Topological quasiparticles and the holographic bulk-edge relation in (2+1)-dimensional string-net models

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String-net models allow us to systematically construct and classify (2+1)-dimensional [(2+1)D] topologically ordered states which can have gapped boundaries. We can use a simple ideal string-net wave function, which is described by a set of F-matrices [or more precisely, a unitary fusion category (UFC)], to study all the universal properties of such a topological order. In this paper, we describe a finite computational method, *Q-algebra approach*, that allows us to compute the non-Abelian statistics of the topological excitations [or more precisely, the unitary modular tensor category (UMTC)], from the string-net wave function (or the UFC). We discuss several examples, including the topological phases described by twisted gauge theory [i.e., twisted quantum double $D^{\alpha}(G)$]. Our result can also be viewed from an angle of holographic bulk-boundary relation. The (1+1)-dimensional [(1+1)D] anomalous topological orders, that can appear as edges of (2+1)D topological states, are classified by UFCs which describe the fusion of quasiparticles in (1+1)D. The (1+1)D anomalous edge topological order uniquely determines the (2+1)D bulk topological order (which are classified by UMTC). Our method allows us to compute this bulk topological order (i.e., the UMTC) from the anomalous edge topological order (i.e., the UFC).

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I. INTRODUCTION

A major problem of physics is to classify phases and phase transitions of matter. The problem was once thought to be completely solved by Landau's theory of symmetry breaking [1], where the phases can be classified by their symmetries. However, the discovery of fractional quantum Hall (FQH) effect [2] indicated that Landau's theory is incomplete. There are different FQH phases with the same symmetry, and the symmetry breaking theory failed to distinguish those phases. FQH states are considered to possess new topological orders [3–5] beyond the symmetry breaking theory.

We know that all the symmetry breaking phases are labeled by two groups (G_H, G_Ψ), where G_H is the symmetry group of the Hamiltonian and G_Ψ is the symmetry group of the ground state. This fact motivates us to search for the complete "label" of topological order.

Here, the "label" that labels a topological order corresponds to a set of universal properties that can fully determine the phase and distinguish it from other phases. Such universal properties should always remain the same as long as there is no phase transition. In particular, they are invariant under any small local perturbations. Such universal properties are called topological invariants in mathematics.

In (2+1) dimensions [(2+1)D], it seems that anyonic quasiparticle statistics, or the modular data T,S matrices, are the universal properties. The set of universal properties that describes quasiparticle statistics is also referred to as unitary modular tensor category (UMTC). T,S matrices (i.e., UMTC) can fully determine the topological phases, up to a bosonic E_8 FQH state [5–9]. In Sec. II we will introduce topological quasiparticle excitations and their statistics, i.e., fusion and braiding data, in (2+1)D topological phases and on (1+1)-dimensional [(1+1)D] gapped edges.

Since the universal properties do not depend on the local details of the system, it is possible to calculate them from a simple renormalization fixed-point model. In this paper, we will concentrate on a class of (2+1)D fixed-point lattice model, the Levin-Wen string-net model [10]. As a fixed-point model, the building blocks of Levin-Wen models are effective degrees of freedom with the form of string-nets. The fixed-point string-net wave function is completely determined by important data: the F-matrices. The F-matrices are also referred to as unitary fusion category (UFC).

Therefore, a central question for string-net models is how to calculate the T,S matrices from F-matrices (or how to calculate the UMTC from the UFC). In Ref. [10] the T,S matrices can be calculated by searching for *string operators*. String operators are determined by a set of nonlinear algebraic equations involving the F-matrices. However, this algorithm is not an efficient one. The equations determining string operators have infinite many solutions and there is no general method to pick up the irreducible solutions. In this sense it is even not guaranteed that one can find all the (irreducible) string operators. In this paper we try to fix this weak point. Motivated by the work of Kitaev and Kong [11,12], we introduce the *Q-algebra* approach to compute quasiparticle statistics. The idea using Q-algebra modules to classify quasiparticles is analog to using group representations to classify particles. It is well known that in a system with certain symmetry the energy eigenspaces, including excited states of particles, form representations of the symmetry group. String-net models are fixed-point models, thus renormalization can be viewed as generalized "symmetry." Moreover we show that renormalization in string-net models can be exactly described by *evaluation* linear maps. This allows us to introduce the Q-algebra, which describes the renormalization of quasiparticle states. Quasiparticles are identified as the invariant subspaces under the action of the Q-algebra, i.e., Q-algebra modules. Roughly speaking, the Q-algebra is the "renormalization group" of quasiparticles in string-net models, a linearized, weakened version of a group. The notions of algebra modules and group representations are almost equivalent. Modules over the group algebra are in one-to-one correspondence with group representations up to similarity transformations. The only difference is that "module" emphasizes on the subspace of states that is invariant under the action of the group or algebra, while "representation" emphasizes on how the group or algebra acts on the "module." The specific algorithm to compute the Q-algebra modules is also analog to that to compute the group representations. For a group, first we write the multiplication rules. Second, we take the multiplication rules as the "canonical representation." Third, we try to simultaneously block-diagonalize the canonical representation. Finally the irreducible blocks correspond to irreducible representations, or simple modules over the group algebra. The canonical representation of a group contains all types of irreducible representations of that group. This is also true for the Q-algebra. The multiplication rules of the Q-algebra are fully determined by the F-matrices [i.e., the UFC, see (49) and (86)]. Therefore, following this block-diagonalization process we have a finite algorithm to calculate the quasiparticle statistics from F-matrices. We are guaranteed to find all types of quasiparticles by block-diagonalizing the canonical representation of the Q-algebra. Simultaneous block-diagonalization is a straightforward algorithm, however, it is not a quite efficient way to decompose the Q-algebra. The algorithm used in this paper is an alternative one, *idempotent decomposition*. The notions of algebra, module, and idempotent play an important role in our discussion and algorithm. On the other hand, we think it a necessary step to proceed from "groups and group representations" to "algebras and modules" since we are trying to extend our understanding from "symmetry breaking phases" to "topologically ordered phases." We provide a brief introduction in Appendix A to these mathematical notions in case the reader is not familiar with them.

Another weak point of the original version of the Levin-Wen model in Ref. [10] is that the F-matrices are assumed to be symmetric under certain index permutation. More precisely, the F-matrices have 10 indices which can be associated to a tetrahedron, 6 indices to the edges and 4 indices to the vertices. If we reflect or rotate the tetrahedron, the indices get permuted and the F-matrices are assumed to remain the same. In this paper we find that such tetrahedral symmetry can be dropped, thus the string-net model is generalized. In Sec. III we will first drop the tetrahedron-reflectional symmetry of the F-matrices but keep the tetrahedron-rotational symmetry and reformulate the string-net model. We keep the tetrahedron-rotational symmetry because in this case the relation between string operators and Q-algebra modules is clear. We give the formula to compute quasiparticle statistics, the T,S matrices from Q-algebra modules, by comparing them to string operators.

Next, in Sec. IV we will drop the tetrahedron-rotational symmetry assumption, and generalize string-net models to arbitrary gauge. In arbitrary gauge the string operators are not naturally defined, but we can still obtain the formula of quasiparticle statistics by requiring the formula to be gauge invariant and reduce to the special case if we choose the tetrahedron-rotation-symmetric gauge.

Finally, in Sec. V we briefly discuss the boundary theory [11] of generalized string-net models which shows the holographic bulk-edge relation. In (2+1)D there are many different kinds of topological orders, classified by the non-Abelian statistics of the quasiparticles plus the chiral central charge of the edge state. Mathematically, the non-Abelian statistics, or the fusion and braiding data of quasiparticles, form a UMTC. On the other hand, in (1+1)D, there is only trivial topological order [13,14]. However, if we consider anomalous topological orders that only appear on the edge of (2+1)D gapped states, we will have nontrivial anomalous (1+1)D topological orders. In these anomalous (1+1)D topological orders, the fusion of quasiparticles is also described by a set of F-matrices. Mathematically, the F-matrices give rise to a UFC, and anomalous (1+1)D topological orders are classified by UFCs. The F-matrices we use to determine a string-net ground state wave function turn out to be the same F-matrices describing the fusion of quasiparticles on one of the edges of the string-net model [11,15]. Thus, our algorithm calculating the bulk quasiparticle statistics (UMTC) from the F-matrices (UFC) can also be understood as calculating the bulk topological order (UMTC) from the anomalous boundary topological order (UFC). Since the same bulk topological order may have different gapped boundaries, it is a natural consistency question: Do these different gapped boundaries lead to the same bulk? The answer is "yes" [11]. Mathematically we give an algorithm to compute the Drinfeld center functor $\mathcal{Z}$ that maps a UFC [that describes a (1+1)D anomalous topological order] to a UMTC **[**that describes a (2+1)D topological order with zero chiral central charge [16]**]**. Different gapped boundaries of a (2+1)D topological phase are described by different UFCs, but they share the same Drinfeld center UMTC. In Appendix E we discuss the twisted $\mathbb{Z}_n$ string-net model in detail to illustrate this holographic relation.

II. QUASIPARTICLE EXCITATIONS

A. Local quasiparticle excitations and topological quasiparticle excitations

Topologically ordered states in (2+1)D are characterized by their unusual particlelike excitations which may carry fractional/non-Abelian statistics. To understand and to classify particlelike excitations in topologically ordered states, it is important to understand the notions of local quasiparticle excitations and topological quasiparticle excitations.

First, we define the notion of "particlelike" excitations. Consider a gapped system with translation symmetry. The ground state has a uniform energy density. If we have a state with an excitation, we can measure the energy distribution of the state over the space. If, for some local area, the energy density is higher than ground state, while for the rest area the energy density is the same as ground state, one may say there is a "particlelike" excitation, or a quasiparticle, in this area (see Fig. 1). Among all the quasiparticle excitations, some

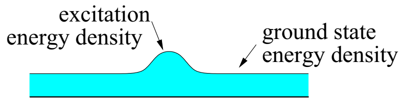


FIG. 1. (Color online) The energy density distribution of a quasiparticle.

can be created or annihilated by local operators, such as a spin flip. This kind of particlelike excitation is called local quasiparticle. However, in topologically ordered systems, there are also quasiparticles that cannot be created or annihilated by any finite number of local operators (in the infinite system size limit). In other words, the higher local energy density cannot be created or removed by *any* local operators in that area. Such quasiparticles are called topological quasiparticles.

From the notions of local quasiparticles and topological quasiparticles, we can further introduce the notion *topological* quasiparticle type or, simply, quasiparticle type. We say that local quasiparticles are of the trivial type, while topological quasiparticles are of nontrivial types. Two topological quasiparticles are of the same type if and only if they differ by local quasiparticles. In other words, we can turn one topological quasiparticle into the other one of the same type by applying some local operators.

B. Simple type and composite type

To understand the notion of simple type and composite type, let us discuss another way to define quasiparticles: consider a gapped local Hamiltonian qubit system defined by a local Hamiltonian H_0 in d-dimensional space M^d without boundary. A collection of quasiparticle excitations labeled by i and located at $\boldsymbol{x}_i$ can be produced as *gapped* ground states of $H_0 + \Delta H$ where ΔH is nonzero only near $\boldsymbol{x}_i$'s. By choosing different ΔH we can create all kinds of quasiparticles. We will use ξ_i to label the type of the quasiparticle at $\boldsymbol{x}_i$.

The gapped ground states of $H_0 + \Delta H$ may have a degeneracy $D(M^d;\xi_1,\xi_2,\ldots)$ which depends on the quasiparticle types ξ_i and the topology of the space M_d. The degeneracy is not exact, but becomes exact in the large space and large particle separation limit. We will use $\mathcal{V}(M^d;\xi_1,\xi_2,\ldots)$ to denote the space of the degenerate ground states.

If the Hamiltonian $H_0 + \Delta H$ is not gapped, we will say $D(M^d;\xi_1,\xi_2,\ldots) = 0$ [i.e., $\mathcal{V}(M^d;\xi_1,\xi_2,\ldots)$ has zero dimension]. If $H_0 + \Delta H$ is gapped, but if ΔH also creates quasiparticles away from $\boldsymbol{x}_i$'s (indicated by the bump in the energy density away from $\boldsymbol{x}_i$'s), we will also say $D(M^d;\xi_1,\xi_2,\ldots) = 0$. (In this case quasiparticles at $\boldsymbol{x}_i$'s do not fuse to trivial quasiparticles.) So, if $D(M^d;\xi_1,\xi_2,\ldots) > 0$, ΔH only creates quasiparticles at $\boldsymbol{x}_i$'s.

If the degeneracy $D(M^d;\xi_1,\xi_2,\ldots)$ cannot not be lifted by any small local perturbation near $\boldsymbol{x}_1$, then the particle type ξ_1 at $\boldsymbol{x}_1$ is said to be simple. Otherwise, the particle type ξ_1 at $\boldsymbol{x}_1$ is said to be composite. The degeneracy $D(M^d;\xi_1,\xi_2,\ldots)$ for simple particle types ξ_i is a universal property (i.e., a topological invariant) of the topologically ordered state.

C. Fusion of quasiparticles

When ξ_1 is composite, the space of the degenerate ground states $\mathcal{V}(M^d;\xi_1,\xi_2,\xi_3,\ldots)$ has a direct sum decomposition:

$$\begin{aligned}\mathcal{V}(M^d;\xi_1,\xi_2,\xi_3,\ldots)&\\ =\mathcal{V}(M^d;\zeta_1,\xi_2,\xi_3,\ldots)&\oplus\mathcal{V}(M^d;\chi_1,\xi_2,\xi_3,\ldots)\\ \oplus\,\mathcal{V}(M^d;\psi_1,\xi_2,\xi_3,\ldots)&\oplus\cdots,\end{aligned} \tag{1}$$

where ζ_1, χ_1, ψ_1, etc. are simple types. To see the above result, we note that when ξ_1 is composite the ground state degeneracy can be split by adding some small perturbations near $\boldsymbol{x}_1$. After splitting, the original degenerate ground states become groups of degenerate states, each group of degenerate states span the space $\mathcal{V}(M^d;\zeta_1,\xi_2,\xi_3,\ldots)$ or $\mathcal{V}(M^d;\chi_1,\xi_2,\xi_3,\ldots)$, etc. which correspond to simple quasiparticle types at $\boldsymbol{x}_1$. We denote the composite type ξ_1 as

$$\xi_1 = \zeta_1 \oplus \chi_1 \oplus \psi_1 \oplus \cdots . \tag{2}$$

When we fuse two simple types of topological particles ξ and ζ together, it may become a topological particle of a composite type:

$$\xi \otimes \zeta = \eta = \chi_1 \oplus \chi_2 \oplus \cdots , \tag{3}$$

where ξ,ζ,χ_i are simple types and η is a composite type. In this paper, we will use an integer tensor $N_{\xi\zeta}^{\chi}$ to describe the quasiparticle fusion, where ξ,ζ,χ label simple types. When $N_{\xi\zeta}^{\chi} = 0$, the fusion of ξ and ζ does not contain χ. When $N_{\xi\zeta}^{\chi} = 1$, the fusion of ξ and ζ contain one χ: $\xi \otimes b = \chi \oplus \chi_1 \oplus \chi_2 \oplus \cdots$. When $N_{\xi\zeta}^{\chi} = 2$, the fusion of ξ and ζ contain two χ's: $\xi \otimes \zeta = \chi \oplus \chi \oplus \chi_1 \oplus \chi_2 \oplus \cdots$. This way, we can denote that fusion of simple types as

$$\xi \otimes \zeta = \oplus_\chi N_{\xi\zeta}^{\chi} \chi . \tag{4}$$

In physics, the quasiparticle types always refer to simple types. The fusion rules $N_{\xi\zeta}^{\chi}$ are a universal property of the topologically ordered state. The degeneracy $D(M^d;\xi_1,\xi_2,\ldots)$ is determined completely by the fusion rules $N_{\xi\zeta}^{\chi}$.

Let us then consider the fusion of three simple quasiparticles ξ,ζ,χ. We may first fuse ξ,ζ, and then with χ, $(\xi \otimes \zeta) \otimes \chi = (\oplus_\alpha N_{\xi\zeta}^{\alpha}\alpha) \otimes \chi = \oplus_\beta(\sum_\alpha N_{\xi\zeta}^{\alpha} N_{\alpha\chi}^{\beta})\beta$. We may also first fuse ζ,χ and then with ξ, $\xi \otimes (\zeta \otimes \chi) = \xi \otimes (\oplus_\alpha N_{\zeta\chi}^{\alpha}\alpha) = \oplus_\beta(\sum_\alpha N_{\xi\alpha}^{\beta} N_{\zeta\chi}^{\alpha})\beta$. This requires that $\sum_\alpha N_{\xi\zeta}^{\alpha} N_{\alpha\chi}^{\beta} = \sum_\alpha N_{\xi\alpha}^{\beta} N_{\zeta\chi}^{\alpha}$. If we further consider the degenerate states $\mathcal{V}(M^d;\xi,\zeta,\chi,\ldots)$, it is not hard to see fusion in different orders means splitting the space $\mathcal{V}(M^d;\xi,\zeta,\chi,\ldots)$ as different direct sums of subspaces. Thus, fusion in different orders differs by basis changes of $\mathcal{V}(M^d;\xi,\zeta,\chi,\ldots)$. The *F-matrices* are nothing but the data to describe such basis changes. For (1+1)D anomalous topological orders [gapped edges of (2+1)D topological orders], the quasiparticles can only fuse but not braiding. So, the fusion rules $N_{\xi\zeta}^{\chi}$ and the F-matrices are enough to describe (1+1)D anomalous topological orders. Later, we will see fusion rules and F-matrices are also used to determine a string-net wave function, which may seem confusing. However, as we have mentioned, this is a natural result of the holographic bulk-edge relation. Intuitively, one may even view the string-net graphs in 2D space as the (1+1)D space-time trajectory of the edge quasiparticles. For

(2+1)D topological orders, the quasiparticles can also braid. We also need data to describe the braiding of the quasiparticles in addition to the fusion rules and the F-matrices, as introduced in the next two subsections.

D. Quasiparticle intrinsic spin

If we twist the quasiparticle at $\boldsymbol{x}_1$ by rotating ΔH at $\boldsymbol{x}_1$ by 360° (note that ΔH at $\boldsymbol{x}_1$ has no rotational symmetry), all the degenerate ground states in $\mathcal{V}(M^d;\xi_1,\xi_2,\xi_3,\ldots)$ will acquire the same geometric phase $e^{i\theta_{\xi_1}}$ provided that the quasiparticle type ξ_1 is a simple type. We will call $e^{i\theta_\xi}$ the intrinsic spin (or simply spin) of the simple type ξ, which is a universal property of the topologically ordered state.

E. Quasiparticle mutual statistics

If we move the quasiparticle ξ_2 at $\boldsymbol{x}_2$ around the quasiparticle ξ_1 at $\boldsymbol{x}_1$, we will generate a non-Abelian geometric phase, a unitary transformation acting on the degenerate ground states in $\mathcal{V}(M^d;\xi_1,\xi_2,\xi_3,\ldots)$. Such a unitary transformation not only depends on the types ξ_1 and ξ_2, but also depends on the quasiparticles at other places. So, here we will consider three quasiparticles of simple types ξ, ζ, χ on a 2D sphere S^2. The ground state degenerate space is $\mathcal{V}(S^2;\xi,\zeta,\chi)$. For some choices of ξ, ζ, χ, $D(S^2;\xi,\zeta,\chi)\geqslant 1$, which is the dimension of $\mathcal{V}(S^2;\xi,\zeta,\chi)$. Now we move the quasiparticle ζ around the quasiparticle ξ. All the degenerate ground states in $\mathcal{V}(S^2;\xi,\zeta,\chi)$ will acquire the same geometric phase $\frac{e^{i\theta_{\chi^*}}}{e^{i\theta_\xi}e^{i\theta_\zeta}}$. This is because, in $\mathcal{V}(S^2;\xi,\zeta,\chi)$, the quasiparticles ξ and ζ fuse into χ^*, the antiquasiparticle of χ. Moving quasiparticle ζ around the quasiparticle ξ plus rotating ξ and ζ, respectively, by 360° is like rotating χ^* by 360°. So moving quasiparticle ζ around the quasiparticle ξ generates a phase $\frac{e^{i\theta_{\chi^*}}}{e^{i\theta_\xi}e^{i\theta_\zeta}}$. We see that the quasiparticle mutual statistics is determined by the quasiparticle spin $e^{i\theta_\xi}$ and the quasiparticle fusion rules $N^\chi_{\xi\zeta}$. For this reason, we call the set of data $(e^{i\theta_\xi},N^\chi_{\xi\zeta})$ quasiparticle statistics.

It is an equivalent way to describe quasiparticle statistics by T,S matrices. The T matrix is a diagonal matrix. The diagonal elements are the quasiparticle spins

$$T_{\xi\zeta}=T_\xi\delta_{\xi\zeta}=e^{i\theta_\xi}\delta_{\xi\zeta}. \tag{5}$$

The S matrix can be determined from the quasiparticle spin $e^{i\theta_\xi}$ and quasiparticle fusion rules $N^\chi_{\xi\zeta}$ [see Eq. (223) in Ref. [17]]:

$$S_{\xi\zeta}=\frac{1}{D_{\mathcal{Z}(\mathcal{C})}}\sum_\chi N^\chi_{\xi\zeta^*}\frac{e^{i\theta_\chi}}{e^{i\theta_\xi}e^{i\theta_\zeta}}d_\chi, \tag{6}$$

where $d_\xi>0$ is the largest eigenvalue of the matrix N_ξ, whose elements are $N_{\xi,\zeta\chi}=N^\chi_{\xi\zeta}$. On the other hand, the S matrix determines the fusion rules $N^\chi_{\xi\zeta}$ via the Verlinde formula [see (60) in Sec. III E]. So, T_ξ and $S_{\xi\zeta}$ fully determine the quasiparticle statistics $(e^{i\theta_\xi},N^\chi_{\xi\zeta})$, and the quasiparticle statistics $(e^{i\theta_\xi},N^\chi_{\xi\zeta})$ fully determines T_ξ and $S_{\xi\zeta}$. We want to emphasize that the fusion rules and F-matrices of bulk quasiparticles and edge quasiparticles are *different*. In this paper we use only the F-matrices of edge quasiparticles, which are also the F-matrices describing the bulk string-net wave functions. Although our Q-algebra module algorithm can be used to compute the F-matrices of bulk quasiparticles, we did not explain in detail how to do this because calculating the T,S matrices is enough to distinguish and classify (2+1)D topological orders with gapped boundaries.

III. STRING-NET MODELS WITH TETRAHEDRON-ROTATIONAL SYMMETRY

The string-net condensation was suggested by Levin and Wen as a mechanism for topological phases [10]. We give a brief review here.

