, f_0 vanish on D to degree at least 2, etc. An immediate consequence is that f, g cannot vanish on Σ_- to degrees 3, 5 or higher, or they would vanish on $\Sigma_- \cap D$ to degrees 4, 6. This means that there cannot be a generic ϵ_7 or ϵ_8 gauge group on Σ_- . Furthermore, if the degrees of vanishing are 3, 4 then g_4 cannot be a perfect square, since if it was then it would vanish to degree two on D and again f, g would vanish to degrees 4, 6 on $\Sigma_- \cap D$. As reviewed in section 3.1, this condition means that in the 3, 4 vanishing case the generic gauge group must be f_4 and not ϵ_6 . Thus, in the non-base-point free cases where an F-theory construction is possible, the generic gauge group cannot be ϵ_6, ϵ_7 , or ϵ_8 . A similar consideration holds for the generic gauge group on Σ_+ when $T \cdot D = -1$.

These conditions can be made more explicit in the toric context. Given a divisor D_i with $\tilde{t}_i = T \cdot D_i = 1$, we have a local set of rays in the toric fan as in (4.12)

$$s_- = (0, 0, 1) \tag{6.5}$$

$$\tilde{w}_{i-1} = (1, 0, 0) \tag{6.6}$$

$$\tilde{w}_i = (0, -1, 0) \tag{6.7}$$

$$\tilde{w}_{i+1} = (-1, -2, 1). \tag{6.8}$$

All monomials $m = (a, b, c) \in N^*$ in \mathcal{F}, \mathcal{G} then satisfy the inequalities (4.14)–(4.16), as well as (4.18), which becomes $c \geq a + 2b - B$ for $B = 4, 6$. In addition, there must be at least one monomial m that satisfies (4.17), $b - c > B$, for $B = 4$ or 6 .

First, we can ask if it is possible to have a gauge group factor on Σ_- associated with a summand ϵ_7 or ϵ_8 . From table 1, this would mean that every monomial $m = (a, b, c)$ would have $c \geq -1$ for both $B = 4$ and $B = 6$. From figure 2, it is clear that this is not possible, however. The only simultaneous solution in (b, c) to $c < b - B, b \leq B, c \geq -1$ is $b = B, c = -1$, and this is ruled out for both values of B by $c \geq a + 2b - B$ since $a \geq -B$. Thus, when the base-point free condition is violated through $\tilde{t}_i = +1$ over a -2 curve, the gauge algebra summand associated with Σ_- cannot be ϵ_7 or ϵ_8 . A similar result follows for Σ_+ when $\tilde{t}_i = -1$.

Now let us consider the possible summands \mathfrak{e}_6 and \mathfrak{f}_4 , both associated with vanishing degrees of 3, 4 for f, g on Σ_- . In this case the constraint for $B = 6$ is $c \geq -2$. There is a simultaneous solution to the inequalities for this value of c , given by $m = (a, b, c) = (-6, 5, -2)$, again as depicted in figure 2. So this combination of vanishing degrees can be realized. The distinction between \mathfrak{e}_6 and \mathfrak{f}_4 can be seen most easily from the leading term in the Weierstrass coefficient $g = g_4 z^4 + g_5 z^5 + \dots$, in a local expansion in coordinates around the relevant divisor, which in this case is Σ_- . If g_4 is a perfect square, the gauge algebra is \mathfrak{e}_6 , and otherwise it is \mathfrak{f}_4 . Since the only allowed monomial at order z^4 is $g_4 = y$, which is not a perfect square, the gauge algebra must always be \mathfrak{f}_4 when the base-point free condition fails on $\eta_1 = 6c_1 + T$ and the degree of vanishing of f, g is 3, 4 on Σ_- . The same result follows for Σ_+ when $\tilde{t}_i = -1$.

The upshot of this analysis is that the only gauge algebras that are possible for the structure group of the bundle in the dual heterotic model when the base-point free condition is violated are the commutants of the possible gauge algebras of the 4D theory, namely $\mathfrak{e}_8, \mathfrak{e}_7, \mathfrak{e}_6, \mathfrak{f}_4, \mathfrak{so}_8$, and \mathfrak{g}_2 . This is in good agreement with what is known of heterotic/F-theory duality in these cases, since the spectral cover construction for $SU(N)$ and $Sp(N)$ structure group bundles is not possible when the base point free condition is violated. Furthermore, these results demonstrate that the base-point-free condition must be necessary for *any* heterotic bundle with structure group $SU(N)$ or $Sp(N)$, independent of the spectral cover construction.

For the gauge groups associated with the more general exceptional algebras, it is expected that other bundle constructions such as the more general “cameral cover” construction [40–42] will exist for the heterotic bundles (though explicit conditions on bundle topology are at present not as well understood in this context as they are in the case of spectral covers) and as a result, the base-point free condition is not necessarily a requirement for the construction of a sensible bundle. The analysis of this section suggests that all F-theory models on \mathbb{P}^1 bundles over B_2 ’s that are generalized del Pezzo surfaces have well-defined heterotic duals, even when the base-point free condition is violated, though new tools may be needed for explicit construction of the appropriate bundles on the heterotic side.

6.5 A note of caution: G-flux

To close this section, we return briefly to a caveat mentioned in section 1 regarding the results presented here that are based purely on F-theory geometry. In deriving the bounds on structure groups and η in sections 3–6 we have *ignored G-flux* which must be taken into account for a full description of the F-theory physics and dual heterotic bundle moduli space. Some general aspects of G-flux in 4D F-theory models, and relevant references are given in section 7.8, section 7.9.

In some cases it may be possible for non-trivial G-flux (in the singular limit of the fourfold geometry) to change the apparent symmetry group that would be inferred from the Weierstrass equation. Although counterintuitive from the perspective of Abelian G-flux in a smooth M-theory limit, such symmetry-breaking by flux can be generic in the singular limit and is expected to occur in a wide range of 4D F-theory models. This has recently been explored in the context of local F-theory models as “T-branes” [111, 112] (or

equivalently “gluing data” [113, 114]) and in the global context in both 4- [115, 116] and 6-dimensional compactifications [101].

The basic mechanism by which G-flux can break an apparent symmetry appearing from the geometric F-theory analysis in terms of a Weierstrass model is most clear in the dual heterotic picture. As discussed in section 3.1, the geometry of the F-theory base \mathcal{B}_3 determines almost all of the topology of the corresponding bundle on the heterotic side. The components ζ_i in (3.2), however, are not determined by the 4-fold geometry. On the heterotic side, non-trivial bundles with a second Chern class entirely in ζ_i can break the gauge group just as effectively as bundles with non-vanishing η_i that have a clear dual in F-theory geometry. The symmetry breaking bundles corresponding to the topology $c_2(V_i) \sim \zeta_i$ will not be visible in the F-theory geometry, and can only be seen in F-theory when G-flux is correctly incorporated.

In the context of the present investigation, we hope to explore the full moduli/vacuum space of the dual theories including G-flux in future work. For now, however, we simply provide an illustration of where the purely geometric criteria may miss solutions involving exotic G-flux, in an example appearing in section 7.9.

7 Examples

We examine some specific examples of F-theory models on bases \mathcal{B}_3 that are \mathbb{P}^1 bundles over various bases B_2 and which illustrate various features discussed in the main part of the paper.

7.1 \mathbb{P}^1 bundles over \mathbb{P}^2

We begin with the simplest example, taking \mathcal{B}_3 to be a \mathbb{P}^1 fibration over $B_2 = \mathbb{P}^2$. The effective divisors on \mathbb{P}^2 are multiples nH of the hyperplane class H with $H \cdot H = 1$. The (anti-)canonical class is $-K_2 = 3H$. The general constraint on the twist $T = tH$ is that $|t| \leq 18$; this is the analogue of the constraints (4.8)–(4.11) for a curve of self-intersection $+1$. This class of F-theory models and their heterotic duals was described in [30, 38].

From the general classification of allowed F-theory models it follows that there is a valid model with $t = 18$ ($-6K_2 - tH = 0$), and that there are valid models for $0 \leq t \leq 12$ ($-4K_2 - tH$ effective). Models with negative t are equivalent under reflection to those with positive t . The twists $13 \leq t \leq 17$ correspond to bases \mathcal{B}_3 in which g_5 vanishes on some curves on the ϵ_8 locus Σ_- , which must be blown up for a smooth threefold base, analogous to the Hirzebruch surfaces $F_{9,10,11}$ in the 6D construction. Since the base B_2 has no curves of self-intersection -2 or below, the base-point free condition is never violated. The resulting 4D supergravity models have a range of gauge group factors according to the value of t . We thus have a total of 14 distinct bases \mathcal{B}_3 corresponding to different twists in a \mathbb{P}^1 bundle over \mathbb{P}^2 giving F-theory models with distinct smooth heterotic duals.

These features can be seen explicitly through a toric construction, where the parts of the fan from B_2 are

$$w_0 = (0, 1, 0), w_1 = (1, 0, 0), w_2 = (-1, -1, t). \tag{7.1}$$

Here we have used linear transformations to set the component of T to 0 for the first two vectors. The F-theory condition that f, g do not automatically vanish to degrees 4, 6 on Σ_- corresponds to the condition that $t \leq 18$ in the toric picture since the plane spanned by w_1, w_2, w_3 intersects the third axis at $(0, 0, t/3)$, so $t/3 \leq 6$. (In general, the condition that g not vanish on a divisor such as Σ_- at degree 6 can be described in any toric case as the condition that the plane spanning the w_i 's intersect the z axis at a value less or equal to 6, as can be verified geometrically.²⁰) The corresponding condition for Σ_+ gives $t \geq -18$.

On the heterotic side, the choice of t corresponds to the bundle decomposition where $\eta_{1,2} = 18 \pm t \geq 0$ is the number of instanton factors in each component of the gauge group, which must be nonnegative associated with the condition that η_i is effective.

Specific examples with generic gauge algebras $\mathfrak{su}(2)$ and $\mathfrak{su}(3)$ arise in the cases $t = 4, 5$. For $t = 4$, f and g vanish to degrees 1, 2 on Σ_- , as can be seen from the fact that f_0, g_0 , and g_1 must all vanish as the corresponding divisors $-nK_2 - (n-k)tH = -4H, -6H, -2H$ are all non-effective. Similarly, for $t = 5$, f, g vanish to degrees (2, 2). Both of these cases correspond to situations where the gauge algebra $\mathfrak{su}(2)$ or $\mathfrak{su}(3)$ cannot be enhanced to an $\mathfrak{su}(5)$, though the algebra can be enhanced to the exceptional series $\mathfrak{g}_2, \mathfrak{f}_4, \mathfrak{e}_6, \mathfrak{e}_7, \mathfrak{e}_8$.

The unique model with an $\text{SO}(32)$ heterotic dual is the model with $t = 6$ ($tH = -2K_2$), with gauge algebra $\mathfrak{so}(8)$, parallel to the 6D model on \mathbb{F}_4 .

7.2 \mathbb{P}^1 bundles over \mathbb{F}_m

Now consider F-theory models where the base B_2 is a Hirzebruch surface. Some of these models were also discussed in [30]. The cases \mathbb{F}_0 and \mathbb{F}_1 are qualitatively similar to the models described in the previous section, and are always base-point free as there are no -2 curves in the base. The cases \mathbb{F}_m with $m > 2$ correspond to singular geometries on the heterotic side. We briefly describe the cases \mathbb{F}_0 and \mathbb{F}_1 and then focus on the case \mathbb{F}_2 . For any Hirzebruch surface \mathbb{F}_m , the cone of effective divisors is generated by divisors S, F with $S \cdot S = -m, F \cdot F = 0, S \cdot F = 1$, and $-K_2 = 2S + (2 + m)F$.

For \mathbb{F}_0 , we have $-K_2 = 2S + 2F$, and we can parameterize $T = aS + bF$. There are symmetries under $T \rightarrow -T$ and $a \leftrightarrow b$. The general constraints on twists over curves of self-intersection 0 give $-12 \leq a, b \leq 12$. There is a single \mathfrak{e}_8 model with $a = b = 12$ ($T = -6K_2$). For all other good bases $-4K_2 \pm T$ are effective, so $-8 \leq a, b \leq 8$. Up to symmetries this gives $81 + 1 = 82$ distinct twists associated with valid F-theory models. As in the models over $B_2 = \mathbb{P}^2$, there are a variety of gauge groups associated with the different twists.

For \mathbb{F}_1 , we have $-K_2 = 2S + 3F$, and we can again parameterize $T = aS + bF$. There is a symmetry under $T \rightarrow -T$. The general constraints on twists over curves of self-intersection 0 and -1 give $|a| = T \cdot F \leq 12, |b - a| = T \cdot S \leq 6$. There is a single \mathfrak{e}_8 model with $a = 12, b = 18$ ($T = -6K_2$). For all other good bases $-4K_2 \pm T$ are effective, so $|a| \leq 8, |b| \leq 12$. Up to the sign symmetry of T this gives $108 + 1 = 109$ distinct twists associated with valid F-theory models. Again, there are a variety of gauge groups associated with the different twists, and all models have $\eta_{\pm} = -6K_2 \pm T$ base-point free.

²⁰Thanks to L. Swanson for discussions on this point.

A situation mentioned in section 4.1.2 occurs for several \mathbb{P}^1 bundles \mathcal{B}_3 over $B_2 = \mathbb{F}_1$, where a codimension two singularity arises on a curve despite the absence of gauge groups from codimension one singularities. A sample example of this occurs for the twist $T = 2F$. In this case, $-nK - (n - k)T = 2nS + (n + 2k)F$ has a negative intersection with S when $n > 2k$, so f_0, f_1, g_0, g_1, g_2 all vanish on S , giving a $(2, 3)$ codimension two singularity type over the curve $S \cap \Sigma_-$ although there is no gauge group on Σ_- as $-nK - (n - k)T$ are all effective for $n = 4, 6, 0 \leq k < n$.

Now we consider models with $B_2 = \mathbb{F}_2$. We have $-K_2 = 2S + 4F$, and we parameterize $T = aS + bF$, with a symmetry under $T \rightarrow -T$ exchanging Σ_{\pm} . The general constraints on twists over curves of self-intersection 0 and -2 constrain $|a| = T \cdot F \leq 12, |b - 2a| = T \cdot S \leq 1$. There is a single \mathfrak{e}_8 model with $a = 12, b = 24$ ($T = -6K_2$). The models with $b = 2a$ are base-point free; from the constraint that $-4K_2 - T$ be effective the base-point free models have $0 \leq a \leq 8$ up to symmetry, so along with the \mathfrak{e}_8 model there are 10 base-point free configurations. For the non-base-point free configurations, up to the sign symmetry on T we can choose $b = 2a + 1$, from which

$$-3K_2 - T = (6 - a)S + (12 - 2a - 1)F \tag{7.2}$$

must be effective, and similar for $-3K_2 + T$. This constrains

$$-8 \leq a \leq 5 \tag{7.3}$$

so there are 14 non-base-point free configurations, with gauge groups up to \mathfrak{f}_4 on Σ_- .

We can describe these cases explicitly in toric language. The toric fan for \mathcal{B}_3 contains the rays

$$s_{\pm} = (0, 0, \pm 1) \tag{7.4}$$

$$w_0 = (0, 1, 0) \tag{7.5}$$

$$w_1 = (1, 0, 0) \tag{7.6}$$

$$w_2 = (0, -1, a) \tag{7.7}$$

$$w_3 = (0, -1, b). \tag{7.8}$$

The twist T is parameterized by the integers a, b . An explicit computation of the monomials in the dual lattice that satisfy $\langle m, w_{\alpha} \rangle \geq -4, -6$ confirms that the cases described above are the only ones for which the F-theory model is acceptable and that in all these cases f, g have acceptable degrees of vanishing on all divisors and curves.

We consider explicitly the cases where $\eta_- = -6K_2 - T$ fails the base-point free condition. When $a < 2, b = 2a + 1$, there is no vanishing of f, g on Σ_- . For the twist combination $(a, b) = (2, 5)$, the vanishing degrees are 1, 2, so the gauge algebra contribution from Σ_- is \mathfrak{su}_2 . This should correspond on the heterotic side to an E_7 structure bundle on the Calabi-Yau described by the generic elliptic fibration over \mathbb{F}_2 that violates the base-point free condition, with $\eta = 6c_1 - T = 10S + 19F$, which is not base-point free since $\eta \cdot S = -1$.

A similar analysis for the twist combination $(a, b) = (3, 7)$ gives vanishing degrees of f, g on Σ_- 2, 3, for a gauge algebra factor of \mathfrak{g}_2 and a dual heterotic bundle structure

group of \mathfrak{f}_4 . In this case the bundle has $\eta = 9S + 17F$. For the twists $(a, b) = (4, 9)$ and $(5, 11)$ the vanishing degrees are 3, 4, so the gauge algebra contribution on Σ_- is \mathfrak{f}_4 , and the heterotic structure group is \mathfrak{g}_2 .

