NUCLEAR ENGINEERING   **MASSACHUSETTS INSTITUTE OF TECHNOLOGY**

Choice of Discount Rate for Cost Levelization

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FOREWORD

The manuscript which follows is in the nature of a working paper. It deals with the mathematically and conceptually correct discount rate and procedure for the computation of a lifetime-levelized cost-of-product for enterprises which are funded by a mixture of debt and equity, and taxed according to representative US practices. It shows that methods of long standing contain a subtle but important, albeit generally small, error. Thus the paper is open to two interpretations: the first is a call for the use of a correct approach in calculations which are represented to be at a state-of-the-art level of complexity - which is also a simple matter when such computations are incorporated in computer programs; the second is the demonstration and quantification of what may be an acceptable degree of inaccuracy for simpler, more approximate back-of-the envelope estimates, and for more transparent pedogogical demonstrations of general principles.

Michael J. Driscoll
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Cambridge, MA, September 20, 1991

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INTRODUCTION

Computation of a levelized revenue expectation or requirement, or unit cost of product, is a central task of the engineering economist. Despite preference for the use of the tax-sheltered cost of corporate capital in authoritative texts such as Ref. [1], backed by detailed analyses, as in Refs. [2] - [4], we show here that there is an abundance of comparable choices, only one of which, the cost of composite capital, properly discounts the periodic cash flow of taxes. Although levelized costs usually differ only slightly with the choice of rate (if applied to properly weighted cash flows), this should be sufficient reason to prefer this particular discount rate in economic analyses.

NOTATION

I_i = capital investment made at the end of the ith year, $

C_i = annual expensed costs at the end of the ith year, $

D_i, D_i' = annual book and tax depreciation at the end of the ith year, $

V_i = undepreciated book investment remaining at the start of year i, $

T_i = annual income taxes paid at the end of year i, $

T_{iy} = annual income taxes paid at the end of year i for the virtual cash flow, \$

R_i = annual revenue requirement at the end of year i, \$

τ = income tax rate, %/100

r = weighted total rate of return = b + s, % per yr/100

b = weighted bond rate of return = $f_b\, r_b$, % per yr/100

s = weighted stock rate of return = $f_s\, r_s$, % per yr/100

r_b = bond interest rate, % per yr/100

r_s = stock rate of return, % per yr/100

f_b = bond fraction of investment

f_s = stock fraction of investment

x = rate of return to the stockholders if the investment is 100% composed of stock in the virtual cash flow, % per yr/100

y = total discount rate for the virtual cash flow, % per yr/100

w = interest rate which could be paid to the bondholders if the investment is 100% composed of bonds in the virtual cash flow, % per yr/100

α = ratio between weighted bond to total return rate for the virtual cash flow

n = economic lifetime, yr.

R_r = levelized revenue for the original cash flow pattern

R_y = levelized revenue for the virtual cash flow pattern

R_x = levelized revenue for the virtual cash flow pattern with $\alpha = 0$ (y = x)

R_w = levelized revenue for the virtual cash flow pattern with $\alpha = 1$ (y = w)

Note: The superscript symbol * identifies levelized cash flows.

THEORY OF LEVELIZED COSTS - PART I

The basic equations and nomenclature of Ref. [5], will be followed with some minor exceptions. The required revenue is the sum of costs, return on capital, depreciation and taxes:

$$R_i = C_i + r V_i + D_i + T_i \tag{1}$$

while taxes are some fraction of revenue less deductions, which include bond payments and depreciation which is not necessarily the same as book depreciation:

$$T_i = \tau \left(R_i - C_i - bV_i - D_i' \right) \tag{2}$$

where

$$V_i = \sum_{j=0}^{i-1} I_j - \sum_{j=1}^{i-1} D_j \tag{3}$$

$$\sum_{i=1}^{n} D_i = \sum_{i=1}^{n} D_i' = \sum_{i=0}^{n} I_i \tag{4}$$

$$r = b + s = f_b \, r_b + f_s \, r_s \tag{5}$$

and

$$f_b + f_s = 1 \tag{6}$$

Substituting Eq. (2) into Eq. (1),

$$R_i = C_i + \frac{1}{1-\tau}\left[(r - b\tau)V_i + D_i\right] - \frac{\tau}{1-\tau}D_i' \tag{7}$$

Let us add to Eq. (7):

$$-\frac{\alpha\tau}{1-\tau}D_i + \frac{\alpha\tau}{1-\tau}D_i$$

where α is a free parameter, then:

$$R_i = C_i + \frac{1-\alpha\tau}{1-\tau}\left[\frac{r-b\tau}{1-\alpha\tau}V_i + D_i\right] - \frac{\tau}{1-\tau}\left(D_i' - \alpha D_i\right)$$

(8)

Defining: $\quad y = \dfrac{r-b\tau}{1-\alpha\tau}$

(9)

and multiplying Eq. (8) by the factor $1/(1+y)^i$ and adding Eqs. (8) from $i = 1$ to $i = n$, one finds:

$$\sum_{i=1}^{n}\frac{R_i}{(1+y)^i} = \sum_{i=1}^{n}\frac{C_i}{(1+y)^i} + \frac{1-\alpha\tau}{1-\tau}\sum_{i=1}^{n}\frac{yV_i+D_i}{(1+y)^i} - \frac{\tau}{1-\tau}\sum_{i=1}^{n}\frac{D_i'-\alpha D_i}{(1+y)^i}$$

(10)

But, from Eqs. (3) and (4), one obtains the following relation (see Appendix I):

$$\sum_{i=1}^{n}\frac{yV_i+D_i}{(1+y)^i} = \sum_{i=0}^{n}\frac{I_i}{(1+y)^i} \quad ; \text{ for } y > -1$$

(11)

Substituting Eq. (11) into Eq. (10):

$$\sum_{i=1}^{n}\frac{R_i}{(1+y)^i} = \sum_{i=1}^{n}\frac{C_i}{(1+y)^i} + \frac{1-\alpha\tau}{1-\tau}\sum_{i=0}^{n}\frac{I_i}{(1+y)^i} - \frac{\tau}{1-\tau}\sum_{i=1}^{n}\frac{D_i'-\alpha D_i}{(1+y)^i}$$