The basic idea of Levin and Wen's construction was to find an ideal fixed-point ground state wave function for topological phases. Such an ideal wave function can be fully determined by a finite amount of data. The idea is not to directly describe the wave function, but to describe some local constraints that the wave function must satisfy. These local constraints can be viewed as a scheme of ground state renormalization.

Let us focus on lattice models. We put the lattice on a sphere so that there are no nontrivial boundary conditions. Since renormalization will change the lattice, we will consider a class of ground states on arbitrary lattices on the sphere. One way to obtain "arbitrary lattices" is to triangulate the sphere in arbitrary ways. There may be physical degrees of freedom on the faces, edges, as well as vertices of the triangles. Any two triangulations can be related by adding, removing vertices and flipping edges. The ideal ground state must renormalize coherently when re-triangulating.

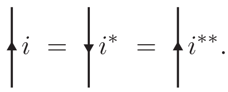
The string-net picture is dual to the triangulation picture. As an intuitive example, one can consider the strings as electric flux lines through the edges of the triangles. Like the triangulation picture, there are some basic local transformations of the string-nets, which we call evaluations. Physically, evaluations are related to the so-called *local unitary transformations* [18], and states related by local unitary transformations belong to the same phase. If we evaluate the whole string-net on the sphere or, in other words, we renormalize the whole string-net so that no degrees of freedom are left, we should obtain just a number. We require that this number remains the same no matter how we evaluate the whole string-net. This gives rise to the desired local constraints of the ideal ground state wave function. We now demonstrate in detail the formulation of the string-net model with the tetrahedron-rotational symmetry.

A. String-net

A string-net is a two-dimensional directed trivalent graph. The vertices and edges (strings) are labeled by some physical degrees of freedom. By convention, we use $i,j,k,\ldots$ for string labels and $\alpha,\beta,\ldots$ for vertex labels. We assume that the string and vertex label sets are finite.

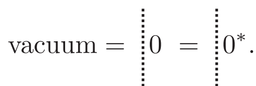
A fully labeled string-net corresponds to a basis vector of the Hilbert space. If a string-net is not labeled, it stands for the ground state subspace in the total Hilbert space spanned by the basis string-nets with all possible labelings. A partially labeled string-net corresponds to the projection of the ground state subspace to the subspace of the total Hilbert space where states on the labeled edges/vertices are given by the fixed labels. This way, we have a graph representation of the ground state subspace, which will help us to actually compute the ground state subspace.

There is an involution of the string label set $i \mapsto i^*$ satisfying $i^{**} = i$, corresponding to reversing the string direction



(7)

When an edge is vacant, or not occupied by any string, we say it is a trivial string. The trivial string is labeled by 0 and $0^* = 0$. Trivial strings are usually omitted or drawn as dashed lines



(8)

In addition, we assume that trivial strings are totally invisible, i.e., can be arbitrarily added, removed, and deformed without affecting the ideal ground state wave function. To understand this point, suppose we have a unlabeled string-net on a graph. It corresponds to a subspace $\mathcal{V}$ of the total Hilbert space $\mathcal{H}$ on the graph. Now we add a trivial string to the string-net which gives us a partially labeled string-net on a new graph (with an extra string carrying the label 0). Such a partially labeled string-net on a new graph corresponds to subspace $\mathcal{V}_0$ of the total Hilbert space $\mathcal{H}_0$ on the new graph. The two subspaces $\mathcal{V}$ and $\mathcal{V}_0$ are very different belonging to different total Hilbert spaces. The statement that trivial strings are totally invisible implies that the two subspaces are isomorphic to each other $\mathcal{V} \cong \mathcal{V}_0$. In other words, there exists a local linear map from $\mathcal{H}_0$ to $\mathcal{H}$, such that the map is unitary when restricted on $\mathcal{V}_0$. Such a map is called an evaluation, which will be discussed in more detail below.

B. Evaluation and F-move

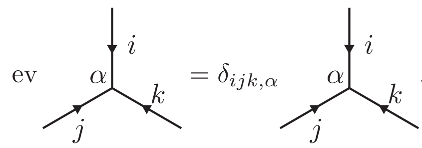
A string-net graph represents a subspace, which corresponds to the ground state subspace on that graph. When we do wave function renormalization, we change the graph on which the string-net is defined. However, the ground state subspace represented by the string-net, in some sense, is not changed since the string-net represents a fixed-point wave function under renormalization. To understand such a fixed-point property of the string-net wave function, we need to compare ground state subspaces on different graphs. This leads to the notion of evaluation.

We do not directly specify the ground state subspace represented by a string-net. Rather, we specify several evaluations (i.e., several local linear maps). Those evaluations will totally fix the ground state subspace of the string-net for every graph.

Consider two graphs with total Hilbert space $\mathcal{H}_1$ and $\mathcal{H}_2$. Assume that the two graphs differ only in a local area and $\dim\mathcal{H}_1 \geqslant \dim\mathcal{H}_2$. An evaluation is a local linear map from $\mathcal{H}_1$ to $\mathcal{H}_2$. Here "local" means that the map is identity on the overlapping part of the two graphs. Note that the evaluation maps a Hilbert space of higher dimension to a Hilbert space of lower dimension. It reduces the degrees of freedom and represents a wave function renormalization.

Although evaluation depends on the two graphs with $\mathcal{H}_1$ and $\mathcal{H}_2$, since the graphs before and after evaluation are normally shown in the equations, we will simply use ev to denote evaluations. We will point out the two graphs only if it is necessary.

Let us list the evaluations that totally fix the ground state subspace. For a single vertex, we have the following evaluation:



(9)

where

$$\delta_{ijk,\alpha} = 0 \text{ or } 1, \tag{10}$$

$$\delta_{ijk,\alpha} = \delta_{kij,\alpha} = \delta_{k^*j^*i^*,\alpha}, \tag{11}$$

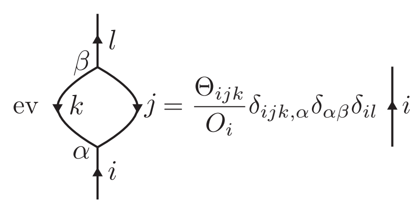
$$\delta_{ij0,\alpha} = \delta_{ij^*}\delta_{0\alpha}, \tag{12}$$

$$\sum_m N_{ijm}N_{m^*kl} = \sum_n N_{inl}N_{n^*jk}. \tag{13}$$

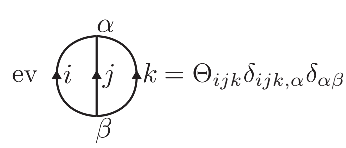
We note that the above evaluation does not change the graph and thus $\mathcal{H}_1 = \mathcal{H}_2$. The evaluation is a projection operator in $\mathcal{H}_1$ whose action on the basis of $\mathcal{H}_1$ is given by (9).

The vertex with $\delta_{ijk,\alpha} = 1$ is called a stable vertex. $N_{ijk} = \sum_\alpha \delta_{ijk,\alpha}$ is the dimension of the stable vertex subspace, called fusion rules. To determine the order of the ijk labels, one should first use (7) to make the three strings going inwards, then read the string labels anticlockwise. If one thinks of strings as electric flux lines, $\delta_{ijk,\alpha}$ enforces the total flux to be zero for the ground state.

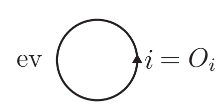
The next few evaluations are for two-edge plaquettes, Θ graphs, and closed loops:



(14)



(15)



(16)

where

$$\Theta_{ijk} = \Theta_{kij} = \Theta_{k^*j^*i^*}, \tag{17}$$

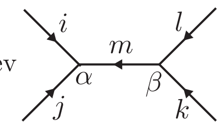
$$\Theta_{ii^*0} = \Theta_{i^*i0} = O_i = O_{i^*}, \tag{18}$$

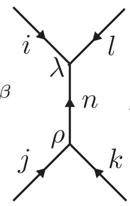
$$O_0 = \text{ev(vacuum)} = 1. \tag{19}$$

$O_i = \varkappa_i d_i$ where $d_i > 0$ is called the quantum dimension of the type i string. When i is self-dual $i = i^*$, the phase factor $\varkappa_i$ corresponds to the Frobenius-Schur indicator. Otherwise,

$\varkappa_i$ can be adjusted to 1 by gauge transformations. $O_i = O_{i^*}, \Theta_{ijk} = \Theta_{kij}$ is because for any closed string-net on the sphere, the half loop on the right can be moved to the left across the other side of the sphere. Those evaluations change the graph. They are described by how every basis vector of $\mathcal{H}_1$ is mapped to a vector in $\mathcal{H}_2$.

The last evaluation is called F-move. It changes the graph. In fact, the F-move is the most basic graph changing operation acting on local areas with two stable vertices. It is given by

ev 

$= \sum_{n\lambda\rho} F^{ijm,\alpha\beta}_{kln,\lambda\rho}$ 

. (20)

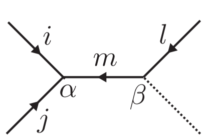
It is equivalent to flipping edges in the triangulation picture. The rank 10 tensor $F^{ijm,\alpha\beta}_{kln,\lambda\rho}$ are called F-matrices. $m,\alpha\beta$ are considered as column indices and $n,\lambda\rho$ as row indices. $F^{ijm,\alpha\beta}_{kln,\lambda\rho}$ is zero if any of the four vertices is unstable. Otherwise, F^{ij}_{kl} is a unitary matrix.

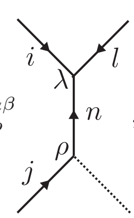
Note that the evaluations can be done recursively. When two graphs within $\mathcal{H}_1$ and $\mathcal{H}_2$ are connected by different sequences of evaluations, the induced maps from $\mathcal{H}_1$ to $\mathcal{H}_2$ by different sequences must be the same. First, the F-matrices must satisfy the well known pentagon equations

$$\sum_{n\tau\lambda\eta} F^{ijq,\alpha\beta}_{kln,\eta\lambda} F^{n^*jk,\lambda\gamma}_{rsp,\tau\mu} F^{lin,\eta\tau}_{pst,\rho\nu} = \sum_{\sigma} F^{lq^*k,\beta\gamma}_{rst,\rho\sigma} F^{ijq,\alpha\sigma}_{rt^*p,\nu\mu}. \tag{21}$$

We also assume the tetrahedron-rotational symmetry. The tetrahedron-rotational symmetry is actually the symmetry of the evaluation, not of the graphs. For example, if one rotates the graphs in (15) by 180°, the result of the evaluation should be $\Theta_{k^*j^*i^*}$ and the tetrahedron-rotational symmetry requires that $\Theta_{ijk} = \Theta_{k^*j^*i^*}$. In general, with tetrahedron-rotational symmetry, doing the evaluation is "rotation invariant." When the evaluation of tetrahedron graphs, and simpler graphs such as Θ graphs or closed loops, is rotation invariant, the evaluation of all graphs is rotation invariant. Therefore, we call it tetrahedron-rotational symmetry.

The tetrahedron-rotational symmetry puts the following constraints on the F-matrices. First, it is necessary that the trivial string is totally invisible. So if in (20) we set the label k to 0, the corresponding F-matrix elements should be 1 when the labels match and 0 otherwise, i.e.,

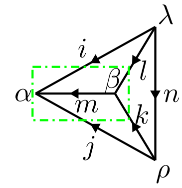
ev 

$= \sum_{n\lambda\rho} F^{ijm,\alpha\beta}_{0ln,\lambda\rho}$ 

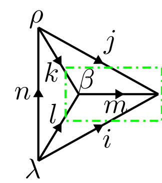
, (22)

$$F^{ijm,\alpha\beta}_{0ln,\lambda\rho} = \delta_{ijl,\alpha}\delta_{\alpha\lambda}\delta_{ml}\delta_{nj}\delta_{\beta 0}\delta_{\rho 0}. \tag{23}$$

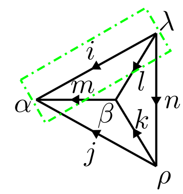
Second, consider the tetrahedron graphs. After one step of F-move, the tetrahedron graphs have only two-edge plaquettes. Thus, the amplitude can be expressed by $F^{ijm,\alpha\beta}_{kln,\lambda\rho}$, Θ_{ijk} and O_i, i.e.,

ev 

$= F^{ijm,\alpha\beta}_{kln,\lambda\rho} \dfrac{\Theta_{nli}\Theta_{n^*jk}}{O_n}$, (24)

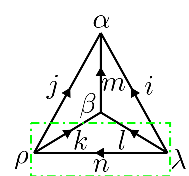
ev 

$= F^{klm^*,\beta\alpha}_{ijn^*,\rho\lambda} \dfrac{\Theta_{nli}\Theta_{n^*jk}}{O_n}$, (25)

ev 

$= F^{jmi,\alpha\lambda}_{l^*n^*k,\rho\beta} \dfrac{\Theta_{m^*kl}\Theta_{n^*jk}}{O_k}$,

(26)

ev 

$= F^{k^*j^*n,\rho\lambda}_{i^*l^*m,\beta\alpha} \dfrac{\Theta_{mij}\Theta_{m^*kl}}{O_m}$, (27)

where the F-move is performed in the boxed area. These four results must be the same. Thus, we got another constraint on the F-matrices:

$$\begin{aligned} F^{ijm,\alpha\beta}_{kln,\lambda\rho} &= F^{klm^*,\beta\alpha}_{ijn^*,\rho\lambda} = F^{jmi,\alpha\lambda}_{l^*n^*k,\rho\beta} \frac{O_n \Theta_{m^*kl}}{O_k \Theta_{nli}} \\ &= F^{k^*j^*n,\rho\lambda}_{i^*l^*m,\beta\alpha} \frac{O_n \Theta_{mij}\Theta_{m^*kl}}{O_m \Theta_{nli}\Theta_{n^*jk}}. \end{aligned} \tag{28}$$

Note that (28) is different from that in Ref. [10] because we do not allow reflection of the tetrahedron. This is necessary to include cases of fusion rules like $N_{ijk} \neq N_{jik}$, for example, the finite group G model with a non-Abelian group G (see Sec. III F 4). It turns out that the conditions above are sufficient for evaluation of any string-net graph to be rotation invariant.

With these consistency conditions, given any two string-net graphs with total Hilbert spaces H_1 and H_2, dim $H_1 \geqslant$ dim H_2, there is a unique evaluation map from H_1 to H_2, given by the compositions of simple evaluations listed above. Thus, evaluation depends on only the graphs before and after, or H_1 and H_2, not on the way we change the graphs. As we mentioned before, usually it is not even necessary to explicitly point out H_1 and H_2 since they are automatically shown in the equations and graphs.

We want to emphasize that the fusion rules (9)–(13), the F-move (20), and the pentagon equation (21) are the most fundamental ones. The rest of the equations (14)–(19), (23), and (28) are either normalization conventions, gauge choices, or conditions of the tetrahedron-rotational symmetry. With the tetrahedron-rotational symmetry, O_i,Θ_{ijk} are encoded in F-matrices. In (28) set some indices to 0, and we have

$$F^{ii^*0,00}_{ii^*0,00} = \frac{1}{O_i}, \tag{29}$$

$$F^{ijk,\alpha\beta}_{j^*i^*0,00} = \frac{\Theta_{ijk}}{O_i O_j}\delta_{ijk,\alpha}\delta_{\alpha\beta}, \tag{30}$$

$$F^{jj^*0,00}_{i^*ik,\alpha\beta} = \frac{O_k}{\Theta_{ijk}}\delta_{ijk,\alpha}\delta_{\alpha\beta}. \tag{31}$$

Moreover, in (21) set r to 0 and one can get

$$\sum_{n\lambda\rho} F^{ijm,\alpha\beta}_{kln,\lambda\rho} F^{jkn^*,\rho\lambda}_{lim',\alpha'\beta'} = \delta_{mm'}\delta_{\alpha\alpha'}\delta_{\beta\beta'}. \tag{32}$$

Thus, O_i satisfies

$$\sum_k N_{ijk} O_k = \sum_{k\alpha\beta} F^{ijk,\alpha\beta}_{j^*i^*0,00} F^{jj^*0,00}_{i^*ik,\alpha\beta} O_i O_j = O_i O_j. \tag{33}$$

This implies that O_i is an eigenvalue of the matrix N_i, whose entries are $N_{i,jk} = N_{ijk}$, and the corresponding eigenvector is $(O_0, O_1, \ldots)^{\mathsf{T}}$.

C. Fixed-point Hamiltonian

Does the evaluation defined above really describe the renormalization of some physical ground states? What is the corresponding Hamiltonian? A sufficient condition for the string-nets to be physical ground states is that the F-move is unitary, or that the F-matrices are unitary:

$$\sum_{n\lambda\rho} F^{ijm,\alpha\beta}_{kln,\lambda\rho} \left(F^{ijm',\alpha'\beta'}_{kln,\lambda\rho}\right)^* = \delta_{mm'}\delta_{\alpha\alpha'}\delta_{\beta\beta'}. \tag{34}$$

This requires a special choice of O_i,Θ_{ijk}. From (34) and (32) we know

$$F^{jkn^*,\rho\lambda}_{lim,\alpha\beta} = \left(F^{ijm,\alpha\beta}_{kln,\lambda\rho}\right)^*, \tag{35}$$

which implies that $F^{ii^*0,00}_{ii^*0,00} = (F^{i^*i0,00}_{i^*i0,00})^*$, $O_i = O_i^*$ are real numbers, or $\varkappa_i = \pm 1$, and $F^{jj^*0,00}_{i^*ik,\alpha\alpha} = (F^{ijk,\alpha\alpha}_{j^*i^*0,00})^*$, i.e., if $N_{ijk} > 0$

$$|\Theta_{ijk}|^2 = O_i O_j O_k = d_i d_j d_k > 0. \tag{36}$$

Moreover, (33) and (36) together imply that

$$\sum_k N_{ijk} d_k = d_i d_j. \tag{37}$$

Hence, d_i has to be the largest eigenvalue (Perron-Frobenius eigenvalue) of the matrix N_i and the corresponding eigenvector is $(d_0,d_1,\ldots)^{\mathsf{T}}$.

To find the corresponding Hamiltonian, note that

$$\begin{aligned} \mathrm{ev}\ \mathrm{ev}^\dagger \uparrow i &= \mathrm{ev} \sum_{jkl\alpha\beta} \frac{\Theta^*_{ijk}}{O_i}\delta_{ijk,\alpha}\delta_{\alpha\beta}\delta_{il} \; \left(\uparrow i \; \overset{\alpha}{\bullet} \; \overleftarrow{k}\;\overrightarrow{j} \; \overset{\beta}{\bullet} \uparrow l\right) \\ &= \sum_{jk} N_{ijk}\frac{|\Theta_{ijk}|^2}{O_i^2} \uparrow i = \sum_k O_k^2 \uparrow i = D_{\mathcal{C}}^2 \uparrow i, \end{aligned} \tag{38}$$

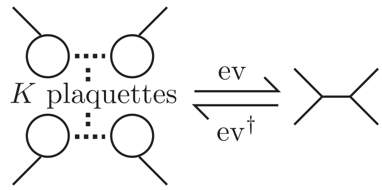


FIG. 2. A local area with K plaquettes and four external legs. The evaluation removes all the plaquettes.

where $D_{\mathcal{C}} = \sqrt{\sum_k O_k^2} = \sqrt{\sum_k d_k^2}$ is the total quantum dimension.

For a local area with K plaquettes, consider the evaluation that removes all the K plaquettes and results in a tree graph, as sketched in Fig. 2. Since F-move does not change the number of plaquettes, we can first use F-move to deform the local area and make all the plaquettes two-edge plaquettes. Thus, we have

$$\frac{\mathrm{ev}\,\mathrm{ev}^\dagger}{D_{\mathcal{C}}^{2K}} = \mathbf{1}. \tag{39}$$

Consider

$$P = \frac{\mathrm{ev}^\dagger\,\mathrm{ev}}{D_{\mathcal{C}}^{2K}}, \tag{40}$$

which means that first use ev to remove all the plaquettes in the local area, and then use ev† to recreate the plaquettes and go back to the original graph. It is easy to see that $P^2 = P$. Thus, P is a Hermitian projection. Like evaluation, P can also act on any local area of the string-net. We can take the Hamiltonian as the sum of local projections acting on every vertex and plaquette

$$H = \sum_{\substack{\text{vertices}\\ \text{plaquettes}}} (\mathbf{1} - P), \tag{41}$$

which is the fixed-point Hamiltonian.

We see that P is exactly the projection onto the ground state subspace. P acting on a single vertex projects onto the stable vertex; P acting on a plaquette is equivalent to the B_p operator [10–12,19]. The B_p operator is more general because there may be "nonlocal" plaquettes, for example, when the string-net is put on a torus, in which case evaluation cannot be performed. But in this paper we will not consider such "nonlocal" plaquettes. Evaluation is enough for our purpose.

If we evaluate the whole string-net, the evaluated tree-graph string-net represents the ground state. For a fixed lattice on the sphere with K plaquettes, the evaluated tree graph is just the void graph, or the vacuum. Therefore, the normalized ground state is

$$|\psi\rangle_{\text{ground}} = \frac{\mathrm{ev}^\dagger}{D_{\mathcal{C}}^K}|\text{vacuum}\rangle. \tag{42}$$

Generically, the ground state subspace is $\mathcal{V} = \mathrm{ev}^\dagger\mathcal{V}_{\text{tree}}$.

D. Cylinder ground states, quasiparticle excitations, and Q-algebra

Now, we have defined the string-net models with tetrahedron-rotational symmetry. We continue to study the quasiparticles excitations.

Let us first discuss the generic properties of quasiparticle excitations from a different point of view. By definition,

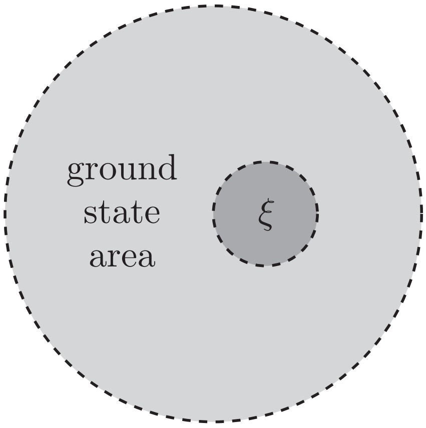


FIG. 3. Quasiparticle ξ: The local energy density is constant in the ground state area but higher in the ξ area.