This gives a number of explicit examples of F-theory constructions that violate the base-point free condition, where the dual heterotic model should nonetheless exist with a bundle having an exceptional structure group.

7.3 An F-theory model over $B_2 = \mathbb{F}_3$ with a codimension two singularity but no gauge group

As discussed in section 3.2, all elliptically fibered Calabi-Yau geometries over the base $B_2 = \mathbb{F}_3$ are singular due to the Kodaira singularity over the -3 curve D in the base with vanishing of f, g to degrees $(2, 2)$ or greater. This means that no F-theory models on any base \mathcal{B}_3 over $B_2 = \mathbb{F}_3$ can have a smooth heterotic dual. In most cases, the absence of the heterotic dual is made particularly clear by the appearance of an additional gauge group factor in the 4D F-theory model over the divisor in \mathcal{B}_3 associated with D . In the singular heterotic dual theory this would correspond to an additional gauge factor arising at the singularity in the Calabi-Yau geometry. It is interesting to note, however, that for certain values of the twist T , a \mathbb{P}^1 bundle \mathcal{B}_3 over $B_2 = \mathbb{F}_3$ can be constructed so that there is no extra nonabelian gauge group factor. For example, with the twist $T = F$, there is no divisor that must carry a gauge group factor, though there is a codimension two singularity where (f, g) vanish to degrees $(3, 4)$ on the curve $S \cap \Sigma_-$. These assertions can easily be checked explicitly using the monomials computed in the toric description.

7.4 dP_2

The second del Pezzo surface, dP_2 , is constructed by blowing up \mathbb{P}^2 at two points, giving a pair of exceptional divisors E_1, E_2 with $E_1 \cdot E_1 = E_2 \cdot E_2 = -1, E_1 \cdot E_2 = 0$. The proper transform of the line passing through the two points is a third -1 curve $F = H - E_1 - E_2$, with $F \cdot E_1 = F \cdot E_2 = 1$. The cone of effective divisors is spanned by F, E_1, E_2 , which we can write as $(1, -1, -1), (0, 1, 0), (0, 0, 1)$ in a basis where the intersection product is $\text{diag}(1, -1, -1)$. dP_2 also has a simple toric presentation, but we use this more abstract formulation for the del Pezzo examples to illustrate how the methods of this paper can be implemented outside the toric context. The (anti-)canonical class of dP_2 is the proper transform of $-K = 3H$ on \mathbb{P}^2 ,

$$-K = 3F + 2E_1 + 2E_2. \tag{7.9}$$

If we parameterize the twist as

$$T = aF + bE_1 + cE_2, \tag{7.10}$$

there are symmetries under $b \leftrightarrow c$ and $(a, b, c) \leftrightarrow (-a, -b, -c)$.

We can now count the set of allowed twists T using the conditions described in section 4.1.3 and section 4.1.5. From the analysis in section 4.1.6, we know this gives a necessary and sufficient set of conditions for the set of allowed \mathcal{B}_3 's. There is a single twist

$T = -6K$ of class (A). There are no -2 curves in the base, so all other \mathcal{B}_3 's are of type (B). The constraints that

$$-4K \pm T = (12 \pm a)F + (8 \pm b)E_1 + (8 \pm c)E_2 \tag{7.11}$$

are effective constrain $|a| \leq 12, |b| \leq 8, |c| \leq 8$. The constraints that $|T \cdot E_1|, |T \cdot E_2|, |T \cdot F| \leq 6$ imply $|a - b| \leq 6, |a - c| \leq 6, |a - b - c| \leq 6$. Up to the symmetries listed above, there are 471 distinct T 's that satisfy these conditions, so the number of distinct \mathbb{P}^1 bundles \mathcal{B}_3 over $B_2 = dP_2$ that give good F-theory models is 472. This agrees with a direct analysis using toric methods, as described in section 9.

7.5 dP_3

The story for dP_3 is similar to that for dP_2 . Blowing up \mathbb{P}^2 at 3 generic points gives 3 exceptional divisors E_1, E_2, E_3 , and three -1 curves $X_1 = H - E_2 - E_3, X_2 = H - E_1 - E_3, X_3 = H - E_1 - E_2$ from the proper transforms of the lines connecting each pair of points. In a basis with intersection form $\text{diag}(1, -1, -1, -1)$ we have

$$X_1 = (1, 0, -1, -1) \quad E_1 = (0, 1, 0, 0) \tag{7.12}$$

$$X_2 = (1, -1, 0, -1) \quad E_2 = (0, 0, 1, 0) \tag{7.13}$$

$$X_3 = (1, -1, -1, 0) \quad E_3 = (0, 0, 0, 1) \tag{7.14}$$

There are symmetries under the 6 permutations of the indices $i = 1, 2, 3$, and under $E_i \leftrightarrow X_i$, which maps

$$(a, -b_1, -b_2, -b_3) \leftrightarrow (2a - b_1 - b_2 - b_3, a - b_2 - b_3, a - b_1 - b_3, a - b_1 - b_2) \tag{7.15}$$

In the toric picture this can be seen as the 12-fold dihedral symmetry group D_6 of the regular hexagon. A divisor $D = (a, -b_1, -b_2, -b_3)$ is effective if $a \geq 0, b_1 + b_2 + b_3 \leq 2a$. The (anti-) canonical class of $B_2 = dP_3$ is again the proper transform of H

$$-K = (3, -1, -1, -1) = \sum_i X_i + \sum_i E_i. \tag{7.16}$$

There is one base \mathcal{B}_3 of type (A), with $T = -6K$. To enumerate bases \mathcal{B}_3 of type (B), following the analysis of section 4.1.6, it is sufficient to identify all twists $T = (a, -b_1, -b_2, -b_3)$ so that the local twist conditions

$$|T \cdot E_i| = |b_i| \leq 6 \tag{7.17}$$

$$|T \cdot X_i| = |a - b_j - b_k| \leq 6, \quad i, j, k \text{ distinct} \in \{1, 2, 3\} \tag{7.18}$$

are satisfied and $-4K \pm T = (12 \pm a, -4 \mp b_1, -4 \mp b_2, -4 \mp b_3)$ is effective, which implies

$$|a| \leq 12, \quad |2a - b_1 - b_2 - b_3| \leq 12. \tag{7.19}$$

A simple enumeration shows that up to the D_6 symmetry group there are 775 solutions of all these conditions, so a total of 776 distinct possible bases \mathcal{B}_3 that are \mathbb{P}^1 bundles over dP_3 . As for dP_2 , this result agrees with the explicit enumeration done using toric methods described in section 9.

7.6 dP₄

For dP_n the analysis is again similar to dP₃ and dP₂, though there is no toric construction, there are more symmetries, and the effectiveness condition is increasingly complicated as the number of -1 curves on the wall of the cone of effective divisors increases.

For dP₄ there are 4 exceptional divisors E_i , $E_1 = (0, 1, 0, 0, 0), \dots E_4 = (0, 0, 0, 0, 1)$, and 6 proper transforms of lines X_{ij} , $X_{12} = (1, -1, -1, 0, 0), \dots$. The intersection form is $\text{diag}(1, -1, -1, -1, -1)$, with (anti-)canonical class $-K = (3, -1, -1, -1, -1)$. There are symmetries under arbitrary permutations of the i 's, as well as additional symmetries of the form

$$E_1 \leftrightarrow X_{23}, \quad E_2 \leftrightarrow X_{13}, \quad E_3 \leftrightarrow X_{12}, \quad X_{i4}, E_4 \text{ fixed} \quad (7.20)$$

The full symmetry group is of order $5! = 120$, and can be seen most clearly by re-defining $\tilde{X}_{0i} = E_i, \tilde{X}_{ij} = \frac{1}{2}|\epsilon_{ijkl}|X_{kl}$, for which the nonzero intersection products are $\tilde{X}_{\mu\nu} \cdot \tilde{X}_{\mu\nu} = -1, \tilde{X}_{\mu\nu} \cdot \tilde{X}_{\lambda\sigma} = 1$ when $\mu, \nu, \lambda, \sigma \in \{0, 1, 2, 3, 4\}$ are distinct; in terms of the \tilde{X} 's, the symmetry group is simply the set of permutations on all 5 possible index values.

In dP₄, a divisor $D = (a, -b_1, -b_2, -b_3, -b_4)$ is effective iff $b_i \leq a \forall i, \sum_i b_i \leq 2a$, which can be seen from the conditions that D must be formed from a positive integral linear combination of X_{ij} 's and E_i 's. The twist $T = (s, t_1, t_2, t_3, t_4)$ must satisfy the conditions

$$|T \cdot E_i| = |t_i| \leq 6, \quad |T \cdot X_{ij}| = |s - t_i - t_j| \leq 6. \quad (7.21)$$

As in the previous del Pezzo examples, there is one solution with $T = -6K$, and we can enumerate all solutions with $-4K \pm T$ effective, which along with the twist conditions (4.8)–(4.11) give necessary and sufficient conditions for a good base \mathcal{B}_3 . An explicit enumeration shows that after taking account of symmetry there are 6976 distinct bases \mathcal{B}_3 of type (B), for a total of 6977 \mathcal{B}_3 that are \mathbb{P}^1 fibrations over dP₄.

For higher dP_n, and for generalized del Pezzo surfaces, the analysis can be carried out in a similar fashion. As n increases, however, the details of the calculation become more complicated. For dP₅, for example, there is an additional -1 curve from a conic passing through all 5 blown up points, $C = 2H - E_1 - E_2 - E_3 - E_4 - E_5$, which complicates the effectiveness condition on divisors. In principle, however, for any base B_2 , the number of twists satisfying the local twist conditions is finite, and the determination of the full set of \mathcal{B}_3 's over B_2 can be done efficiently and explicitly.

7.7 An example of an upper and lower bound on η

One of the more novel observations of this study is the fact that for certain fourfold geometries there exist generic symmetries that *can be neither broken (Higgsed) or enhanced at any points in the complex structure moduli space of Y_4* . These restrictions arise because of a variety of features, however all the failures of “enhancement” occur because of too-high a degree of vanishing of (f, g) on divisors and curves as described in section 6.

The consequences of having a twist T of the \mathbb{P}^1 -fibered base \mathcal{B}_3 on the F-theory side that gives rise to such a restrictive condition on the gauge group corresponds in the heterotic geometry to a choice of partial bundle topology η for which only one structure group H is possible (subject once again to the caveats arising from ignoring G-flux, see section 6.5)

Let us consider here an example of this type for which only one symmetry is possible and all Higgsing/enhancing is forbidden. This is the case for the base \mathcal{B}_3 defined by the \mathbb{P}^1 bundle with twist $T = 5S + 11F$ on \mathbb{F}_2 . Constructing the generic Weierstrass model over this base \mathcal{B}_3 , it is straightforward to verify that this Y_4 manifests a generic F_4 symmetry.

In the dual heterotic theory, this corresponds to a G_2 bundle over the threefold X_3 with $\eta = 7S + 13F$. More precisely, the heterotic bundle V_1 has $\eta_- = 7S + 13F$ and is a G_2 bundle. The second bundle, associated to η_+ , has generic E_8 structure group and hence one E_8 factor is generically completely broken and will not concern us further. More explicitly, we have the following Weierstrass equation for Y_4 :

$$Y^2 = X^3 + (f_3z^3 + f_4z^4 + \dots)X + (g_4z^4 + g_5z^5 + g_6z^6 + \dots) \quad (7.22)$$

(Here $(X, Y, Z = 1)$ are the coordinates on the elliptic fiber of the CY_4 , while $z = 0$ defines the 7-brane locus (i.e., the section Σ_-) inside of \mathcal{B}_3 and is of E_6/F_4 type according to Kodaira-Tate.) As mentioned in previous sections, if g_4 is a perfect square then the symmetry is E_6 , and for more general polynomials it is F_4 .

For the twist $T = 5S + 11F$, (7.22) is the generic form of the Weierstrass model for arbitrary complex structure. The fact that the F_4 symmetry is generic (i.e., cannot be Higgsed) from the point of view of the F-theory Weierstrass model has the same natural low-energy 4D interpretation on both sides of the duality — there is simply no charged matter available to get a vev in vacuum.

For the given twist, all additional tunings of the complex structure that might increase the gauge group on Σ_- induce non-CY singularities. As an example, consider the specialization of $g_4 = \alpha^2$ for some polynomial α of the appropriate degree. Here the vanishing degree of (f, g, Δ) increases from the generic values of $(4, 5, 10)$ on the curve $\Sigma_- \cap S$ to $(4, 6, 12)$ on the same curve, and hence the singularity cannot be resolved without going to a different F-theory base \mathcal{B}_3 by blowing up the curve.

In this case, the restriction on enhancement has a clear interpretation in terms of the heterotic bundle geometry. An enhancement of the symmetry of $F_4 \rightarrow E_6$ for example, corresponds in the heterotic theory to a *reduction in rank* of the associated bundle from $G_2 \rightarrow \text{SU}(3)$. For the case at hand this would indicate that by tuning the complex structure of Y_4 we were inducing a “splitting” of the vector bundle. In terms of the associated vector bundles the following reduction of representations in $G_2 \rightarrow \text{SU}(3)$,

$$\mathbf{7} \rightarrow \mathbf{3} + \bar{\mathbf{3}} + \mathbf{1} \quad (7.23)$$

would lead to

$$V_7 = V_3 \oplus V_3^\vee \oplus \mathcal{O}_{X_3} \quad (7.24)$$

the fact, however, that this tuning leads to a badly singular Y_4 indicates that a generic G_2 bundle with $\eta = 7S + 13F$ cannot be decomposed as in (7.24) for *smooth* $\text{SU}(3)$ bundles V_3 . Instead, any such decomposition must lead to non-locally free sheaves (i.e., heterotic “small instantons” [117–119]) and a degenerate limit of the theory.

For this choice of twist, we have an additional confirmation of this heterotic result in the fact that η is not base-point-free. Thus by the arguments of section 6, we cannot define

any smooth $SU(3)$ spectral cover bundle to play the role of V_3 in (7.24). Although the spectral cover construction is not guaranteed to be representative for the bundle moduli space in general, the consistency conditions on Y_4 applied to these “non-enhanceable” geometries indicate that if a generic symmetry F_4 is not base-point free, it will be impossible to enhance the symmetry for special values of the complex structure (compatible with a CY resolution). This provides an interesting window into the moduli space of all such G_2 -bundles by providing general restrictions on possible decompositions like the one given above.

7.8 Examples with non-trivial chiral matter

In the previous examples we have seen that the F-theory fourfold geometry frequently encodes otherwise hard to obtain information about the moduli space of vector bundles on heterotic CY threefolds. In this section, we use the heterotic theory to obtain new information about the matter spectrum of a 4D effective F-theory. To accomplish this, we return to the formulas for chiral matter given in section 5.2.

As an example, let us consider E_6 theories in the dual heterotic/F-theory geometry. As discussed in section 5.3.2, for an $SU(N)$ bundle described via a spectral cover the chiral index is [45, 103]

$$\text{Ind}(V) = -h^1(X, V) + h^1(X, V^\vee) = \lambda[\eta] \cdot [\eta + NK_2] \tag{7.25}$$

where λ is defined by (5.27) and (5.28). Thus, the chiral index is proportional to a simple geometric intersection of the curve $[\eta]$ in the base with the matter curve $[\eta + NK_2]$.

To understand the significance of this geometry, the case of E_6 theories is particularly interesting because for $SU(3)$ bundles described as spectral covers we can guarantee that the constant λ is non-vanishing. Recall from (5.28) that for some integer m

$$\lambda = \begin{cases} m + \frac{1}{2}, & \text{if } N \text{ is odd} \\ m, & \text{if } N \text{ is even} \end{cases} \tag{7.26}$$

Thus, for $SU(3)$ bundles/ E_6 theories it is required that $\lambda \neq 0$ and the question of whether or not the theory has chiral matter can be reduced to a question of intersection theory for the matter curve $[\eta + 3K_2]$ in the 2-fold base. We will be interested in whether or not this curve is reducible and whether or not it has non-trivial intersection with $[\eta]$.

