

Bulk Entanglement Spectrum Reveals Quantum Criticality within a Topological State

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A quantum phase transition is usually achieved by tuning physical parameters in a Hamiltonian at zero temperature. Here, we show that the ground state of a topological phase itself encodes critical properties of its transition to a trivial phase. To extract this information, we introduce an extensive partition of the system into two subsystems both of which extend throughout the bulk in all directions. The resulting bulk entanglement spectrum has a low-lying part that resembles the excitation spectrum of a bulk Hamiltonian, which allows us to probe a topological phase transition from a single wave function by tuning either the geometry of the partition or the entanglement temperature. As an example, this remarkable correspondence between the topological phase transition and the entanglement criticality is rigorously established for integer quantum Hall states.

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Topological phases of matter are characterized by quantized physical properties that arise from topological quantum numbers. For instance, the quantized Hall conductance of an integer quantum Hall state is determined by its Chern number [1], the quantized magnetoelectric response of a topological insulator is governed by its Z_2 topological invariant [2–4], and the quasiparticle charge in a fractional quantum Hall state is deeply related to its topological degeneracy [5]. Remarkably, the complete set of topological quantum numbers, or topological order [6], is entirely encoded in the ground state wave function, which can be computed either directly [1,7] or from the topological entanglement entropy [8–13].

To extract more information about a topological phase from its ground state wave function, Li and Haldane [14] considered the full entanglement spectrum of the reduced density matrix upon tracing out a subsystem. When a topologically nontrivial ground state is spatially divided into two halves, the resulting entanglement spectrum bears a remarkable similarity to the edge state spectrum of the system in the presence of a physical boundary [16–22]. Given this capability of the entanglement spectrum to simulate edge excitations, one may wonder if universal bulk properties of topological phases can also be obtained via entanglement.

In this work, we show that the ground state of a topological phase (a single wave function) encodes information on its phase transition to a trivial product state, despite the fact that the system itself is away from the phase transition. To expose this “hidden” topological phase transition, we introduce a new type of real-space partitions, which divide the system into two parts that are extensive with system size in all directions, as shown in Fig. 1. The entanglement Hamiltonian obtained from such an extensive partition is a bulk entity, which we term the bulk entanglement Hamiltonian. The corresponding bulk entanglement

spectrum (BES) has a low-lying part that resembles the excitation spectrum of a physical system in the bulk. This enables us to probe bulk properties of topological phases and topological phase transitions, which cannot be accessed from the left-right partition.

Our results can be understood from the Chalker-Coddington network model [15], which describes the transition between integer quantum Hall states in terms of the percolation of chiral edge states. Our extensive partition generates a periodic array of boundaries. In the case of the $\nu = 1$ quantum Hall state, each boundary introduces a corresponding edge mode in the bulk entanglement spectrum. By varying the geometry of the partition as shown in Fig. 1, these edge modes interconnect and percolate throughout the entire system, which mirrors edge state percolation in an actual quantum Hall transition. This correspondence explains why universal critical properties of a topological phase transition are encoded in a single wave function away from criticality, and how they can be extracted from the bulk entanglement spectrum.

The entanglement between two parts A and B of a many-body ground state $|\Psi\rangle$ is characterized by the reduced density matrix ρ_A . ρ_A can be formally written as the thermal density matrix of an entanglement Hamiltonian H_A at temperature $T = 1$:

$$\rho_A = \text{Tr}_B |\Psi\rangle\langle\Psi| \equiv e^{-H_A}.$$

The full set of eigenvalues of H_A , denoted by $\{\xi_i\}$ with $1 \leq i \leq \dim[\mathcal{H}_A]$, constitutes the entanglement spectrum of subsystem A . These eigenvalues are directly related to the coefficients in the Schmidt decomposition of the ground state:

$$|\Psi\rangle = \sum_i e^{-\xi_i/2} |\psi_i\rangle_A \otimes |\tilde{\psi}_i\rangle_B. \quad (1)$$

Among all states in subsystem A , we will pay attention to those “dominant states” that have small eigenvalues and hence large weight in the ground state $|\Psi\rangle$.