(12)

Defining:

$$R_y = \left(\frac{\sum\limits_{i=1}^{n} \dfrac{R_i}{(1+y)^i}}{\sum\limits_{i=1}^{n} \dfrac{1}{(1+y)^i}} \right)$$

(13)

Then:

$$R_y = \frac{\sum\limits_{i=1}^{n} \dfrac{C_i}{(1+y)^i}}{\sum\limits_{i=1}^{n} \dfrac{1}{(1+y)^i}} + \frac{1-\alpha\tau}{1-\tau} \frac{\sum\limits_{i=0}^{n} \dfrac{I_i}{(1+y)^i}}{\sum\limits_{i=1}^{n} \dfrac{1}{(1+y)^i}} - \frac{\tau}{1-\tau} \frac{\sum\limits_{i=1}^{n} \dfrac{D_i' - \alpha D_i}{(1+y)^i}}{\sum\limits_{i=1}^{n} \dfrac{1}{(1+y)^i}}$$

(14)

R_y is the levelized revenue with the discount rate equal to y. Then R_r, R_x and R_w are found with y = r, y = x and y = w (or α = b/r, α = 0 and α = 1), respectively:

(a) $y = r = b + s \rightarrow \alpha = b/r$

(15)

$$R_r = \frac{\sum_{i=1}^{n} \dfrac{C_i}{(1+r)^i}}{\sum_{i=1}^{n} \dfrac{1}{(1+r)^i}} + \frac{1-\dfrac{b}{r}\tau}{1-\tau}\,\frac{\sum_{i=0}^{n} \dfrac{I_i}{(1+r)^i}}{\sum_{i=1}^{n} \dfrac{1}{(1+r)^i}} - \frac{\tau}{1-\tau}\,\frac{\sum_{i=1}^{n} \dfrac{D_i' - \dfrac{b}{r}D_i}{(1+r)^i}}{\sum_{i=1}^{n} \dfrac{1}{(1+r)^i}}$$

(16)

(b) $\quad y = x = r - b\tau \;\rightarrow\; \alpha = 0$

(17)

$$R_x = \frac{\sum_{i=1}^{n} \dfrac{C_i}{(1+x)^i}}{\sum_{i=1}^{n} \dfrac{1}{(1+x)^i}} + \frac{1}{1-\tau}\,\frac{\sum_{i=0}^{n} \dfrac{I_i}{(1+x)^i}}{\sum_{i=1}^{n} \dfrac{1}{(1+x)^i}} - \frac{\tau}{1-\tau}\,\frac{\sum_{i=1}^{n} \dfrac{D_i'}{(1+x)^i}}{\sum_{i=1}^{n} \dfrac{1}{(1+x)^i}}$$

(18)

(c) $\quad y = w = \dfrac{r - b\tau}{1-\tau} = \dfrac{x}{(1-\tau)} \;\rightarrow\; \alpha = 1$

(19)

$$R_w = \frac{\sum_{i=1}^{n} \dfrac{C_i}{(1+w)^i}}{\sum_{i=1}^{n} \dfrac{1}{(1+w)^i}} + \frac{\sum_{i=0}^{n} \dfrac{I_i}{(1+w)^i}}{\sum_{i=1}^{n} \dfrac{1}{(1+w)^i}} - \frac{\tau}{1-\tau}\,\frac{\sum_{i=1}^{n} \dfrac{D_i' - D_i}{(1+w)^i}}{\sum_{i=1}^{n} \dfrac{1}{(1+w)^i}}$$

(20)

Expression (14) also holds for y > -1 or, equivalently, for $\alpha < 1/\tau$ (with $\tau > 0$) or $\alpha >$

$(r - b\tau + 1)/\tau$ (with $\tau > 0$). This can be clearly seen in Figure 1. The condition $\tau = 0$ implies

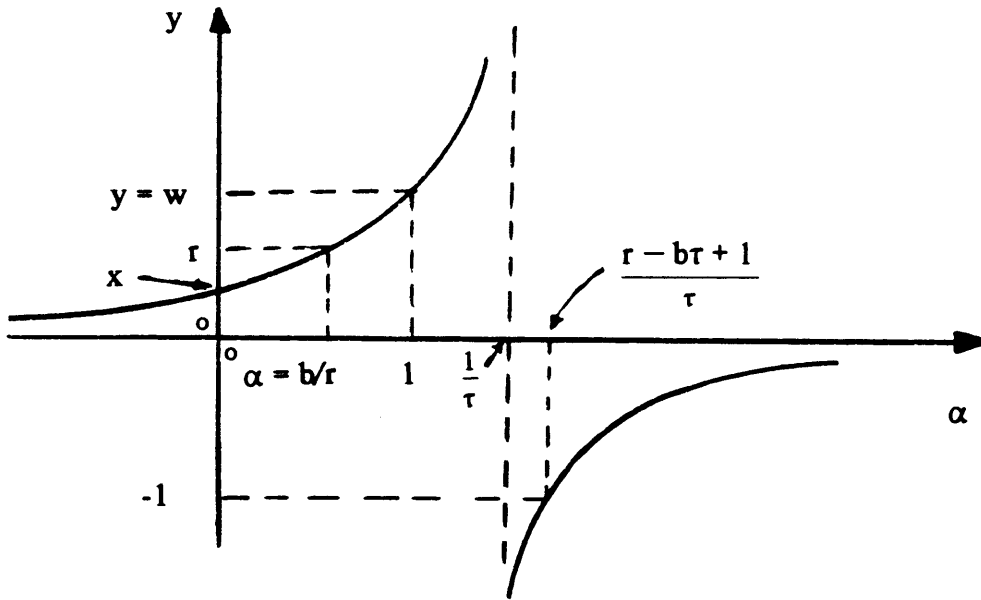that y has only one value: $y = r$.

Figure 1. Variation of y with α (for $0 < \tau < 1$).