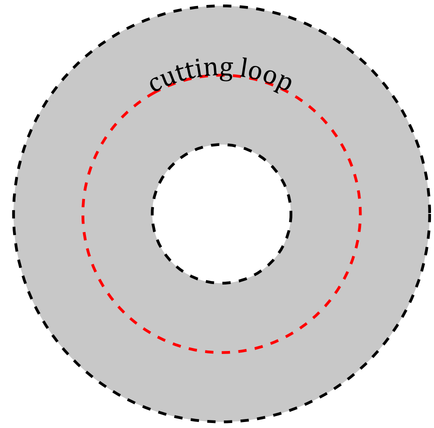


FIG. 4. (Color online) Cut a cylinder into two cylinders. The entanglement between the two cylinders is only in the neighborhood of the cutting loop.

a quasiparticle is a local area with higher energy density, labeled by ξ, surrounded by the ground state area (see Fig. 3). We want to point out that a topological quasiparticle is scale invariant. If we zoom out, put the ξ area and ground state area together, and view the larger area as a single quasiparticle area ξ', then ξ' should be the same type as ξ. Moreover, if we are considering a fixed-point model such as the string-net model, the excited states of the quasiparticle will not even change no matter how much surrounding ground state area is included. Intuitively, we may view this renormalization process as "gluing" a cylinder ground state to the quasiparticle area. "Gluing a cylinder ground state" is then an element of the "renormalization group" that acts on (renormalizes) the quasiparticle states. Thus, quasiparticle states form "representations" of the "renormalization group." Of course, "renormalization group" is not a group at all, but the idea to identify quasiparticles as "representations" still works. We develop this idea rigorously in the following. We will define the "gluing" operation, introduce the algebra induced by gluing cylinder ground states, and show that quasiparticles are representations of, or modules over, this algebra. This algebra is nothing but the "renormalization group."

Since any local operators acting inside the ξ area will not change the quasiparticle type, we do not quite care about the degrees of freedom inside the ξ area. Instead, the entanglement between the ground state area and the ξ area is much more important, and should capture all the information about the quasiparticle types and statistics. Since we are considering systems with local Hamiltonians, the entanglement should be only in the neighborhood of the boundary between the ground state area and the ξ area.

To make things clear, we would first forget about the entanglement and study the properties of ground states on a cylinder with the open boundary condition. Here, open boundary condition means that setting all boundary Hamiltonian terms to zero thus strings on the boundary are free to be in any state. Later we will put the entanglement back by "gluing" boundaries and adding back the Hamiltonian terms near the "glued" boundaries.

On a cylinder with the open boundary condition, the ground states form a subspace $\mathcal{V}_{\text{cyl}}$ of the total Hilbert space. $\mathcal{V}_{\text{cyl}}$ should be scale invariant, i.e., not depend on the size of the cylinder. We want to show that, the fixed-point cylinder ground states in $\mathcal{V}_{\text{cyl}}$ allows a cut-and-glue operation.

Given a cylinder, we can cut it into two cylinders with a loop, as in Fig. 4. The states in the two cylinders are entangled with each other; but again, the entanglement is only near the cutting loop. If we ignore the entanglement for the moment, in other words, imposing open boundary conditions for both cylinders, by scale invariance, the ground state subspaces on the two cylinders should be both $\mathcal{V}_{\text{cyl}}$. Next, we add back the entanglement (this can be done, e.g., by applying proper local projections in the neighborhood of the cutting loop), which is like "gluing" the two cylinders along the cutting loop, and we should obtain the ground states on the bigger cylinder before cutting, but still states in $\mathcal{V}_{\text{cyl}}$. Therefore, gluing two cylinders by adding the entanglement back gives a map

$$\mathcal{V}_{\text{cyl}} \otimes \mathcal{V}_{\text{cyl}} \xrightarrow{\text{glue}} \mathcal{V}_{\text{cyl}}, \quad h_1 \otimes h_2 \longmapsto h_1 h_2. \tag{43}$$

It is a natural physical requirement that such gluing is associative, $(h_1h_2)h_3 = h_1(h_2h_3)$. Thus, it can be viewed as a multiplication. Now, the cylinder ground state subspace $\mathcal{V}_{\text{cyl}}$ is equipped with a multiplication, the gluing map. Mathematically, $\mathcal{V}_{\text{cyl}}$ forms an *algebra* (see Appendix A).

We can also enlarge a cylinder by gluing another cylinder onto it. Note that when two cylinders are cut from a larger one as in Fig. 4, there is a natural way to put them back together, however, when we arbitrarily pick two cylinders, simply putting them together may not work. To glue or enforce entanglements between two cylinders, we need to first put them in such a way that there is an overlapping area between their glued boundaries (see Fig. 5). In this overlapping area, we identify degrees of freedom from one cylinder with those from the other cylinder; this way we "connect and match" the boundaries. Next, we apply proper local projections in the neighborhood of the overlapping area, such that the two cylinders are well glued. But, the ground state subspace remains the same, i.e., "multiplying" $\mathcal{V}_{\text{cyl}}$ by $\mathcal{V}_{\text{cyl}}$ is still $\mathcal{V}_{\text{cyl}}$:

$$\mathcal{V}_{\text{cyl}}\mathcal{V}_{\text{cyl}} = \left\{ \sum_k c^{(k)} h_1^{(k)} h_2^{(k)} \middle| \begin{matrix} k \in \mathbb{N}, & h_1^{(k)} \in \mathcal{V}_{\text{cyl}}, \\ c^{(k)} \in \mathbb{C}, & h_2^{(k)} \in \mathcal{V}_{\text{cyl}} \end{matrix} \right\}$$
$$= \mathcal{V}_{\text{cyl}}. \tag{44}$$

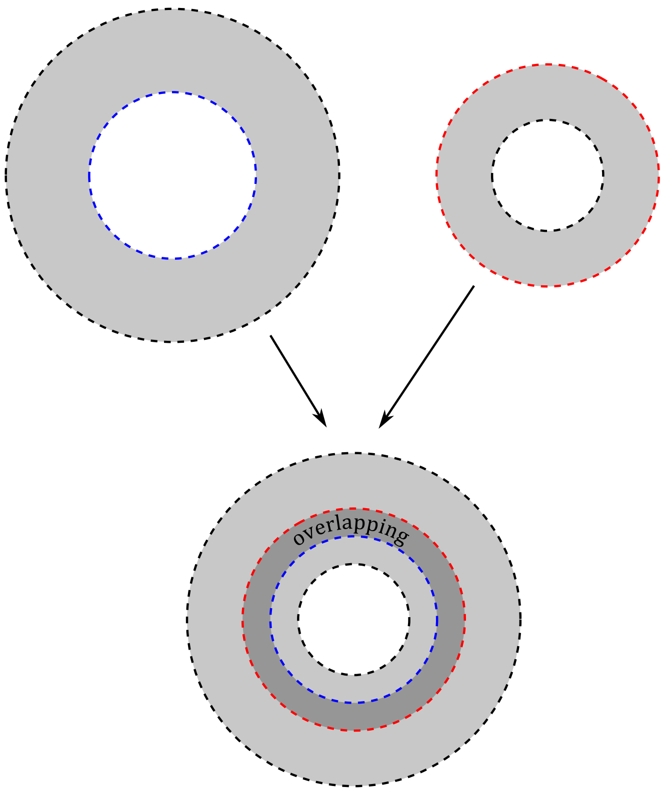


FIG. 5. (Color online) Gluing two cylinders: make sure there is an overlapping area between the glued boundaries (red and blue).

Now, we put back the quasiparticle ξ. Since the entanglement between ξ and the ground state area is restricted in the neighborhood of the boundary, it can be viewed as imposing some nontrivial boundary conditions on the cylinder. Equivalently, we may say that the quasiparticle ξ picks a subspace M_ξ of $\mathcal{V}_{\text{cyl}}$. M_ξ should also be scale invariant. If we enlarge the area by gluing a cylinder onto it, in other words, multiply M_ξ by $\mathcal{V}_{\text{cyl}}$, M_ξ remains the same, $\mathcal{V}_{\text{cyl}} M_\xi = M_\xi$. Mathematically, M_ξ is a *module* over the algebra $\mathcal{V}_{\text{cyl}}$. In this way, the quasiparticle ξ is identified with the module M_ξ over the algebra $\mathcal{V}_{\text{cyl}}$. A reducible module corresponds to a composite type of quasiparticle, and an irreducible module corresponds to a simple type of quasiparticle (see Sec. II B).

As for string-net models, recall that ground state subspaces can be represented by evaluated tree graphs. The actual ground state subspace can always be obtained by applying $\text{ev}^\dagger$ to the space of evaluated tree graphs. Thus, we can find out $\mathcal{V}_{\text{cyl}}$ by examining the possible tree graphs on a cylinder. A typical tree graph on a cylinder is like Fig. 6. Assuming that there are a legs on the outer boundary and b legs on the inner boundary, we denote the space of these graphs by V_b^a. As evaluated graphs, all the vertices in the graphs in V_b^a must be stable. In principle, a,b can take any integer numbers. But note that if $c < a$, we can add $a-c$ trivial legs on the outer boundary, and V_b^c can be viewed as a subspace of V_b^a. Similarly $V_c^a \subset V_b^a$ for $c < b$. Therefore, we know the largest space is $\mathcal{V}_{\text{cyl}} = \text{ev}^\dagger V_\infty^\infty$.

We find that the gluing of cylinder ground states can be captured by the spaces V_b^a. The gluing is nothing but adding back the entanglement. For string-net model the proper local projections are just $\text{ev}^\dagger\text{ev}$. But before doing the evaluation

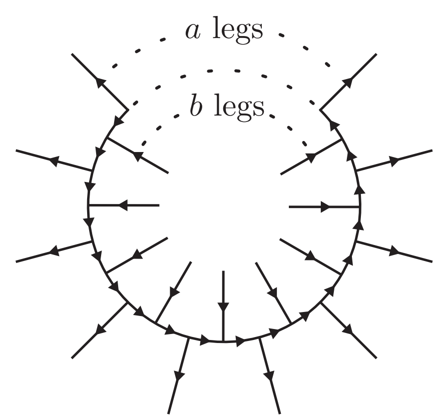


FIG. 6. A typical tree graph on a cylinder. Here the dashed lines stand for the omitted part of the graph, but not trivial strings.

we have to "connect and match" the boundaries, i.e., make sure the strings are well connected. Note that $\text{ev}^\dagger$ and ev acting inside each cylinder do not affect the boundary legs. $(\text{ev}^\dagger V_b^a)$ can be glued onto $(\text{ev}^\dagger V_d^c)$ from the outer side only if $b = c$. We need to first connect the legs on the inner boundary of $(\text{ev}^\dagger V_b^a)$ with those on the outer boundary of $(\text{ev}^\dagger V_d^b)$ and make their labels match each other's; broken strings are not allowed inside a ground state area. This defines a map $p : (\text{ev}^\dagger V_b^a) \otimes (\text{ev}^\dagger V_d^b) \to (\text{ev}^\dagger V_b^a) \otimes (\text{ev}^\dagger V_d^b)|_{\text{w.c.}}$, where w.c. means restriction to the subspace in which the strings are well connected. Thus, $\frac{\text{ev}^\dagger\text{ev}}{D_C^{2K}} p : (\text{ev}^\dagger V_b^a) \otimes (\text{ev}^\dagger V_d^b) \to (\text{ev}^\dagger V_d^a)$ is the desired gluing if there are K plaquettes in $(\text{ev}^\dagger V_d^a)$. Recall that evaluation can be performed in any sequence. We know the following diagram:

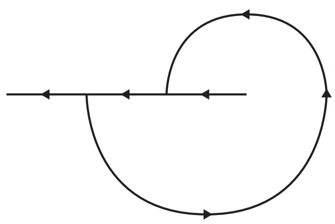
$$\begin{tikzcd}
(\text{ev}^\dagger V_b^a) \otimes (\text{ev}^\dagger V_d^b) \arrow[r, "\text{ev}\otimes\text{ev}"] \arrow[d, "p"] & V_b^a \otimes V_d^b \arrow[d, "p"] \\
(\text{ev}^\dagger V_b^a) \otimes (\text{ev}^\dagger V_d^b)|_{\text{w.c.}} \arrow[r, "\text{ev}\otimes\text{ev}"] \arrow[d, "\frac{\text{ev}^\dagger\text{ev}}{D_C^{2K}}"] \arrow[dr, "\text{ev}"] & V_b^a \otimes V_d^b|_{\text{w.c.}} \arrow[d, "\text{ev}"] \\
(\text{ev}^\dagger V_d^a) & V_d^a \arrow[l, "\frac{\text{ev}^\dagger}{D_C^{2K}}"]
\end{tikzcd} \tag{45}$$

commutes. Thus, gluing $(\text{ev}^\dagger V_b^a)$ with $(\text{ev}^\dagger V_d^c)$ to obtain ground states in $(\text{ev}^\dagger V_d^a)$ can be done by first considering the evaluation of the tree graphs, $V_b^a \otimes V_d^c \xrightarrow{p} V_b^a \otimes V_d^c|_{\text{w.c.}} \xrightarrow{\text{ev}} V_d^a$ and then applying $\text{ev}^\dagger$ to get the actual ground states.

However, it is impossible to deal with an infinite-dimensional algebra $\mathcal{V}_{\text{cyl}} = \text{ev}^\dagger V_\infty^\infty$. We want to reduce it to an algebra of finite dimension. Again our idea is to do renormalization. When we glue the cylinder ground states, we renormalize along the radial direction. Now we renormalize along the tangential direction, or reduce the number of boundary legs, to reduce the dimension of the algebra.

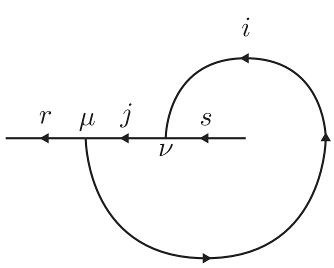
More rigorously, our goal is to study the quasiparticles, which correspond to modules over $\mathcal{V}_{\text{cyl}}$, rather than the algebra $\mathcal{V}_{\text{cyl}}$ itself. So, if we can find some algebra such that its modules are the "same" as those over $\mathcal{V}_{\text{cyl}}$ (here "same" means that the categories of modules are equivalent), this algebra can also be used to study the quasiparticles. Mathematically, two algebras are called *Morita equivalent* [12,20] if they have the "same" modules. Thus, we want to find finite-dimensional algebras that are Morita equivalent to $\mathcal{V}_{\text{cyl}}$.

Note that V_a^a with the multiplication $\text{ev}\,p : V_a^a \otimes V_a^a \xrightarrow{p} V_a^a \otimes V_a^a|_{\text{w.c.}} \xrightarrow{\text{ev}} V_a^a$ forms an algebra. From (45) we also know that $(\text{ev}^\dagger V_a^a)$ and V_a^a are isomorphic algebras (the isomorphisms are just ev and $\text{ev}^\dagger$). It turns out that all the algebras V_a^a are Morita equivalent for $a = 1,2,\ldots$ (see Sec. VI). Thus, we know V_1^1 and $\mathcal{V}_{\text{cyl}} = \text{ev}^\dagger V_\infty^\infty$ have the "same" modules. We choose the algebra V_1^1 to study the quasiparticles of string-net models for V_1^1 has the lowest dimension among the algebras V_a^a. Now we reduce the infinite-dimensional algebra $\mathcal{V}_{\text{cyl}}$ to the finite-dimensional V_1^1. Since a graph in V_1^1 is like a letter Q, and V_1^1 describes the physics of quasiparticles, we name it the *Q-algebra*, denoted by

$$Q = V_1^1 =$$  . (46)

The subtlety of Morita equivalence will be discussed further in Sec. VI.

In detail, the natural basis of Q is

$$Q_{rsj}^{i,\mu\nu} =$$  . (47)

The notation $Q_{rsj}^{i,\mu\nu}$ looks like a tensor. But $Q_{rsj}^{i,\mu\nu}$ denotes a basis vector rather than a number. On one hand, $Q_{rsj}^{i,\mu\nu}$ represents a cylinder ground state $\text{ev}^\dagger|Q_{rsj}^{i,\mu\nu}\rangle$; on the other hand, when glued onto other cylinder ground states, $Q_{rsj}^{i,\mu\nu}$ can be viewed as a linear operator $\hat{Q}_{rsj}^{i,\mu\nu}$. Both of $|Q_{rsj}^{i,\mu\nu}\rangle$ and $\hat{Q}_{rsj}^{i,\mu\nu}$ are incomplete and misleading. That is why we choose the simple notation $Q_{rsj}^{i,\mu\nu}$; just keep in mind that it stands for a vector/operator. As an evaluated graph, the two vertices are stable, $\delta_{rj^*i,\mu} = \delta_{sij^*,\nu} = 1$. Thus, the dimension of the Q-algebra is

$$\dim Q = \sum_{rsij} N_{rj^*i} N_{sij^*} = \sum_{rs} \text{Tr}(N_r N_s). \tag{48}$$

In terms of the natural basis, the multiplication is

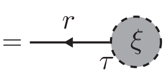
$$\begin{aligned} Q_{rsj}^{i,\mu\nu} Q_{s'tl}^{k,\sigma\tau} &= \text{ev}\,p\big(Q_{rsj}^{i,\mu\nu} \otimes Q_{s'tl}^{k,\sigma\tau}\big) \\ &= \delta_{ss'} \sum_{mn\lambda\rho} Q_{rtn}^{m,\lambda\rho} \sum_{\alpha\beta\gamma} F_{k^*ln^*,\alpha\beta}^{ij^*s,\nu\sigma} F_{k^*nm^*,\lambda\gamma}^{r^*i^*j,\mu\beta} F_{in^*m,\rho\gamma}^{tkl^*,\tau\alpha} \\ &\quad \times \frac{\Theta_{kim^*}}{O_m}. \end{aligned} \tag{49}$$

We know that the identity is

$$\mathbf{1} = \sum_r Q_{rrr}^{0,00} = \sum_r \longleftarrow r. \tag{50}$$

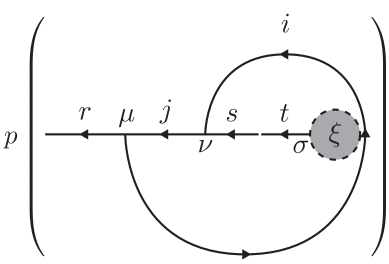
We can study the quasiparticles by decomposing the Q-algebra. The simple quasiparticle types correspond to simple Q-modules. The number of quasiparticle types is just the number of different simple Q-modules. As of the Morita equivalence of V_a^a algebras, we also want to mention that the *centers* (see Appendix A) of Morita equivalent algebras are isomorphic. Thus, the center $Z(Q) \cong Z(V_a^a) \cong Z(\mathcal{V}_{\text{cyl}})$ is an *invariant*. We argue that $Z(Q)$ is exactly the ground state subspace on a torus and $\dim[Z(Q)]$ is the torus ground state degeneracy, also the number of quasiparticle types. We give a more detailed discussion on the Q-algebra in Appendix B.

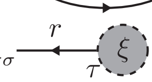
Assume that we have obtained the module M_ξ over the Q-algebra, or the invariant subspace $M_\xi \subset Q$, that corresponds to the quasiparticle ξ. Since $M_\xi = \mathbf{1}M_\xi = \oplus_r Q_{rrr}^{0,00} M_\xi$, it is possible to choose the basis vectors of M_ξ from $Q_{rrr}^{0,00} M_\xi$, respectively. Such a basis vector can be labeled by r,τ, namely,

$$e_{r\tau}^\xi =$$  $$\in Q_{rrr}^{0,00} M_\xi. \tag{51}$$

Then we can calculate the representation matrix of $Q_{rsj}^{i,\mu\nu}$ with respect to this basis

$$Q_{rsj}^{i,\mu\nu} e_{t\sigma}^\xi = \sum_{q\tau} M_{\xi,rsj,q\tau t\sigma}^{i,\mu\nu} e_{q\tau}^\xi$$

$$= \text{ev}\,p$$  (52)

$$= \delta_{st} \sum_\tau M_{\xi,rsj,\tau\sigma}^{i,\mu\nu}$$ 

$$= \delta_{st} \sum_\tau M_{\xi,rsj,\tau\sigma}^{i,\mu\nu} e_{r\tau}^\xi,$$

where p is still the map that connects legs and matches labels. We know that the representation matrix of $Q_{rsj}^{i,\mu\nu}$ is $M_{\xi,rsj,q\tau t\sigma}^{i,\mu\nu} = \delta_{rq}\delta_{st} M_{\xi,rsj,\tau\sigma}^{i,\mu\nu}$, which is a block matrix. And since $Q_{rrr}^{0,00}$ is an idempotent, $M_{\xi,rrr,\tau\sigma}^{0,00} = \delta_{\tau\sigma}$. Later we will see that the representation matrices $M_{\xi,rsj,\tau\sigma}^{i,\mu\nu}$ are closely related to the string operators, and can be used to calculate the quasiparticle statistics.

E. String operators and quasiparticle statistics

The string operator [10] is yet another way to study the quasiparticles. A string operator creates a pair of quasiparticles at its ends (see Fig. 7). It is also the hopping operator of

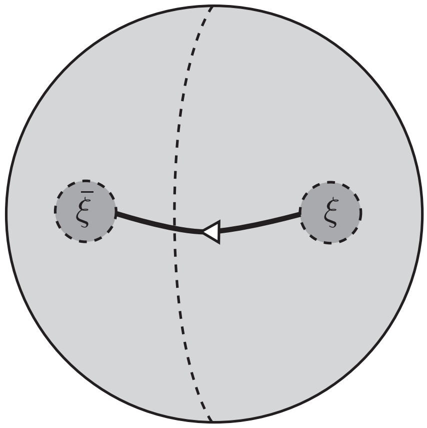
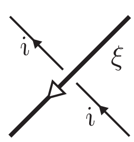
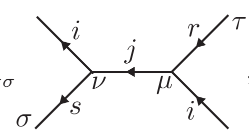


FIG. 7. A string operator on the sphere

the quasiparticles, i.e., a quasiparticle can be moved around with the corresponding string operator. First recall the matrix representations of string operators. For consistency we still label the string operator with ξ,



$$= \sum_{rsj\mu\nu\tau\sigma} \Omega^{i,\mu\nu}_{\xi,rsj,\tau\sigma}$$

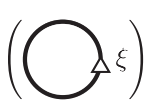


$$, \tag{53}$$

where $\Omega^{i,\mu\nu}_{\xi,rsj,\tau\sigma}$ is zero when either vertex is unstable.

For a longer string operator, one can apply (53) piece by piece, and contract the r,τ or s,σ labels at the connections. In particular, $\Omega^{0,00}_{\xi,rrr,\tau\sigma} = \delta_{\tau\sigma}$ since $\Omega^{0,00}_{\xi,rrr}$ means simply extend the string operator. We define $N_{\xi,r} = \mathrm{Tr}(\Omega^{0,00}_{\xi,rrr})$, which means the number of type r strings the string operator ξ decomposes to.