To illustrate the possibilities, we can consider the four generic E_6 theories over the base $B_2 = \mathbb{F}_1$. There the twists

$$T = nS + 9F \quad , \quad 3 \leq n \leq 6 \tag{7.27}$$

all give rise to E_6 symmetries on Σ_- . This is easy to check since the coefficient g_4 in the Weierstrass equation

$$Y^2 = X^3 + (f_3z^3 + f_4z^4 + \dots)X + (g_4z^4 + g_5z^5 + g_6z^6 + \dots) \tag{7.28}$$

satisfies $g_4 \in H^0(B_2, \mathcal{O}(\eta)^{\otimes 2} \otimes K_2^{\otimes 6}) = H^0(B_2, \mathcal{O}(-6K_2 - 2T)) = H^0(\mathbb{F}_1, 2(6 - n)S)$, which indicates that g_4 is a perfect square in these cases²¹ and hence (as described in the previous

²¹Note that the line bundle cohomology over \mathbb{F}_n can be shown to satisfy: $h^0(\mathbb{F}_n, \mathcal{O}(S)) = 1$ for the divisor $S^2 = -n$. Hence if s is the toric coordinate associated to the divisor S , the generic (only) element of $H^0(\mathbb{F}_n, nS)$ is s^n .

section) the fiber type is split to E_6 (rather than the generic, non-split F_4). Moreover, for $3 \leq n \leq 6$ we have the heterotic topology:

$$\eta = (12 - n)S + 9F \quad \text{and} \quad \eta + 3K_2 = (6 - n)S \quad (7.29)$$

For these $SU(3)$ bundles, η is effective and base-point-free, $\eta - 3c_1$ is effective, and it is straightforward to verify that simple line bundles L_S of the form (5.27) can be found with $\lambda \neq 0$. Thus, we are guaranteed that smooth spectral cover bundles exist (stable in the appropriate adiabatic region in Kähler moduli space).

In the case that $g_4 = \alpha^2$ the polynomials in (7.28) appear as a $SU(3)$ spectral cover inside the heterotic CY_3 of the form

$$g_5 \hat{Z}^3 + f_3 \hat{X} \hat{Z} + \alpha \hat{Y} = 0 \quad (7.30)$$

where $(\hat{X}, \hat{Y}, \hat{Z})$ are coordinates on the CY_3 elliptic fiber (the above equation gives three points on the elliptic fiber for each point on the \mathbb{F}_2 base, as expected for an $SU(3)$ spectral cover). As usual, f_4, g_6 in the fourfold Weierstrass equation above appear as the coefficients in the CY_3 Weierstrass

$$\hat{Y}^2 = \hat{X}^3 + f_4 \hat{X} + g_6 \quad (7.31)$$

Recall from the arguments of section 5.3.2 that the coefficient $\alpha \in H^0(\mathcal{O}(\eta) \otimes K_2^{\otimes 3}) = H^0(\mathbb{F}_2, \mathcal{O}((6 - n)S))$ in (7.30) defines the “matter curve”, $\alpha = 0$. This is where the $\mathbf{27}$ -type matter is localized in both the heterotic/F-theory geometries.

Thus, for this class of bundles (5.28) and (5.32) can be used to straightforwardly compute

$$\text{Ind}(V) = \text{no. of } \overline{\mathbf{27}}'s - \text{no. of } \mathbf{27}'s = (6 - n)(n - 3) \quad (7.32)$$

From this we see that the cases $n = 3$ and $n = 6$ have chiral index zero, but for $n = 4, 5$ the theory *must* have chiral matter. Given the full defining data of the bundle, the exact multiplicity of the $\mathbf{27}$'s and $\overline{\mathbf{27}}$'s could be computed using Leray spectral sequences [98], but even at this preliminary level, the results of the chiral index are intriguing. For these dual geometries, given a value of η , it can immediately be determined whether or not the theory contains chiral matter.

Of course, we derived the necessity of a non-vanishing chiral index for an $SU(3)$ bundle described as a spectral cover, but the third Chern class (and hence the Chiral index) is a topological invariant in the bundle moduli space. As a result, so long as a good $SU(3)$ spectral cover bundle exists, we can use it as a probe to extract the structure of the full moduli space of hermitian bundles, all of which must have non-vanishing index! As discussed in section 6.4, all CY fourfold geometries with generic E_6 symmetry and smooth heterotic duals satisfy the base-point-free condition and can be described by well-behaved spectral cover bundles as in section 5.3.

It would be interesting to investigate the question of chiral matter in this context more directly on the F-theory side in the future, in particular by including G-flux. Some general aspects of how G-flux can be incorporated in 4D F-theory models are described in [120–123]. Progress has been made in understanding how chiral matter in F-theory

models can be determined in the presence of G-flux based on aspects of the spectral cover construction [20, 124–130] and more directly from the M-theory description [131], but a more complete and directly computable formulation is desirable. An inspection of (7.28) shows that the points determined by the intersection

$$[\eta] \cdot [\eta + NK_2] \tag{7.33}$$

in the F-theory geometry corresponds exactly to the simultaneous vanishing of g_4 and g_5 at points in the base — that is, the chiral index in these cases is counted by co-dimension 3 singular loci in the 4-fold geometry. The observation that chiral matter and co-dimension 3 singularities (and associated G -flux) could be linked (at least in $K3$ -fibered 4-folds) has been observed for some time [124, 125] and has been used more recently in F-theory model building [20, 132].

It would be interesting to study more generally whether simple correlations such as those between (5.21) and (5.28) and (5.32) exist between η and the chiral index, independent of the existence of a heterotic dual. In fact, it is possible that the F-theory 4-fold could explicitly give indications of such correlations through its topology. For example, it is well known that some 4-folds cannot be good F-theory vacua without including non-trivial G-flux. In these cases their second Chern class (or more generally Wu class) is incompatible with trivial G-flux in the presence of quantization conditions. A study of the topology of Y_4 and its links to intersection structure such as that in (7.25) could yield important information along these lines (for similar investigations see [133, 134] which explore Chern and Wu classes of Y_4 with simple singularities). We hope to explore this in future work.

7.9 Generic G-flux that breaks gauge symmetry

In section 6.5, we discussed how G-flux can break the gauge group associated with a purely geometric construction, through the structure of the second Chern class of the dual heterotic bundle. As an example of this mechanism, consider the case of base $B_2 = \mathbb{P}^1 \times \mathbb{P}^1$ with twist $T = 6c_1(B_2)$; i.e., $\eta_- = 0$. According to the arguments of section 3, we would naturally have determined from the F-theory geometry Y_4 that Σ_- carried an E_8 symmetry (i.e., the fiber degeneration is type II^*) and hence that there were *no smooth bundles* V_1 on X_3 with $\eta_- = 0$ (i.e., that V_1 is trivial and V_2 satisfies $\eta_+ = 12c_1(B_2)$). However, this is too quick since this argument ignores the fact that η alone is not enough to determine even $c_2(V_1)$. To see this, consider the following smooth heterotic geometry.

On the base B_2 , consider the poly-stable rank 2 vector bundle defined as a kernel (of the map m) via the following short exact sequence (i.e., a “monad” bundle [108]):

$$0 \rightarrow \mathcal{V}_1 \rightarrow \mathcal{O}(0, 1)^{\oplus 2} \oplus \mathcal{O}(1, 0)^{\oplus 2} \xrightarrow{m} \mathcal{O}(1, 1)^{\oplus 2} \rightarrow 0 \tag{7.34}$$

(where $\mathcal{O}(a, b) = \mathcal{O}(aS + bF)$ and S, F are the hyperplanes in each \mathbb{P}^1 factor). For an appropriately block-diagonal choice of the map m , \mathcal{V}_1 is a simple twist of the poly-stable tangent bundle of \mathbb{F}_0 with vanishing slope; i.e., $\mathcal{V}_1 = T\mathbb{P}^1 \otimes \mathcal{O}(1, 1)$. For generic choices of m this bundle is slope-stable for all of the Kähler cone of $\mathbb{P}^1 \times \mathbb{P}^1$. Over the entire elliptically fibered threefold, $\pi : X_3 \rightarrow B_2$, we can likewise define the stable, slope-zero pull-back

bundle $\pi^*(\mathcal{V}_1) = V_1$ with $c_1(V_1) = 0$ and $c_2(V_1)$ non-trivial solely *from the pull-back of the (2,2)-form on the base*, $\pi^*(\zeta)$

$$c_2(V_1) = \pi^*(\zeta) = \pi^*(4\omega_1\omega_2) \Rightarrow \eta_- = 0 \tag{7.35}$$

where ω_i are the (1,1)-forms dual to the divisors S, F in \mathbb{F}_0 .

The pull-back bundle defined by (7.34) is an example of a smooth, everywhere stable $SU(2)$ bundle which breaks $E_8 \rightarrow E_7$ in the heterotic effective theory. Thus, in contradiction to the conventional indication of the F-theory E_8 -type Weierstrass equation, it is clear that this is an everywhere well-defined E_7 theory. In the other E_8 factor a generic bundle with $c_2(V_2) = 12c_1(\mathbb{F}_0) + 88\pi^*(\omega_1\omega_2)$ (as required by anomaly-cancellation, see (5.2) and (A.7)) breaks all the symmetry. At first pass it would seem that the bundle in (7.34) cannot be naively be described by a smooth spectral cover and that as a result the Heterotic/F-theory dictionary is unclear. For spectral covers $\eta = 0$ indicates that $[\mathcal{S}] = 2[\sigma]$ from (5.24) and we would be tempted to conclude here that the spectral cover (5.25) with $\eta = 0$ described only the Fourier-Mukai transform of the trivial rank 2 bundle $\mathcal{O}^{\oplus 2}$. However, this is forgetting half of the data of the Fourier-Mukai transform: in particular the rank 1 sheaf $L_{\mathcal{S}}$ over \mathcal{S} (see section 5.3 and (5.27)). Taking into account the possibility of rank 1 sheaves on the non-reduced scheme \mathcal{S} which arise from higher rank sheaves (in this case rank 2) on $\sigma = 0$ [135, 136] it is clear that more general bundles V_1 are possible after FM transform.²² In the standard heterotic/F-theory dictionary, the data of these rank 1 sheaves (whether ordinary line bundles or higher rank sheaves in the non-reduced or reducible case as above) is mapped into G-flux [20, 42, 80].

As this example illustrates, such possibilities must be taken into account if one hopes to fully determine the properties of heterotic vector bundle moduli space from its F-theory dual. For now, we consider only the data of Y_4 itself, focusing on purely geometric structure and properties, and leave an investigation of the intriguing possibilities of G-flux for future work.

8 Consequences for heterotic bundles

Many of the new results in this paper are conclusions/constraints regarding properties of the moduli space of bundles (more precisely, the moduli space of semi-stable sheaves) arising in heterotic theories and links between bundle topology and structure group. In this section we provide a brief summary of these results.

Unlike in six dimensions, four-dimensional heterotic/F-theory duality provides new and non-trivial insight into the structure of the heterotic moduli space $\mathcal{M}_{\omega}(c(V))$ of semi-stable sheaves with fixed topology on X_3 . At present, very few techniques are known for determining the dimension and structure of \mathcal{M}_{ω} on Calabi-Yau threefolds and there are many open questions which are of interest to both physics and mathematics. These include applications to string phenomenology (for example the large scale scans for “Standard

²²As mentioned in section section 5.3.3, the holomorphic tangent bundle to an elliptically fibered threefold, TX_3 , is frequently found to have a degenerate spectral cover description of this type (i.e., reducible or non-reduced \mathcal{S}) [110, 137].

Model” bundles undertaken in [15–17]) as well as more mathematical questions such as the possible existence of new mathematical rules for linking topology ($c(V)$) to conditions for vanishing/triviality of $\mathcal{M}_\omega(c(V))$ and the computation of higher rank Donaldson-Thomas invariants. For elliptically fibered Calabi-Yau threefolds, heterotic/F-theory duality provides a rich set of new computational tools and we view this work as a preliminary step in using these tools to determine the full structure of \mathcal{M}_ω . We briefly summarize here the main new results:

Effectiveness conditions on η .

- As discussed in section 5.1 and section 5.3, the parameter η determining part of the second Chern class of the heterotic bundles must obey several effectiveness constraints in different contexts. η must be effective in the stable degeneration limit, and $\eta - Nc_1(B_2)$ must be effective for a spectral cover construction of an $SU(N)$ bundle. We have found that these constraints are more general. For any F-theory construction with a smooth heterotic dual, η must be effective, and $\eta - Nc_1$ must be effective for $N = 2, 3$ for gauge groups E_7, E_6 (or smaller) corresponding to heterotic theories with structure bundles $SU(2), SU(3)$, independent of the stable degeneration limit or method of bundle construction.

Base-point-freeness and bundles with exceptional structure group.

- Previous work aimed at describing vector bundles over Calabi-Yau threefolds in the context of heterotic/F-theory duality, such as [19], has focused on bundles with $SU(N)$ and $Sp(N)$ structure groups, constructed using spectral covers in the stable degeneration limit. Here we have considered consistency conditions on topology in a construction-independent way and demonstrated that the base-point-freeness condition on η is necessary for $SU(N)$ and $Sp(N)$ structure groups, independent of the the method of bundle construction.

Moreover, our study has shown that these constraints on the topology of the vector bundle do not seem to be universal. We have considered a broader class of heterotic/F-theory dual models and identified a large range of models in which the base-point free condition on the components η of the second Chern class need not be satisfied in the dual F-theory model. In all these models the structure group on the heterotic side is an exceptional group or $SO(8)$. Thus, F-theory allows us to identify the conditions on the second Chern class that are necessary, and apparently sufficient, for vector bundles with exceptional and $SO(8)$ structure group to be constructed over a broad class of Calabi-Yau threefolds. These results could be mathematically useful in explicitly constructing or characterizing such bundles. In particular, it would be intriguing to utilize these conditions in formulating topological consistency conditions for bundles constructed through the cameral cover construction (which are at present not as explicitly described as those for the spectral cover construction).

- One interesting feature of the heterotic models with F-theory duals where the base-point free condition is violated is that they all involve elliptic fibrations over generalized del Pezzo surfaces that contain curves of self-intersection -2 . These surfaces are limits of usual del Pezzo surfaces where the points where \mathbb{P}^2 is blown up are brought together in specific ways. In principle, the Calabi-Yau threefolds formed over generalized del Pezzo surfaces should simply be special limits in the moduli space of the generic elliptically fibered Calabi-Yau threefold over the corresponding del Pezzo. This limit can be controlled precisely and may provide an avenue for the explicit construction of the vector bundles with exceptional structure groups that arise in these cases. More precisely, in these limits as the complex structure of B_2 is tuned to produce the generalized del Pezzo surfaces, the Mori cone of effective divisors in B_2 jumps discontinuously (though $h^{1,1}(X_3)$ remains unchanged), and the Kähler cone of the Calabi-Yau threefold decreases correspondingly. This change in the Kähler cone impacts the properties of the moduli space of stable bundles that can arise, and also seems to restrict the existing bundles to have exceptional structure groups in many cases.²³

SO(32) heterotic/F-theory duality and the connectivity of string moduli space.

- By using topological terms in the 4D effective supergravity action to characterize heterotic F-theory duality [9], we can identify topologically which F-theory models are dual to heterotic models for SO(32) as well as $E_8 \times E_8$ models without requiring a stable degeneration limit. We have explicitly identified those F-theory models that are dual to SO(32) heterotic string theory over a general smooth elliptically fibered Calabi-Yau threefold base, and shown that in all such cases the generic model has a gauge group of SO(8), which cannot be broken further by Higgsing.
- On the F-theory side, all the models we have considered are connected in a smooth geometric moduli space. Over each base \mathcal{B}_3 there is a moduli space of Weierstrass models that provides a nonperturbative completion of the perturbative heterotic moduli space of bundles (sheaves) over the dual heterotic elliptically fibered Calabi-Yau threefold. Furthermore, the distinct bases are connected by tensionless string transitions that correspond to small instanton transitions on the heterotic side. For those F-theory bases with SO(32) heterotic duals, there are also $E_8 \times E_8$ duals; F-theory/heterotic duality may illuminate the connection between these two distinct heterotic perturbative limits and the resulting relationship on Calabi-Yau threefolds between the moduli spaces of SO(8) structure group bundles and other bundle structures associated with the $E_8 \times E_8$ theory. This extends to four dimensions results on the geometry of $E_8 \times E_8/\text{SO}(32)$ dual heterotic pairs that were previously understood in higher-dimensional contexts [21, 22, 56, 143–146].

²³See [103, 138–142] for similar “Noether-Lefschetz” type-problems and “jumping” in complex structure/bundle moduli space.

Upper bounds on H .