Consequently, there are an infinite number of levelized revenues R_y; some of the more

interesting R_y are as follows:

$\alpha = b/r$ $\rightarrow$ $y = r$ $\rightarrow$ $R_y = R_r$

$\alpha = 0$ $\rightarrow$ $y = x = r - b\tau$ $\rightarrow$ $R_y = R_x$

$\alpha = 1$ $\rightarrow$ $y = w = \dfrac{r - b\tau}{1 - \tau} = \dfrac{x}{1 - \tau}$ $\rightarrow$ $R_y = R_w$

$\alpha \rightarrow -\infty$ $\rightarrow$ $y \rightarrow 0$ $\rightarrow$ $R_y \rightarrow \dfrac{1}{n} \sum_{i=1}^{n} R_i$

$$\alpha \rightarrow \frac{1}{\tau} \bigg|_{-} \qquad \rightarrow \quad y \rightarrow \infty \qquad\qquad \rightarrow \qquad R_y \rightarrow R_1$$

$$\alpha \rightarrow \frac{r - b\tau + 1}{\tau} \bigg|_{+} \rightarrow \quad y \rightarrow -1 \qquad\qquad \rightarrow \qquad R_y \rightarrow R_n$$

Other R_y could also be derived, such as those for $y = r_b$ or $y = r_s$, for example. But, the central question is now the following: which levelized revenue R_y should be used to compare similar economic alternatives?

The levelized cost R_y being sought is one that, if charged uniformly throughout the life of the plant, will just pay for all current expenses, taxes, return on and return of investment. The levelized cash flow that satisfies these assumptions is the following:

$$R_i^* = C_i^* + r V_i^* + D_i^* + T_i^* \tag{21}$$

$$T_i^* = \tau \left(R_i^* - C_i^* - b V_i^* - D_i^{'*} \right) \tag{22}$$

$$C_i^* = C_i \tag{23}$$

$$R_i^* = R_y \tag{24}$$

$$V_i^* = \sum_{j=0}^{i-1} I_j - \sum_{j=1}^{i-1} D_j^* \tag{25}$$

$$D_i'^* - \alpha D_i^* = D_i' - \alpha D_i \tag{26}$$

In this cash flow, there are 6n unknowns: R_i^*, C_i^*, V_i^*, D_i^*, T_i^*, and $D_i'^*$; and 6n equations (Eq. 21 to Eq. 26).

It must be proven now that:

$$\sum_{i=1}^{n} D_i^* = \sum_{i=1}^{n} D_i'^* = \sum_{i=0}^{n} I_i$$

Following the steps to obtain Eq. (8) leads to:

$$R_i^* = C_i^* + \frac{1-\alpha\tau}{1-\tau}\left[y\,V_i^* + D_i^*\right] - \frac{\tau}{1-\tau}\left(D_i'^* - \alpha D_i^*\right) \tag{27}$$

Subtracting Eq.(8) from Eq.(27), multiplying by the factor $1/(1+y)^i$ and adding equations from $i = 1$ to $i = n$, gives:

$$\sum_{i=1}^{n} \frac{R_i^* - R_i}{(1+y)^i} = \sum_{i=1}^{n} \frac{C_i^* - C_i}{(1+y)^i} + \frac{1-\alpha\tau}{1-\tau} \left[\sum_{i=1}^{n} \frac{yV_i^* + D_i^*}{(1+y)^i} - \sum_{i=1}^{n} \frac{yV_i + D_i}{(1+y)^i} \right]$$
$$- \frac{\tau}{1-\tau} \sum_{i=1}^{n} \frac{\left(D_i^{'*} - \alpha D_i^*\right) - \left(D_i^{'} - \alpha D_i\right)}{(1+y)^i} \tag{28}$$

From Eqs.(23), (24) and (26):

$$\sum_{i=1}^{n} \frac{yV_i^* + D_i^*}{(1+y)^i} = \sum_{i=1}^{n} \frac{yV_i + D_i}{(1+y)^i} \tag{29}$$

From Eq. (11):

$$\sum_{i=1}^{n} \frac{yV_i^* + D_i^*}{(1+y)^i} = \sum_{i=0}^{n} \frac{I_i}{(1+y)^i} \tag{30}$$

From Eqs. (30) and (25), it can be shown that (see Appendix II)

$$\sum_{i=1}^{n} D_i^* = \sum_{i=0}^{n} I_i \tag{31}$$

From Eqs. (31) and (26) and knowing that $\displaystyle\sum_{i=1}^{n} D_i = \sum_{i=1}^{n} D_i' = \sum_{i=0}^{n} I_i$, it follows that

$$\sum_{i=1}^{n} D_i'^* = \sum_{i=0}^{n} I_i \tag{32}$$

The "levelized cash flow" defined as having a constant revenue R_y, pays for all current expenses C_i and return on and return of investment. On the other hand, it is impossible to have $T_i^* = T_i$ without disobeying other equations, such as (32) for example.

For $\alpha = 0$, from Eq. (26), it follows that $D_i'^* = D_i'$. This is the reason why the approach (choice of x and cash flow weighting) embodied in Eq. (18) is said to be the one which pays for all current expenses C_i, return on and of investment, while keeping $D_i'^* = D_i'$.