When A and B are the left and right halves of a quantum Hall state, the entanglement spectrum is gapless, and its low-lying part resembles the excitation spectrum of a physical edge [14,16–22]. Because of the inherent boundary-local nature [23,24], the entanglement spectrum from the left-right partition does not directly reveal properties of the bulk.

We introduce an extensive partition to study the bulk: a system is divided into two subsystems A and B consisting of a periodic array of blocks. See Fig. 1 for examples. The shape of each block does not matter; the defining characteristic of such extensive partitions is that the entire boundary between A and B extends throughout the bulk of the system in all directions. Because (i) quantum entanglement in a gapped system comes mostly from boundary degrees of freedom and (ii) the boundary between A and B in an extensive partition is itself extensive, the low-lying part of the corresponding entanglement spectrum contains information about the bulk.

To demonstrate this bulk nature explicitly, we note that the entanglement entropy of an extensive partition $S = -\text{Tr}(\rho_A \log \rho_A)$ scales as the total area of the boundary: $L^{d-1}N$, where L is the linear size of each block, d is the spatial dimension of the system, and N is the number of

blocks in the entire system. In the thermodynamic limit, $N \rightarrow \infty$ while L is fixed. Therefore, S is proportional to the volume of the system: $S \sim V/L \propto V$. In this case, it is meaningful and instructive to draw analogies between the reduced density matrix ρ_A and the thermal density matrix of a d -dimensional physical system defined on A (a superlattice), and between the entanglement Hamiltonian H_A and the physical Hamiltonian.

We will study a sequence of extensive partitions for a topological state. Depending on the actual partition, the bulk entanglement spectrum can be either gapped or gapless. When the BES is gapped, its lowest eigenvalue ξ_0 is separated from the rest by a finite amount in the thermodynamic limit. In this case, the corresponding product state $|\psi_0\rangle_A \otimes |\tilde{\psi}_0\rangle_B$, which has the largest weight in the Schmidt decomposition, stands out as the most dominant component in the ground state $|\Psi\rangle$. Here $|\psi_0\rangle_A$ is the lowest eigenstate of the entanglement Hamiltonian of subsystem A , which is defined by $H_A|\psi_0\rangle_A = \xi_0|\psi_0\rangle_A$ and hereafter referred to as the entanglement ground state; likewise, $|\tilde{\psi}_0\rangle_B$ is the entanglement ground state of subsystem B . In contrast, when the BES is gapless, no individual product state can be singled out as the most dominant component in $|\Psi\rangle$.

We now derive the main result of this work: by varying the extensive partition between two limits defined below,