- As first explored in [37, 38], the presence of generic, non-Higgsable symmetries for singular Y_4 geometries indicates that for a given η there is an upper bound on the size of the structure group H for any bundle in the moduli space. As explained in section 6, if a generic symmetry G cannot be Higgsed in the 4-dimensional effective theory, this implies that for the given topology (η) there exist *no bundles with structure group larger than H* , the commutant of G in E_8 . Phrased differently, in order to define a bundle with structure group H over the elliptically fibered CY threefold, there is a minimum “size” for η . These conditions (given in the absence of G -flux) are listed in table 3.
- We further observe that these rules are at present only a first step in determining $\mathcal{M}_\omega(c(V))$ and its constraints. As described in section 6.5, for some choices of bundle/4-fold topology it may be that generic G -flux breaks the apparent symmetry G indicated by the Weierstrass equation of Y_4 . In these cases, the bundle structure group may be bigger than indicated by the bounds on η in table 3. When this occurs, it must involve non-trivial values for ζ in (3.2); this indicates a new layer of structure linking not only η with H , but also with $(\zeta, c_3(V))$ — the two integer values specifying the remaining bundle topology not studied in this work. Such a correlation would involve a finer level of structure linking H and $c(V)$ than has been so far explored in the literature. We hope to explore these issues in further work.

Lower bounds on structure group, H . In addition to the non-Higgsable symmetries described above and their heterotic consequences, in the F-theory geometry we have seen many examples of Y_4 with a generic symmetry G which *cannot be consistently enhanced* (see section 6). In the dual heterotic geometry, these geometric observations provide constraints on when a given vector bundle can be consistently decomposed into a reducible sum:

$$V \rightarrow \mathcal{V}_1 \oplus \mathcal{V}_2 \oplus \mathcal{O} \oplus \dots \tag{8.1}$$

with a smaller (reduced) structure group.

- The results of section 6.4 indicate that the base-point-free condition described in section 5.3 and section 6.4 cannot be consistently violated in the case of $H = \text{SU}(N)$ (i.e., for a hermitian bundle, regardless of the method of construction). As a result, for any technology applied to a bundle associated with a consistent *non-base-point free choice of η* , it is clear that there is no way to reduce the structure group to $\text{SU}(N)$. We find that all the non-base-point free examples of η correspond to structure groups $H = \text{SO}(8), G_2, F_4, E_6, E_7$ or E_8 . In these cases, the bundle structure group can never be consistently decomposed $H \rightarrow H_1 \times H_2 \dots$ with $H_i \subset H$ hermitian; in these cases the gauge group G thus cannot be enhanced to E_6 or E_7 .
- In a similar spirit, for many examples (see section 4.2.1) there are generic $\text{SU}(3)$ or $\text{SU}(2)$ symmetries on Y_4 , with (f, g) vanishing to degree $(2, 2)$ or $(1, 2)$ and no possible enhancements to higher $\text{SU}(N)$ gauge symmetries are possible (this would

require (f, g) non-vanishing and Δ vanishing to degree N). As in the case of the non-base-point free examples above, this constrains the ways in which bundles with exceptional structure groups can be decomposed into hermitian factors.

- We find examples for which there appear to be both upper and lower bounds on H ; in these cases for the given value of η the moduli space of semi-stable sheaves can contain bundles with exactly one allowed structure group only. See for example section 7.7, where an example of a topology is given for which F_4 is the only consistent structure group.
- The lower bounds on H constrain the structure of possible sub-sheaves $\mathcal{F} \subset V$, and determine a bound below which V cannot decompose as $V = \mathcal{F} \oplus V/\mathcal{F}$, etc. The determination of such substructure has important consequences for the Harder-Narasimhan filtration of V , and the group quotient structure of \mathcal{M} itself [72].
- Finally, these “lower” bounds on H may be strengthened by the presence of G -flux. While G -flux cannot enhance the gauge group in a way that violates these lower bounds on H , it could act to reduce the generic gauge symmetry arising purely from geometry, which could lead to even stronger lower bounds on H in some cases.

9 Enumeration of heterotic/F-theory dual pairs with toric bases B_2

We have systematically analyzed all toric F-theory bases \mathcal{B}_3 (constructed as \mathbb{P}^1 bundles) that have smooth heterotic duals on Calabi-Yau threefolds that are elliptically fibered with section. The toric bases B_2 that support such models are the generalized del Pezzo (gdP) surfaces, a subset of 16 of the complete set of 61,539 toric surfaces enumerated in [60] that can act as bases for elliptically fibered Calabi-Yau manifolds with section. Over the 16 toric gdP bases we find 4962 distinct \mathcal{B}_3 's that have smooth heterotic duals. Each choice of \mathcal{B}_3 corresponds to a specific Calabi-Yau threefold X_3 giving the generic elliptic fibration over B_2 , with a particular choice in the part of the bundle topology characterized by η_i in (3.2). The analysis was performed by considering all possible twists T compatible with the bounds (4.8)–(4.10), and explicitly analyzing the monomial and singularity structure of the resulting Weierstrass model in the toric description. For each model that does not have a (4, 6) singularity on a divisor or curve, we determine the gauge group content and whether η_1, η_2 are base-point free. The resulting enumeration of \mathcal{B}_3 's for some specific B_2 's matches that described in the previous section using the more general constraints described in section 4. The results of the toric analysis are listed in tables 4, 5. For each base B_2 we have indicated the number of distinct \mathcal{B}_3 's over that base (the number of possible “twists” T in the \mathbb{P}^1 bundle giving acceptable \mathcal{B}_3 's), and the number of these \mathcal{B}_3 's that violate the base-point free condition for one or both gauge group factors. A subset of the F-theory bases \mathcal{B}_3 tabulated in table 4 have been explored in previous work; in particular, the \mathbb{P}^1 bundles over \mathbb{P}^2 and \mathbb{F}_m described in the first four lines of the table, and the 18 toric Fano varieties were also described in [30, 137, 147].

One interesting observation from the data in table 4 is that more than half of the possible toric F-theory geometries violate the base-point free condition on at least one

base B_2	$h_{1,1}$	# \mathcal{B}_3 's	NB (1)	NB (2)	# cod 3
(1, 1, 1) (\mathbb{P}^2)	1	14	0	0	0
(0, 0, 0, 0) (\mathbb{F}_0)	2	82	0	0	0
(1, 0, -1, 0) (\mathbb{F}_1)	2	109	0	0	0
(2, 0, -2, 0) (\mathbb{F}_2)	2	24	14	0	0
(0, 0, -1, -1, -1) (dP_2)	3	472	0	0	0
(1, -1, -1, -2, 0)	3	173	100	0	0
(-1, -1, -1, -1, -1, -1) (dP_3)	4	776	0	0	0
(0, -1, -1, -2, -1, -1)	4	729	396	0	0
(0, 0, -2, -1, -2, -1)	4	312	213	42	0
(1, 0, -2, -2, -1, -2)	4	62	31	25	32
(-1, -1, -2, -1, -2, -1, -1)	5	1119	755	140	0
(0, -1, -1, -2, -2, -1, -2)	5	406	219	150	217
(-1, -1, -2, -1, -2, -2, -1, -2)	6	351	149	185	173
(-1, -2, -1, -2, -1, -2, -1, -2)	6	214	119	69	0
(0, -2, -1, -2, -2, -2, -1, -2)	6	83	18	59	45
(-1, -2, -2, -1, -2, -2, -1, -2, -2)	7	36	8	26	29
total		4962	2022	696	496

Table 4. Table of all smooth F-theory bases with smooth heterotic duals that are \mathbb{P}^1 bundles over toric bases B_2 . The base B_2 is characterized by the sequence of self-intersections of toric divisors. NB (Non-Base point free) indicates the number of bases \mathcal{B}_3 that violate the base-point free condition on one (1) or both (2) sides Σ_{\pm} . The final column is the number of models that have a toric codimension 3 locus where f, g vanish to degrees 4, 6.

of the gauge factors, so that more than half of the corresponding heterotic models have generic bundles with exceptional or $SO(8)$ structure group. The only models that can violate the base-point free condition are those with generalized del Pezzo bases having -2 curves, and for such bases a very high fraction of models violate the base-point free condition on at least one side. Since the vast majority of the hundreds of possible non-toric bases B_2 compatible with a smooth heterotic dual are generalized del Pezzo's, we expect that the fraction of all models with smooth heterotic duals that have exceptional or $SO(8)$ structure group is quite high.

In table 5 we tabulate the number of models in the full set that have each of the possible distinct gauge algebras $\mathcal{G}_1 \oplus \mathcal{G}_2$. Note that this gauge algebra represents the minimal (most generic) gauge algebra for each base. For each \mathcal{B}_3 , tuning Weierstrass monomials can lead to enhanced gauge groups through “unHiggsing,” corresponding on the heterotic side to special loci in bundle moduli space where the structure group H becomes smaller and G correspondingly larger. The minimal gauge algebra summands \mathcal{G}_i for each base \mathcal{B}_3

\times	\cdot	\mathfrak{su}_2	\mathfrak{su}_3	\mathfrak{g}_2	\mathfrak{so}_8	\mathfrak{f}_4	\mathfrak{e}_6	\mathfrak{e}_7	\mathfrak{e}_8
\cdot	712								
\mathfrak{su}_2	499	47							
\mathfrak{su}_3	121	11	2						
\mathfrak{g}_2	589	62	7	34					
\mathfrak{so}_8	276	14	1	12	3				
\mathfrak{f}_4	1245	74	6	54	9	32			
\mathfrak{e}_6	184	2	0	2	0	2	0		
\mathfrak{e}_7	890	24	0	14	2	13	0	4	
\mathfrak{e}_8	15	0	0	0	0	0	0	0	0

Table 5. Gauge algebras $\mathcal{G}_1 \oplus \mathcal{G}_2$ arising in generic models for the 4962 F-theory bases \mathcal{B}_3 with toric B_2 and smooth heterotic duals. Note that for many bases \mathcal{B}_3 a variety of distinct models with enhanced gauge groups can be realized when moduli are tuned to specific loci, corresponding to distinct elliptically fibered fourfolds in the F-theory picture.

are determined from the dual monomials in f, g in the toric picture using table 1. In places where the degrees of vanishing of f, g do not uniquely determine the gauge algebra type, the gauge algebra is fixed by the monodromy around the codimension one divisor, which can be read off from the structure of the monomials following the discussion in section 3.1. When the vanishing degrees of f, g, Δ are 2, 2, 4, the gauge algebra is \mathfrak{su}_2 unless $g = g_2(u, v)z^2 + \mathcal{O}(z^3)$ with $g_2(u, v)$ a perfect square, where $z = 0$ on the divisor locus in question and u, v are coordinates on the divisor. When there is no restriction other than the vanishing of certain monomials, $g_2(u, v)$ is only guaranteed to be a perfect square if it contains only a single even monomial $u^{2n}v^{2m}$. Similarly, when f, g, Δ vanish to degrees 3, 4, 8, the gauge algebra factor is only \mathfrak{e}_6 when the leading part of g is a perfect square, which again is only possible when it is a single even monomial. For vanishing degrees 2, 3, 6 the story is slightly more subtle, but again easy to analyze in terms of the monomials. The generic gauge algebra factor is \mathfrak{g}_2 . The algebra becomes \mathfrak{so}_8 when $f_2^3 = cg_3^2$, with c an overall constant (complex) coefficient, which is only possible when each contains only a single monomial $f_2 = au^{3n}v^{3m}, g_3 = bu^{2n}v^{2m}$. In principle, the gauge algebra could be \mathfrak{so}_7 , which can occur when $g_3 = 0$ and $f_2(u, v)$ is not a perfect square, but this does not occur for any of the 3D bases \mathcal{B}_3 considered here. (Note, however, that for more general bases that do not have smooth heterotic duals we do expect \mathfrak{so}_7 to arise as a generic gauge algebra component. This occurs, for example, in 6D models when the base B_2 contains intersecting curves of self-intersection $-2, -3, -2$, which support the gauge algebra $\mathfrak{su}_2 \oplus \mathfrak{so}_7 \oplus \mathfrak{su}_2$ [8].)

In table 5 we see that the great majority of models (85%) have some gauge group automatically imposed from the geometry, which cannot be removed by Higgsing charged moduli fields. Furthermore, most of the models either have gauge factors that are not subgroups of $SU(5)$, or contain $SU(2)$ or $SU(3)$ factors that cannot be enhanced to $SU(5)$ as discussed in the previous sections. Note that the gauge group described here is purely

that determined by the geometry. As discussed in section 6.5, in some situations the gauge group may be modified when G-flux is taken into account. We leave further investigation of this effect to future work.

Note that there are many toric \mathcal{B}_3 's that are \mathbb{P}^1 bundles over toric B_2 bases for which gauge algebra summands \mathfrak{e}_8 arise that are not included in this tabulation because there are codimension two curves in \mathcal{B}_3 living in the \mathfrak{e}_8 locus where the degree of vanishing of f, g reaches 4, 6. These are a special class of examples of situations where a curve in the base \mathcal{B}_3 must be blown up to have a base \mathcal{B}'_3 that can act as the base of an elliptically fibered Calabi-Yau fourfold. After this blow-up, the F-theory model no longer has a smooth heterotic dual and is not included in this analysis. In this situation the blown up base is also generically non-toric. These cases are closely analogous to base surfaces B_2 for 6D F-theory models that contain $-9, -10$, and -11 curves; along such curves there is an \mathfrak{e}_8 gauge algebra summand and 3, 2, or 1 points where f, g vanish to degrees 4, 6 and the base must be blown up for a smooth F-theory model [8, 60, 61].

Several unusual features arise in many of the 4962 models we have constructed. As mentioned in section 4.1, in some models that are otherwise well-behaved there are codimension three singularities of order (4, 6). It is not known whether these singularities herald a sickness of the associated 4D supergravity theories [89]; as discussed in section 7.8, such codimension three singularities may also be associated with chiral matter, G-flux, or abelian gauge symmetries. In some cases, a codimension three (4, 6) singularity arises at a toric point given by the intersection of three toric divisors. There are a total of 496 models with this feature (or bug) in the toric set; we have tabulated the number of threefolds \mathcal{B}_3 where this occurs for each base surface B_2 in the last column of table 4. One of the simplest examples of a threefold with this property is the \mathbb{P}^1 bundle over the 10th base B_2 in table 4, characterized by divisors in the base with self-intersections (1, 0, -2, -2, -1, -2) and a twist divisor $T = D_5$, where D_5 is the divisor in B_2 with self-intersection -1. This model has no codimension one singularities associated with nonabelian gauge groups, but f and g vanish to degrees (4, 6) at the point $\Sigma_- \cap D_3 \cap D_4$. Note that, as mentioned in section 4.1.6 there can also be codimension three (4, 6) singularities arising at non-toric points, such as generically occurs on curves where f, g vanish to degrees (4, 5). We have not attempted to classify the models with such singularities here.

Another feature that can arise is a codimension two singularity on a curve that does not lie on any divisor carrying a gauge group. While in general codimension two singularities indicate matter charged under the nonabelian gauge groups of the corresponding divisors, in this situation this interpretation is not possible. It is possible that these singularities are simply cusps in the discriminant locus with no physical meaning, or they may herald the presence of abelian U(1) factors. This occurs in roughly half (2495 of the 4962 total) of the threefold bases \mathcal{B}_3 . The simplest example is the \mathbb{P}^1 bundle over \mathbb{F}_1 with twist $T = 2F$; this base gives no codimension one singularity associated with nonabelian gauge groups, but has a (2, 3) vanishing of f, g on $S \cap \Sigma_-$. We leave further investigation of the models with these features to future work.

10 Conclusions and open questions

In this paper we have given a global characterization of a broad class of 4D $\mathcal{N} = 1$ string vacua that admit both a heterotic description and a dual F-theory description. The class of vacua we have considered are described in the heterotic theory through compactification on a smooth Calabi-Yau threefold that is elliptically fibered with section over a base B_2 and carries a smooth vector bundle (and in some cases, 5-branes wrapping the elliptic fiber), and in F-theory through compactification on an elliptic fibration over a base threefold that is itself a \mathbb{P}^1 bundle with section over the same base B_2 . We have shown that the number of topologically distinct vacua in this class is finite, and we have explicitly enumerated all models where the base B_2 is toric.

By focusing on the underlying geometrical and topological structure of the theories, we have developed tools and identified features of these models that do not depend on specific limits or bundle constructions on the heterotic side of the duality. We have identified from the F-theory side a simple set of constraints that are necessary and sufficient for the existence of a Calabi-Yau compactification geometry; these constraints are expressed in terms of the “twist” defining the \mathbb{P}^1 bundle on the F-theory side and related components of the second Chern class of the bundles on the heterotic side. These constraints give a detailed characterization of the circumstances under which slope-stable bundles with general structure groups should exist both for heterotic $E_8 \times E_8$ and $SO(32)$ theories on smooth Calabi-Yau threefolds. The structure of chiral matter on the heterotic side has implications for the interplay between chiral matter and G-flux on the F-theory side.