But, if it is impossible to keep $T_i^* = T_i$, at least one can look for an R_y such that

$$\sum_{i=1}^{n} \frac{T_i^*}{(1+y)^i} = \sum_{i=1}^{n} \frac{T_i}{(1+y)^i}$$

From Eqs. (22) and (2):

$$T_i^* - T_i = \tau\left(R_i^* - R_i\right) - \tau\left(C_i^* - C_i\right) - \tau\left(bV_i^* - bV_i\right) - \tau\left(D_i'^* - D_i'\right) \tag{33}$$

Multiplying by $1/(1 + y)^i$ and adding terms:

$$\sum_{i=1}^{n} \frac{T_i^* - T_i}{(1+y)^i} = \tau \sum_{i=1}^{n} \frac{R_i^* - R_i}{(1+y)^i} - \tau \sum_{i=1}^{n} \frac{C_i^* - C_i}{(1+y)^i} - \tau \sum_{i=1}^{n} \frac{\left(bV_i^* - bV_i\right) + \left(D_i'^* - D_i'\right)}{(1+y)^i} \tag{34}$$

Using Eqs. (23), (24) and (26)

$$\sum_{i=1}^{n} \frac{T_i^* - T_i}{(1+y)^i} = -\tau \sum_{i=1}^{n} \frac{bV_i^* - bV_i + \alpha D_i^* - \alpha D_i}{(1+y)^i} \tag{35}$$

$$\sum_{i=1}^{n} \frac{T_i^* - T_i}{(1+y)^i} = -\alpha\tau \left[\sum_{i=1}^{n} \frac{\dfrac{b}{\alpha} V_i^* + D_i^*}{(1+y)^i} - \sum_{i=1}^{n} \frac{\dfrac{b}{\alpha} V_i + D_i}{(1+y)^i} \right] \tag{36}$$

in order to have: $\displaystyle\sum_{i=1}^{n} \frac{T_i^* - T_i}{(1+y)^i} = 0,$

then: either   (a) $\tau = 0$

    or   (b) $\alpha = 0$ and $b = 0$

    or   (c) $\alpha = 1$ and $s = 0$

    or   (d) $y = \dfrac{b}{\alpha}$ $\rightarrow$ $y = r$ and $\alpha = \dfrac{b}{r}$, (from Eq. 9).

(for $\alpha \rightarrow -\infty$ ($y \rightarrow 0$) this relation is also satisfied, but the practical range of α is $0 \le \alpha \le 1$ as will be shown in Part II of this article).

Note that alternative (d) always includes the other alternatives (a), (b) and (c).

One must conclude then that the levelized cost (revenue) R_r with $y = r$ should be used to compare similar alternatives.

To find all unknowns of each levelized cash flow given by Eqs. (21) to (26), it is convenient to proceed in the following way:

$$V_1^* = I_0 \tag{37}$$

From Eqs. (27) and (26):

$$D_i^* = \frac{1-\tau}{1-\alpha\tau}\left[R_i^* - C_i^* + \frac{\tau}{1-\tau}\left(D_i' - \alpha D_i\right)\right] - yV_i^* \tag{38}$$

From Eq. (26):

$$D_i'^* = D_i' + \alpha\left(D_i^* - D_i\right) \tag{39}$$

Using Eqs. (3), (37), (38) and (39), each levelized cash flow can be developed in turn, as defined for each value of y (or α). The example in Table I shows that all levelized cash flows for y = r, y = x and y = w, keep $C_i^* = C_i$ and that for y = x, $D_i^{'*} = D_i'$; but only for y = r does

$$\sum_{i=1}^{n} \frac{T_i^*}{(1+y)^i} = \sum_{i=1}^{n} \frac{T_i}{(1+y)^i}.$$

While the differences in R_y are small, case to case, other examples can be constructed in which larger discrepancies are displayed. Hence we believe that the use of y = r is not a trivial requirement.

THEORY OF LEVELIZED COSTS - PART II

The Meaning of R_y

The true meaning of R_y is the following: R_y is the levelized revenue of a virtual cash flow given by Equations (40) and (41), which is very similar to the original cash flow given by Equations (1) and (2). In this virtual cash flow, the total rate of return of the project is y instead of r and the weighted bond rate of return is αy instead of b. Comparing both cash flows, it is seen that R_i, C_i, V_i, D_i, D_i', and I_i are the same, while differences in income taxes are given by Eq. (42).

$$R_i = C_i + y V_i + D_i + T_{iy} \tag{40}$$

$$T_{iy} = \tau\left(R_i - C_i - \alpha y V_i - D_i'\right) \tag{41}$$

$$T_{iy} - T_i = \tau\left(b - \alpha y\right) V_i = \left(r - y\right) V_i \tag{42}$$

TABLE I. Numerical Example Showing an Original Cash Flow and Some Levelized Cash Flows (for y=r, y=x and y=w).

	i	0	1	2	3	4
Actual	R_i	0	6.6	7.3	7.2	5.2
cash	C_i	0	2	3	3.5	4
flow	V_i	0	8	6.5	3.5	1
with	D_i	0	3.5	3	2.5	1
$r = 0.12$	T_i	0	0.14	0.52	0.78	0.08
$b = 0.04$	D_i'	0	4	3	2	1
$\tau = 0.5$	I_i	8	2	0	0	0
Levelized	$R_i^* = R_r$	0	6.631	6.631	6.631	6.631
cash	V_i^*	0	8	6.481	3.880	1.767
flow	D_i^*	0	3.519	2.601	2.113	1.767
for	T_i^*	0	0.153	0.253	0.553	0.652
$y = r$	$D_i'^*$	0	4.006	2.867	1.871	1.256
Levelized	$R_i^* = R_x$	0	6.623	6.623	6.623	6.623
cash	V_i^*	0	8	6.488	3.826	1.647
flow	D_i^*	0	3.511	2.663	2.179	1.647
for	T_i^*	0	0.152	0.182	0.485	0.779
$y = x$	$D_i'^*$	0	4	3	2	1

TABLE I. (CONTINUED) Numerical Example Showing an Original Cash Flow and Some Levelized Cash Flows (for y=r, y=x and y=w).

i	0	1	2	3	4
$R_i^* = R_w$	0	6.661	6.661	6.661	6.661
V_i^*	0	8	6.439	4.066	2.218
D_i^*	0	3.561	2.373	1.848	2.218
T_i^*	0	0.140	0.515	0.825	0.177
$D_i'^*$	0	4.061	2.373	1.348	2.218

Levelized cash flow for y = w (bracketed grouping of the above rows)

Notes

a) Note that in the levelized cash flows above: $C_i^* = C_i$ and $I_i^* = I_i$

b) Small differences in cash flow balances are due to roundoff.