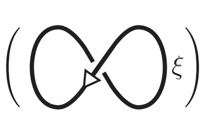
Consider a closed string operator ξ:

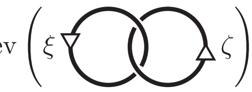
$$\mathrm{ev}\Big($$  $$\Big) = \varkappa_\xi d_\xi = \sum_r N_{\xi,r} O_r = \sum_r N_{\xi,r}\varkappa_r d_r. \tag{54}$$

If ξ is simple and $N_{\xi,r} > 0$, $N_{\xi,s} > 0$, there must be some i,j,μ,ν,τ,σ such that $\Omega^{i,\mu\nu}_{\xi,rsj,\tau\sigma} \neq 0$. Otherwise, ξ is reducible, $\xi = \xi_1 \oplus \xi_2$ where ξ_1 does not contain s, $N_{\xi_1,s} = 0$, and ξ_2 does not contain r, $N_{\xi_2,r} = 0$. $\Omega^{i,\mu\nu}_{\xi,rsj,\tau\sigma} \neq 0$ implies that $N_{i^*s^*j} > 0$, $N_{irj^*} > 0$, and due to (36), $O_r O_i O_j > 0$, $O_s O_i O_j > 0$. Thus, we have $O_r O_s > 0$, $\varkappa_r = \varkappa_s$, which is also the same as $\varkappa_\xi$. In other words, when ξ is simple, $\varkappa_\xi = \varkappa_r$ for $N_{\xi,r} > 0$. Therefore, the quantum dimension of simple quasiparticle ξ is

$$d_\xi = \sum_r N_{\xi,r} d_r. \tag{55}$$

The quasiparticle spin and S matrix can be expressed in terms of string operators. For simple quasiparticles ξ,ζ,

$$\overline{T_\xi} = \mathrm{e}^{-\mathrm{i}\theta_\xi} = \frac{1}{d_\xi}\,\mathrm{ev}\Big($$  $$\Big), \tag{56}$$

$$S_{\xi\zeta} = \frac{1}{D_{\mathcal{Z}(\mathcal{C})}}\,\mathrm{ev}\Big($$  $$\Big), \tag{57}$$

where $D_{\mathcal{Z}(\mathcal{C})} = \sqrt{\sum_\xi d_\xi^2}$ is the total quantum dimension of the quasiparticles. Applying (53) we have

$$\overline{T_\xi} = \mathrm{e}^{-\mathrm{i}\theta_\xi} = \frac{1}{d_\xi}\sum_r O_r^2 \mathrm{Tr}\big(\Omega^{r^*,00}_{\xi,rr0}\big), \tag{58}$$

$$S_{\xi\zeta} = \frac{1}{D_{\mathcal{Z}(\mathcal{C})}} \sum_{rst\mu\nu} \frac{\Theta_{rst}\Theta_{srt}}{O_t} \mathrm{Tr}\big(\Omega^{s,\mu\nu}_{\xi,rrt^*}\big)\mathrm{Tr}\big(\Omega^{r^*,\mu\nu}_{\zeta,s^*s^*t}\big). \tag{59}$$

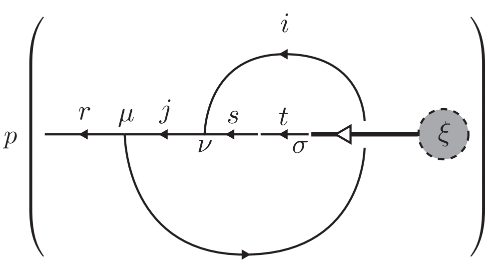
One can find that for some ξ, $S_{\xi\zeta} = \frac{d_\zeta}{D_{\mathcal{Z}(\mathcal{C})}}$. Such ξ is the trivial quasiparticle, and later will be labeled by 1. The quasiparticle fusion rules $N^\chi_{\xi\zeta}$ can be determined from $S_{\xi\zeta}$, which is known as the Verlinde formula [21]:

$$N^\chi_{\xi\zeta} = \sum_\psi \frac{S_{\xi\psi}S_{\zeta\psi}\overline{S_{\chi\psi}}}{S_{1\psi}}, \tag{60}$$

and then we can then identify the antiquasiparticle ξ^* of ξ, which satisfies $N^1_{\zeta\xi^*} = N^1_{\xi^*\zeta} = \delta_{\xi\zeta}$.

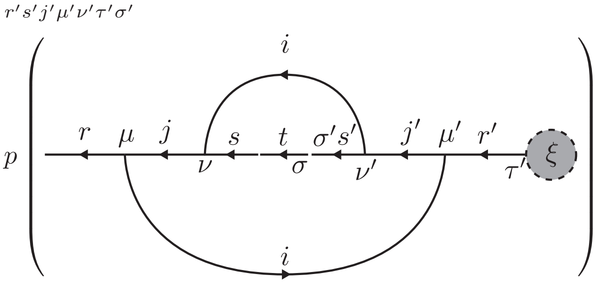
Now look at the graph in (52). We can also use string operator ξ to move the quasiparticle out of the loop, and then do the evaluation. The result should be the same as the representation matrix $M^{i,\mu\nu}_{\xi,rsj,\tau\sigma}$:

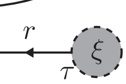
$$\mathrm{ev}\,p$$



$$= \sum_{r's'j'\mu'\nu'\tau'\sigma'} \Omega^{i,\mu'\nu'}_{\xi,r's'j',\tau'\sigma'} \times$$

$$\mathrm{ev}\,p$$



$$= \delta_{st}\sum_\tau \frac{\Theta_{rj^*i}\Theta_{sij^*}}{O_r O_j}\Omega^{i,\mu\nu}_{\xi,rsj,\tau\sigma}$$  $$, \tag{61}$$

Comparing the two results (52) and (61), we get the relations between the module M_ξ and the string operator ξ:

$$M^{i,\mu\nu}_{\xi,rsj,\tau\sigma} = \frac{\Theta_{rj^*i}\Theta_{sij^*}}{O_r O_j}\Omega^{i,\mu\nu}_{\xi,rsj,\tau\sigma}, \tag{62}$$

$$N_{\xi,r} = \mathrm{Tr}\big(M^{0,00}_{\xi,rrr}\big) = \dim\big(Q^{0,00}_{rrr}M_\xi\big). \tag{63}$$

It turns out that the matrix representations of Q-algebra modules and the string operators differ by only some normalizing factors. The statistics in terms of Q-algebra modules is

$$\overline{T_\xi} = \frac{1}{d_\xi}\sum_r O_r \mathrm{Tr}\big(M^{r^*,00}_{\xi,rr0}\big), \tag{64}$$

$$S_{\xi\zeta} = \frac{1}{D_{\mathcal{Z}(\mathcal{C})}}\sum_{rst\mu\nu}\frac{O_r O_s O_t}{\Theta_{rst}\Theta_{srt}}\mathrm{Tr}\big(M^{s,\mu\nu}_{\xi,rrt^*}\big)\mathrm{Tr}\big(M^{r^*,\mu\nu}_{\zeta,s^*s^*t}\big). \tag{65}$$

F. Examples

In the following examples, there are no extra degrees of freedom on the vertices $N_{ijk} \leqslant 1$. Such fusion rules are called multiplicity free. We can omit all the vertex labels. We will first list the necessary data (N_{ijk}, F^{ijm}_{kln}) to define a specific rotation-invariant string-net model. The tensor elements not explicitly given are either 0 or can be calculated from the constraints given in Sec. III B. Second, we give the corresponding Q-algebra. The multiplication is given as a table

	e_b
e_a	e_ae_b

.

In the end we calculate the simple modules over the Q-algebra and $N_{\xi,r}, d_\xi, T_\xi, S_{\xi\zeta}$.

1. Toric code ($\mathbb{Z}_2$) model

The toric code model [22] is the most simple string-net model. We have the following:

(i) Two types of strings, labeled by 0,1 and $1^* = 1$.

(ii) $N_{011} = 1$, $F^{110}_{110} = 1$, $O_1 = 1$.

The Q-algebra is four dimensional. The natural basis is

$$e_{00} = Q^0_{000}, \quad e_{01} = Q^1_{001}, \quad e_{10} = Q^0_{111}, \quad e_{11} = Q^1_{110}.$$

The multiplication is

	e_{00}	e_{01}	e_{10}	e_{11}
e_{00}	e_{00}	e_{01}	0	0
e_{01}	e_{01}	e_{00}	0	0
e_{10}	0	0	e_{10}	e_{11}
e_{11}	0	0	e_{11}	e_{10}

It is easy to see this is the direct sum of two group algebras of $\mathbb{Z}_2$. There are four one-dimensional simple modules:

ξ	1	2	3	4
Basis	$\frac{e_{00}+e_{01}}{2}$	$\frac{e_{00}-e_{01}}{2}$	$\frac{e_{10}+e_{11}}{2}$	$\frac{e_{10}-e_{11}}{2}$
$M^0_{\xi,000}$	1	1	0	0
$M^1_{\xi,001}$	1	−1	0	0
$M^0_{\xi,111}$	0	0	1	1
$M^1_{\xi,110}$	0	0	1	−1
$N_{\xi,0}$	1	1	0	0
$N_{\xi,1}$	0	0	1	1
d_ξ	1	1	1	1
T_ξ	1	1	1	−1

and

$$S = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1\\ 1 & 1 & -1 & -1\\ 1 & -1 & 1 & -1\\ 1 & -1 & -1 & 1\end{pmatrix}.$$

2. Double-semion model

We have the following:

(i) Two types of strings, labeled by 0,1 and $1^* = 1$.

(ii) $N_{011} = 1$, $F^{110}_{110} = -1$, $O_1 = -1$.

The Q-algebra is four dimensional. The natural basis is

$$e_{00} = Q^0_{000}, \quad e_{01} = Q^1_{001},$$
$$e_{10} = Q^0_{111}, \quad e_{11} = Q^1_{110}.$$

The multiplication is

	e_{00}	e_{01}	e_{10}	e_{11}
e_{00}	e_{00}	e_{01}	0	0
e_{01}	e_{01}	e_{00}	0	0
e_{10}	0	0	e_{10}	e_{11}
e_{11}	0	0	e_{11}	$-e_{10}$

If we change the basis $e_{11} \mapsto -\mathrm{i}e_{11}$, this is still the direct sum of two group algebras of $\mathbb{Z}_2$. There are four one-dimensional simple modules:

ξ	1	2	3	4
Basis	$\frac{e_{00}+e_{01}}{2}$	$\frac{e_{00}-e_{01}}{2}$	$\frac{e_{10}-\mathrm{i}e_{11}}{2}$	$\frac{e_{10}+\mathrm{i}e_{11}}{2}$
$M^0_{\xi,000}$	1	1	0	0
$M^1_{\xi,001}$	1	−1	0	0
$M^0_{\xi,111}$	0	0	1	1
$M^1_{\xi,110}$	0	0	i	−i
$N_{\xi,0}$	1	1	0	0
$N_{\xi,1}$	0	0	1	1
d_ξ	1	1	1	1
T_ξ	1	1	i	−i

and

$$S = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1\\ 1 & 1 & -1 & -1\\ 1 & -1 & -1 & 1\\ 1 & -1 & 1 & -1\end{pmatrix}.$$

3. $\mathbb{Z}_N$ model

We have the following:

(i) N types of strings, labeled by $0,1,\ldots,N-1$ and $i^* = N - i$.

(ii) We use $\langle\ldots\rangle_N$ to denote the residual modulo N.

(iii) $N_{ijk} = 1$ iff $\langle i+j+k\rangle_N = 0$.

(iv) $F^{ijm}_{kln} = 1$ iff $m = \langle k+l\rangle_N$, $n = \langle j+k\rangle_N$, $\langle i+j+k+l\rangle_N = 0$.

The Q-algebra is N^2 dimensional. The natural basis is

$$e_{ri} = Q^{\langle -i\rangle_N}_{rr\langle r-i\rangle_N}.$$

The multiplication is

$$e_{ri}e_{sj} = \delta_{rs}e_{r\langle i+j\rangle_N}.$$

This is the direct sum of N group algebras of $\mathbb{Z}_N$. There are N^2 one-dimensional simple modules. We use $\lfloor\ldots\rceil$ to denote a composite label. The simple modules can be labeled with two numbers $\lfloor ri\rceil$. The basis is

$$M^r_{\lfloor ri\rceil} = \frac{1}{N}\sum_{k=0}^{N-1} \mathrm{e}^{-\frac{2\pi\mathrm{i}}{N}ik}e_{rk}.$$

The matrix representations are

$$M^j_{\lfloor ri\rceil,ss\langle s+j\rangle_N} = \delta_{rs}\mathrm{e}^{-\frac{2\pi\mathrm{i}}{N}ij}.$$

Then, we get

$$N_{\lfloor ri\rceil,s} = \delta_{rs},\quad d_{\lfloor ri\rceil} = 1,$$
$$T_{\lfloor ri\rceil} = \mathrm{e}^{-\frac{2\pi\mathrm{i}}{N}ri},\quad S_{\lfloor ri\rceil\lfloor sj\rceil} = \frac{1}{N}\mathrm{e}^{\frac{2\pi\mathrm{i}}{N}(rj+si)}.$$

4. Finite group G model

Similar to the $\mathbb{Z}_N$ case we can define the rotation-invariant string-net model for a finite group G:

(i) $|G|$ types of strings, labeled by the group elements $g \in G$ and $g^* = g^{-1}$.

(ii) The trivial string is now labeled by $\mathbf{1}$, the identity element of G.

(iii) $N_{g_1g_2g_3} = 1$ iff $g_1g_2g_3 = \mathbf{1}$.

(iv) $F^{g_1g_2g_5}_{g_3g_4g_6} = 1$ iff $g_5 = g_3g_4$, $g_6 = g_2g_3$, $g_1g_2g_3g_4 = \mathbf{1}$.

The Q-algebra is $|G|^2$ dimensional and the natural basis is

$$e_{gh} = Q^{\lfloor h^{-1}\rceil}_{g\lfloor h^{-1}gh\rceil\lfloor h^{-1}g\rceil}.$$

The multiplication is

$$e_{gh}e_{g'h'} = \delta_{g\lfloor hg'h^{-1}\rceil}e_{g\lfloor hh'\rceil}.$$

It turns out that the Q-algebra is the Drinfeld double $D(G)$ of the finite group G. The modules over $D(G)$ have been well studied. Some examples of the T,S matrices of $D(G)$ can be found in Refs. [23,24]. In particular if G is Abelian, $D(G)$ is the direct sum of $|G|$ group algebras of G, and there are $|G|^2$ one-dimensional simple modules.

5. Doubled Fibonacci phase

We have the following:

(i) Two types of strings, labeled by 0,1 and $1^* = 1$.

(ii) $N_{011} = N_{111} = 1$, $O_1 = \gamma = \frac{1+\sqrt{5}}{2}$.

(iii) $F^{110}_{110} = \gamma^{-1}, F^{111}_{110} = F^{110}_{111} = \gamma^{-1/2}, F^{111}_{111} = -\gamma^{-1}$.

The Q-algebra is seven dimensional. The natural basis is

$$e_1 = Q^0_{000},\quad e_2 = Q^1_{001},\quad e_3 = Q^0_{111},$$
$$e_4 = Q^1_{110},\quad e_5 = Q^1_{111},\quad e_6 = Q^1_{011},\quad e_7 = Q^1_{101}.$$

The multiplication is

	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	e_1	e_2	0	0	0	e_6	0
e_2	e_2	e_1+e_2	0	0	0	$-\frac{1}{\gamma}e_6$	0
e_3	0	0	e_3	e_4	e_5	0	e_7
e_4	0	0	e_4	$\frac{1}{\gamma}e_3+\frac{1}{\sqrt{\gamma}}e_5$	$\frac{1}{\sqrt{\gamma}}e_3-\frac{1}{\gamma}e_5$	0	e_7
e_5	0	0	e_5	$\frac{1}{\sqrt{\gamma}}e_3-\frac{1}{\gamma}e_5$	$-\frac{1}{\gamma}e_3+e_4-\frac{1}{\gamma^2\sqrt{\gamma}}e_5$	0	$\frac{1}{\gamma\sqrt{\gamma}}e_7$
e_6	0	0	e_6	e_6	$\frac{1}{\gamma\sqrt{\gamma}}e_6$	0	$\sqrt{\gamma}e_1-\frac{1}{\sqrt{\gamma}}e_2$
e_7	e_7	$-\frac{1}{\gamma}e_7$	0	0	0	$\frac{1}{\sqrt{\gamma}}e_3+\frac{1}{\sqrt{\gamma}}e_4+\frac{1}{\gamma^2}e_5$	0

To decompose this algebra we can do idempotent decomposition. First, $\mathbf{1} = e_1 + e_3$. Second, as stated in Appendix B, since $Q_{01} = \langle e_6\rangle, Q_{10} = \langle e_7\rangle$, we immediately obtain two primitive orthogonal idempotents

$$h_1 = \sqrt{\frac{\gamma}{5}}e_6e_7 = \frac{1}{\sqrt{5}\gamma}(\gamma^2e_1 - \gamma e_2),$$
$$h_2 = \sqrt{\frac{\gamma}{5}}e_7e_6 = \frac{1}{\sqrt{5}\gamma}\left(\gamma e_3 + \gamma e_4 + \frac{1}{\sqrt{\gamma}}e_5\right).$$

Third, since $\dim[(e_1 - h_1)Q(e_1 - h_1)] = 1$,

$$h_3 = e_1 - h_1 = \frac{1}{\sqrt{5}\gamma}(e_1 + \gamma e_2)$$

is another primitive orthogonal idempotent. Fourth, since $\dim[(e_3 - h_2)Q(e_3 - h_2)] = 2$ we can solve for the two primitive orthogonal idempotents in $(e_3 - h_2)Q(e_3 - h_2)$:

$$h_4 + h_5 = e_3 - h_2,$$
$$h_4 = \frac{1}{\sqrt{5}\gamma}\left(e_3 + \mathrm{e}^{-\frac{4\pi\mathrm{i}}{5}}e_4 + \sqrt{\gamma}\mathrm{e}^{\frac{3\pi\mathrm{i}}{5}}e_5\right),$$
$$h_5 = \frac{1}{\sqrt{5}\gamma}\left(e_3 + \mathrm{e}^{\frac{4\pi\mathrm{i}}{5}}e_4 + \sqrt{\gamma}\mathrm{e}^{-\frac{3\pi\mathrm{i}}{5}}e_5\right).$$

The final primitive orthogonal idempotent decomposition is

$$\mathbf{1} = h_1 + h_2 + h_3 + h_4 + h_5.$$

The Q-algebra can now be decomposed as it own module

$$Q = Qh_1 \oplus Qh_2 \oplus Qh_3 \oplus Qh_4 \oplus Qh_5$$

and the two two-dimensional modules are isomorphic

$$Qh_1 = \langle h_1, e_7 \rangle \cong Qh_2 = \langle e_6, h_2 \rangle.$$

We have made a special choice of the basis so that the representation matrices look nice. However, this is not necessary. The statistics depends on only the traces:

ξ	1	2	3	4
Basis	h_3	h_4	h_5	$h_1, \sqrt[4]{\gamma/5}e_7$ or $\sqrt[4]{\gamma/5}e_6, h_2$
$M^0_{\xi,000}$	1	0	0	$\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}$
$M^1_{\xi,001}$	γ	0	0	$\begin{pmatrix}-\gamma^{-1} & 0\\ 0 & 0\end{pmatrix}$
$M^0_{\xi,111}$	0	1	1	$\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}$
$M^1_{\xi,110}$	0	$\mathrm{e}^{\frac{4\pi\mathrm{i}}{5}}$	$\mathrm{e}^{-\frac{4\pi\mathrm{i}}{5}}$	$\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}$
$M^1_{\xi,111}$	0	$\sqrt{\gamma}\mathrm{e}^{-\frac{3\pi\mathrm{i}}{5}}$	$\sqrt{\gamma}\mathrm{e}^{\frac{3\pi\mathrm{i}}{5}}$	$\begin{pmatrix}0 & 0\\ 0 & \gamma^{-3/2}\end{pmatrix}$
$M^1_{\xi,011}$	0	0	0	$\begin{pmatrix}0 & \sqrt[4]{5/\gamma}\\ 0 & 0\end{pmatrix}$
$M^1_{\xi,101}$	0	0	0	$\begin{pmatrix}0 & 0\\ \sqrt[4]{5/\gamma} & 0\end{pmatrix}$
$N_{\xi,0}$	1	0	0	1
$N_{\xi,1}$	0	1	1	1
d_ξ	1	γ	γ	γ^2
T_ξ	1	$\mathrm{e}^{-\frac{4\pi\mathrm{i}}{5}}$	$\mathrm{e}^{\frac{4\pi\mathrm{i}}{5}}$	1

and

$$S = \frac{1}{\sqrt{5}\gamma}\begin{pmatrix} 1 & \gamma & \gamma & \gamma^2 \\ \gamma & -1 & \gamma^2 & -\gamma \\ \gamma & \gamma^2 & -1 & -\gamma \\ \gamma^2 & -\gamma & -\gamma & 1 \end{pmatrix}.$$

IV. GENERALIZED STRING-NET MODELS

From now on we will drop the assumption that evaluation of the string-nets is rotation invariant. We are going to choose a preferred orientation of the string-nets, from bottom to top, and we can then safely drop the arrows in the graphs. We will also change our notations of fusion rules and F-matrices to a less symmetric version $N^k_{ij}, F^{ijk}_{l;nm}$. The trivial string type is not assumed to be totally invisible.

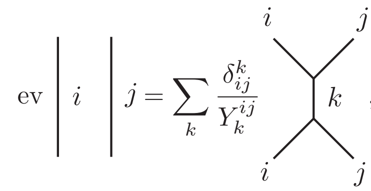
The generalized string-net model in arbitrary gauge is defined as follows.

A. String types and fusion rules

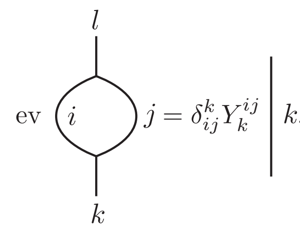
The string types are given by a label set L. Strings can fuse and split. For simplicity we consider multiplicity-free fusion rules $N^k_{ij} = \delta^k_{ij} \in \{0,1\}$ in this section, so there are no vertex labels. But it is quite straightforward to generalize to fusion rules with multiplicity, as in the previous section. The fusion rules satisfy

$$\sum_m N^m_{ij} N^l_{mk} = \sum_n N^l_{in} N^n_{jk}. \tag{66}$$

For each splitting or fusion vertex, there is a nonzero number Y^{ij}_k:



(67)

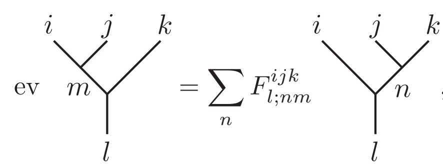


(68)

There is a trivial string type, labeled by 0, and $N^k_{0i} = N^k_{i0} = \delta_{ik}$, and there is an involution of the label set L, $i \mapsto i^*$. i^* is called the dual type of i, and $N^0_{ij} = N^0_{ji} = \delta_{ij^*}$.