The results described in this work represent a small step towards a systematic characterization of the broader class of $\mathcal{N} = 1$ 4D supergravity theories that can be realized in string theory. There are clearly many directions in which this work could be expanded further. Many more detailed aspects of the physics of the large class of models described here can be explored further using the tools described here. This work also provides a basis for a further systematic expansion of our understanding of heterotic/F-theory duality, as well as tools for expanding the range of applicability of both heterotic and F-theory approaches to string compactification. We conclude with a brief summary of some of these possible future directions.

10.1 Detailed physics of smooth heterotic/F-theory dual pairs

In this work, following [9], we used topological structure, in the form of axion-curvature squared terms in the 4D supergravity theory, to identify dual heterotic and F-theory geometries. This gives an association between F-theory constructions and heterotic bundles that is independent of the stable degeneration limit [19] in which the duality has been most thoroughly studied; the approach taken here has enabled a systematic classification of all smooth dual geometries where the F-theory model is described in terms of a threefold base \mathcal{B}_3 that is a \mathbb{P}^1 bundle over a base B_2 . Within this class of smooth dual geometries, there are many questions that could be explored further.

The roughly 5000 models where B_2 is toric provide an extensive dataset of dual heterotic/F-theory constructions that may be useful in a variety of contexts. To aid further

development in this direction, we have provided the details of this set of F-theory compactification geometries in a file that can be downloaded.²⁴ This file contains a listing for each of the bases B_2 of the complete set of allowed twists T describing a \mathbb{P}^1 bundle over that B_2 that does not have (4, 6) codimension one or two singularities, as well as the generic gauge algebras $\mathcal{G}_1 \oplus \mathcal{G}_2$ generated by for codimension one singularities over that base (SO(8) in those cases that have an SO(32) heterotic dual). The methods of this paper can be applied more generally for any of the several hundred generalized del Pezzo bases that support a smooth elliptically fibered Calabi-Yau threefold. One natural extension of the work here would be the explicit construction and classification of this broader class of (generically non-toric) bases, along the lines of the example dP_4 worked out in section 7.6 (which by itself already gives rise to roughly 7000 additional \mathcal{B}'_3 s; other non-toric generalized del Pezzo surfaces are expected to similarly generate large numbers of additional examples).

For the models considered and enumerated here, many more detailed questions remain to be addressed. For each of the ~ 5000 toric \mathcal{B}_3 's, there are many branches of the moduli space in which the generic gauge group is enhanced by “unHiggsing”, corresponding to a tuning of Weierstrass moduli in the F-theory picture and special loci in bundle moduli space on the heterotic side. Many general aspects of the branching structure of these moduli spaces remain to be investigated. We have identified from the F-theory side specific conditions under which the dual heterotic model should admit a bundle with exceptional structure group; in many of these cases there is no explicit mathematical construction known for such models, finding such constructions represents another class of open problems related to this work. And, as mentioned throughout the text, we have not incorporated the effects of G-flux on the F-theory side; this mechanism will in general lift many geometric F-theory moduli and produce chiral matter. We hope that the explicit correspondence we have developed here will help in elucidating these issues further.

Although we have focused in this paper on general aspects of heterotic and F-theory constructions that are independent of specific models, some lessons have emerged that may be relevant for more phenomenological “model building”. One general lesson from the systematic study of F-theory models both in 6D and in 4D, illustrated particularly clearly in six dimensions [60, 61], is that a large fraction of the elliptically fibered Calabi-Yau manifolds that can be used to compactify F-theory give rise to large “non-Higgsable” gauge groups. While a clear understanding of the connection between geometrically non-Higgsable gauge groups and 4D physics requires a better incorporation of the effects of G-flux, the models we have studied here are among those in which the minimal geometric gauge groups are smallest, and may provide the most promising candidates for realistic models of physics. In terms of potentially phenomenologically relevant gauge groups, we have found that many F-theory geometries contain geometric SU(2) and/or SU(3) factors that cannot be enhanced to SU(5), but that can for example be enhanced to SO(10), E_6 , or E_7 . Much work has been done in constructing phenomenologically oriented F-theory models based on an SU(5) unification structure (see [149–151] for a review of some of this work, and [34, 152–158] for

²⁴The list of 4962 toric bases \mathcal{B}_3 and associated minimal gauge groups described in section 9 is available online in an ancillary mathematica format text file at [148] or at <http://ctp.lns.mit.edu/wati/data/het-F-dual-bases.m>.

some specific global GUT models). It would be interesting to study more broadly how the generic gauge group structures that we have explored here might play into more general model building approaches, perhaps in the context of GUT groups other than $SU(5)$. As we have focused on generic geometric structure in the models studied here, we have also not investigated the tuning of abelian gauge group factors. Much recent work [82, 159–174] has focused on the role of global $U(1)$ factors in F-theory models. In most cases such $U(1)$ factors arise only at very special tuned loci in the Weierstrass moduli space over any given F-theory base; it was recently found, however, that $U(1)$ factors can be generic over certain special F-theory bases in 6D [61], including bases related to non-toric generalized del Pezzo bases considered here. Further investigation of $U(1)$ factors in the class of dual heterotic/F-theory models provides another interesting direction for further work.

At present, the explicit heterotic/F-theory dual “dictionary” has been most fully determined in a corner of moduli space in which the heterotic bundles can be described via spectral (more generally cameral) covers [19, 41, 42, 80]. However, the results of this work shows that many consistent, perturbative heterotic theories cannot be described by a naive application of these constructions. For example, the bundle with $\eta = 0$ of (7.34) can clearly not be described as an ordinary, smooth spectral cover (see section 7.9 for a discussion). Other specific examples arise in the class of models we have analyzed here. For instance, considering the set of 4962 toric dual pairs described in section 9, there appear to be many good F-theory geometries for which we cannot directly construct the heterotic dual bundles with standard tools, even when the heterotic structure bundle is $SU(N)$. For example, of the 947 4-folds with a generic E_7 symmetry (on at least one patch), 897 of these fail to satisfy the parity condition (5.29) for $SU(2)$ bundles constructed as generic, irreducible spectral covers. This indicates that if good heterotic duals exist they must either a) not be constructible as ordinary $SU(2)$ spectral covers (i.e., they possess one of the limitations described in section 5.3.3) or b) the bundle moduli space contains no irreducible spectral covers at all (see [103] for example, for more exotic possibilities). It is also possible that this parity condition may be a more general constraint and may indicate some problem with the associated F-theory models, for example that may indicate a conflict with the existence of a consistent choice of G-flux. In any case, there remains something to be understood in the explicit moduli mapping of the heterotic/F-theory dual pair which could yield important new insights into both theories.

As mentioned above, we have not considered here the effects of G-flux in modifying the underlying Calabi-Yau geometry of F-theory. One of the most interesting aspects of 4-dimensional heterotic/F-theory duality is the fact that deformations that change the gauge symmetry (i.e., deformations of the complex structure of Y_4 or of the bundles V_i on X_3) can be obstructed. These obstructions can appear both through D- and F-term contributions to the potential in the low energy theory. A better understanding of this potential would have impact not only on the problem of moduli stabilization in heterotic/F-theory effective theories but could also lead to novel dynamical effects in the 4-dimensional theories — including, for example, the obstruction of tensionless string/small-instanton transitions and possible duality to non-commutative $D3$ branes. We hope to explore these topics in future work.

10.2 Expanding heterotic/F-theory duality

In this work we have focused on the simplest class of heterotic/F-theory dualities, where both sides have elliptically fibered Calabi-Yau geometries with a single section, and the F-theory base \mathcal{B}_3 is a \mathbb{P}^1 bundle that also has a single section. Since we expect that the heterotic and F-theory constructions are simply different mathematical approaches to describing the same physical theory in distinct limits, we expect that it should be possible to extend this duality to a much broader class of vacua, possibly at the expense of needing to introduce more complicated mathematical objects in the theory on one or both sides.

One clear question is the extent to which heterotic/F-theory duality can be systematically described when the heterotic Calabi-Yau geometry and/or bundle structure becomes singular, leading mathematically to a description in terms of more singular objects such as sheaves. While many examples of this have been studied in the literature, the appearance of structure such as enhanced gauge groups and additional geometric moduli arising through tensionless string transitions is more transparent geometrically from the point of view of Weierstrass models on the F-theory side. The framework developed here may provide a useful context in which to systematically extend the duality in these directions. The simple geometric framework of F-theory has the potential to clarify some of the mathematical questions that are rather subtle in this context on the heterotic side.

More generally, there are classes of geometries that are slightly more general than those considered here in which heterotic/F-theory duality is not understood. These include bases \mathcal{B}_3 built as more general \mathbb{P}^1 fibrations (rather than \mathbb{P}^1 bundles) and situations where either the heterotic or F-theory elliptic fibrations have multiple sections (higher rank Mordell-Weil group) or no section at all. In particular, for Calabi-Yau geometries that are elliptically fibered but have more than one section, or a multi-section, the story is not yet completely clear. F-theory models with multiple sections (higher rank Mordell-Weil group) are understood simply as models with additional U(1) factors, which generally should have natural heterotic duals. F-theory models with a multi-section but no global section have recently been incorporated into the global moduli space of Weierstrass models [81, 82]. It is less clear, however, how to construct an F-theory dual for a heterotic model on a Calabi-Yau threefold with multiple sections or a multi-section. While in principle such threefolds can be realized as special limits in the Weierstrass moduli space of elliptic fibrations (using the Jacobian fibration associated with threefolds having a multi-section, as in [82]), which should give a corresponding construction on the F-theory side, the details of the physics of this correspondence have not been worked out. In this paper we considered only cases where the F-theory threefold base \mathcal{B}_3 is itself a P^1 bundle with section. It is also possible to consider situations where \mathcal{B}_3 is a P^1 fibration without a section or indeed even more general geometries [66, 175].

Another example of a situation where heterotic/F-theory duality is not well understood comes from the fact that for Calabi-Yau threefolds, the moduli space of bundles \mathcal{M} can have multiple components (see appendix B). If it is not known how F-theory duals to such situations can be understood. A natural hypothesis is that for each component of the heterotic moduli space there would exist topologically identical, non-diffeomorphic Calabi-Yau 4-folds on the F-theory side. This is another interesting avenue for further investigation.

Finally, on the heterotic side there are compactifications on Calabi-Yau threefolds that are not elliptically fibered. Such geometries can be reached by nonperturbative transitions from elliptically fibered Calabi-Yau threefolds, so should in principle be connected to the underlying geometric moduli space of F-theory compactifications. At present, there is no known mechanism by which F-theory can include such vacua. F-theory is at present still an incomplete physical theory, however; there is no direct action principle for the theory that incorporates all degrees of freedom. In an optimistic scenario, further development of heterotic/F-theory duality may provide some insight into a more complete formulation of the theory and a broader and more unified characterization of the full space of $\mathcal{N} = 1$ 4D supersymmetric string theory vacua.

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A Properties of elliptically fibered Calabi-Yau three- and fourfolds

In this appendix we briefly review a collection of useful results regarding the geometry and topology of elliptically fibered Calabi-Yau manifolds (see [19, 30] for a more complete treatment). We focus on smooth elliptically fibered Calabi-Yau threefolds, $\pi : X_3 \rightarrow B_2$ with a single section (which defines B_2 as an algebraic sub-manifold within X_3). If X_3 is in Weierstrass form, a minimal set of divisors²⁵ that span the Picard group of X_3 is given by the zero section, D_0 and divisors pulled back from the base of the form $D_\alpha = \pi^*(D_\alpha^{base})$, where D_α^{base} , $\alpha = 1, \dots, h^{1,1}(B_2)$ is an ample divisor on B_2 . For such smooth, minimal elliptic fibrations, $h^{1,1}(X_3) = h^{1,1}(B_2) + 1$. We will denote the basis of $\{1, 1\}$ -forms dual to the divisors above as $\{\omega_0, \omega_\alpha\}$.

By virtue of this simple fibration structure, the triple intersection numbers of these divisors exhibit a universal behavior. First, since the base is a 2-fold it is clear that

$$D_\alpha \cap D_\beta \cap D_\gamma = 0 \tag{A.1}$$

Moreover, from the very definition of what it means for D_0 to be section (and not a multisection) it is guaranteed that for any two-form ζ on B_2 (dual to a single point),

²⁵We focus here on a minimal form of elliptically fibered threefold in which all exceptional curves in the fiber have been blown down.

$D_0 \cap \zeta = 1$ (that is, the zero section intersects each elliptic fiber precisely once). It follows from this fact that

$$D_0 \cap D_\alpha \cap D_\beta = m_{\alpha\beta} \tag{A.2}$$

where $m_{\alpha\beta} = D_\alpha^{base} \cap D_\beta^{base}$. These facts are enough to derive the following important cohomological identity on $\{2, 2\}$ forms,

$$\omega_0 \wedge \omega_0 = K \wedge \omega_0 \tag{A.3}$$

where K is the canonical class of the base, $K = -c_1(B_2) = K^\alpha \omega_\alpha$. With these results, the triple intersection numbers of X_3

$$d_{ABC} = \int_{X_3} \omega_A \wedge \omega_B \wedge \omega_C \tag{A.4}$$

where $\omega_A = \{\omega_0, \omega_\alpha\}$, are given by

$$\begin{aligned} d_{000} &= m_{\alpha\beta} K^\alpha K^\beta & d_{00\alpha} &= m_{\alpha\beta} K^\beta \\ d_{0\alpha\beta} &= m_{\alpha\beta} & d_{\alpha\beta\gamma} &= 0 \end{aligned} \tag{A.5}$$

With these intersection numbers and a chosen Kähler form $\omega = t^0 \omega_0 + t^\alpha \omega_\alpha$, the volume of X_3 takes the form

$$\text{Vol}(X_3) = \frac{1}{3!} \int_X \omega \wedge \omega \wedge \omega = \frac{1}{3!} \left(d_{000} (t^0)^3 + 3d_{00\alpha} (t^0)^2 t^\alpha + 3d_{0\alpha\beta} t^0 t^\alpha t^\beta \right) \tag{A.6}$$

The fibration structure guarantees that the second Chern class of X_3 can be written as [19]

$$c_2(TX_3) = 12c_1(B_2) \wedge \omega_0 + c_2(B_2) + 11c_1(B_2)^2 \tag{A.7}$$

where in addition the topology of B_2 satisfies

$$\chi(B_2) = \int_{B_2} c_2(B_2) = 2 + h^{1,1}(B_2) \quad , \quad \int_{B_2} c_1^2(B_2) = K^\alpha K^\beta m_{\alpha\beta} = 10 - h^{1,1}(B_2) \tag{A.8}$$

Finally, using the redundancy relation on $\{2, 2\}$ forms in (A.3) it is possible to write the second Chern class of any bundle, V , on X_3

$$c_2(V) = \eta \wedge \omega_0 + \zeta \tag{A.9}$$

where η and ζ are pullbacks through π of $\{1, 1\}$ $\{2, 2\}$ forms from B_2 .

B A brief exploration of rigid bundles

A novel feature of four-dimensional compactifications of heterotic string theory/F-theory is the possibility of multiple components in the dual (vector bundle/fourfold) moduli spaces. In the case of heterotic/F-theory duality, such multiple components to the moduli space have not yet been studied in detail. Indeed, thus far in the literature the correspondence

between vector bundle moduli spaces in heterotic theories and the complex structure moduli space of Calabi-Yau fourfolds has only been studied in the case of single, connected components [19].

For the moduli space of stable sheaves on Calabi-Yau threefolds a natural case of interest is given when the local moduli space in fact contains an isolated, rigid component. In general, given two bundles with the same topology, it is difficult to decide whether or not they reside in the same component of a global bundle moduli space. However, if one bundle is rigid (and the other has local deformation moduli), it is clear that the rigid bundle consists of its own distinct component to bundle moduli space. As a result, a search for rigid bundles is one of the simplest probes for multiple components of bundle moduli spaces.