c) Tax cash flows have present worths as follows:

	y = r	y = x	y = w
$\sum_{i=1}^{n} \dfrac{T_i}{(1+y)^i}$	1.146	1.198	0.968
$\sum_{i=1}^{n} \dfrac{T_i^*}{(1+y)^i}$	1.146	1.185	1.037

Note that when $\alpha = b/r$, $y = r$, and the original cash flow is obtained. Similarly, when $\alpha = 0$, $y = x$, and the following virtual cash flow is obtained:

$$R_i = C_i + x\,V_i + D_i + T_{ix} \tag{43}$$

$$T_{ix} = \tau\left(R_i - C_i - D_i'\right) \tag{44}$$

In the same way, when $\alpha = 1$, $y = w$, another virtual cash flow is obtained:

$$R_i = C_i + w\,V_i + D_i + T_{iw} \tag{45}$$

$$T_{iw} = \tau\left(R_i - C_i - w\,V_i - D_i'\right) \tag{46}$$

Table II repeats the original cash flow of Table I and gives its virtual cash flows for x ($\alpha = 0$) and for w ($\alpha = 1$). Note the consistency of the virtual cash flows. The similarity between the original and the virtual cash flows has caused great confusion in the choice of the proper discount rate for cost levelization, with many authors [1, 2, 3, 4] incorrectly favoring the use of the so called "effective discount rate x", instead of the total rate of return r, to calculate the levelized revenue of the original cash flow given by Eqs. (1) and (2).

Note that, consistent with this definition of the virtual cash flow, for each value of y, there can be associated a corresponding (and unique) levelized-virtual-cash flow by exchanging y for r, αy for b and T_{iy}^* for T_i^* in Eqs. (21), (22) and their derivations. Making these modifications to Eq. (36) and using Eqs. (11) and (30), it follows that:

$$\sum_{i=1}^{n} \frac{T_{iy}^*}{(1+y)^i} = \sum_{i=1}^{n} \frac{T_{iy}}{(1+y)^i} \tag{47}$$

Table II also gives some examples of levelized virtual cash flows. Note that Eq. (47) is always satisfied for all values of y. Observe again the internal coherence of the virtual cash-flow model.

TABLE II. Numerical Example of the Original Cash Flow of Table I and Some of its Virtual Cash Flows and Levelized Virtual Cash Flows.

i	0	1	2	3	4
R_i	0	6.6	7.3	7.2	5.2
C_i	0	2	3	3.5	4
V_i	0	8	6.5	3.5	1
D_i	0	3.5	3	2.5	1
D'_i	0	4	3	2	1
I_i	8	2	0	0	0
rV_i	0	0.96	0.78	0.42	0.12
T_i	0	0.14	0.52	0.78	0.08

Original cash flow with
r = 0.12
b = 0.04
τ = 0.5

i	0	1	2	3	4
xV_i	0	0.8	0.65	0.35	0.1
T_{ix}	0	0.30	0.65	0.85	0.10

Virtual cash flow for y = x

x = 0.10

(R_i, C_i, V_i, D_i, D'_i and I_i are the same as in the original cash flow above).

i	0	1	2	3	4
wV_i	0	1.6	1.3	0.7	0.2
T_{iw}	0	-0.5	0	0.5	0

Virtual cash flow for y = w

w = 0.20

(R_i, C_i, V_i, D_i, D'_i and I_i are the same as in the original cash flow above).

Note that $\sum_{i=1}^{n} T_{iw} = 0$

Note that $rV_i + T_i = x V_i + T_{ix} = wV_i + T_{iw}$

$$= y V_i + T_{iy}$$

TABLE II. (CONTINUED) Numerical Example of the Original Cash Flow of Table I and Some of its Virtual Cash Flows and Levelized Virtual Cash Flows.

Levelized	i	1	2	3	4
virtual	$R_i^* = R_x$	6.623	6.623	6.623	6.623
cash flow	V_i^*	8	6.488	3.826	1.647
for	D_i^*	3.511	2.663	2.179	1.647
$y = x$	$D_i^{'*}$	4	3	2	1
$x = 0.10$	xV_i^*	0.800	0.649	0.383	0.165
	T_{ix}^*	0.312	0.312	0.562	0.812
Levelized	$R_i^* = R_w$	6.661	6.661	6.661	6.661
virtual	V_i^*	8	6.439	4.066	2.218
cash flow	D_i^*	3.561	2.373	1.848	2.218
for	$D_i^{'*}$	4.061	2.373	1.348	2.218
$y = w$	wV_i^*	1.600	1.288	0.813	0.444
$w = 0.20$	T_{iw}^*	-0.500	0.000	0.500	0.000

Note: a) In the levelized cash flows above, $C_i^* = C_i$ and $I_i^* = I_i$, as in the original cash flow

b) Small differences in cash flow balances of this table are due to roundoff.

c) Tax cash flows have present worths as follows:

	$y = r$	$y = x$	$y = w$
$\sum_{i=1}^{n} \dfrac{T_{iy}}{(1+y)^i}$	1.146	1.517	-0.127
$\sum_{i=1}^{n} \dfrac{T_{iy}^*}{(1+y)^i}$	1.146	1.517	-0.127

It should then be repeated here that R_r, given by Eq. (16) is the levelized revenue associated with the original cash flow given by Eqs. (1) and (2). R_x, given by Eq. (18) is the levelized revenue associated with the virtual cash flow given by Eqs. (43) and (44). R_w, given by Eq. (20) is the levelized revenue which corresponds to the virtual cash flow when $y = w$ ($\alpha = 1$), given by Eqs. (45) and (46). Finally, for each value of y, R_y, given by Eq. (14) is the levelized revenue corresponding to the virtual cash flow defined by Eqs. (40) and (41).

Meaning of x and R_x

From Eqs. (43) and (44), R_x may be interpreted as being the levelized revenue of its corresponding virtual cash flow, for 0% (zero percent) participation of bondholders in the investments. In other words, x is the rate of return to the stockholders if the investment is 100% composed of stock.