B. F-move and pentagon equations

We only need to assume one kind of F-move



(69)

$F^{ijk}_{l;nm} = 0$ if $N^m_{ij} N^l_{mk} N^l_{in} N^n_{jk} = 0$. F^{ijk}_l are invertible matrices and satisfy the pentagon equations

$$\sum_n F^{jkl}_{q;pn} F^{inl}_{s;qr} F^{ijk}_{r;nm} = F^{ijp}_{s;qm} F^{mkl}_{s;pr}. \tag{70}$$

We see that $F^{ijk}_{0;nm} = \omega^{ijk}\delta_{in^*}\delta_{km^*}\delta^{k^*}_{ij}$ is just a number. And we can express the invertible matrix of F^{ijk}_l in terms of F^{ijk}_l, ω^{ijk}. Consider the following pentagon equations:

$$\sum_n F^{jkl^*}_{i^*;pn} F^{inl^*}_{0;i^*l} F^{ijk}_{l;nm} = F^{ijp}_{0;i^*m} F^{mkl^*}_{0;pl}, \tag{71}$$

$$\sum_m F^{ijk}_{l;nm} F^{l^*mk}_{0;lk^*} F^{l^*ij}_{k^*;mp} = F^{l^*in}_{0;lp} F^{pjk}_{0;nk^*}, \tag{72}$$

we have

$$\left(F^{ijk}_l\right)^{-1}_{mn} = \frac{\omega^{inl^*}}{\omega^{ijm^*}\omega^{mkl^*}} F^{jkl^*}_{i^*;m^*n} = \frac{\omega^{l^*mk}}{\omega^{l^*in}\omega^{n^*jk}} F^{l^*ij}_{k^*;mn^*}. \tag{73}$$

This is like "rotating" the F-matrix by 90°. We see that evaluation is no longer rotation invariant, and the difference after rotations is controlled by F^{ijk}_0. This explains why we have to assume that the trivial strings are not totally invisible. Trivial strings can still be added, removed, or deformed, which will introduce isomorphisms between different ground state subspaces. But, unlike the rotation-invariant case, these isomorphisms can be highly nontrivial.

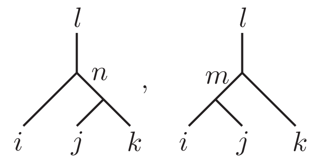
If we "rotate" once more, we find

$$F^{ijk}_{l;nm} = \frac{\omega^{jm^*i}\omega^{ijm^*}\omega^{mkl^*}}{\omega^{jkn^*}\omega^{nl^*i}\omega^{inl^*}} F^{kl^*i}_{j^*;n^*m^*}, \tag{74}$$

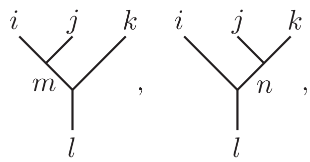
thus 360° rotation implies

$$\frac{\omega^{jm^*i}\omega^{ijm^*}\omega^{mkl^*}\omega^{l^*mk}\omega^{kl^*m}\omega^{m^*ij}}{\omega^{jkn^*}\omega^{nl^*i}\omega^{inl^*}\omega^{l^*in}\omega^{n^*jk}\omega^{kn^*j}} = 1. \tag{75}$$

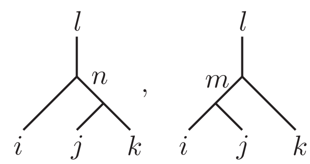
Since we choose a special orientation for the generalized string-net model, there are also other kinds of F-moves. If we stack



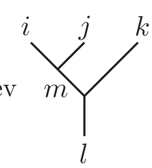
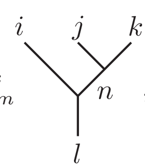
onto

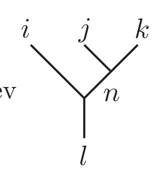
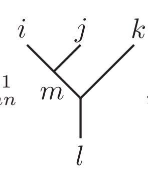


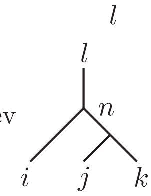
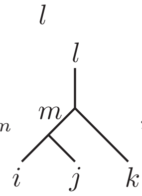
we can evaluate the amplitude using F^{ijk}_l and Y^{ij}_k. This way we can find out what should the F-moves between

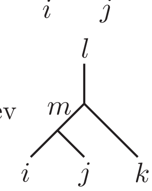
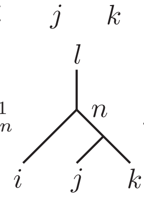


look like. Now we have four kinds of F-move:

ev  $= \sum_n F^{ijk}_{l;nm}$  , (76)

ev  $= \sum_m (F^{ijk}_l)^{-1}_{mn}$  , (77)

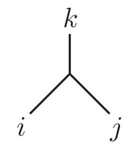
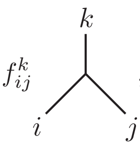
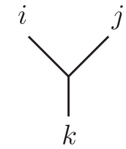
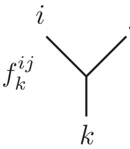
ev  $= \sum_m \frac{Y^{jk}_n Y^{in}_l}{Y^{ij}_m Y^{mk}_l} F^{ijk}_{l;nm}$  , (78)

ev  $= \sum_n \frac{Y^{ij}_m Y^{mk}_l}{Y^{jk}_n Y^{in}_l} (F^{ijk}_l)^{-1}_{mn}$  .

(79)

C. Gauge transformation and quantum dimension

A gauge of the string-net model is a choice of fusion or splitting vertices. Thus, a gauge transformation is nothing but a change of basis. For the case of multiplicity-free fusion rules, it can be given by a set of nonzero numbers f^{ij}_k, f^k_{ij}:

 $\mapsto f^k_{ij}$  ,  $\mapsto f^{ij}_k$  ,

(80)

$$Y^{ij}_k \mapsto \tilde{Y}^{ij}_k = f^{ij}_k f^k_{ij} Y^{ij}_k, \tag{81}$$

$$F^{ijk}_{l;nm} \mapsto \tilde{F}^{ijk}_{l;nm} = \frac{f^{ij}_m f^{mk}_l}{f^{jk}_n f^{in}_l} F^{ijk}_{l;nm}. \tag{82}$$

Gauge transformations should not affect the physics of the system. Physical quantities, such as the T,S matrices, should be gauge invariant.

In addition, we assume that $F^{ijk}_l = \mathbf{1}$ if any of i,j,k is the trivial type 0 and $Y^{i0}_i = Y^{0i}_i = 1$. But, essentially these correspond to a convenient gauge choice (see Appendix C). With this assumption the gauge transformation is slightly restricted:

$$f^{i0}_i = f^{0i}_i = f^{00}_0, \quad f^i_{i0} = f^i_{0i} = f^0_{00} = \left(f^{00}_0\right)^{-1}. \tag{83}$$

We want to point out that, by choosing a special direction, the string-net model with tetrahedron-rotational symmetry can be mapped to the generalized string-net model with a rotation-invariant gauge. And, one can see that the resulting rotation-invariant gauge must satisfy $N^k_{ij} = N_{ijk^*}$, $Y^{i^*i}_0 = O_i$, $Y^{ij}_k = \frac{\Theta_{ijk^*}}{O_k}$, $F^{ijk}_{l;nm} = F^{j^*i^*m}_{lk^*n}$ and other conditions of the tetrahedron-rotational symmetry. A generalized string-net model may not always allow a rotation-invariant gauge.

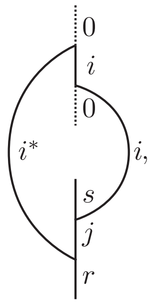
In the rotation-invariant case, we assumed that F-matrices are unitary, which is a physical requirement. But, now we allow arbitrary gauge transformations, which may break the unitary condition of F-matrices. Thus, we slightly weaken the condition: There exists a *unitary gauge* such that F^{ijk}_l are unitary matrices. For generalized string-net models, we prefer to work in the unitary gauge where all $Y^{ij}_k = 1$. Note that in a unitary gauge $F^{i^*ii^*}_{i^*;00} = (F^{ii^*i}_i)^{-1}_{00} = \overline{F^{ii^*i}_{i;00}}$. We can define a gauge invariant quantity

$$d_i = \frac{1}{\sqrt{F^{ii^*i}_{i;00} F^{i^*ii^*}_{i^*;00}}}, \tag{84}$$

which is the quantum dimension of the type i string. Thus, it is also required that for any string type i, $F^{ii^*i}_{i;00} \neq 0$, which is necessary for defining quantum dimensions.

D. Q-algebra and quasiparticle statistics

The Q-algebra in arbitrary gauge is

$$Q^i_{rsj} =$$ 

(85)

$$Q^i_{rsj}\,Q^k_{s'tl} = \delta_{ss'}\sum_{mn} F^{k^*li}_{j;ns}F^{tki}_{n;ml}\big(F^{i^*k^*n}_{r}\big)^{-1}_{m^*j} \times \big(F^{k^*ki}_{i}\big)^{-1}_{0m}F^{i^*k^*m}_{0;im^*}\frac{Y^{k^*k}_0 Y^{i^*i}_0}{Y^{m^*m}_0}\,Q^m_{rtn}. \quad (86)$$

In the rotation-invariant gauge the string operators are well defined and can be obtained from the matrix representations of the Q-algebra. But, in arbitrary gauge, since there is a preferred direction, it is not quite obvious how to construct a closed string operator. However, note that different gauges just mean that we choose different bases of the Q-algebra, we know that the difference between string operators and matrix representations of the Q-algebra is at most some factors depending on the choice of gauge.

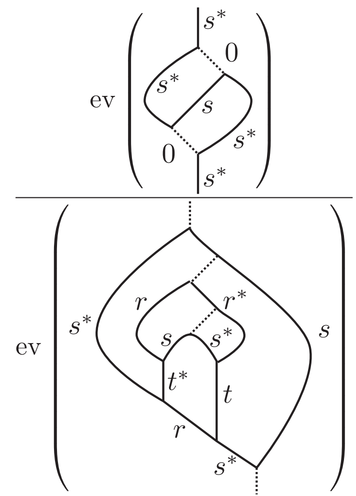
Therefore, similarly to the rotation-invariant case, if we have found the irreducible matrix representations of the Q-algebra, we can calculate the quasiparticle statistics. The number of quasiparticle types is just the number of different irreducible representations up to similarity transformation. We can also calculate the T,S matrices. Use ξ to label irreducible representations, assuming the representation matrix of Q^i_{rsj} is $M^i_{\xi,rsj}$, and we have

$$\overline{T_\xi} = \frac{1}{d_\xi}\sum_r d_r^2 C(T,r)\mathrm{Tr}\big(M^{r^*}_{\xi,rr0}\big), \quad (87)$$

$$S_{\xi\zeta} = \frac{1}{D_{\mathcal{Z}(\mathcal{C})}}\sum_{rst} d_r d_s C(S,r,s,t) \times \mathrm{Tr}\big(M^{s}_{\xi,rrt^*}\big)\mathrm{Tr}\big(M^{r^*}_{\zeta,s^*s^*t}\big), \quad (88)$$

where $d_\xi = \sum_r \mathrm{Tr}(M^0_{\xi,rrr})d_r$, $D_{\mathcal{Z}(\mathcal{C})} = \sqrt{\sum_\xi d_\xi^2}$, and $C(T,r), C(S,r,s,t)$ are undetermined factors that make the expressions gauge invariant. To determine $C(T,r), C(S,r,s,t)$, the basic idea is to use the vertices in Q^i_{rsj} to rebuild a ground state graph, whose gauge transformation will cancel that of the trace term. The result should agree with the special case (64) and (65) in the rotation-invariant gauge.

The graph to cancel the gauge transformation of $\mathrm{Tr}(M^{r^*}_{\xi,rr0})$ is easy to find, simply a closed r loop. Thus, we have $C(T,r) = 1/Y^{rr^*}_0$. However, there are two graphs for $C(S,r,s,t)$. One is



$$= \frac{F^{s^*ss^*}_{s^*;00}F^{t^*s^*r^*}_{0;tr}}{F^{s^*t^*t}_{s^*;0r}F^{rss^*}_{r;0t^*}Y^{rr^*}_0 Y^{s^*s}_0}.$$

(89)

The other can be obtained by permuting the labels $r\to s^*, s\to r^*, t\to t^*$. The two graphs should give the same amplitude, i.e.,

$$\frac{F^{s^*ss^*}_{s^*;00}F^{t^*s^*r^*}_{0;tr}}{F^{s^*t^*t}_{s^*;0r}F^{rss^*}_{r;0t^*}} = \frac{F^{rr^*r}_{r;00}F^{trs}_{0;t^*s^*}}{F^{rtt^*}_{r;0s^*}F^{s^*r^*r}_{s^*;0t}}. \quad (90)$$

Amazingly, this is true due to the pentagon equations. One can prove this using (74) and (75) and the following pentagon equation:

$$F^{srt}_{0;s^*t^*}F^{s^*t^*t}_{s^*;0r}F^{s^*sr}_{r;t^*0} = F^{s^*ss^*}_{s^*;00}. \quad (91)$$

Finally, we obtain the gauge invariant formulas of T,S matrices

$$\overline{T_\xi} = \frac{1}{d_\xi}\sum_r d_r^2\frac{1}{Y^{rr^*}_0}\mathrm{Tr}\big(M^{r^*}_{\xi,rr0}\big), \quad (92)$$

$$S_{\xi\zeta} = \frac{1}{D_{\mathcal{Z}(\mathcal{C})}}\sum_{rst} d_r d_s \frac{F^{s^*ss^*}_{s^*;00}F^{t^*s^*r^*}_{0;tr}}{F^{s^*t^*t}_{s^*;0r}F^{rss^*}_{r;0t^*}Y^{rr^*}_0 Y^{s^*s}_0} \times \mathrm{Tr}\big(M^{s}_{\xi,rrt^*}\big)\mathrm{Tr}\big(M^{r^*}_{\zeta,s^*s^*t}\big). \quad (93)$$

We want to mention that the mathematical structure underlying generalized string-net models is *category theory*. After generalizing to arbitrary gauge, the data $(N^k_{ij}, F^{ijk}_{l;nm})$ of a generalized string-net model correspond to a fusion category $\mathcal{C}$. Moreover, with the unitary assumption, $\mathcal{C}$ is UFC. The Q-algebra modules correspond to the Drinfeld center $\mathcal{Z}(\mathcal{C})$, which is the unitary modular tensor category that describes the fusion and braiding of the quasiparticles.

E. Example: Twisted quantum double

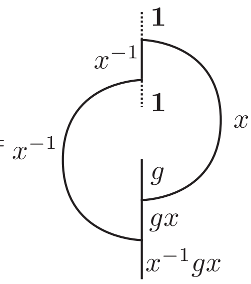
Now, we give a simple example built on a finite group G and its 3-cocycles $H^3[G,U(1)]$:

(i) Label set $L = G$, $N^c_{ab} = \delta_{\lfloor ab\rceil c}$, $Y^{ab}_c = 1$.

(ii) $F^{abc}_{\lfloor abc\rceil;\lfloor bc\rceil\lfloor ab\rceil} = \alpha_{abc}$. $\alpha_{abc} \in H^3[G,U(1)]$ is the 3-cocycle. $\alpha_{abc} = 1$ if any of a,b,c is identity. α_{abc} satisfies the cocycle condition

$$\alpha_{abc}\alpha_{a\lfloor bc\rceil d}\alpha_{bcd} = \alpha_{ab\lfloor cd\rceil}\alpha_{\lfloor ab\rceil cd}. \tag{94}$$

A basis of the Q-algebra is

$$Q^x_g = \tag{95}$$


and

$$Q^y_h Q^x_g = \frac{\alpha_{\lfloor y^{-1}\rceil\lfloor x^{-1}\rceil x}\alpha_{\lfloor x^{-1}\rceil\lfloor gx\rceil y}\alpha_{gxy}}{\alpha_{\lfloor y^{-1}\rceil\lfloor x^{-1}\rceil\lfloor gxy\rceil}\alpha_{\lfloor y^{-1}x^{-1}\rceil xy}}\delta_{\lfloor x^{-1}gx\rceil h}Q^{\lfloor xy\rceil}_g. \tag{96}$$

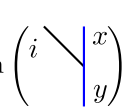
It turns out that $Q^{\rm op}$ (the same algebra Q with the multiplication performed in the reverse order) is isomorphic to the twisted quantum double $D^\alpha(G)$. See Appendix D for the proof.

It is well known that 2D symmetry protected topological (SPT) phases are classified by the 3-cocycles $H^3[G,U(1)]$ [25]. While in this example, when the fusion rules are given by the group G, the generalized string-net models, up to gauge transformations, are also in one-to-one correspondence with 3-cocycles in $H^3[G,U(1)]$. This example indicates that there may be deeper relations between generalized string-net models and SPT phases [26–28].

V. BOUNDARY THEORY OF STRING-NET MODELS

We have used tensors $(N^k_{ij},F^{ijk}_{l;mn})$ to label different string-net models processing different topological orders. Here, we like to follow a similar scheme as in the bulk to construct the (gapped) boundary theory of string-net models [11]. In particular, we want to find the tensors that label different types of boundaries for a given bulk string-net model labeled by the UFC $\mathcal{C}$, or $(N^k_{ij},F^{ijk}_{l;mn})_{\mathcal{C}}$.

First, we still assume the degrees of freedom at the boundary have the form of string-nets. We need a label set B to label the boundary string types. To distinguish from the bulk string types, we add a underline to the boundary string type labels: $\underline{x},\underline{y},\ldots \in B$. Again, the bulk strings can fuse with the boundary strings. There are fusion rules

$$N^{\underline{y}}_{i\underline{x}} = \dim \left(\quad \right), \tag{97}$$


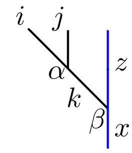
satisfying

$$\sum_y N^{\underline{y}}_{i\underline{x}}N^{\underline{z}}_{j\underline{y}} = \sum_k N^k_{ij}N^{\underline{z}}_{k\underline{x}}, \quad N^{\underline{y}}_{0\underline{x}} = \delta_{xy}, \tag{98}$$

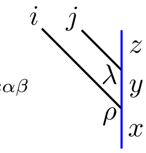
or in matrix form

$$N_iN_j = \sum_k N^k_{ij}N_k, \quad N_0 = \mathbf{1}, \tag{99}$$

where the entries of matrix N_i are $N_{i,xy} = N^{\underline{y}}_{i\underline{x}}$. There are similar F-moves on the boundary



$$= \sum_{y\lambda\rho} F^{ij\underline{z}}_{\underline{x};\underline{y}\lambda\rho,k\alpha\beta} \tag{100}$$

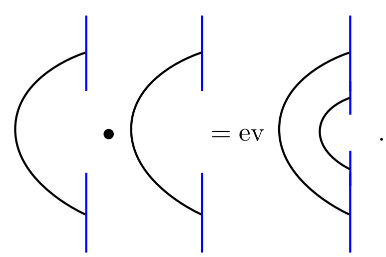


which also satisfy the pentagon equations

$$\sum_{n\tau\lambda\eta} F^{ijk}_{l;n\eta\lambda,m\alpha\beta}F^{in\underline{z}}_{\underline{w};\underline{x}\tau\mu,l\lambda\gamma}F^{jk\underline{z}}_{\underline{x};\underline{y}\rho\nu,n\eta\tau} = \sum_\sigma F^{mk\underline{z}}_{\underline{w};\underline{y}\rho\sigma,l\beta\gamma}F^{ij\underline{y}}_{\underline{w};\underline{x}\nu\mu,m\alpha\sigma}. \tag{101}$$

With the boundary fusion rules $N^{\underline{y}}_{i\underline{x}}$ and the boundary F-matrices $F^{ij\underline{z}}_{\underline{x};\underline{y}\lambda\rho,k\alpha\beta}$ we can similarly define evaluation maps and then the Hamiltonians on the boundary as what we did in the bulk. This way we have a gapped boundary theory of the string-net model, labeled by $(N^{\underline{y}}_{i\underline{x}},F^{ij\underline{z}}_{\underline{x};\underline{y}\lambda\rho,k\alpha\beta})$.

The boundary quasiparticles can also be classified by modules over the boundary Q-algebra [11,12] shown in the following sketch graph:



$$\tag{102}$$

The modules over the boundary Q-algebra form a fusion category $\mathcal{B}$, with another set of data $(N^k_{ij},F^{ijk}_{l;mn})_{\mathcal{B}}$. $\mathcal{B}$ describes the fusion of the boundary quasiparticles. And $\mathcal{B}$ can also be used to construct a string-net model. It is interesting that no matter which boundary we choose, such string-net model constructed from $\mathcal{B}$ always describes the same bulk phase constructed from $\mathcal{C}$, or $\mathcal{Z}(\mathcal{B}) \cong \mathcal{Z}(\mathcal{C})$ [11].

We provide an example of this. Consider the bulk phase described by $\mathbb{Z}_N$ string-net model as in Sec. III F 3. The gapped boundaries and boundary quasiparticles of the $\mathbb{Z}_N$ model are easy to find. (In Ref. [19] this has been done using the language of module category theory.) The boundaries are classified by the integer factors of N. For each integer factor M of N, there is a gapped boundary:

(i) The boundary string type label set is $B = \{0,1,\ldots,M-1\}$.

(ii) The boundary fusion rules are $N^{\underline{y}}_{i\underline{x}} = 1$ iff $y = i + x$ mod M, otherwise $N^{\underline{y}}_{i\underline{x}} = 0$.

(iii) The boundary F-matrices are $F^{ij\underline{z}}_{\underline{x};\underline{y}k} = 1$ for all stable vertices.

There are N types of boundary quasiparticles on this M boundary. The string-net model given by the fusion category of these boundary quasiparticles is as follows:

(i) The string type label set is $L = \mathbb{Z}_M \times \mathbb{Z}_{\frac{N}{M}}$. More precisely the labels are (x,y), $x = 0,1,\ldots,M-1, y = 0,1,\ldots,\frac{N}{M}-1$.

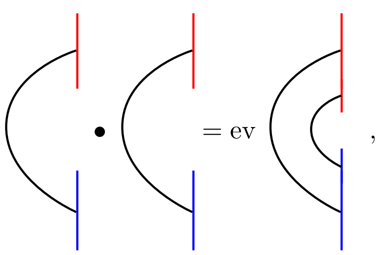
(ii) The fusion rules are given by the group $\mathbb{Z}_M \times \mathbb{Z}_{\frac{N}{M}}$, or $N^{(a_3,b_3)}_{(a_1,b_1)(a_2,b_2)} = \delta_{a_3\langle a_1+a_2\rangle_M}\delta_{b_3\langle b_1+b_2\rangle_{\frac{N}{M}}}$.