For elliptically fibered Calabi-Yau threefolds, it is straightforward to come by examples of such rigid vector bundles. For instance, over the Calabi-Yau threefold defined as a single degree $\{3, 3\}$ hypersurface in the product, $\mathbb{P}^2 \times \mathbb{P}^2$, of two projective spaces, consider the following poly-stable $SU(2)$ bundle

$$V_2 = L + L^\vee = \mathcal{O}(-H_1 + H_2) + \mathcal{O}(H_1 - H_2) \tag{B.1}$$

where H_1, H_2 are the restrictions of the hyperplanes of each ambient \mathbb{P}^2 factor to X_3 . On this space, V_2 is rigid, since the dimension of the space of bundle-valued singlets is given by

$$h^1(X_3, L^{\otimes 2}) = h^1(X_3, L^{\vee \otimes 2}) = 0 \tag{B.2}$$

for generic values of the complex structure of X_3 [99]. Geometrically the spaces $H^1(X_3, L^{\otimes 2}), H^1(X_3, L^{\vee \otimes 2})$ constitute the space of non-trivial extensions (for example $H^1(X_3, L^{\otimes 2})$ parameterizes the space of non-trivial extensions $0 \rightarrow L \rightarrow V_2 \rightarrow L^\vee \rightarrow 0$) which parameterize how L, L^\vee may be non-trivially “glued” back into an indecomposable $SU(2)$ bundle. Since the space of such extensions vanishes in these cases, the split bundle $L + L^\vee$ has no infinitesimal deformations, that is, it is rigid.

Although the example above is interesting from the point of view of vector bundle moduli spaces, it is not clear what the impact of such examples will be in the heterotic-F-theory pairs constructed in this work. Although the threefold above is elliptically fibered, it has no section and cannot be written in Weierstrass form. At present its F-theory dual (if any) is unknown.

In this appendix, we make a tentative exploration of whether it is possible to obtain isolated components to the moduli space of $SU(2)$ bundles, such as the one described above, over the class of Calabi-Yau threefolds considered here — that is, elliptically fibered threefolds with a single section, obeying the topological identities listed in appendix A. Once again, we can search for bundles of the form

$$L + L^\vee \tag{B.3}$$

where L is a holomorphic line bundle on X_3 satisfying $\mu(L) = 0$ in the Kähler cone, as required for supersymmetry by (5.6). This reducible bundle in (B.3) will be rigid if $h^1(X, L^{\otimes 2}) = h^1(X, (L^\vee)^{\otimes 2}) = 0$

As we will argue below, however, for the geometry in consideration in this work, such examples appear to be rare and we have obtained no explicit examples. This result is fully consistent with the fact that in the dual F-theory geometry we find a single Calabi-Yau fourfold (with a single connected (and non-trivial) component to its complex structure moduli space) for each choice of vector bundle topology/twisting parameter, T .

To see this, we must consider the line bundle cohomology of $L^{\otimes 2}$ on $\pi : X_3 \rightarrow B_2$. Here we are aided by the formalism of Leray Spectral sequences [98]. According to this spectral sequence for the fibration $\pi : X_3 \rightarrow B_2$, we have a natural bi-grading such that for any bundle V on X_3 ,

$$H^p(X, V) = \sum_{p=l+m} E_{\infty}^{l,m} \tag{B.4}$$

where

$$E_1^{l,m} = H^l(B_2, R^m \pi_*(V)) \tag{B.5}$$

and $R^m \pi_*(V)$ is the m -th direct image sheaf of the bundle V (pushed forward under the fibration π). We need not concern ourselves with the iteration of the sequence via the maps $d_r : E_r^{p,q} \rightarrow E_r^{p+r, q-r+1}$, since the spectral sequence terminates at E_1 . To see this, note that on any open set \mathcal{U} on \mathbb{P}^1 , the m -th direct image sheaf, $R^m \pi_*(V)$ can be locally represented by the pre-sheaf

$$\mathcal{U} \rightarrow H^m(f^{-1}(\mathcal{U}), R^m \pi_*(V)) \tag{B.6}$$

For elliptic fibrations, however, the fiber (locally isomorphic to $f^{-1}(\mathcal{U})$) is one dimensional. As a result, $R^m \pi_*(V)$ is non-vanishing only for $m = 0, 1$ and the spectral sequence terminates at

$$E_{\infty} = E_1 \tag{B.7}$$

To analyze the cohomology, we further need to observe how a line bundle of the form

$$L = \mathcal{O}(a\sigma + b^{\alpha} \pi^*(D_{\alpha})) \tag{B.8}$$

behaves under the push-forward functor. The first useful observation is that line bundles of the form $\mathcal{O}(b^{\alpha} D_{\alpha})$ built from divisors pulled back from the base B_2 satisfies $\pi^* \mathcal{O}_{B_2}(b^{\alpha} D_{\alpha}) = \mathcal{O}_{X_3}(b^{\alpha} \pi^*(D_{\alpha}))$. This implies that under push-forwards we have the so-called ‘‘projection formula’’. For a fibration $\pi : X_3 \rightarrow B_2$, and any bundles V on X and U on B ,

$$R^q \pi_*(V \otimes \pi^* U) = R^q \pi_*(V) \otimes U \tag{B.9}$$

To determine the cohomology of $L = \mathcal{O}_{X_3}(a\sigma + b^{\alpha} \pi^*(D_{\alpha}))$ then, we need only consider $R^i \pi_*(\mathcal{O}_{X_3}(a\sigma + b^{\alpha} \pi^*(D_{\alpha}))) = R^i \pi_*(\mathcal{O}_{X_3}(a\sigma)) \otimes \mathcal{O}_{B_2}(b^{\alpha} D_{\alpha})$. Moreover, the structure of $R^i \pi_*(\mathcal{O}_{X_3}(a\sigma))$ can straightforwardly be determined by considering the Koszul sequence:

$$0 \rightarrow \mathcal{O}_{X_3}(-\sigma) \rightarrow \mathcal{O}_{X_3} \rightarrow \mathcal{O}|_{\sigma=B_2} \rightarrow 0 \tag{B.10}$$

Twisting this by $\mathcal{O}_{X_3}(\sigma)$ yields

$$0 \rightarrow \mathcal{O}_{X_3} \rightarrow \mathcal{O}_{X_3}(\sigma) \rightarrow \mathcal{O}(\sigma \cdot \sigma)|_{\sigma=B_2} \rightarrow 0 \tag{B.11}$$

But by (A.3) and the definition of σ as the zero-section this is simply

$$0 \rightarrow \mathcal{O}_{X_3} \rightarrow \mathcal{O}_{X_3}(\sigma) \rightarrow \mathcal{O}(K_2 \cdot \sigma)|_{\sigma=B_2} \rightarrow 0 \quad (\text{B.12})$$

and pushing forward to B_2 gives the short exact sequence

$$0 \rightarrow \mathcal{O}_{B_2} \rightarrow R^0\pi_*(\mathcal{O}_{X_3}(\sigma)) \rightarrow K_2 \rightarrow 0 \quad (\text{B.13})$$

For B_2 the base of a CY threefold, the above sequence splits and we have determined the direct image sheaf:

$$R^0\pi_*(\mathcal{O}_{X_3}(\sigma)) = \mathcal{O}_{B_2} \oplus K_2 \quad (\text{B.14})$$

The calculation outlined above can be iterated inductively to find the higher direct image sheaves $R^i\pi_*(\mathcal{O}_{X_3}(a\sigma))$ for $a > 1$ in a similar manner. It is straightforward to demonstrate that

$$R^0\pi_*(\mathcal{O}_{X_3}(2\sigma)) = \mathcal{O}_{B_2} \oplus K_2 \oplus K_2^{\otimes 2} \quad R^1\pi_*(\mathcal{O}_{X_3}(2\sigma)) = 0 \quad (\text{B.15})$$

$$R^0\pi_*(\mathcal{O}_{X_3}(3\sigma)) = \mathcal{O}_{B_2} \oplus K_2 \oplus K_2^{\otimes 2} \oplus K_2^{\otimes 3} \quad R^1\pi_*(\mathcal{O}_{X_3}(3\sigma)) = 0 \quad (\text{B.16})$$

$$\vdots \quad \vdots \quad (\text{B.17})$$

$$R^0\pi_*(\mathcal{O}_{X_3}(a\sigma)) = \text{Sym}^a(\mathcal{O}_{B_2} \oplus K_2) \quad R^1\pi_*(\mathcal{O}_{X_3}(a\sigma)) = 0 \quad (\text{B.18})$$

Similar results for $a < 0$ can be found by using Grothendieck duality: for any sheaf F on X_3 , the push-forward functors obey the following relation:

$$R^{1-i}\pi_*(F^\vee \otimes \omega_{X_3|B_2}) = (R^i\pi_*F)^\vee \quad , \quad i = 0, 1 \quad (\text{B.19})$$

where $\omega_{X_3|B_2} = K_{X_3} \otimes \pi^*(K_2^\vee) = \pi^*(K_2^\vee)$ is the ‘‘dualizing sheaf’’ [98]. With these results in hand, we can now in principle calculate all line bundle cohomology on X_3 . In order to build reducible, rigid SU(2) bundles like those above we must note that for the line bundles of interest, $\mu(L) = 0$, and hence [99]

$$H^0(X, L) = H^3(X, L) = 0 \quad (\text{B.20})$$

Thus, to construct a rigid SU(2) bundle we must use the results above for line bundle cohomology and further ask, for what values of a, b in (B.8) can we have $H^1(X, L) = H^1(X, L^\vee) = 0$? By (B.4) and (B.5) it is clear that we require

$$H^1(B_2, R^0\pi_*L) = H^0(B_2, R^1\pi_*L) = 0 \quad (\text{B.21})$$

in order to satisfy $H^1(X, L) = 0$ and

$$H^2(B_2, R^0\pi_*L) = H^1(B_2, R^1\pi_*L) = 0 \quad (\text{B.22})$$

for $H^2(X, L) = H^1(X, L^\vee) = 0$ (by Serre duality). This, coupled with (B.20) means that the direct image sheaves $R^0\pi_*L$ and $R^1\pi_*L$ must have entirely vanishing cohomology on B_2 . However, as we will see below, this does not occur for simple threefolds of the type we are considering here.

To see that it is rare for $H^i(B_2, R^0\pi_*L) = 0 \forall i$, it is useful to consider the index of $R^0\pi_*L$ using the Riemann-Roch theorem. With L as in (B.8)

$$R^0\pi_*L = (\mathcal{O}_{B_2} \oplus K_2 \oplus \dots K_2^{\otimes a}) \otimes \mathcal{O}_{B_2}(b^\alpha D_\alpha) \tag{B.23}$$

we have that for each term in the sum, the index is additive. Thus

$$\chi(R^0\pi_*L) = \chi(\mathcal{O}_{B_2}(b^\alpha D_\alpha) + \chi(K_2 \otimes \mathcal{O}_{B_2}(b^\alpha D_\alpha)) \dots \tag{B.24}$$

Letting $D = b^\alpha D_\alpha$ and using the fact that for any divisor, $A \subset B_2$,

$$\chi(A) = 1 + \frac{1}{2}A \cdot (A - K_2) \tag{B.25}$$

we have

$$\chi(R^0\pi_*L) = (a + 1) + D \cdot D \tag{B.26}$$

If we demand that the index vanishes as a necessary condition for entirely vanishing cohomology, we require $D \cdot D = -(a + 1)$ for some curve $D \subset B_2$. Putting this together with other geometric constraints in the problem, we see that the Bogomolov bound of section 5.1 places a positivity condition on $c_2(V)$. For $V = L \oplus L^\vee$ this enforces that $a \geq 0$. Finally, it can be noted that for the case $a = 0$, the line bundles can be verified to have non-vanishing cohomology. Thus, here we will consider $a > 0$.

For the geometries considered in this work, we have at most -2 curves, thus without loss of generality we can restrict ourselves to line bundles of the form $\mathcal{O}(\sigma + b^\alpha D_\alpha)$ where $D = b^\alpha D_\alpha$ is a -2 curve. Although there do exist curves of this type (for example the divisor S in \mathbb{F}_2), it can be verified on a case-by-case basis that here the necessary condition is not in fact sufficient and $H^1(B_2, R^0\pi_*L) \neq 0$. Although we have not rigorously ruled out all possible -2 curves in our set of base manifolds B_2 , systematic searches have found no examples with entirely vanishing cohomology. Thus, we expect that for the simple geometries outlined in appendix A, no reducible $SU(2)$ bundles of the form shown in (B.3) exist as rigid components in the moduli space. It would be nice, however, to have a more general abstract proof of this result. If rigid bundles can be found within the context of heterotic/F-theory duality, it would be interesting to investigate the dual F-theory constructions. Some possibly related F-theory models may exist; for 6D compactifications over $B_2 = \mathbb{P}^2$, some models with exotic matter were identified in [68] that are similarly rigid in the sense that they have no moduli that preserve the gauge group and matter content.

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References

- [1] M.B. Green, J.H. Schwarz and E. Witten, *Superstring theory*, volume 1 and 2, Cambridge Univ. Pr., Cambridge U.K. (1989).
- [2] J. Polchinski, *String theory*, volume 1 and 2, Cambridge Univ. Pr., Cambridge U.K. (1998).

- [3] L. Susskind, *The anthropic landscape of string theory*, [hep-th/0302219](#) [INSPIRE].
- [4] M. Graña, *Flux compactifications in string theory: a comprehensive review*, *Phys. Rept.* **423** (2006) 91 [[hep-th/0509003](#)] [INSPIRE].
- [5] F. Denef, M.R. Douglas and S. Kachru, *Physics of string flux compactifications*, *Ann. Rev. Nucl. Part. Sci.* **57** (2007) 119 [[hep-th/0701050](#)] [INSPIRE].
- [6] V. Kumar and W. Taylor, *String universality in six dimensions*, *Adv. Theor. Math. Phys.* **15** (2011) 325 [[arXiv:0906.0987](#)] [INSPIRE].
- [7] V. Kumar, D.R. Morrison and W. Taylor, *Global aspects of the space of 6D $N = 1$ supergravities*, *JHEP* **11** (2010) 118 [[arXiv:1008.1062](#)] [INSPIRE].
- [8] D.R. Morrison and W. Taylor, *Classifying bases for 6D F-theory models*, *Central Eur. J. Phys.* **10** (2012) 1072 [[arXiv:1201.1943](#)] [INSPIRE].
- [9] T.W. Grimm and W. Taylor, *Structure in 6D and 4D $N = 1$ supergravity theories from F-theory*, *JHEP* **10** (2012) 105 [[arXiv:1204.3092](#)] [INSPIRE].
- [10] R. Donagi, B.A. Ovrut, T. Pantev and D. Waldram, *Standard model bundles on nonsimply connected Calabi-Yau threefolds*, *JHEP* **08** (2001) 053 [[hep-th/0008008](#)] [INSPIRE].
- [11] R. Donagi, B.A. Ovrut, T. Pantev and D. Waldram, *Standard model bundles*, *Adv. Theor. Math. Phys.* **5** (2002) 563 [[math/0008010](#)] [INSPIRE].
- [12] L.B. Anderson, Y.-H. He and A. Lukas, *Heterotic compactification, an algorithmic approach*, *JHEP* **07** (2007) 049 [[hep-th/0702210](#)] [INSPIRE].
- [13] L.B. Anderson, Y.-H. He and A. Lukas, *Monad bundles in heterotic string compactifications*, *JHEP* **07** (2008) 104 [[arXiv:0805.2875](#)] [INSPIRE].
- [14] L.B. Anderson, J. Gray, Y.-H. He and A. Lukas, *Exploring positive monad bundles and a new heterotic standard model*, *JHEP* **02** (2010) 054 [[arXiv:0911.1569](#)] [INSPIRE].
- [15] L.B. Anderson, A. Constantin, J. Gray, A. Lukas and E. Palti, *A comprehensive scan for heterotic SU(5) GUT models*, *JHEP* **01** (2014) 047 [[arXiv:1307.4787](#)] [INSPIRE].
- [16] L.B. Anderson, J. Gray, A. Lukas and E. Palti, *Heterotic line bundle standard models*, *JHEP* **06** (2012) 113 [[arXiv:1202.1757](#)] [INSPIRE].
- [17] L.B. Anderson, J. Gray, A. Lukas and E. Palti, *Two hundred heterotic standard models on smooth Calabi-Yau threefolds*, *Phys. Rev. D* **84** (2011) 106005 [[arXiv:1106.4804](#)] [INSPIRE].
- [18] M. Gabella, Y.-H. He and A. Lukas, *An abundance of heterotic vacua*, *JHEP* **12** (2008) 027 [[arXiv:0808.2142](#)] [INSPIRE].
- [19] R. Friedman, J. Morgan and E. Witten, *Vector bundles and F-theory*, *Commun. Math. Phys.* **187** (1997) 679 [[hep-th/9701162](#)] [INSPIRE].
- [20] H. Hayashi, R. Tatar, Y. Toda, T. Watari and M. Yamazaki, *New aspects of heterotic/F-theory duality*, *Nucl. Phys. B* **806** (2009) 224 [[arXiv:0805.1057](#)] [INSPIRE].
- [21] D.R. Morrison and C. Vafa, *Compactifications of F-theory on Calabi-Yau threefolds. 1*, *Nucl. Phys. B* **473** (1996) 74 [[hep-th/9602114](#)] [INSPIRE].
- [22] D.R. Morrison and C. Vafa, *Compactifications of F-theory on Calabi-Yau threefolds. 2*, *Nucl. Phys. B* **476** (1996) 437 [[hep-th/9603161](#)] [INSPIRE].