Income taxes given by Eq. (48), which was derived from Eqs. (40) and (41), are maximum in this case, where $\alpha = 0$.

$$T_{iy} = \frac{\tau}{1-\tau}\left[\left(1-\alpha\right)y\,V_i + D_i - D_i'\right]$$

(48)

Meaning of w and R_w

From Eqs. (45) and (46), R_w may be interpreted as being the levelized revenue, for its corresponding virtual cash flow, for 100% participation of bondholders in the investments. In other words, w is the interest rate which could be paid to the bondholders if the investment was 100% composed of bonds. Income taxes given by Eq. (48) are minimum in this case, where $\alpha = 1$. In fact, the net income taxes paid throughout the life of the project would be equal to zero (see numerical example in Table II). Then, one would be tempted to interpret w as being

equivalent to the internal rate of return of the project before taxes, although this interpretation is correct only when $D_i \equiv D_i'$; for all $i = 1$ to n.

Range of significance of α and y

Since α may be interpreted as being the ratio between the weighted bond to total rate of return of the virtual cash flow, in order to have physical significance, the following relation must be satisfied:

$$0 \leq \alpha \leq 1 \tag{49}$$

Consequently, from Eqs. (9), (17), (19) and (49), (see Figure 1):

$$x \leq y \leq w \tag{50}$$

Relationship between r, y, x and w

As the original and the virtual cash flows were defined, r, x, and w are fixed parameters while y is variable. From Eqs. (9) and (17), y may be expressed as follows:

$$y = \frac{x}{1 - \alpha\tau} \tag{51}$$

y then increases from $y = x$, when $\alpha = 0$, to $y = w$, when $\alpha = 1$, passing through $y = r$, when $\alpha = b/r$ (see Figure 1).

When $y = r$ ($\alpha = b/r$), Eq. (51) transforms into Eq. (52)

$$r = x + b\tau \tag{52}$$

Some authors [3, 4] have used Eq. (52) as an argument in favor of the use of x instead of r to calculate the levelized cost of the original cash flow. They have argued that the $b\tau$ portion of the rate r can be deducted from income taxes with the difference x being the "effective after-tax

cost of money". In fact, it has been shown here that r and x are discount rates corresponding to different cash flows. From Eqs. (9) and (19), the following relationship may be derived:

$$w = y + \frac{\tau}{1 - \tau} \left(1 - \alpha \right) y \quad ,$$

(53)

which may be interpreted in the following way: for each value of y, the maximum rate of return w (which is fixed), of the set of appropriate virtual cash flows, is divided between the investors through the total rate of return y (bondholders receive the rate αy and stockholders receive the rate $(1 - \alpha)$ y), and the government, through the rate of return for "fictitious income", $\frac{\tau}{1 - \tau} \left(1 - \alpha \right) y$ (see Equation 48). Equation (53) clearly shows that y increases as α increases (and/or τ decreases!).

Some practical applications of x, r and w in economic analyses

(a) To find the levelized revenue when one is given the cash flows $\{I_i, C_i, D_i, D'_i, i = 1$ to n, with r, b and τ also known$\}$, Eq. (16) should be used. The original cash flow of the numerical example in Table I was constructed with the above vectors and parameters given, and using Eq. (3) to find V_i, Eq. (7) to find R_i and Eq. (2) to find T_i. The levelized cash flow corresponding to this original cash flow is also given in Table I for y = r with R_r being the levelized revenue.

(b) To maximize the rate of return r_s to the stockholders, given the cash flow: $\{R_i, I_i, C_i, D'_i, i = 1$ to n, with τ and r_b also known$\}$, use Eq. (12) with $\alpha = 0$ to find x and then Eq. (19) to find w. Compare the bond rate of return r_b with w through Eq. (54) (which was derived from Eqs. (5), (6) and (19).

$$r_s = \frac{1 - \tau}{1 - f_b} \left(w - f_b r_b \right)$$

(54)

If r_b < w, maximize the bond fraction f_b in the investments in order to maximize the rate of return r_s to the stockholders. If r_b = w, r_s does not depend on f_b. If r_b > w and if the project must be implemented anyway, minimize f_b to maximize r_s. Figure 2 shows these relationships.
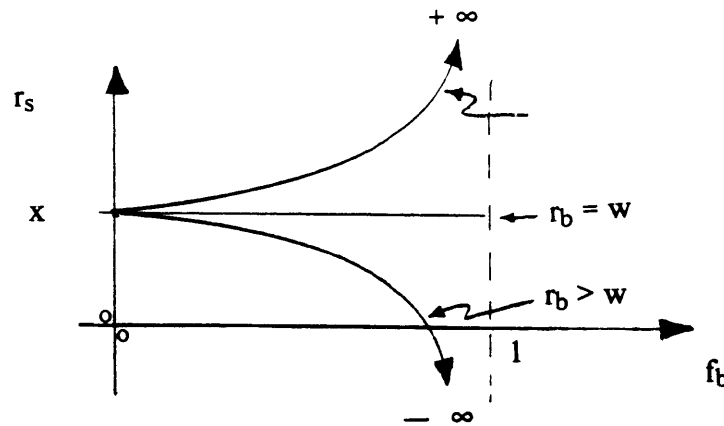

Figure 2. **Dependence of r_s on f_b, r_b and w.**

To find the vector D_i, use Eqs. (3) and (7) after finding x.

Numerical Example

Given the cash flow of a project (Table III) and knowing that the income tax rate is 50%, find the minimum bond fraction f_b in the investments such that the minimum stockholders rate of return r_s be 0.22. Assume that the market bond rate is 0.11.