(iii) The F-matrices, as in Sec. IV E, are given by the nontrivial 3-cocycle

$$\alpha_{(a_1,b_1)(a_2,b_2)(a_3,b_3)} = \mathrm{e}^{-2\pi \mathrm{i}\frac{a_1}{N}[b_2+b_3-\langle b_2+b_3\rangle_{\frac{N}{M}}]}.$$

By straightforward calculation, one can show that the modular data T,S of the above string-net model are the same as that of the $\mathbb{Z}_N$ model. This relation is independent of the choice of boundary type M.

We can even extend our method to study the boundary changing operators. They should be classified by modules over the boundary changing Q-algebras at the junction of two different boundaries, as the sketch (color online)



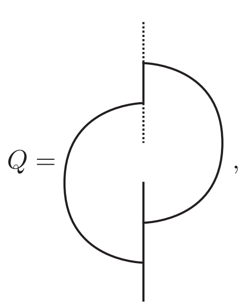
(103)

where the upper red lines and the lower blue lines represent different boundaries.

The formulation of the Q-algebras at the boundaries is very much similar to that in the bulk. We will not elaborate on general formulas of the Q-algebras in this section. Instead, we will give a rather detailed discussion about the twisted $(\mathbb{Z}_N,p)$ string-net model and its boundary theory in Appendix E, which we expect to be helpful for the readers to understand this subject.

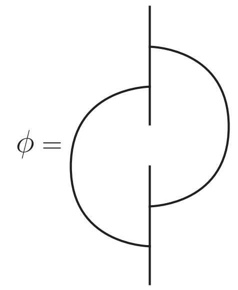
VI. MORITA EQUIVALENCE AND FUSION OF EXCITATIONS

In Sec. IV D we discussed the Q-algebra



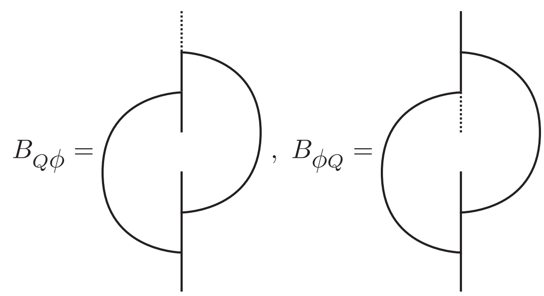
(104)

but as we have mentioned, the Q-algebra is not the only one that is related to quasiparticle excitations. We should also consider, for example, the ϕ-algebra



(105)

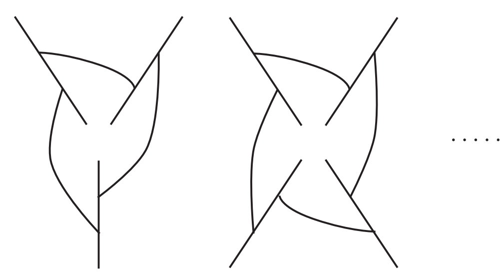
Q-modules are in one-to-one correspondence with ϕ-modules. To see this, consider the following subspaces of ϕ:



(106)

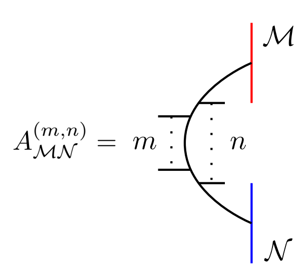
It is not difficult to check that $B_{Q\phi}B_{\phi Q} = Q$ and $B_{\phi Q}B_{Q\phi} = \phi$. Therefore, for a ϕ-module M_ϕ, $B_{Q\phi}M_\phi$ is a Q-module, and for a Q-module M_Q, $B_{\phi Q}M_Q$ is a ϕ-module. Such maps of modules are invertible, $B_{Q\phi}B_{\phi Q}M_Q = M_Q$ and $B_{\phi Q}B_{Q\phi}M_\phi = M_\phi$.

Moreover, there are more complicated algebras, such as



One can similarly show that the modules over these algebras are in one-to-one correspondence; these algebras are Morita equivalent. Therefore, one can take any of these algebras to study the quasiparticles. The physical properties of the quasiparticles do not depend on the choice of algebras.

There are similar Morita equivalent relations for the local operator algebras on boundaries. The most general case is the graph



(107)

where m,n are the number of legs (not string labels). $A^{(0,0)}_{\mathcal{M}\mathcal{M}}$ and $A^{(0,0)}_{\mathcal{M}\mathcal{N}}$ are the boundary Q-algebra and boundary changing Q-algebra discussed before. According to Ref. [12], Lemma 2, $A^{(m,m)}_{\mathcal{MN}}$ and $A^{(n,n)}_{\mathcal{MN}}$ are Morita equivalent algebras; the $A^{(m,m)}_{\mathcal{MN}}$-$A^{(n,n)}_{\mathcal{MN}}$-bimodule $A^{(m,n)}_{\mathcal{MN}}$ is invertible and defines the Morita equivalence, i.e., $A^{(m,n)}_{\mathcal{MN}} \otimes_{A^{(n,n)}_{\mathcal{MN}}} A^{(n,m)}_{\mathcal{MN}} \cong A^{(m,m)}_{\mathcal{MN}}$ as $A^{(m,m)}_{\mathcal{MN}}$-$A^{(m,m)}_{\mathcal{MN}}$-bimodules.

Moreover, it was pointed out by Kong that there are comultiplicationlike maps

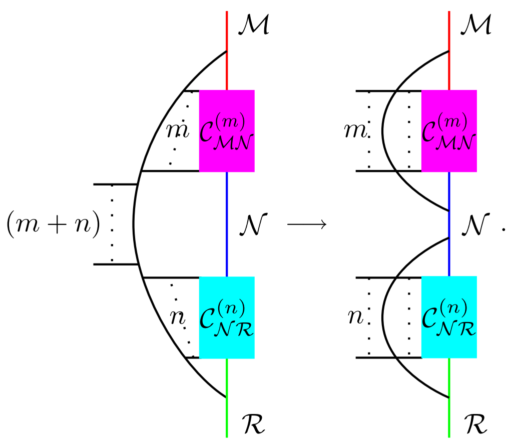
$$\Delta_{m,n,\mathcal{M},\mathcal{N},\mathcal{R}} : A^{(m+n,m+n)}_{\mathcal{MR}} \to A^{(m,m)}_{\mathcal{MN}} \otimes A^{(n,n)}_{\mathcal{NR}}, \tag{108}$$

which control the fusion of boundary quasiparticles or boundary changing operators

$$\otimes_{m,n,\mathcal{M},\mathcal{N},\mathcal{R}} : \mathcal{C}^{(m)}_{\mathcal{MN}} \times \mathcal{C}^{(n)}_{\mathcal{NR}} \to \mathcal{C}^{(m+n)}_{\mathcal{MR}}, \tag{109}$$

where $\mathcal{C}^{(m)}_{\mathcal{MN}}$ is the category of modules over $A^{(m,m)}_{\mathcal{MN}}$.

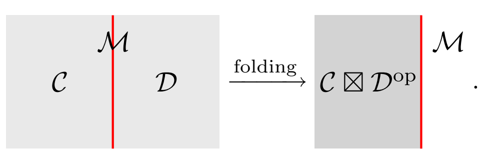
Graphically (color online),



(110)

This picture can be used to compute the F-matrices of the quasiparticles on the boundary.

By the folding trick [11], a $\mathcal{C}$-$\mathcal{D}$ domain wall $\mathcal{M}$ can be viewed as a $\mathcal{C} \boxtimes \mathcal{D}^{\text{op}}$ boundary $\mathcal{M}$:



(111)

As a special case, the ϕ-algebra in the $\mathcal{C}$ bulk can be viewed as the boundary Q-algebra on the $\mathcal{C} \boxtimes \mathcal{C}^{\text{op}}$ boundary $\mathcal{C}$. Therefore, the bulk quasiparticle excitations can also be studied via boundary quasiparticles, as in Ref. [11]. But, for bulk quasiparticles we already know how to compute the T,S matrices, using the simpler Q-algebra, which fully determines the quasiparticle statistics. This approach is only useful if we also want to compute, e.g., the F-matrices and braiding R-matrices of the UMTC $\mathcal{Z}(\mathcal{C})$ that describe the bulk quasiparticles.

VII. MATHEMATICAL STRUCTURE OF OUR CONSTRUCTION

We start with a *unitary fusion category* (UFC). In this paper, a UFC $\mathcal{C}$ is given by the *fusion rules* and *F-matrices*, which satisfy a series of self-consistent conditions. We then use the UFC $\mathcal{C}$ to construct the fixed-point ground state wave function, and the corresponding Levin-Wen Hamiltonian, i.e., a string-net model.

TABLE I. Mathematical structure of string-net models: the excitations are obtained by taking modules over Q-algebras.

	(2+1)D bulk	(1+1)D boundary
Ground states	UFC $\mathcal{C}$	$\mathcal{C}$ module $\mathcal{M}$
Excitations	UMTC $\mathcal{Z}(\mathcal{C}) \cong \mathcal{Z}(\mathcal{C}_\mathcal{M})$	UFC $\mathcal{C}_\mathcal{M}$

The quasiparticle excitations of such a model are given by the Drinfeld center $\mathcal{Z}(\mathcal{C})$ of the UFC $\mathcal{C}$, which is a UMTC. One can take the definition of $\mathcal{Z}(\mathcal{C})$ and solve the corresponding conditions to search for the quasiparticles. However, this is not a finite algorithm. Instead, we introduce a finite algorithm, the Q-algebra approach, to calculate $\mathcal{Z}(\mathcal{C})$. We use the data of $\mathcal{C}$ to construct the Q-algebra, and the quasiparticles correspond to the modules over the Q-algebra. In other words, the UMTC $\mathcal{Z}(\mathcal{C})$ is equivalent to the category of modules over Q-algebra. Our Q-algebra approach to compute the Drinfeld center functor may be a special case of *annularization* [29].

Then, we consider the "natural" boundary of a string-net model given by UFC $\mathcal{C}$. The ground state wave function of the "natural" boundary, similarly, is given by *boundary fusion rules* and *boundary F-matrices*, which are compatible with those in the bulk. Mathematically, such a boundary corresponds to a *module category* $\mathcal{M}$ over $\mathcal{C}$. (Note that a module category over a tensor category is a different notion from a category of modules.) One can use a similar Q-algebra approach to study the quasiparticle excitations on the boundary, i.e., the boundary quasiparticles are modules over the boundary Q-algebra. It turns out that the category of excitations on the $\mathcal{M}$ boundary is again a UFC $\mathcal{C}_\mathcal{M}$, and string-net models given by $\mathcal{C}$ and $\mathcal{C}_\mathcal{M}$ describe the same phase. In other words, $\mathcal{Z}(\mathcal{C}) \cong \mathcal{Z}(\mathcal{C}_\mathcal{M})$, and $\mathcal{M}$ is an invertible $\mathcal{C}$-$\mathcal{C}_\mathcal{M}$-bimodule (or a transparent domain wall between $\mathcal{C}$ and $\mathcal{C}_\mathcal{M}$). Moreover, $\mathcal{C}$ is naturally a $\mathcal{C}$-module, and we know that $\mathcal{C}_\mathcal{C} \cong \mathcal{C}$. That is to say, the UFC $\mathcal{C}$ which we start with can be viewed as a boundary theory of the $\mathcal{Z}(\mathcal{C})$ bulk. The data of excitations on 1D boundaries can be used to construct the 2D bulk string-net ground states. This is the boundary-bulk duality of string-net models. We conclude the discussion above with Table I.

We also want to point out that the boundary changing operators can also be calculated using the Q-algebra approach. The boundary changing operators between boundary $\mathcal{M}$ and boundary $\mathcal{N}$ are the modules over the Q-algebra at the junction of $\mathcal{M}$,$\mathcal{N}$, and form a category which is the invertible $\mathcal{C}_\mathcal{M}$-$\mathcal{C}_\mathcal{N}$-bimodule $\mathcal{C}_{\mathcal{MN}}$. This provides us another holographic picture: The zero-dimensional (0D) boundary changing operators can be used to construct the 1D transparent domain walls. We conclude the holographic relation in Fig. 8. The $\mathcal{C}_\mathcal{M}$, $\mathcal{C}_\mathcal{N}$,

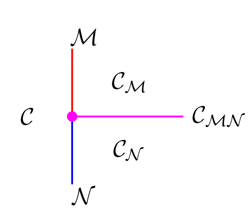


FIG. 8. (Color online) Holographic relation.

and $\mathcal{C}_{\mathcal{MN}}$ on the right side can be viewed either as boundary quasiparticles on $\mathcal{M}$, $\mathcal{N}$ and boundary changing operators between them, or as bulk string-net models and the transparent domain wall between them. In particular, if we take $\mathcal{M}=\mathcal{N}$, the boundary changing operators reduce to the boundary quasiparticles on $\mathcal{M}$, i.e., $\mathcal{C}_{\mathcal{MM}}\cong\mathcal{C}_{\mathcal{M}}$ is a UFC. Also recall that $\mathcal{C}_{\mathcal{C}}\cong\mathcal{C}$, we have $\mathcal{M}\cong\mathcal{C}_{\mathcal{CM}}$. Since string-net models given by $\mathcal{C}$ and $\mathcal{C}_{\mathcal{M}}$ are equivalent, we may start with $\mathcal{C}'=\mathcal{C}_{\mathcal{M}}$ instead of $\mathcal{C}$, but in the end we should arrive at exactly the same structure, in particular, $\mathcal{C}_{\mathcal{MN}}=\mathcal{C}'_{\mathcal{MN}}$. All in all, we conclude all the information of string-net models are included in the pointlike objects, either boundary changing operators or excitations, in the categories $\mathcal{C}_{\mathcal{MN}}$.

VIII. CONCLUSION

Given a many-body ground state wave function and its Hamiltonian, how to compute the topological excitations and their properties? This is one of the fundamental problems in the theory of topologically ordered states. In this paper, we address this issue in a simple situation: We compute the topological excitations and their properties from an ideal many-body ground state wave function (and its ideal Hamiltonian).

The ideal ground state wave function and its ideal Hamiltonian (i.e., the string-net model) is constructed on the data of a UFC, i.e., fusion rules and F-matrices. They satisfy a series of consistent conditions. Using the data of the UFC, we can construct the Q-algebra. We showed that the topological excitations in a string-net model can be classified by the modules over the corresponding Q-algebra. The dimensions of Q-algebras are finite. Like the groups, the canonical representation of the Q-algebra contains all types of irreducible representations. In other words, the Q-algebra contains all types of simple modules as its subspaces. So, it is an efficient approach to study the properties of the quasiparticles by studying the Q-algebra and its modules. Using this approach we calculated the modular data T,S of the quasiparticles. Since the topological excitations are described by a UMTC which is the Drinfeld center of the UFC describing the ground state, our Q-algebra approach can also be viewed as an efficient method to compute the Drinfeld center of a UFC.

The whole scheme to construct string-net models is very general, systematic, and can be naturally generalized to construct the boundary theory. The boundary quasiparticles and boundary changing operators can also be studied via Q-algebras at the boundaries.

It is interesting to note that the particlelike excitations at the boundary of a string-net model are also described by a UFC. The boundary UFC fully determines the bulk, including the UMTC that describe the bulk topological quasiparticles [15]. The bulk UMTC is again given by the Drinfeld center of the boundary UFC. Thus, our Q-algebra approach is an efficient method to compute the bulk properties from the edge properties. It is also a concrete example of the holographic relation between topological orders in different dimensions [15].

ACKNOWLEDGMENTS

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APPENDIX A: A BRIEF INTRODUCTION TO ALGEBRAS AND MODULES

An *algebra* A is a vector space equipped with a multiplication

$$A\otimes A\to A,\quad a\otimes b\mapsto ab. \tag{A1}$$

The multiplication must be bilinear and associative. The identity of the multiplication must exist, i.e., there exists $\mathbf{1}\in A$ such that $\forall\, a\in A, \mathbf{1}a=a\mathbf{1}=a$.

Given an algebra A, we can define the multiplication of the subspaces of A. Let A_1 and A_2 be subspaces of A:

$$A_1A_2:=\left\{\sum_k c^{(k)}a_1^{(k)}a_2^{(k)}\,\middle|\,\begin{matrix}k\in\mathbb{N}, & a_1^{(k)}\in A_1,\\ c^{(k)}\in\mathbb{C}, & a_2^{(k)}\in A_2\end{matrix}\right\} \tag{A2}$$

is still a subspace of A. This is analog to the multiplication of subgroups, but note that here we need to take linear combinations.

Another important notion is the idempotent, which is analog to projection operators. An *idempotent* h in an algebra A is a vector such that $hh=h$. Two idempotents h_1,h_2 are *orthogonal* iff $h_1h_2=h_2h_1=0$. Note that the sum of orthogonal idempotents $h=h_1+h_2$ is still an idempotent. An idempotent h is *primitive* iff it can not be written as sum of nontrivial (i.e., not 0 or h itself) orthogonal idempotents.

We also like to consider *central* elements. A vector a in A is central if it commutes with all other vectors, $ab=ba,\forall\, b\in A$. The *center* of A is the subspace formed by all central elements, denoted by $Z(A)$.

The most simple example is the matrix algebra. Consider the $n\times n$ square matrices $\mathbb{M}_n$. Under usual matrix multiplication $\mathbb{M}_n$ forms an algebra. The identity matrix I_n is the identity of the algebra. A canonical basis of $\mathbb{M}_n$ is E_{ab}. E_{ab} is the matrix with only the (a,b) entry 1 and other entries 0. Then, the matrix multiplication can be written as

$$E_{ab}E_{b'c}=\delta_{bb'}E_{ac}. \tag{A3}$$

$\{E_{aa}\}$ is a set of primitive orthogonal idempotents

$$I_n=\sum_{a=1}^n E_{aa}. \tag{A4}$$

A slightly more complicated case is the direct sum of matrix algebras. Assume that $A=\mathbb{M}_{n_1}\oplus\mathbb{M}_{n_2}\oplus\cdots\oplus\mathbb{M}_{n_\xi}\oplus\cdots\oplus\mathbb{M}_{n_K}$. We know the dimension of A satisfies

$$\dim A=\sum_{\xi=1}^K n_\xi^2. \tag{A5}$$

The elements of A can be written as $(A_1,A_2,\dots,A_\xi,\dots,A_K)$, $A_\xi\in\mathbb{M}_{n_\xi}$. The multiplication is componentwise $(\dots,A_\xi,\dots)(\dots,B_\xi,\dots)=(\dots,A_\xi B_\xi,\dots)$. Equivalently, one may think the elements of A as block-diagonal matrices, with K blocks and the ξth block is $n_\xi\times n_\xi$. Similarly we have a canonical basis $E^{\xi}_{ab} = (0,\ldots,0,E_{ab},0,\ldots,0)$, where E_{ab} is the ξth component

$$E^{\xi}_{ab}E^{\zeta}_{b'c} = \delta_{\xi\zeta}\delta_{bb'}E^{\xi}_{ac}. \tag{A6}$$

$\{E^{\xi}_{aa}\}$ is a set of primitive orthogonal idempotents

$$\mathbf{1} = \left(I_{n_1},\ldots,I_{n_K}\right) = \sum_{\xi=1}^{K}\sum_{a=1}^{n_\xi} E^{\xi}_{aa}. \tag{A7}$$

Note that $(0,\ldots,0,I_{n_\xi},0,\ldots,0)$ are *central* primitive orthogonal idempotents, and $Z(A) = \mathbb{C}(I_{n_1}) \oplus \cdots \oplus \mathbb{C}(I_{n_K})$.

If an algebra A is isomorphic to the direct sum of matrix algebras, we say A is *semisimple*. In other words, if A is semisimple, there exists a basis e^{ξ}_{ab} of A, satisfying

$$e^{\xi}_{ab}e^{\zeta}_{b'c} = \delta_{\xi\zeta}\delta_{bb'}e^{\xi}_{ac}. \tag{A8}$$

We call such basis e^{ξ}_{ab} *canonical*. Finding a canonical basis means that we fully decomposed the algebra, which is usually a nontrivial task. But, we can do idempotent decomposition, i.e., decomposing the identity as the sum of primitive orthogonal idempotents. Each set of primitive orthogonal idempotents corresponds to the "diagonal elements" of a canonical basis e^{ξ}_{aa}. The "off-diagonal elements" e^{ξ}_{ab} can be picked out from $e^{\xi}_{aa}Ae^{\xi}_{bb}$.

A *module* over an algebra A is a vector space M equipped with an A action. A action means that the elements of A can act on M as linear transformations of M. We also require that the A action is linear and associative, and that the identity of A acts on M as the identity transformation. M is invariant under the A action. It is obvious that A can be considered as the module over itself.

Equivalently, we can say there is an algebra homomorphism from A to the linear transformations of M. After choosing a basis of M, one can represent the elements of A by matrices. The matrix representations are equivalent up to basis changes of M, or up to similarity transformations. If we take A as the module over itself, the corresponding matrix representation is called the *canonical representation*.

It is possible that M has some subspace V that is invariant under the A action. Such V is a submodule of M. If M has no submodules other than 0 and itself, we say M is a *simple* module over A.

It is easy to check that, up to isomorphism, the matrix algebra $\mathbb{M}_n$ has only one simple module, the n-dimensional vector space, or the column vector space $\mathbb{M}_{n\times 1}$. If we choose the canonical basis of $\mathbb{M}_{n\times 1}$, the matrix representation is just $\mathbb{M}_n$ itself. We can also think $\mathbb{M}_n$ as it own module. As $\mathbb{M}_n$ module, $\mathbb{M}_n$ is the direct sum of n column vector spaces $\mathbb{M}_{n\times 1}$. The corresponding matrix representation has dimension $n^2 \times n^2$, and is block-diagonal with n blocks of dimension $n \times n$, if we choose the canonical basis E_{ab} of $\mathbb{M}_n$.

Now, we can easily get the properties of modules over semisimple algebras. Assuming that A is a semisimple algebra, $A \cong \mathbb{M}_{n_1} \oplus \cdots \oplus \mathbb{M}_{n_K}$. We know that up to isomorphism, A has K different simple modules of dimension $n_1,\ldots,n_K$. And, A as its own module is the direct sum of these simple modules, in which the simple module of dimension n_ξ appears n_ξ times. Thus, we have the "sum of squares" law $\dim A = \sum_\xi n_\xi^2$. One can also easily check that $\dim[Z(A)]$ = number of central primitive orthogonal idempotents = number of different simple modules.