- [23] M.J. Duff, R. Minasian and E. Witten, *Evidence for heterotic/heterotic duality*, *Nucl. Phys. B* **465** (1996) 413 [[hep-th/9601036](#)] [[INSPIRE](#)].
- [24] N. Seiberg and E. Witten, *Comments on string dynamics in six-dimensions*, *Nucl. Phys. B* **471** (1996) 121 [[hep-th/9603003](#)] [[INSPIRE](#)].
- [25] M. Gross, *A finiteness theorem for elliptic Calabi-Yau threefolds*, *Duke Math. J.* **74** (1994) 271 [[alg-geom/9305002](#)] [[INSPIRE](#)].
- [26] M. Maruyama, *On boundedness of torsion free sheaves*, *J. Math. Kyoto Univ.* **21** (1981) 673.
- [27] A. Langer, *Semistable sheaves in positive characteristic*, *Ann. Math.* **159** (2004) 251 [*Addendum ibid.* **160** (2004) 1211].
- [28] M. Demazure, H.C. Pinkham and B. Teissier eds., *Seminaire sur les singularités des surfaces* (in French), *Lect. Notes Math.* **777**, Springer, Germany (1980).
- [29] M. Cvetič, J. Halverson, D. Klevers and P. Song, *On finiteness of type IIB compactifications: magnetized branes on elliptic Calabi-Yau threefolds*, *JHEP* **06** (2014) 138 [[arXiv:1403.4943](#)] [[INSPIRE](#)].
- [30] A. Klemm, B. Lian, S.S. Roan and S.-T. Yau, *Calabi-Yau fourfolds for M-theory and F-theory compactifications*, *Nucl. Phys. B* **518** (1998) 515 [[hep-th/9701023](#)] [[INSPIRE](#)].
- [31] I. Brunner and R. Schimmrigk, *F-theory on Calabi-Yau fourfolds*, *Phys. Lett. B* **387** (1996) 750 [[hep-th/9606148](#)] [[INSPIRE](#)].
- [32] I. Brunner, M. Lynker and R. Schimmrigk, *Unification of M-theory and F-theory Calabi-Yau fourfold vacua*, *Nucl. Phys. B* **498** (1997) 156 [[hep-th/9610195](#)] [[INSPIRE](#)].
- [33] I. Brunner, M. Lynker and R. Schimmrigk, *Dualities and phase transitions for Calabi-Yau threefolds and fourfolds*, *Nucl. Phys. Proc. Suppl.* **56B** (1997) 120 [[hep-th/9703182](#)] [[INSPIRE](#)].
- [34] J. Knapp, M. Kreuzer, C. Mayrhofer and N.-O. Walliser, *Toric construction of global F-theory GUTs*, *JHEP* **03** (2011) 138 [[arXiv:1101.4908](#)] [[INSPIRE](#)].
- [35] J. Gray, A.S. Haupt and A. Lukas, *All complete intersection Calabi-Yau four-folds*, *JHEP* **07** (2013) 070 [[arXiv:1303.1832](#)] [[INSPIRE](#)].
- [36] J. Gray, A.S. Haupt and A. Lukas, *Topological invariants and fibration structure of complete intersection Calabi-Yau four-folds*, [arXiv:1405.2073](#) [[INSPIRE](#)].
- [37] G. Rajesh, *Toric geometry and F-theory/heterotic duality in four-dimensions*, *JHEP* **12** (1998) 018 [[hep-th/9811240](#)] [[INSPIRE](#)].
- [38] P. Berglund and P. Mayr, *Stability of vector bundles from F-theory*, *JHEP* **12** (1999) 009 [[hep-th/9904114](#)] [[INSPIRE](#)].
- [39] R. Friedman, J.W. Morgan and E. Witten, *Vector bundles over elliptic fibrations*, [alg-geom/9709029](#) [[INSPIRE](#)].
- [40] R.Y. Donagi, *Principal bundles on elliptic fibrations*, *Asian J. Math* **1** (1997) 214 [[alg-geom/9702002](#)] [[INSPIRE](#)].
- [41] R. Donagi, *Spectral covers*, in *Current topics in complex algebraic geometry*, *MSRI pub.* **28** (1992) 65 [[alg-geom/9505009](#)].
- [42] R. Donagi, *Heterotic/F-theory duality: ICMP lecture*, in *Mathematical physics*, Brisbane Australia (1997), pg. 206 [[hep-th/9802093](#)] [[INSPIRE](#)].

- [43] E. Looijenga, *Root systems and elliptic curves*, *Invent. Math.* **38** (1977) 17.
- [44] E. Looijenga, *Invariant theory for generalized root systems*, *Invent. Math.* **61** (1980) 1.
- [45] G. Curio, *Chiral matter and transitions in heterotic string models*, *Phys. Lett. B* **435** (1998) 39 [[hep-th/9803224](#)] [[INSPIRE](#)].
- [46] C. Vafa, *Evidence for F-theory*, *Nucl. Phys. B* **469** (1996) 403 [[hep-th/9602022](#)] [[INSPIRE](#)].
- [47] F. Denef, *Les Houches lectures on constructing string vacua*, [arXiv:0803.1194](#) [[INSPIRE](#)].
- [48] A. Clingher and J.W. Morgan, *Mathematics underlying the F-theory/heterotic string duality in eight-dimensions*, *Commun. Math. Phys.* **254** (2005) 513 [[math/0308106](#)] [[INSPIRE](#)].
- [49] G. Lopes Cardoso, G. Curio, D. Lüst and T. Mohaupt, *On the duality between the heterotic string and F-theory in eight-dimensions*, *Phys. Lett. B* **389** (1996) 479 [[hep-th/9609111](#)] [[INSPIRE](#)].
- [50] W. Lerche and S. Stieberger, *Prepotential, mirror map and F-theory on K3*, *Adv. Theor. Math. Phys.* **2** (1998) 1105 [*Erratum ibid.* **3** (1999) 1199] [[hep-th/9804176](#)] [[INSPIRE](#)].
- [51] J. McOrist, D.R. Morrison and S. Sethi, *Geometries, non-geometries and fluxes*, *Adv. Theor. Math. Phys.* **14** (2010) [[arXiv:1004.5447](#)] [[INSPIRE](#)].
- [52] K. Kodaira, *On compact analytic surfaces. II*, *Ann. Math.* **77** (1963) 563.
- [53] K. Kodaira, *On compact analytic surfaces. III*, *Ann. Math.* **78** (1963) 1.
- [54] O.J. Ganor, D.R. Morrison and N. Seiberg, *Branes, Calabi-Yau spaces and toroidal compactification of the $N = 1$ six-dimensional E_8 theory*, *Nucl. Phys. B* **487** (1997) 93 [[hep-th/9610251](#)] [[INSPIRE](#)].
- [55] W. Taylor, *TASI lectures on supergravity and string vacua in various dimensions*, [arXiv:1104.2051](#) [[INSPIRE](#)].
- [56] M. Bershadsky et al., *Geometric singularities and enhanced gauge symmetries*, *Nucl. Phys. B* **481** (1996) 215 [[hep-th/9605200](#)] [[INSPIRE](#)].
- [57] W.P. Barth, K. Hulek, C.A.M. Peters and A. Van de Ven, *Compact complex surfaces*, Springer, Germany (2004).
- [58] M. Reid, *Chapters on algebraic surfaces*, in *Complex algebraic geometry, Park City U.S.A.* (1993), *IAS/Park City Math. Ser.* **3** (1997) 3 [[alg-geom/9602006](#)].
- [59] A. Grassi, *On minimal models of elliptic threefolds*, *Math. Ann.* **290** (1991) 287.
- [60] D.R. Morrison and W. Taylor, *Toric bases for 6D F-theory models*, *Fortsch. Phys.* **60** (2012) 1187 [[arXiv:1204.0283](#)] [[INSPIRE](#)].
- [61] G. Martini and W. Taylor, *6D F-theory models and elliptically fibered Calabi-Yau threefolds over semi-toric base surfaces*, [arXiv:1404.6300](#) [[INSPIRE](#)].
- [62] S. Donaldson, *Anti self-dual Yang-Mills connections over complex algebraic surfaces and stable vector bundles*, *Proc. London Math. Soc.* **3** (1985) 1.
- [63] K. Uhlenbeck and S.T. Yau, *On the existence of Hermitian Yang-Mills connections in stable bundles*, *Comm. Pure App. Math.* **39** (1986) 257.
- [64] K. Uhlenbeck and S.T. Yau, *A note on our previous paper: on the existence of Hermitian Yang-Mills connections in stable vector bundles*, *Comm. Pure App. Math.* **42** (1986) 703.

- [65] P.S. Aspinwall, *Aspects of the hypermultiplet moduli space in string duality*, *JHEP* **04** (1998) 019 [[hep-th/9802194](#)] [[INSPIRE](#)].
- [66] R. Donagi, S. Katz and M. Wijnholt, *Weak coupling, degeneration and log Calabi-Yau spaces*, [arXiv:1212.0553](#) [[INSPIRE](#)].
- [67] V. Sadov, *Generalized Green-Schwarz mechanism in F-theory*, *Phys. Lett. B* **388** (1996) 45 [[hep-th/9606008](#)] [[INSPIRE](#)].
- [68] V. Kumar, D.S. Park and W. Taylor, *6D supergravity without tensor multiplets*, *JHEP* **04** (2011) 080 [[arXiv:1011.0726](#)] [[INSPIRE](#)].
- [69] D.R. Morrison and W. Taylor, *Matter and singularities*, *JHEP* **01** (2012) 022 [[arXiv:1106.3563](#)] [[INSPIRE](#)].
- [70] P.S. Aspinwall and D.R. Morrison, *Point-like instantons on K3 orbifolds*, *Nucl. Phys. B* **503** (1997) 533 [[hep-th/9705104](#)] [[INSPIRE](#)].
- [71] P.S. Aspinwall and R.Y. Donagi, *The heterotic string, the tangent bundle and derived categories*, *Adv. Theor. Math. Phys.* **2** (1998) 1041 [[hep-th/9806094](#)] [[INSPIRE](#)].
- [72] D. Huybrechts and M. Lehn, *The geometry of moduli spaces of sheaves*, Cambridge University Press, Cambridge U.K. (2010).
- [73] G. Ellingsrud and L. Gottsche, *Variation of moduli spaces and Donaldson invariants under change of polarization*, *J. reine angew. Math.* **467** (1995) 1 [[alg-geom/9410005](#)].
- [74] Z. Qin, *Birational properties of moduli spaces of stable locally free rank-2 sheaves on algebraic surfaces*, *Manuscripta Math.* **72** (1991) 163.
- [75] W.P. Li and Z. Qin, *Donaldson-Thomas invariants of certain Calabi-Yau 3-folds*, [arXiv:1002.4080](#).
- [76] W.P. Li and Z. Qin, *Stable rank-2 bundles on Calabi-Yau manifolds*, *Internat. J. Math.* **14** (2003) 1097.
- [77] R. Friedman and Z. Qin, *Flips of moduli spaces and transition formulas for Donaldson polynomial invariants of rational surfaces*, *Commun. Anal. Geom.* **3** (1995) 11 [[alg-geom/9410007](#)].
- [78] L.B. Anderson, J. Gray and B.A. Ovrut, *Transitions in the web of heterotic vacua*, *Fortsch. Phys.* **59** (2011) 327 [[arXiv:1012.3179](#)] [[INSPIRE](#)].
- [79] S.K. Donaldson and R.P. Thomas, *Gauge theory in higher dimensions*, in *The geometric universe*, Oxford Univ. Press, Oxford U.K. (1998), pg. 31.
- [80] G. Curio and R.Y. Donagi, *Moduli in $N = 1$ heterotic/F-theory duality*, *Nucl. Phys. B* **518** (1998) 603 [[hep-th/9801057](#)] [[INSPIRE](#)].
- [81] V. Braun and D.R. Morrison, *F-theory on genus-one fibrations*, [arXiv:1401.7844](#) [[INSPIRE](#)].
- [82] D.R. Morrison and W. Taylor, *Sections, multisections and U(1) fields in F-theory*, [arXiv:1404.1527](#) [[INSPIRE](#)].
- [83] S. Katz, D.R. Morrison, S. Schäfer-Nameki and J. Sully, *Tate's algorithm and F-theory*, *JHEP* **08** (2011) 094 [[arXiv:1106.3854](#)] [[INSPIRE](#)].
- [84] H.C. Pinkham, *Simple elliptic singularities, Del Pezzo surfaces and Cremona transformations*, *Proc. Sympos. Pure Math.* **30** (1977) 69, Amer. Math. Soc., U.S.A. (1977).

- [85] U. Persson, *Configurations of Kodaira fibers on rational elliptic surfaces*, *Math. Z.* **205** (1990) 1.
- [86] R. Miranda, *Persson's list of singular fibers for a rational elliptic surface*, *Math. Z.* **205** (1990) 191.
- [87] W. Fulton, *Introduction to toric varieties*, *Ann. Math. Study* **131**, Princeton University Press, Princeton U.S.A. (1993).
- [88] J. Knapp and M. Kreuzer, *Toric methods in F-theory model building*, *Adv. High Energy Phys.* **2011** (2011) 513436 [[arXiv:1103.3358](#)] [[INSPIRE](#)].
- [89] P. Candelas, D.-E. Diaconescu, B. Florea, D.R. Morrison and G. Rajesh, *Codimension three bundle singularities in F-theory*, *JHEP* **06** (2002) 014 [[hep-th/0009228](#)] [[INSPIRE](#)].
- [90] A. Grassi and D.R. Morrison, *Group representations and the Euler characteristic of elliptically fibered Calabi-Yau threefolds*, *J. Alg. Geom.* **12** (2003) 321 [[math/0005196](#)] [[INSPIRE](#)].
- [91] V. Braun, T.W. Grimm and J. Keitel, *Geometric engineering in toric F-theory and GUTs with U(1) gauge factors*, *JHEP* **12** (2013) 069 [[arXiv:1306.0577](#)] [[INSPIRE](#)].
- [92] M.R. Douglas, R.L. Karp, S. Lukic and R. Reinbacher, *Numerical solution to the hermitian Yang-Mills equation on the Fermat quintic*, *JHEP* **12** (2007) 083 [[hep-th/0606261](#)] [[INSPIRE](#)].
- [93] L.B. Anderson, V. Braun, R.L. Karp and B.A. Ovrut, *Numerical Hermitian Yang-Mills connections and vector bundle stability in heterotic theories*, *JHEP* **06** (2010) 107 [[arXiv:1004.4399](#)] [[INSPIRE](#)].
- [94] L.B. Anderson, V. Braun and B.A. Ovrut, *Numerical Hermitian Yang-Mills connections and Kähler cone substructure*, *JHEP* **01** (2012) 014 [[arXiv:1103.3041](#)] [[INSPIRE](#)].
- [95] W.P. Li and Z. Qin, *Donaldson-Thomas invariants of certain Calabi-Yau 3-folds*, [arXiv:1002.4080](#).
- [96] W.P. Li and Z. Qin, *Stable rank-2 bundles on Calabi-Yau manifolds*, *Internat. J. Math.* **14** (2003) 1097.
- [97] M. Lübke and A. Teleman, *The Kobayashi-Hitchin correspondence*, World Scientific, Singapore (1995).
- [98] R. Hartshorne, *Algebraic geometry*, Graduate Texts in Mathematics, Springer, Germany (1977).
- [99] L.B. Anderson, *Heterotic and M-theory compactifications for string phenomenology*, [arXiv:0808.3621](#) [[INSPIRE](#)].
- [100] R. Friedman and J. Morgan, *Smooth four-manifolds and complex surfaces*, Springer, New York U.S.A. (1994).
- [101] L.B. Anderson, J.J. Heckman and S. Katz, *T-branes and geometry*, *JHEP* **05** (2014) 080 [[arXiv:1310.1931](#)] [[INSPIRE](#)].
- [102] R. Donagi, Y.-H. He, B.A. Ovrut and R. Reinbacher, *The particle spectrum of heterotic compactifications*, *JHEP* **12** (2004) 054 [[hep-th/0405014](#)] [[INSPIRE](#)].
- [103] G. Curio, *Moduli restriction and chiral matter in heterotic string compactifications*, *JHEP* **01** (2012) 015 [[arXiv:1110.6315](#)] [[INSPIRE](#)].