Using Eq. (12) with α = 0, one finds x = 0.183. Consequently, using Eq. (19), w = $\dfrac{x}{1-\tau}$ = 0.367. In order to have r_s = 0.22, through Eq. (54) one finds the required f_b = 0.222 = 22%. Note that r_s may be increased by increasing the bond fraction f_b to its maximum allowed value. The total rate of return of the project is then r = r_b f_b + r_s f_s = 0.196. The

TABLE III. Cash Flow Used as Input for Numerical Example of Item (b).

i	0	1	2	3	4
R_i	0	7	7	7.5	8
C_i	0	2	2.5	3	3.5
D_i'	0	4	3	2	1
I_i	9	1	0	0	0

levelized revenue is found using Eq. (16), or Eq. (13) with $y = r$, giving $R_r = 7.300$. The vector D_i is found using Eqs. (3) and (7):

$$Di \equiv \{2.850/2.439/2.387/2.324\}$$

(c) To maximize the rate of return, r_s, to the stockholders, given the cash flow $\{R_i, C_i, I_i, D_i \equiv D_i', \text{ but without knowing numerical values of } D_i \text{ and } D_i', i = 1 \text{ to } n\}$, find w using Eq. (12) with $\alpha = 1$. In this case, Eq. (12) becomes:

$$\sum_{i=1}^{n} \frac{R_i}{(1+w)^i} = \sum_{i=1}^{n} \frac{C_i}{(1+w)^i} + \sum_{i=0}^{n} \frac{I_i}{(1+w)^i} \tag{55}$$

Given τ and r_b, proceed as in part (b) to choose f_b and r_s.

<u>Numerical Example</u>

Given the estimated cash flow of a project (Table IV) and $D_i \equiv D_i'$, determines the feasibility of the project assuming $r_b = 0.12$ and the minimum $r_s = 0.18$. Assume an income tax rate equal to 50%.

TABLE IV. Cash Flow Used as Input for Numerical Example of Item (c).

i	0	1	2	3	4	5
R_i	0	5	5	5	5	5
C_i	0	2	2	2.5	2.5	3
I_i	10	0	0	0	0	0

Using Eq. (55) one finds $w = 0.101$. As $r_b = 0.12 > w = 0.101$, capital borrowed should be minimized. As $x = w(1 - \tau) = 0.051 < r_s = 0.18$, this project is not feasible and should be abandoned (see Figure 2). Note that $D_i \equiv D'_i$ may be caculated using Eqs. (3) and (7).

$$D_i \equiv D'_i \equiv \{1.985/2.187/1.909/2.102/1.816\}$$

(d) Although the range of physical significance of α is $0 \leq \alpha \leq 1$, other parts of Figure 1 may be studied in detail to derive useful relations, such as the one given by Eq. (56) which was derived from Eq. (12) as $\alpha \rightarrow -\infty$ ($y \rightarrow 0$) (see Appendix III). Note that while Eq. (12) with $\alpha = 0$ ($y = x$), requires the knowledge of D'_i in order to find x, Eq. (56) requires D_i to be given to find w (Note also that w given by Eq. (56) does not assume $D_i \equiv D'_i$, as Eq. (55) does!)

$$w = \frac{\sum\limits_{i=1}^{n} R_i - \sum\limits_{i=1}^{n} C_i - \sum\limits_{i=0}^{n} I_i}{\sum\limits_{i=1}^{n} i(D_i - I_i)}$$

(56)

A practical use of Eq. (56) is the following. Given a cash flow $\{R_i, I_i, C_i, D_i, i = 1 \text{ to } n\}$ find w using Eq. (56). Now, given τ and r_b, choose f_b (and then r_s) in the same way as in (b) above.

<u>Numerical example</u>

Given the following cash flow of a project (Table V), find the bond fraction f_b in the investments such that the stockholders rate of return r_s is 0.22. Assume that the market bond rate r_b is 0.12 and that the income tax rate is 50%; and also if the income tax rate is 30%.

TABLE V. Cash Flow Used as Input for Numerical Example of Item (d).

i	0	1	2	3	4
R_i	0	6	5.8	5.4	5.2
C_i	0	1	2	3	4
D_i	0	4	3	2	1
I_i	8	2	0	0	0

Using Eq. (56), one finds $w = 0.133$. Through Eq. (54), $f_b = 0.958$. With $\tau = 0.3$, using Eq. (54) again, $f_b = 0.931$.

The vector D_i' may be calculated using Eqs. (3) and (7). $D_i' \equiv \{4.067/3.000/2.000/0.933\}$.

(One can check the consistency of D_i' by finding x from Eq. (12) with $\alpha = 0$).

<u>CONCLUSION</u>

There is a general formula from which a wide variety of levelized cost conventions can be derived. (In fact, there are an infinite number of types of levelized costs.) But there is only one levelized revenue, which, if charged uniformly throughout the life of the plant, will just pay for all current expenses, return on and return of investment <u>and keep coherence on taxes</u>. This

preferred discount rate and cash flow weighting are embodied in Eq. (16) of the present exposition.

APPENDIX I

The objective is to prove relation (11) given Eqs. (3) and (4). Equation (3) can be used to generate the following development:

$$\sum_{i=1}^{n} \frac{y\,V_i + D_i}{(1+y)^i} = \sum_{i=1}^{n} \frac{y}{(1+y)^i}\left[\sum_{j=0}^{i-1} I_j - \sum_{j=1}^{i-1} D_j\right] + \sum_{i=1}^{n} \frac{D_i}{(1+y)^i}$$

$$= \sum_{j=0}^{n-1}\sum_{i=j+1}^{n} \frac{y}{(1+y)^i} I_j - \sum_{j=1}^{n-1}\sum_{i=j+1}^{n} \frac{y}{(1+y)^i} D_j + \sum_{i=1}^{n} \frac{D_i}{(1+y)^i}$$

$$= \sum_{j=0}^{n-1} y \cdot \frac{1}{y}\left[\frac{1}{(1+y)^j} - \frac{1}{(1+y)^n}\right] I_j - \sum_{j=1}^{n-1} y \cdot \frac{1}{y}\left[\frac{1}{(1+y)^j} - \frac{1}{(1+y)^n}\right] D_j + \sum_{i=1}^{n} \frac{D_i}{(1+y)^i}$$

$$= \sum_{j=0}^{n-1} \frac{I_j}{(1+y)^j} - \frac{1}{(1+y)^n}\sum_{j=0}^{n-1} I_j - \sum_{j=1}^{n-1} \frac{D_j}{(1+y)^j} + \frac{1}{(1+y)^n}\sum_{j=1}^{n-1} D_j + \sum_{i=1}^{n} \frac{D_i}{(1+y)^i}$$

$$= \sum_{i=0}^{n} \frac{I_i}{(1+y)^i} - \sum_{i=1}^{n} \frac{D_i}{(1+y)^i} + \frac{1}{(1+y)^n}\left[\sum_{i=1}^{n} D_i - \sum_{i=1}^{n} I_i\right] + \sum_{i=1}^{n} \frac{D_i}{(1+y)^i}$$

and hence, by Eq. (4)

$$= \sum_{i=0}^{n} \frac{I_i}{(1+y)^i}$$

The resulting relation:

$$\sum_{i=1}^{n} \frac{y\,V_i + D_i}{(1+y)^i} = \sum_{i=0}^{n} \frac{I_i}{(1+y)^i}$$

holds for $y > -1$.