APPENDIX B: Q-ALGEBRAS IN STRING-NET MODELS WITH TETRAHEDRON-ROTATIONAL SYMMETRY

We discuss the Q-algebra in a string-net model with the tetrahedron-rotational symmetry in detail in this section. We know that the Q-algebra is semisimple [16]. Immediately, we get the powerful "sum of squares" law. Let ξ be the label of simple quasiparticles, and M_ξ be the corresponding simple module

$$\dim Q = \sum_\xi (\dim M_\xi)^2. \tag{B1}$$

This puts a strict constraint on the number of simple quasiparticle types. For example, in doubled Fibonacci phase there are two types of strings and the fusion rules are $N_{000} = N_{011} = N_{111} = 1$. The Q-algebra has dimension 7. Since $7 = 7 \times 1 = 3 \times 1 + 1 \times 2^2$, we know the number of simple quasiparticle types in doubled Fibonacci phase can be only either 7 or 4. Moreover, since the Q-algebra of doubled Fibonacci phase is not a commutative algebra, we must have $\dim[Z(Q)] < 7$, therefore double Fibonacci phase must have four types of quasiparticles.

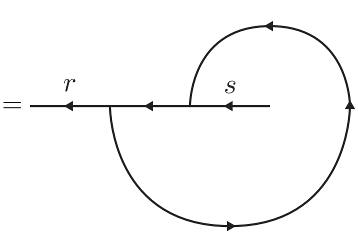
How do we decompose the Q-algebra? A straightforward approach is trying to simultaneously block-diagonalize the representation matrices. But, this is tedious and impractical. A better way is to do idempotent decomposition. Decomposing the algebra is equivalent to decomposing its identity as the sum of primitive orthogonal idempotents

$$\mathbf{1} = \sum_a h_a, \quad h_a h_b = \delta_{ab} h_a \tag{B2}$$

and h_a cannot be further decomposed. With such idempotent decomposition, Qh_a are simple modules and $Q = \oplus_a Qh_a$.

Still, it is not recommended to search for all the idempotents and then try to decompose the identity. As long as the algebra has simple modules of dimension 2 or more, there are infinite many idempotents. It is more practical to decompose the idempotents recursively. Given an idempotent h, if by any means we find an idempotent $h' \in hQh, h' \neq h$, we can decompose h as $h = h' + (h - h')$; otherwise, if such h' does not exist, h is primitive. This way we only need to find one idempotent in hQh, so it is much more efficient. We can always do this recursive decomposition numerically.

Also, note that the identity of subalgebras are essentially idempotents. We can as well search for subalgebras of hQh. For the Q-algebra case this is very useful. To see this, we first define the following subspace of Q:

$Q_{rs} =$ 

(B3)

and

$$Q_{rs}Q_{s't} \subseteq \delta_{ss'}Q_{rt}, \tag{B4}$$

$$\dim Q_{rs} = \sum_{ij} N_{rji}N_{sij} = \mathrm{Tr}(N_rN_s). \tag{B5}$$

Q_{rr} are subalgebras of Q. The identity in Q_{rr} is

$$Q^{0,00}_{rrr} = \longleftarrow r. \tag{B6}$$

And we know that the identity of Q can be decomposed as

$$\mathbf{1} = \sum_r Q^{0,00}_{rrr} = \sum_r \longleftarrow r. \tag{B7}$$

Such decomposition is almost trivial. But, imagine if we continue to decompose $Q^{0,00}_{rrr}$, eventually we will arrive at a "canonical" basis $e^{\xi}_{rs,ab}$ of Q such that

$$e^{\xi}_{rs,ab} \in Q_{rs}, \tag{B8}$$

$$e^{\xi}_{rs,ab}e^{\zeta}_{s't,b'c} = \delta_{\xi\zeta}\delta_{ss'}\delta_{bb'}e^{\xi}_{rt,ac}. \tag{B9}$$

$\{e^{\xi}_{rr,aa}\}$ is a set of primitive orthogonal idempotents

$$\mathbf{1} = \sum_{\xi ra} e^{\xi}_{rr,aa}. \tag{B10}$$

If we fix the labels ξ,s,b in $e^{\xi}_{rs,ab}$ and let r,a vary, they span a subspace $Qe^{\xi}_{ss,bb}$ which is a simple module corresponding to the simple quasiparticle type ξ.

Although for now we cannot explicitly calculate the canonical basis, we know they exist. The existence of such nice basis is significantly helpful for understanding the structure of the Q-algebra. For example, if for some string labels $r \neq s$, $\dim Q_{rs} = 1$, we know that $Q_{sr}Q_{rs}$ and $Q_{rs}Q_{sr}$ are subalgebras of dimension 1. Thus, we obtain two primitive orthogonal idempotents $h_1 \in Q_{sr}Q_{rs}$, $h_2 \in Q_{rs}Q_{sr}$, which are identities of $Q_{sr}Q_{rs}$ and $Q_{rs}Q_{sr}$. We can immediately construct two isomorphic simple modules $Qh_1 \cong Qh_2$. The dimension of Qh_1 or Qh_2 is at least 2.

For doubled Fibonacci phase, it is exactly this case. The dimensions of Q_{rs} subspaces are $\dim Q_{00} = 2$, $\dim Q_{11} = 3$, $\dim Q_{01} = \dim Q_{10} = 1$. Therefore, the Q-algebra of doubled Fibonacci phase has simple modules of dimension at least 2, and due to the "sum of squares" law (B1) the number of simple quasiparticle types must be 4, as we claimed. With the help of the two primitive orthogonal idempotents obtained from $Q_{01}Q_{10}$ and $Q_{10}Q_{01}$, it becomes very easy to do further idempotent decomposition and find out the rest of three simple modules of dimension 1. More details about the doubled Fibonacci phase can be found in Sec. III F 5. We see the power of the Q-algebra approach. By simply examining the dimensions of the Q-algebra and its subspaces, which depend on only the fusion rules N_{ijk}, we obtain the number of simple quasiparticle types of doubled Fibonacci phase. For complicated phases at least we can restrict the number of simple quasiparticle types to several possible values. To get full information of the quasiparticles, such as string operators and the statistics, we still need to fully decompose the algebra and explicitly calculate the simple modules.

APPENDIX C: THE GAUGE TRANSFORMATION THAT FIXES $F^{ijk}_l = 1$ FOR i, j OR k TRIVIAL AND $Y^{0i}_i = Y^{i0}_i = 1$

Recall the pentagon equation (70). Set indices j,k or i,j or k,l to 0, and we have

$$F^{00l}_{l;l0}F^{i0l}_{s;li}F^{i00}_{i;0i} = F^{i0l}_{s;li}F^{i0l}_{s;li}, \tag{C1}$$

$$F^{0kl}_{q;qk}F^{0kl}_{q;qk}F^{00k}_{k;k0} = F^{00q}_{q;q0}F^{0kl}_{q;qk}, \tag{C2}$$

$$F^{j00}_{j;0j}F^{ij0}_{r;jr}F^{ij0}_{r;jr} = F^{ij0}_{r;jr}F^{r00}_{r;0r}. \tag{C3}$$

Thus, we know

$$F^{i0j}_{k;ji} = F^{00j}_{j;j0}F^{i00}_{i;0i}, \tag{C4}$$

$$F^{0ij}_{k;kj} = \frac{F^{00k}_{k;k0}}{F^{00i}_{i;i0}}, \tag{C5}$$

$$F^{ij0}_{k;jk} = \frac{F^{k00}_{k;0k}}{F^{j00}_{j;0j}}. \tag{C6}$$

Therefore, we just to need transform $F^{00i}_{i;0i}$ and $F^{i00}_{i;i0}$ to 1 for all i, then all F^{ijk}_l with i, j, or k trivial will be transformed to $\mathbf{1}$ automatically. Since

$$Y^{i0}_i \mapsto \tilde{Y}^{i0}_i = f^{i0}_i f^i_{i0} Y^{i0}_i, \tag{C7}$$

$$Y^{0i}_i \mapsto \tilde{Y}^{0i}_i = f^{0i}_i f^i_{0i} Y^{0i}_i, \tag{C8}$$

$$F^{i00}_{i;0i} \mapsto \tilde{F}^{i00}_{i;0i} = \frac{f^{i0}_i}{f^{00}_0}F^{i00}_{i;0i}, \tag{C9}$$

$$F^{00i}_{i;i0} \mapsto \tilde{F}^{00i}_{i;i0} = \frac{f^{00}_0}{f^{0i}_i}F^{00i}_{i;i0}, \tag{C10}$$

choosing

$$f^{i0}_i = f^{00}_0\left(F^{i00}_{i;0i}\right)^{-1}, \tag{C11}$$

$$f^{0i}_i = f^{00}_0 F^{00i}_{i;i0}, \tag{C12}$$

$$f^i_{i0} = \left(f^{i0}_i Y^{i0}_i\right)^{-1}, \tag{C13}$$

$$f^i_{0i} = \left(f^{0i}_i Y^{0i}_i\right)^{-1}, \tag{C14}$$

we see that $\tilde{Y}^{i0}_i = \tilde{Y}^{0i}_i = \tilde{F}^{i00}_{i;0i} = \tilde{F}^{00i}_{i;i0} = 1$. But this does not totally fix $f^{i0}_i, f^{0i}_i, f^i_{i0}, f^{0i}_i$; we can still choose arbitrary f^{00}_0. This degree of freedom can be covered by further gauge transformations satisfying (83).

APPENDIX D: ISOMORPHISM BETWEEN Q-ALGEBRA IN SEC. IV E AND TWISTED QUANTUM DOUBLE $D^{\alpha}(G)$

Recall the definition of $D^{\alpha}(G)$. The underlining vector space is $(\mathbb{C}G)^* \otimes \mathbb{C}G$ and the multiplication is given by

$$(g^* \otimes x)(h^* \otimes y) = \delta_{g\lfloor xhx^{-1}\rceil}\frac{\alpha_{gxy}\alpha_{xy\lfloor (xy)^{-1}gxy\rceil}}{\alpha_{x\lfloor x^{-1}gx\rceil y}}g^* \otimes xy. \tag{D1}$$

(Note that here the * symbol denotes dual vectors, but not dual string types.)

By multiplying the following cocycle conditions

$$\alpha_{xy\lfloor y^{-1}\rceil}\alpha_{y\lfloor y^{-1}\rceil\lfloor x^{-1}gxy\rceil} = \alpha_{\lfloor xy\rceil\lfloor y^{-1}\rceil\lfloor x^{-1}gxy\rceil}\alpha_{xy\lfloor (xy)^{-1}gxy\rceil}, \tag{D2}$$

$$\alpha_{x\lfloor x^{-1}\rceil\lfloor gx\rceil}\alpha_{x\lfloor x^{-1}gx\rceil y}\alpha_{\lfloor x^{-1}\rceil\lfloor gx\rceil y} = \alpha_{x\lfloor x^{-1}\rceil\lfloor gxy\rceil}, \tag{D3}$$

$$\alpha_{x\lfloor x^{-1}\rceil\lfloor gxy\rceil}\alpha_{\lfloor xy\rceil\lfloor y^{-1}\rceil\lfloor x^{-1}gxy\rceil} = \alpha_{\lfloor xy\rceil\lfloor y^{-1}\rceil\lfloor x^{-1}\rceil}\alpha_{\lfloor xy\rceil\lfloor (xy)^{-1}\rceil\lfloor gxy\rceil}\alpha_{\lfloor y^{-1}\rceil\lfloor x^{-1}\rceil\lfloor gxy\rceil}, \tag{D4}$$

one can get

$$\alpha_{xy\lfloor y^{-1}\rceil}\alpha_{x\lfloor x^{-1}\rceil\lfloor gx\rceil}\alpha_{\lfloor x^{-1}\rceil\lfloor gx\rceil y}\alpha_{x\lfloor x^{-1}gx\rceil y}\alpha_{y\lfloor y^{-1}\rceil\lfloor x^{-1}gxy\rceil} = \alpha_{\lfloor xy\rceil\lfloor y^{-1}\rceil\lfloor x^{-1}\rceil}\alpha_{\lfloor xy\rceil\lfloor (xy)^{-1}\rceil\lfloor gxy\rceil}\alpha_{\lfloor y^{-1}\rceil\lfloor x^{-1}\rceil\lfloor gxy\rceil}\alpha_{xy\lfloor (xy)^{-1}gxy\rceil} \tag{D5}$$

and thus

$$\frac{\alpha_{\lfloor x^{-1}\rceil\lfloor gx\rceil y}}{\alpha_{\lfloor y^{-1}\rceil\lfloor x^{-1}\rceil\lfloor gxy\rceil}} = \frac{\alpha_{xy\lfloor (xy)^{-1}gxy\rceil}}{\alpha_{x\lfloor x^{-1}gx\rceil y}}\frac{\alpha_{\lfloor xy\rceil\lfloor (xy)^{-1}\rceil\lfloor gxy\rceil}}{\alpha_{x\lfloor x^{-1}\rceil\lfloor gx\rceil}\alpha_{y\lfloor y^{-1}\rceil\lfloor x^{-1}gxy\rceil}} \times \frac{\alpha_{\lfloor xy\rceil\lfloor y^{-1}\rceil\lfloor x^{-1}\rceil}}{\alpha_{xy\lfloor y^{-1}\rceil}}. \tag{D6}$$

Similarly, the following cocycle condition

$$\alpha_{\lfloor y^{-1}\rceil\lfloor x^{-1}\rceil x}\alpha_{\lfloor x^{-1}\rceil xy} = \alpha_{\lfloor y^{-1}x^{-1}\rceil xy}\alpha_{\lfloor y^{-1}\rceil\lfloor x^{-1}\rceil\lfloor xy\rceil} \tag{D7}$$

implies that

$$\frac{\alpha_{\lfloor y^{-1}\rceil\lfloor x^{-1}\rceil x}}{\alpha_{\lfloor y^{-1}x^{-1}\rceil xy}} = \left(\frac{\alpha_{\lfloor x^{-1}\rceil\lfloor x\rceil y}}{\alpha_{\lfloor y^{-1}\rceil\lfloor x^{-1}\rceil\lfloor xy\rceil}}\right)^{-1} = \left(\frac{\alpha_{\lfloor x^{-1}\rceil\lfloor gx\rceil y}}{\alpha_{\lfloor y^{-1}\rceil\lfloor x^{-1}\rceil\lfloor gxy\rceil}}\bigg|_{g=\mathbf{1}}\right)^{-1} = \left(\frac{\alpha_{\lfloor xy\rceil\lfloor (xy)^{-1}\rceil\lfloor xy\rceil}}{\alpha_{x\lfloor x^{-1}\rceil\lfloor x\rceil}\alpha_{y\lfloor y^{-1}\rceil\lfloor x^{-1}xy\rceil}}\frac{\alpha_{\lfloor xy\rceil\lfloor y^{-1}\rceil\lfloor x^{-1}\rceil}}{\alpha_{xy\lfloor y^{-1}\rceil}}\right)^{-1}. \tag{D8}$$

Therefore, we know that

$$\left(\frac{\alpha_{y\lfloor y^{-1}\rceil\lfloor hy\rceil}}{\alpha_{y\lfloor y^{-1}\rceil y}}Q_h^y\right)\left(\frac{\alpha_{x\lfloor x^{-1}\rceil\lfloor gx\rceil}}{\alpha_{x\lfloor x^{-1}\rceil x}}Q_g^x\right) = \delta_{g\lfloor xhx^{-1}\rceil}\frac{\alpha_{gxy}\alpha_{xy\lfloor (xy)^{-1}gxy\rceil}}{\alpha_{x\lfloor x^{-1}gx\rceil y}}\left(\frac{\alpha_{\lfloor xy\rceil\lfloor (xy)^{-1}\rceil\lfloor gxy\rceil}}{\alpha_{\lfloor xy\rceil\lfloor (xy)^{-1}\rceil\lfloor xy\rceil}}Q_g^{\lfloor xy\rceil}\right), \tag{D9}$$

which means that $Q^{\rm op}\cong D^\alpha(G)$ as algebras. Actually, both Q and $D^\alpha(G)$ are quasi-Hopf algebras. One may further check that they are isomorphic as quasi-Hopf algebras.

APPENDIX E: TWISTED $(\mathbb{Z}_N,p)$ STRING-NET MODEL

In this section, we discuss the twisted $(\mathbb{Z}_N,p)$ string-net model and its boundary theory in detail. We know that the generator in $H^3[\mathbb{Z}_N,U(1)]$ is

$$\alpha_{ijk} = e^{2\pi i\frac{1}{N^2}i[j+k-\langle j+k\rangle_N]}. \tag{E1}$$

This model is given by $\mathbb{Z}_N$ fusion rule with the pth 3-cocycle α^p_{ijk}, i.e.,

(i) string label set $L=\mathbb{Z}_N$,
(ii) fusion rule $N^k_{ij}=\delta_{k\langle i+j\rangle_N}$,
(iii) F-matrices $F^{ijk}_{\langle i+j+k\rangle_N;\langle j+k\rangle_N\langle i+j\rangle_N}=\alpha^p_{ijk}$.

1. Q-algebra and bulk quasiparticle excitations

As discussed in Sec. IV E, the Q-algebra of twisted $(\mathbb{Z}_N,p)$ model is given by

$$Q_s^jQ_r^i = \delta_{rs}e^{2\pi i\frac{p}{N^2}i(2r+i-\langle r+i\rangle_N)}e^{2\pi i\frac{p}{N^2}j(2r+j-\langle r+j\rangle_N)} \times e^{-2\pi i\frac{p}{N^2}\langle i+j\rangle_N(2r+\langle i+j\rangle_N-\langle r+i+j\rangle_N)}Q_r^{\langle i+j\rangle_N}. \tag{E2}$$

If we choose the basis

$$\tilde{Q}_r^i = e^{-2\pi i\frac{p}{N^2}i(2r+i-\langle r+i\rangle_N)}Q_r^i, \tag{E3}$$

we see that $\tilde{Q}_s^j\tilde{Q}_r^i=\delta_{rs}\tilde{Q}_r^{\langle i+j\rangle_N}$. Therefore, we find the irreducible representations (labeled by $\lfloor ri\rceil$)

$$M^j_{\lfloor ri\rceil,s} = \delta_{rs}e^{(-2\pi i\frac{ij}{N})}e^{[2\pi i\frac{p}{N^2}j(2r+j-\langle r+j\rangle_N)]}. \tag{E4}$$

Applying (92) and (93) we get

$$T_{\lfloor ri\rceil} = e^{-2\pi i(\frac{ri}{N}-\frac{pr^2}{N^2})}, \tag{E5}$$

$$S_{\lfloor ri\rceil\lfloor sj\rceil} = \frac{1}{N}e^{2\pi i(\frac{rj+si}{N}-\frac{2prs}{N^2})}. \tag{E6}$$

When $p\neq 0$, the fusion rule of the quasiparticles is not simply $\mathbb{Z}_N\times\mathbb{Z}_N$. Using the Verlinde formula,

$$N^{\lfloor tk\rceil}_{\lfloor ri\rceil\lfloor sj\rceil} = \sum_{ql}\frac{S_{\lfloor ri\rceil\lfloor ql\rceil}S_{\lfloor sj\rceil\lfloor ql\rceil}\overline{S_{\lfloor tk\rceil\lfloor ql\rceil}}}{S_{\lfloor 00\rceil\lfloor ql\rceil}} = \delta_{0\langle r+s-t\rangle_N}\delta_{0\langle i+j-k-2p\frac{r+s-t}{N}\rangle_N}. \tag{E7}$$

We also see the equivalent relations of the quasiparticles are

$$\lfloor ri\rceil\sim\lfloor r'i'\rceil \iff r'=r+k_1N,\quad i'=i+2k_1p+k_2N, \tag{E8}$$

where k_1,k_2 are integers.

2. Boundary types

First, we search for possible boundary fusion rules. Note that (99) now becomes

$$N_iN_j = N_{\langle i+j\rangle_N},\quad N_0=\mathbf{1}, \tag{E9}$$

thus it suffices to work out N_1, which is a matrix with non-negative integer entries and $(N_1)^N=N_0=\mathbf{1}$.

We may write the conditions explicitly

$$\sum_{x_1,\ldots,x_{N-1}} N^{x_1}_{1\underline{x_0}}N^{x_2}_{1\underline{x_1}}\ldots N^{x_N}_{1\underline{x_{N-1}}} = \delta_{x_0x_N}. \tag{E10}$$

This is like a "path integral.'. Since all the entries $N^{x_{i+1}}_{1\underline{x_i}}$ are non-negative integers, we know that, starting from a fixed boundary string label $x_0=X_0$, there is only one path $(X_0,X_1,\ldots,X_{N-1},X_N=X_0)$ with $N^{X_{i+1}}_{1\underline{X_i}}=1$, and for all other paths $(X_0,x_1,\ldots,x_{N-1},x_N)$, there is at least one segment $N^{x_{i+1}}_{1\underline{x_i}}=0$.

Similarly, we may start from Y_0 and find a path $(Y_0,Y_1,\ldots,Y_{N-1},Y_N=Y_0)$, $N^{Y_{i+1}}_{1\underline{Y_i}}=1$. Consider the path $(X_0,Y_0,Y_1,\ldots,Y_{N-1})$. It is a different path from

$(X_0,X_1,\ldots,X_{N-1},X_0)$ as long as $Y_0 \neq X_1$, and we have $N^{Y_0}_{1\underline{X_0}} = 0$. Considering the path $(Y_0,Y_1,\ldots,Y_{N-1},X_0)$ we know that if $Y_{N-1} \neq X_{N-1}$, $N^{X_0}_{1\underline{Y_{N-1}}} = 0$. Therefore, there is only one x satisfying $N^{x}_{1\underline{X_0}} = 1$ and only one y satisfying $N^{X_0}_{1\underline{y}} = 1$. We may say two labels x,y are *1-step-connected* if $N^{y}_{1\underline{x}} = 1$. Then, X_0 is only 1-step-connected to X_1 forwards and only 1-step-connected to X_{N-1} backwards. Such analysis applies to any label X_0.

The connection of labels forms an equivalent relation. The discussions above then imply that connected labels form closed paths. If M is the number of different labels in $(X_0,X_1,\ldots,X_{N-1},X_0)$, we know that $X_M = X_0, X_{M+1} = X_1,\ldots$, in general, $X_i = X_{\langle i\rangle_M}$, and since $X_N = X_0$, M must be a factor of N, i.e., $M|N$. Since different closed paths have no intersections, for an *indecomposable* boundary, it suffices to consider the boundary fusion rules

(i) boundary string label set $B = \{0,1,2,\ldots,M-1\}$, where $M|N$,

(ii) boundary fusion rules $N^{y}_{i\underline{x}} = \delta_{y\langle i+x\rangle_M}$.