- [104] S. Guerra, *Spectral cover construction for associated bundles*, Ph.D. dissertation, <http://repository.upenn.edu/dissertations/AAI3260911>, University of Pennsylvania, U.S.A. January 1 2007.
- [105] L.B. Anderson, J. Gray, A. Lukas and B. Ovrut, *The edge of supersymmetry: stability walls in heterotic theory*, *Phys. Lett. B* **677** (2009) 190 [[arXiv:0903.5088](#)] [[INSPIRE](#)].
- [106] L.B. Anderson, J. Gray, A. Lukas and B. Ovrut, *Stability walls in heterotic theories*, *JHEP* **09** (2009) 026 [[arXiv:0905.1748](#)] [[INSPIRE](#)].
- [107] M. Atiyah, *Vector bundles over an elliptic curve*, *Proc. London Math. Soc.* **7** (1957) 414.
- [108] J. Distler and B.R. Greene, *Aspects of (2,0) string compactifications*, *Nucl. Phys. B* **304** (1988) 1 [[INSPIRE](#)].
- [109] O. Okonek, M. Schneider and H. Spindler, *Vector bundles on complex projective spaces*, Birkhauser, Boston U.S.A. (1980).
- [110] M. Bershadsky, T.M. Chiang, B.R. Greene, A. Johansen and C.I. Lazaroiu, *F-theory and linear σ -models*, *Nucl. Phys. B* **527** (1998) 531 [[hep-th/9712023](#)] [[INSPIRE](#)].
- [111] R. Donagi, S. Katz and E. Sharpe, *Spectra of D-branes with Higgs vevs*, *Adv. Theor. Math. Phys.* **8** (2005) 813 [[hep-th/0309270](#)] [[INSPIRE](#)].
- [112] S. Cecotti, C. Cordova, J.J. Heckman and C. Vafa, *T-branes and monodromy*, *JHEP* **07** (2011) 030 [[arXiv:1010.5780](#)] [[INSPIRE](#)].
- [113] R. Donagi and M. Wijnholt, *Gluing branes, I*, *JHEP* **05** (2013) 068 [[arXiv:1104.2610](#)] [[INSPIRE](#)].
- [114] R. Donagi and M. Wijnholt, *Gluing branes, II: flavour physics and string duality*, *JHEP* **05** (2013) 092 [[arXiv:1112.4854](#)] [[INSPIRE](#)].
- [115] J. Marsano, N. Saulina and S. Schäfer-Nameki, *Global gluing and G-flux*, *JHEP* **08** (2013) 001 [[arXiv:1211.1097](#)] [[INSPIRE](#)].
- [116] A.P. Braun and T. Watari, *On singular fibres in F-theory*, *JHEP* **07** (2013) 031 [[arXiv:1301.5814](#)] [[INSPIRE](#)].
- [117] R. Donagi, B.A. Ovrut and D. Waldram, *Moduli spaces of five-branes on elliptic Calabi-Yau threefolds*, *JHEP* **11** (1999) 030 [[hep-th/9904054](#)] [[INSPIRE](#)].
- [118] B.A. Ovrut, T. Pantev and J. Park, *Small instanton transitions in heterotic M-theory*, *JHEP* **05** (2000) 045 [[hep-th/0001133](#)] [[INSPIRE](#)].
- [119] E. Buchbinder, R. Donagi and B.A. Ovrut, *Vector bundle moduli and small instanton transitions*, *JHEP* **06** (2002) 054 [[hep-th/0202084](#)] [[INSPIRE](#)].
- [120] T.W. Grimm, T.-W. Ha, A. Klemm and D. Klevers, *Computing brane and flux superpotentials in F-theory compactifications*, *JHEP* **04** (2010) 015 [[arXiv:0909.2025](#)] [[INSPIRE](#)].
- [121] T.W. Grimm, T.-W. Ha, A. Klemm and D. Klevers, *Five-brane superpotentials and heterotic/F-theory duality*, *Nucl. Phys. B* **838** (2010) 458 [[arXiv:0912.3250](#)] [[INSPIRE](#)].
- [122] M. Bies, C. Mayrhofer, C. Pehle and T. Weigand, *Chow groups, Deligne cohomology and massless matter in F-theory*, [arXiv:1402.5144](#) [[INSPIRE](#)].
- [123] N.C. Bizet, A. Klemm and D.V. Lopes, *Landscaping with fluxes and the E_8 Yukawa point in F-theory*, [arXiv:1404.7645](#) [[INSPIRE](#)].

- [124] B. Andreas and G. Curio, *On discrete twist and four flux in $N = 1$ heterotic/ F -theory compactifications*, *Adv. Theor. Math. Phys.* **3** (1999) 1325 [[hep-th/9908193](#)] [[INSPIRE](#)].
- [125] P. Candelas, D.-E. Diaconescu, B. Florea, D.R. Morrison and G. Rajesh, *Codimension three bundle singularities in F -theory*, *JHEP* **06** (2002) 014 [[hep-th/0009228](#)] [[INSPIRE](#)].
- [126] R. Donagi and M. Wijnholt, *Model building with F -theory*, *Adv. Theor. Math. Phys.* **15** (2011) 1237 [[arXiv:0802.2969](#)] [[INSPIRE](#)].
- [127] C. Beasley, J.J. Heckman and C. Vafa, *GUTs and exceptional branes in F -theory — I*, *JHEP* **01** (2009) 058 [[arXiv:0802.3391](#)] [[INSPIRE](#)].
- [128] A.P. Braun, A. Collinucci and R. Valandro, *G -flux in F -theory and algebraic cycles*, *Nucl. Phys.* **B 856** (2012) 129 [[arXiv:1107.5337](#)] [[INSPIRE](#)].
- [129] J. Marsano and S. Schäfer-Nameki, *Yukawas, G -flux and spectral covers from resolved Calabi-Yau's*, *JHEP* **11** (2011) 098 [[arXiv:1108.1794](#)] [[INSPIRE](#)].
- [130] S. Krause, C. Mayrhofer and T. Weigand, *G_4 flux, chiral matter and singularity resolution in F -theory compactifications*, *Nucl. Phys.* **B 858** (2012) 1 [[arXiv:1109.3454](#)] [[INSPIRE](#)].
- [131] T.W. Grimm and H. Hayashi, *F -theory fluxes, chirality and Chern-Simons theories*, *JHEP* **03** (2012) 027 [[arXiv:1111.1232](#)] [[INSPIRE](#)].
- [132] H. Hayashi, T. Kawano, R. Tatar and T. Watari, *Codimension-3 singularities and Yukawa couplings in F -theory*, *Nucl. Phys.* **B 823** (2009) 47 [[arXiv:0901.4941](#)] [[INSPIRE](#)].
- [133] A. Collinucci and R. Savelli, *On flux quantization in F -theory*, *JHEP* **02** (2012) 015 [[arXiv:1011.6388](#)] [[INSPIRE](#)].
- [134] A. Collinucci and R. Savelli, *On flux quantization in F -theory II: unitary and symplectic gauge groups*, *JHEP* **08** (2012) 094 [[arXiv:1203.4542](#)] [[INSPIRE](#)].
- [135] R. Donagi and E. Markman, *Spectral curves, algebraically completely integrable Hamiltonian systems and moduli of bundles*, [alg-geom/9507017](#) [[INSPIRE](#)].
- [136] R. Donagi, L. Ein and R. Lazarsfeld, *Nilpotent cones and sheaves on $K3$ surfaces*, *Contemp. Math.* **207** (1997) 51.
- [137] P. Berglund and P. Mayr, *Heterotic string/ F -theory duality from mirror symmetry*, *Adv. Theor. Math. Phys.* **2** (1999) 1307 [[hep-th/9811217](#)] [[INSPIRE](#)].
- [138] R. Donagi and M. Wijnholt, *Higgs bundles and UV completion in F -theory*, *Commun. Math. Phys.* **326** (2014) 287 [[arXiv:0904.1218](#)] [[INSPIRE](#)].
- [139] L.B. Anderson, J. Gray, A. Lukas and B. Ovrut, *Stabilizing the complex structure in heterotic Calabi-Yau vacua*, *JHEP* **02** (2011) 088 [[arXiv:1010.0255](#)] [[INSPIRE](#)].
- [140] L.B. Anderson, J. Gray, A. Lukas and B. Ovrut, *The Atiyah class and complex structure stabilization in heterotic Calabi-Yau compactifications*, *JHEP* **10** (2011) 032 [[arXiv:1107.5076](#)] [[INSPIRE](#)].
- [141] L.B. Anderson, J. Gray, A. Lukas and B. Ovrut, *Vacuum varieties, holomorphic bundles and complex structure stabilization in heterotic theories*, *JHEP* **07** (2013) 017 [[arXiv:1304.2704](#)] [[INSPIRE](#)].
- [142] G. Curio, *Complete bundle moduli reduction in heterotic string compactifications*, *JHEP* **05** (2012) 075 [[arXiv:1203.4961](#)] [[INSPIRE](#)].

- [143] K.S. Narain, *New heterotic string theories in uncompactified dimensions < 10* , *Phys. Lett. B* **169** (1986) 41 [[INSPIRE](#)].
- [144] P.H. Ginsparg, *Comment on toroidal compactification of heterotic superstrings*, *Phys. Rev. D* **35** (1987) 648 [[INSPIRE](#)].
- [145] E. Witten, *Small instantons in string theory*, *Nucl. Phys. B* **460** (1996) 541 [[hep-th/9511030](#)] [[INSPIRE](#)].
- [146] P.S. Aspinwall and M. Gross, *The SO(32) heterotic string on a K3 surface*, *Phys. Lett. B* **387** (1996) 735 [[hep-th/9605131](#)] [[INSPIRE](#)].
- [147] K. Mohri, *F-theory vacua in four-dimensions and toric threefolds*, *Int. J. Mod. Phys. A* **14** (1999) 845 [[hep-th/9701147](#)] [[INSPIRE](#)].
- [148] L.B. Anderson and W. Taylor, *Geometric constraints in dual F-theory and heterotic string compactifications*, [arXiv:1405.2074](#) [[INSPIRE](#)].
- [149] J.J. Heckman, *Particle physics implications of F-theory*, *Ann. Rev. Nucl. Part. Sci.* **60** (2010) 237 [[arXiv:1001.0577](#)] [[INSPIRE](#)].
- [150] T. Weigand, *Lectures on F-theory compactifications and model building*, *Class. Quant. Grav.* **27** (2010) 214004 [[arXiv:1009.3497](#)] [[INSPIRE](#)].
- [151] A. Maharana and E. Palti, *Models of particle physics from Type IIB string theory and F-theory: a review*, *Int. J. Mod. Phys. A* **28** (2013) 1330005 [[arXiv:1212.0555](#)] [[INSPIRE](#)].
- [152] J. Marsano, N. Saulina and S. Schäfer-Nameki, *F-theory compactifications for supersymmetric GUTs*, *JHEP* **08** (2009) 030 [[arXiv:0904.3932](#)] [[INSPIRE](#)].
- [153] J. Marsano, N. Saulina and S. Schäfer-Nameki, *Monodromies, fluxes and compact three-generation F-theory GUTs*, *JHEP* **08** (2009) 046 [[arXiv:0906.4672](#)] [[INSPIRE](#)].
- [154] R. Blumenhagen, T.W. Grimm, B. Jurke and T. Weigand, *Global F-theory GUTs*, *Nucl. Phys. B* **829** (2010) 325 [[arXiv:0908.1784](#)] [[INSPIRE](#)].
- [155] T.W. Grimm, S. Krause and T. Weigand, *F-theory GUT vacua on compact Calabi-Yau fourfolds*, *JHEP* **07** (2010) 037 [[arXiv:0912.3524](#)] [[INSPIRE](#)].
- [156] M. Cvetič, I. Garcia-Etxebarria and J. Halverson, *Global F-theory models: instantons and gauge dynamics*, *JHEP* **01** (2011) 073 [[arXiv:1003.5337](#)] [[INSPIRE](#)].
- [157] C.-M. Chen, J. Knapp, M. Kreuzer and C. Mayrhofer, *Global SO(10) F-theory GUTs*, *JHEP* **10** (2010) 057 [[arXiv:1005.5735](#)] [[INSPIRE](#)].
- [158] J. Marsano, H. Clemens, T. Pantev, S. Raby and H.-H. Tseng, *A global SU(5) F-theory model with Wilson line breaking*, *JHEP* **01** (2013) 150 [[arXiv:1206.6132](#)] [[INSPIRE](#)].
- [159] T.W. Grimm and T. Weigand, *On Abelian gauge symmetries and proton decay in global F-theory GUTs*, *Phys. Rev. D* **82** (2010) 086009 [[arXiv:1006.0226](#)] [[INSPIRE](#)].
- [160] T.W. Grimm, M. Kerstan, E. Palti and T. Weigand, *Massive Abelian gauge symmetries and fluxes in F-theory*, *JHEP* **12** (2011) 004 [[arXiv:1107.3842](#)] [[INSPIRE](#)].
- [161] D.S. Park and W. Taylor, *Constraints on 6D supergravity theories with Abelian gauge symmetry*, *JHEP* **01** (2012) 141 [[arXiv:1110.5916](#)] [[INSPIRE](#)].
- [162] D.S. Park, *Anomaly equations and intersection theory*, *JHEP* **01** (2012) 093 [[arXiv:1111.2351](#)] [[INSPIRE](#)].

- [163] D.R. Morrison and D.S. Park, *F-theory and the Mordell-Weil group of elliptically-fibered Calabi-Yau threefolds*, *JHEP* **10** (2012) 128 [[arXiv:1208.2695](#)] [[INSPIRE](#)].
- [164] M. Cvetič, T.W. Grimm and D. Klevers, *Anomaly cancellation and Abelian gauge symmetries in F-theory*, *JHEP* **02** (2013) 101 [[arXiv:1210.6034](#)] [[INSPIRE](#)].
- [165] C. Mayrhofer, E. Palti and T. Weigand, *U(1) symmetries in F-theory GUTs with multiple sections*, *JHEP* **03** (2013) 098 [[arXiv:1211.6742](#)] [[INSPIRE](#)].
- [166] V. Braun, T.W. Grimm and J. Keitel, *New global F-theory GUTs with U(1) symmetries*, *JHEP* **09** (2013) 154 [[arXiv:1302.1854](#)] [[INSPIRE](#)].
- [167] J. Borchmann, C. Mayrhofer, E. Palti and T. Weigand, *Elliptic fibrations for $SU(5) \times U(1) \times U(1)$ F-theory vacua*, *Phys. Rev. D* **88** (2013) 046005 [[arXiv:1303.5054](#)] [[INSPIRE](#)].
- [168] M. Cvetič, D. Klevers and H. Piragua, *F-theory compactifications with multiple U(1)-factors: constructing elliptic fibrations with rational sections*, *JHEP* **06** (2013) 067 [[arXiv:1303.6970](#)] [[INSPIRE](#)].
- [169] V. Braun, T.W. Grimm and J. Keitel, *Geometric engineering in toric F-theory and GUTs with U(1) gauge factors*, *JHEP* **12** (2013) 069 [[arXiv:1306.0577](#)] [[INSPIRE](#)].
- [170] M. Cvetič, A. Grassi, D. Klevers and H. Piragua, *Chiral four-dimensional F-theory compactifications with SU(5) and multiple U(1)-factors*, *JHEP* **04** (2014) 010 [[arXiv:1306.3987](#)] [[INSPIRE](#)].
- [171] J. Borchmann, C. Mayrhofer, E. Palti and T. Weigand, *SU(5) tops with multiple U(1)s in F-theory*, *Nucl. Phys. B* **882** (2014) 1 [[arXiv:1307.2902](#)] [[INSPIRE](#)].
- [172] M. Cvetič, D. Klevers, H. Piragua and P. Song, *Elliptic fibrations with rank three Mordell-Weil group: F-theory with $U(1) \times U(1) \times U(1)$ gauge symmetry*, *JHEP* **03** (2014) 021 [[arXiv:1310.0463](#)] [[INSPIRE](#)].
- [173] A.P. Braun, A. Collinucci and R. Valandro, *The fate of U(1)'s at strong coupling in F-theory*, *JHEP* **07** (2014) 028 [[arXiv:1402.4054](#)] [[INSPIRE](#)].
- [174] M.R. Douglas, D.S. Park and C. Schnell, *The Cremmer-Scherk mechanism in F-theory compactifications on K3 manifolds*, *JHEP* **05** (2014) 135 [[arXiv:1403.1595](#)] [[INSPIRE](#)].
- [175] J.J. Heckman, H. Lin and S.-T. Yau, *Building blocks for generalized heterotic/F-theory duality*, [arXiv:1311.6477](#) [[INSPIRE](#)].