APPENDIX II

Starting with Eq. (25)

$$V_i^* = \sum_{j=0}^{i-1} I_j - \sum_{j=1}^{i-1} D_j^*$$

define: $\quad \Theta = \dfrac{1}{1+y}$ and $y = \dfrac{1-\Theta}{\Theta}$

$$\sum_{i=1}^{n} V_i^*\,\Theta^i = \sum_{i=1}^{n} \sum_{j=0}^{i-1} I_j\,\Theta^i - \sum_{i=1}^{n} \sum_{j=1}^{i-1} D_j^*\,\Theta^i \qquad \Rightarrow$$

$$\sum_{i=1}^{n} V_i^*\,\Theta^i = \sum_{j=0}^{n-1} \sum_{i=j+1}^{n} I_j\,\Theta^i - \sum_{j=1}^{n-1} \sum_{i=j+1}^{n} D_j^*\,\Theta^i \qquad \Rightarrow$$

$$\sum_{i=1}^{n} V_i^* \Theta^i = I_0 \frac{\Theta^{n+1} - \Theta}{\Theta - 1} + \sum_{j=1}^{n-1} \left(I_j - D_j^*\right) \frac{\Theta^{n+1} - \Theta^{j+1}}{\Theta - 1}$$

But, from Eq. (30):

$$\sum_{i=1}^{n} \left[y V_i^* + D_i^*\right] \Theta^i = \sum_{i=0}^{n} I_i \Theta^i$$

Then:

$$\frac{1-\Theta}{\Theta} I_0 \frac{\Theta}{\Theta-1}\left[\Theta^n - 1\right] + \frac{1-\Theta}{\Theta} \sum_{i=1}^{n-1} \left(I_i - D_i^*\right) \frac{\Theta\left(\Theta^n - \Theta^i\right)}{\Theta-1} + \sum_{i=1}^{n} D_i^* \Theta^i = \sum_{i=0}^{n} I_i \Theta^i \Rightarrow$$

$$-\sum_{i=0}^{n-1} I_i\left(\Theta^n - \Theta^i\right) + \sum_{i=1}^{n-1} D_i^*\left(\Theta^n - \Theta^i\right) + \sum_{i=1}^{n} D_i^* \Theta^i = \sum_{i=0}^{n} I_i \Theta^i \qquad \Rightarrow$$

$$-\sum_{i=0}^{n} I_i \Theta^n + \sum_{i=1}^{n} D_i^* \Theta^n = 0$$

hence

$$\sum_{i=1}^{n} D_i^* = \sum_{i=0}^{n} I_i$$

APPENDIX III

From Eq. (9), as $\alpha \to -\infty$, $y \to 0$ (see Figure 1). Taking the limit of Eq. (12) as $\alpha \to -\infty$ ($y \to 0$), one has:

$$\sum_{i=1}^{n} R_i = \sum_{i=1}^{n} C_i + \frac{1}{1-\tau} \sum_{i=0}^{n} I_i - \frac{\tau}{1-\tau} \sum_{i=1}^{n} D_i'$$

$$- \lim_{\substack{\alpha \to -\infty \\ (y \to 0)}} \frac{\alpha\tau}{1-\tau} \left[\sum_{i=0}^{n} \frac{I_i}{(1+y)^i} - \sum_{i=1}^{n} \frac{D_i}{(1+y)^i} \right]$$

Since from Eq. (4): $\displaystyle \sum_{i=1}^{n} D_i = \sum_{i=1}^{n} D_i' = \sum_{i=0}^{n} I_i$,

and using Eq. (9) to obtain $\alpha\tau = 1 - \dfrac{r-b\tau}{y}$,

$$\sum_{i=1}^{n} R_i = \sum_{i=1}^{n} C_i + \sum_{i=0}^{n} I_i - \lim_{y \to 0} \left\{ \frac{1}{1-\tau} \left(1 - \frac{r-b\tau}{y}\right). \right.$$

$$[I_0 + (I_1 - D_1)(1-y) + (I_2 - D_2)(1-2y) +$$

$$\left. + (I_3 - D_3)(1-3y) + \ldots + (I_n - D_n)(1-ny)]\right\} \qquad \Rightarrow$$

$$\sum_{i=1}^{n} R_i = \sum_{i=1}^{n} C_i + \sum_{i=0}^{n} I_i - \lim_{y \to 0} \left\{ \frac{1}{1-\tau} \left(1 - \frac{r-b\tau}{y}\right). \right.$$

$$\left. \left[\sum_{i=0}^{n} I_i - \sum_{i=1}^{n} D_i - \sum_{i=1}^{n} iy(I_i - D_i) \right] \right\} \qquad \Rightarrow$$

$$\sum_{i=1}^{n} R_i = \sum_{i=1}^{n} C_i + \sum_{i=0}^{n} I_i - \frac{r-b\tau}{1-\tau} \sum_{i=1}^{n} i\left(I_i - D_i\right)$$

Then:

$$w = \frac{r-b\tau}{1-\tau} = \frac{\displaystyle\sum_{i=1}^{n} R_i - \sum_{i=1}^{n} C_i - \sum_{i=0}^{n} I_i}{\displaystyle\sum_{i=1}^{n} i\left(D_i - I_i\right)}$$

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