However, this not the end of story. We need to find the solutions to the boundary pentagon equations (101). With the boundary fusion rules above we may simplify our notation of the boundary F-matrices $F^{ij\underline{x}}_{\underline{\langle i+j+x\rangle_M};\underline{\langle j+x\rangle_M}\langle i+j\rangle_N} = \beta_{ij\underline{x}}$. The pentagon equations (101) become

$$\alpha^{p}_{ijk}\beta_{i\langle j+k\rangle_N\underline{x}}\beta_{jk\underline{x}} = \beta_{\langle i+j\rangle_N k\underline{x}}\beta_{ij\underline{\langle k+x\rangle_M}}. \tag{E11}$$

There are not always solutions to (E11). To see this, we multiply the following M equations

$$\begin{aligned}
\alpha^{p}_{ijk}\beta_{i\langle j+k\rangle_N\underline{x}}\beta_{jk\underline{x}} &= \beta_{\langle i+j\rangle_N k\underline{x}}\beta_{ij\underline{\langle k+x\rangle_M}},\\
\alpha^{p}_{ijk}\beta_{i\langle j+k\rangle_N\underline{\langle x+1\rangle_M}}\beta_{jk\underline{\langle x+1\rangle_M}} &= \beta_{\langle i+j\rangle_N k\underline{\langle x+1\rangle_M}}\beta_{ij\underline{\langle k+x+1\rangle_M}},\\
\alpha^{p}_{ijk}\beta_{i\langle j+k\rangle_N\underline{\langle x+2\rangle_M}}\beta_{jk\underline{\langle x+2\rangle_M}} &= \beta_{\langle i+j\rangle_N k\underline{\langle x+2\rangle_M}}\beta_{ij\underline{\langle k+x+2\rangle_M}},\\
&\vdots\\
\alpha^{p}_{ijk}\beta_{i\langle j+k\rangle_N\underline{\langle x+M-1\rangle_M}}\beta_{jk\underline{\langle x+M-1\rangle_M}} &= \beta_{\langle i+j\rangle_N k\underline{\langle x+M-1\rangle_M}}\\
&\quad\times \beta_{ij\underline{\langle k+x+M-1\rangle_M}}
\end{aligned} \tag{E12}$$

and obtain

$$\alpha^{pM}_{ijk} f_{i\langle j+k\rangle_N} f_{jk} = f_{\langle i+j\rangle_N k} f_{ij}, \tag{E13}$$

where $f_{ij} = \prod_{x=0}^{M-1}\beta_{ij\underline{x}}$. This implies that α^{pM}_{ijk} is equivalent to the trivial cocycle. Therefore, we know M must also satisfy $N|pM$.

On the other hand, for any integer M satisfying $N|pM$ and $M|N$, (E11) does have solutions. But, as in the bulk, there are gauge transformations between equivalent solutions. It is not hard to check that for each M there is only one equivalent class of solutions. We pick a canonical form of the solutions

$$\beta_{ij\underline{x}} = \mathrm{e}^{2\pi\mathrm{i}\frac{p}{N^2}i(j+x-\langle j+x\rangle_M)}. \tag{E14}$$

To conclude, the boundary types of twisted $(\mathbb{Z}_N,p)$ model are classified by integers M satisfying $N|pM, M|N$. The M boundary is given by

(i) boundary string label set $B = \{0,1,2,\ldots,M-1\}$;

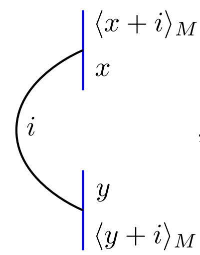
(ii) boundary fusion rules $N^{y}_{i\underline{x}} = \delta_{y\langle i+x\rangle_M}$;

(iii) boundary F-matrices

$$F^{ij\underline{x}}_{\underline{\langle i+j+x\rangle_M};\underline{\langle j+x\rangle_M}\langle i+j\rangle_N} = \beta_{ij\underline{x}}.$$

3. Boundary quasiparticles

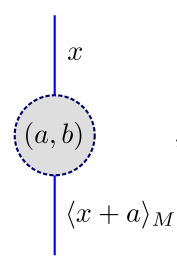
For the M boundary of the $(\mathbb{Z}_N,p)$ model, we classify the boundary quasiparticles by studying the modules over the boundary Q-algebra

$Q^{i}_{xy} =$  $, \quad x = 0,1,\ldots,M-1, \quad y = 0,1,\ldots,M-1, \quad i = 0,1,\ldots,N-1,$ (E15)

$$\begin{aligned}
Q^{i}_{x'y'}Q^{j}_{xy} &= \delta_{x'\langle j+x\rangle_M}\delta_{y'\langle j+y\rangle_M}\frac{\beta_{ij\underline{x}}}{\beta_{ij\underline{y}}}Q^{\langle i+j\rangle_N}_{xy}\\
&= \mathrm{e}^{2\pi\mathrm{i}\frac{p}{N^2}i(x-y+\langle j+y\rangle_M-\langle j+x\rangle_M)}\\
&\quad\times\delta_{x'\langle j+x\rangle_M}\delta_{y'\langle j+y\rangle_M}Q^{\langle i+j\rangle_N}_{xy}.
\end{aligned} \tag{E16}$$

The dimension of this Q-algebra is NM^2. It is easy to get N different M-dimensional simple modules via a bit of observation, guess, and calculation. We know that these are all the simple modules.

In the multiplication rule, $\langle x'-y'\rangle_M = \langle x-y\rangle_M$. Thus, we guess that a simple module can be labeled by (a,b), where a corresponds to the difference between x,y, and b corresponds to the choice of the phase factors. The basis of the (a,b) module is

$e^{(a,b)}_{x} =$  $, \quad x = 0,1,\ldots,M-1,$ (E17)

and the dimension of the (a,b) module is M. The algebra action on the module is

$$Q^{i}_{x'y'}e^{(a,b)}_{x} = \mathrm{e}^{2\pi\mathrm{i}\frac{bi}{N}}\mathrm{e}^{2\pi\mathrm{i}\frac{p}{N^2}i(y'-x'-a)}\delta_{x'x}\delta_{y'\langle x+a\rangle_M}e^{(a,b)}_{\langle x+i\rangle_M}. \tag{E18}$$

It is not hard to check that two modules (a,b) and (a',b') are isomorphic iff

$$a' = a + k_1 M, \quad b' = b + k_1\frac{pM}{N} + k_2\frac{N}{M}. \tag{E19}$$

Thus, we have N different modules and also we got all the possible simple modules. In other words, we got all the boundary quasiparticle types.

We can consider the fusion of the boundary quasiparticles, given by the tensor product of the modules

$$\begin{aligned}&(a_1,b_1)\otimes(a_2,b_2)\to(a_3,b_3),\\ &e_x^{(a_1,b_1)}\otimes e_{x'}^{(a_2,b_2)}\mapsto \mathrm{e}^{-2\pi\mathrm{i}\frac{x}{N}[b_1+b_2-b_3-\frac{p}{N}(a_1+a_2-a_3)]}\\ &\qquad\times\delta_{x'\langle x+a_1\rangle_M}e_x^{(a_3,b_3)},\end{aligned}\tag{E20}$$

where

$$\begin{aligned}a_3&=\langle a_1+a_2\rangle_M\,,\\ b_3&=\left\langle b_1+b_2-\frac{p}{N}(a_1+a_2-a_3)\right\rangle_{\frac{N}{M}}.\end{aligned}\tag{E21}$$

Thus, the fusion category $\mathcal{B}_M$ of the excitations on the M boundary is the following:

(i) Fusion rule

$$N_{(a_1,b_1)(a_2,b_2)}^{(a_3,b_3)}=\delta_{a_3\langle a_1+a_2\rangle_M}\delta_{b_3\langle b_1+b_2-\frac{p}{N}(a_1+a_2-a_3)\rangle_{\frac{N}{M}}}.$$

(ii) For stable vertices, F-matrices

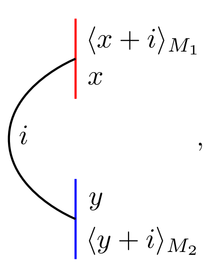
$$F_{(a_4,b_4);(a_6,b_6)(a_5,b_5)}^{(a_1,b_1)(a_2,b_2)(a_3,b_3)}=\mathrm{e}^{-2\pi\mathrm{i}\frac{a_1}{N}[b_2+b_3-b_6-\frac{p}{N}(a_2+a_3-a_6)]}.$$

One can calculate the modular data T,S of $\mathcal{B}_M$ string-net model, which are always the same as those of $(\mathbb{Z}_N,p)$ model, no matter which M boundary we choose. Therefore, $(\mathbb{Z}_N,p)$ and $\mathcal{B}_M$ string-net models describe the same physical phase. Moreover, the M boundary is actually the *transparent domain wall* (mathematically, the invertible bimodule category) between $(\mathbb{Z}_N,p)$ and $\mathcal{B}_M$.

4. Boundary changing operators

Similarly we can find the boundary changing operators via the Q-algebra approach. We now focus at the junction of M_1 boundary (red line) and M_2 boundary (blue line). The corresponding Q-algebra is

$Q^i_{xy}=$



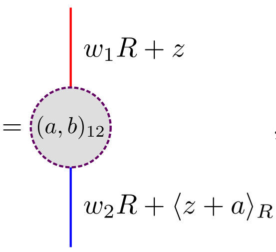
$$x=0,1,\ldots,M_1-1,\quad y=0,1,\ldots,M_2-1,\quad i=0,1,\ldots,N-1,\tag{E22}$$

$$\begin{aligned}Q^i_{x'y'}Q^j_{xy}&=\delta_{x'\langle j+x\rangle_{M_1}}\delta_{y'\langle j+y\rangle_{M_2}}\frac{\beta^{(M_1)}_{ij\underline{x}}}{\beta^{(M_2)}_{ij\underline{y}}}Q^{\langle i+j\rangle_N}_{xy}\\ &=\mathrm{e}^{2\pi\mathrm{i}\frac{p}{N^2}i(x-y+\langle j+y\rangle_{M_2}-\langle j+x\rangle_{M_1})}\\ &\quad\times\delta_{x'\langle j+x\rangle_{M_1}}\delta_{y'\langle j+y\rangle_{M_2}}Q^{\langle i+j\rangle_N}_{xy}.\end{aligned}\tag{E23}$$

The dimension of the Q-algebra is NM_1M_2.

Let R be the greatest common divisor of M_1,M_2, denoted by $R=\gcd(M_1,M_2)$; we can write the basis of a simple module $(a,b)_{12}$

$e^{(a,b)_{12}}_{w_1w_2z}=$



$$w_1=0,1,\ldots,\tfrac{M_1}{R}-1,\quad w_2=0,1,\ldots,\tfrac{M_2}{R}-1,\quad z=0,1,\ldots,R-1,\tag{E24}$$

$$\begin{aligned}Q^i_{xy}e^{(a,b)}_{w_1w_2z}&=\mathrm{e}^{2\pi\mathrm{i}\frac{bi}{N}}\mathrm{e}^{2\pi\mathrm{i}\frac{p}{N^2}i(y-x-a)}\\ &\quad\times\delta_{x\langle w_1R+z\rangle_{M_1}}\delta_{y\langle w_2R+\langle z+a\rangle_R\rangle_{M_2}}\\ &\quad\times e^{(a,b)}_{\langle w_1+\frac{z+i-\langle z+i\rangle_R}{R}\rangle_{\frac{M_1}{R}}\langle w_2+\frac{\langle z+a\rangle_R+i-\langle z+a+i\rangle_R}{R}\rangle_{\frac{M_2}{R}}\langle z+i\rangle_R}.\end{aligned}\tag{E25}$$

We see the dimension of the module $(a,b)_{12}$ is $\frac{M_1M_2}{R}$.

Two simple modules $(a,b)_{12}$ and $(a',b')_{12}$ are isomorphic iff

$$a'=a+k_1R,\quad b'=b+k_1\frac{pR}{N}+k_2\frac{NR}{M_1M_2}.\tag{E26}$$

Therefore, there are $\frac{NR^2}{M_1M_2}$ different simple modules, which satisfies the sum of squares law: $NM_1M_2=\frac{NR^2}{M_1M_2}(\frac{M_1M_2}{R})^2$. We know the $(a,b)_{12}$ modules are all the possible simple modules.

Again, we can say the modules $(a,b)_{12}$ form a category $\mathcal{D}_{12}$. One can always fuse the boundary quasiparticles $(a_1,b_1)_1$ on M_1 boundary and $(a_2,b_2)_2$ on M_2 boundary with the boundary changing operator $(a,b)_{12}$ to get new composite boundary changing operators. Mathematically, this means the tensor products $(a_1,b_1)_1\otimes(a,b)_{12}$ and $(a,b)_{12}\otimes(a_2,b_2)_2$ are still modules in $\mathcal{D}_{12}$. Therefore, $\mathcal{D}_{12}$ is a $\mathcal{B}_{M_1}$-$\mathcal{B}_{M_2}$-bimodule category.

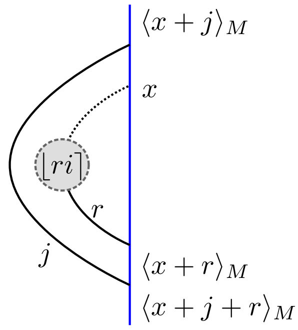
5. Quasiparticles condensing to the boundary: Relation to Lagrangian subgroup

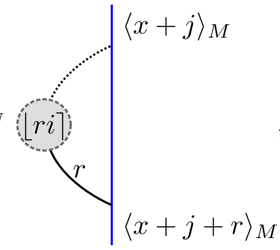
A given topologically ordered state can have many different types of boundaries [11,30–34]. A boundary can be understood in the following way. We can always move a bulk quasiparticle excitation to the boundary, and obtain a boundary quasiparticle. If a quasiparticle moves to the boundary and becomes a trivial boundary quasiparticle, we say the quasiparticle *condenses* [35–37] to the boundary. For Abelian topological phases, it is believed that quasiparticles that can condense to a boundary form a *Lagrangian subgroup*, and Lagrangian subgroups are in one-to-one correspondence to boundary types [31–34,36,38,39]. We will show this correspondence explicitly for the $(\mathbb{Z}_N,p)$ string-net models.

A Lagrangian subgroup $\mathcal{K}$ is a subset of quasiparticle types, such that

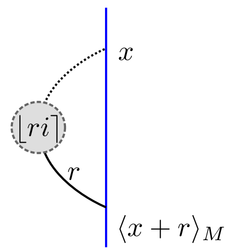
$$\begin{aligned}&\forall\,\xi,\,\zeta\in\mathcal{K},\quad T_\xi=1,\quad D_{\mathcal{Z}(\mathcal{C})}S_{\xi\zeta}=1,\\ &\forall\,\xi'\notin\mathcal{K},\quad\exists\xi\in\mathcal{K},\quad D_{\mathcal{Z}(\mathcal{C})}S_{\xi\xi'}\neq1.\end{aligned}\tag{E27}$$

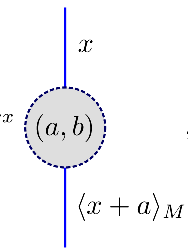
For the $(\mathbb{Z}_N,p)$ model case, moving a quasiparticle $\lfloor ri\rceil$ to the M boundary, we should get a boundary quasiparticle (a,b), as shown in the following graphs:

ev 

$$= \frac{\alpha^p_{j\langle -j\rangle_N j}\alpha^p_{jr\langle -j\rangle_N}}{\alpha^p_{\langle j+r\rangle_N\langle -j\rangle_N j}}\frac{\beta_{rj\underline{x}}}{\beta_{jr\underline{x}}}M^{\langle -j\rangle_N}_{\lfloor ri\rceil,r}$$


, (E28)



$\longmapsto$ $\mathrm{e}^{2\pi \mathrm{i}\frac{p}{N^2}rx}$


, (E29)

where

$$a=\langle r\rangle_M\,,\quad b=\left\langle i-\frac{p(r-\langle r\rangle_M)}{N}\right\rangle_{\frac{N}{M}}. \tag{E30}$$

Let $\mathcal{K}_M$ be the set of quasiparticle types $\lfloor ri\rceil$ that maps to the trivial boundary quasiparticle (0,0), and we see that

$$\mathcal{K}_M=\left\{\lfloor ri\rceil | r=k_1M,\, i=k_2\frac{N}{M}+\frac{pr}{N}\right\}, \tag{E31}$$

where k_1,k_2 are integers. One can easily check that $\mathcal{K}_M$ is indeed a Lagrangian subgroup.

The next question is as follows: Do all the Lagrangian subgroups of $(\mathbb{Z}_N,p)$ model have the form of (E31)? The answer is "yes."

First, note that $T_{\lfloor ri\rceil}=1$ requires $\frac{ri}{N}-\frac{pr^2}{N^2}$ to be some integer number k, i.e.,

$$Nri-pr^2=kN^2. \tag{E32}$$

Let $m=\gcd(r,N)$, and $N=um,r=vm,\gcd(u,v)=1$, we have

$$uvi-pv^2=ku^2, \tag{E33}$$

which implies that $u|pv^2,v|ku^2$. Since $\gcd(u,v)=1$ we know that $u|p,v|k$ and $N=um|pm$. Thus,

$$r=vm,\quad i=\frac{k}{v}\frac{N}{m}+\frac{pr}{N} \tag{E34}$$

or, equivalently,

$$i=t+\frac{pr}{N},\quad N|rt,\, N|pr. \tag{E35}$$

Then, we can show that any Lagrangian subgroup $\mathcal{K}$ must be equal to some $\mathcal{K}_M$. For convenience, say $\mathcal{K}=\{\lfloor s_1j_1\rceil,\lfloor s_2j_2\rceil,\ldots,\lfloor s_{|\mathcal{K}|}j_{|\mathcal{K}|}\rceil\}$, where $|\mathcal{K}|$ is the number of different quasiparticle types in $\mathcal{K}$. As discussed above, $T_{\lfloor s_nj_n\rceil}=1$ requires that

$$j_n=t_n+\frac{ps_n}{N},\quad N|s_nt_n,\, N|ps_n. \tag{E36}$$

Let

$$M=\gcd(N,s_1,s_2,\ldots,s_{|\mathcal{K}|}), \tag{E37}$$

$$P=\gcd(N,t_1,t_2,\ldots,t_{|\mathcal{K}|}). \tag{E38}$$

We have

$$\begin{gathered} s_n=k_nM,\, t_n=l_nP,\\ \gcd\left(\frac{N}{M},k_1,k_2,\ldots,k_{|\mathcal{K}|}\right)=\gcd\left(\frac{N}{P},l_1,l_2,\ldots,l_{|\mathcal{K}|}\right)=1. \end{gathered} \tag{E39}$$

We have $N|PMk_nl_n$. $D_{\mathcal{Z}(\mathcal{C})}S_{\lfloor s_nj_n\rceil\lfloor s_mj_m\rceil}=1$ requires that $N|PM(k_nl_m+k_ml_n)$. With these constraints we can show $N|PM$:

$$\begin{aligned} N|PM(k_nl_m+k_ml_n) &\Rightarrow N|k_nPM(k_nl_m+k_ml_n)\\ &\Rightarrow N|PMk_n^2l_m\\ &\Rightarrow N|PMk_n^2\gcd\left(\frac{N}{P},l_1,l_2,\ldots,l_{|\mathcal{K}|}\right)\\ &\Rightarrow N|PMk_n^2\\ &\Rightarrow N|PM\gcd\left(\frac{N}{M},k_1^2,k_2^2,\ldots,k_{|\mathcal{K}|}^2\right)\\ &\Rightarrow N|PM. \end{aligned}$$

We then have $PM=uN$ for some integer u. We see that

$$s_n=k_nM,\quad j_n=l_nu\frac{N}{M}+\frac{ps_n}{N}, \tag{E40}$$

and we know that $\lfloor s_n j_n \rceil \in \mathcal{K}_M$; in other words, $\mathcal{K} \subseteq \mathcal{K}_M$. Due to the properties of Lagrangian subgroups, this is the same as $\mathcal{K} = \mathcal{K}_M$. This can be proved by contradiction: Suppose there is a quasiparticle $\xi \in \mathcal{K}_M$ but $\xi \notin \mathcal{K}$. $\xi \notin \mathcal{K}$ means that there should exist a quasiparticle $\zeta \in \mathcal{K}$, such that $D_{\mathcal{Z}(\mathcal{C})} S_{\xi\zeta} \neq 1$. But, for $\mathcal{K} \subseteq \mathcal{K}_M$, both ξ,ζ are in $\mathcal{K}_M$ and $\xi \neq \zeta$, we should also have $D_{\mathcal{Z}(\mathcal{C})} S_{\xi\zeta} = 1$. Contradiction.

Now, we have shown that for the $(\mathbb{Z}_N, p)$ model, each M boundary will give a Lagrangian subgroup $\mathcal{K}_M$ and these $\mathcal{K}_M$ are all the possible Lagrangian subgroups. The Lagrangian subgroups are indeed in one-to-one correspondence to boundary types.

There is also correspondence between boundary quasiparticles, boundary changing operators, and Lagrangian subgroups. Roughly speaking, if we use $\mathcal{Z}(\mathcal{C})$ to denote the set of all bulk quasiparticle types and $\mathcal{K}$ a Lagrangian subgroup, then $\mathcal{Z}(\mathcal{C})/\mathcal{K}$ are the quasiparticles on the $\mathcal{K}$ boundary that survive the condensation. Similarly, the boundary changing operators between the $\mathcal{K}_1$ boundary and $\mathcal{K}_2$ boundary should be given by $\mathcal{Z}(\mathcal{C})/\mathcal{K}_1 \boxtimes \mathcal{K}_2$, where $\mathcal{K}_1 \boxtimes \mathcal{K}_2$ are the quasiparticles fused by quasiparticles in $\mathcal{K}_1$ and $\mathcal{K}_2$.

For the $(\mathbb{Z}_N, p)$ case, suppose $\mathcal{K}_{M_1}, \mathcal{K}_{M_2}$ are two Lagrangian subgroups. Quasiparticles $\lfloor ri \rceil$ in $\mathcal{K}_{M_1} \boxtimes \mathcal{K}_{M_2}$ are

$$r = k_1 M_1 + k_2 M_2, \quad i = l_1 \frac{N}{M_1} + l_2 \frac{N}{M_2} + \frac{pr}{N}. \tag{E41}$$

It is easy to see

$$|\mathcal{K}_{M_1} \boxtimes \mathcal{K}_{M_2}| = \frac{N}{\gcd(M_1, M_2)} \frac{N}{\gcd\left(\frac{N}{M_1}, \frac{N}{M_2}\right)}. \tag{E42}$$

Let $R = \gcd(M_1, M_2)$, $|\mathcal{K}_{M_1} \boxtimes \mathcal{K}_{M_2}| = \frac{N M_1 M_2}{R^2}$. Thus,

$$|\mathcal{Z}(\mathcal{C})/\mathcal{K}_{M_1} \boxtimes \mathcal{K}_{M_2}| = \frac{N R^2}{M_1 M_2} \tag{E43}$$

is the number of different boundary changing operators between the M_1 boundary and M_2 boundary, which agrees with our previous results obtained via the Q-algebra approach.

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