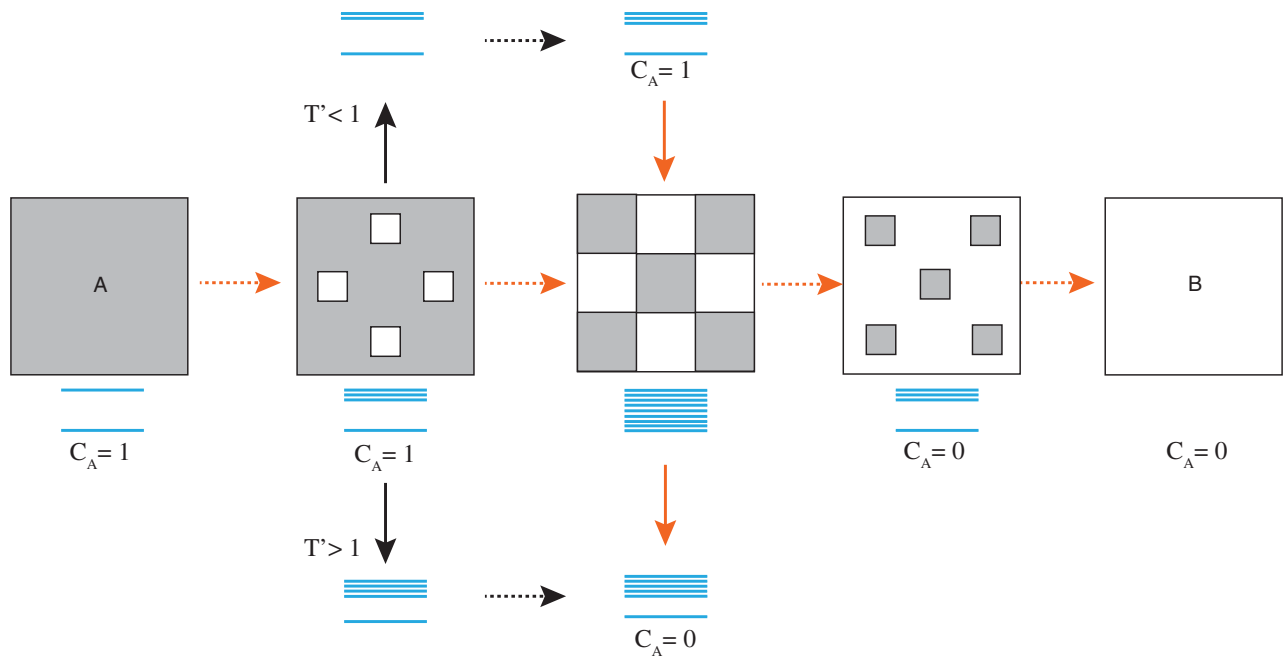


FIG. 1 (color online). Extensive partition of a topological state yields a topological phase transition in the bulk entanglement Hamiltonian H_A of a subsystem. The horizontal and vertical sequences represent two ways of realizing the transition: the horizontal sequence denotes geometrically tuning the partition towards quantum criticality at the symmetric point (center); the vertical sequence comes from changing the entanglement temperature ($T = 1 \rightarrow T'$) at an earlier stage of an asymmetric partition. Dotted arrows always indicate the tracing out procedure. The schematic bulk entanglement spectrum and topological invariant for A are shown below every stage of partition. $C_A = 1$ denotes the topological order of the original topological ground state, and $C_A = 0$ denotes topologically trivial order.

the bulk entanglement Hamiltonian H_A of a given subsystem in a topological state undergoes a gap-closing transition, in which the entanglement ground state $|\psi_0\rangle_A$ changes from being topologically equivalent to the non-trivial ground state $|\Psi\rangle$ to being trivial. Throughout the following general discussion, it will be useful to keep in mind a concrete example of a topological phase with gapless edge excitations, such as a quantum Hall state. For such phases, the low-lying states in the entanglement spectrum of subsystem A arise from boundary degrees of freedom that correspond to a network of chiral edge states, which would occur if B had been physically removed [Fig. 2(a)]. Varying the extensive partition changes how these edge modes interact with each other, and we will refer to this “edge picture” when relevant.

First, consider extensive partitions in two extremely asymmetric limits: (i) a percolating sea of region A with an array of small, isolated islands of region B , and (ii) a percolating sea of region B with an array of small, isolated islands of region A . We require that different islands are separated by a distance L that is greater than the correlation length ξ in the ground state. Since a given island has a finite number of degrees of freedom, the entanglement spectrum of this island, obtained by tracing out the rest of the system, generically has a unique ground state with a finite gap. Moreover, since $L \gg \xi$, different islands contribute to the entanglement between A and B independently. Therefore, the BES from extensive partitions (i) and (ii) are both

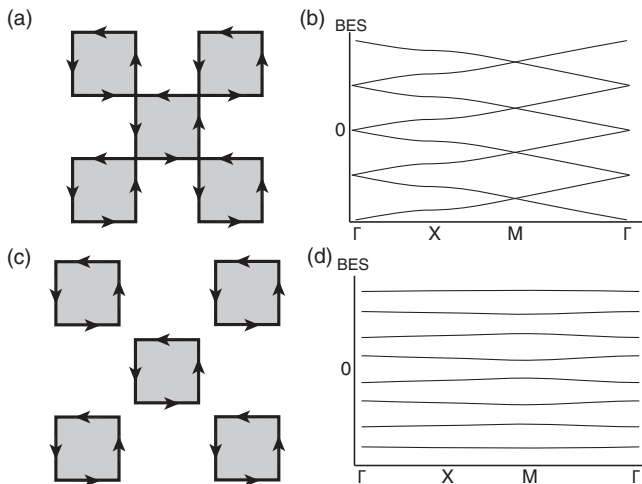


FIG. 2. (a) At the symmetric partition of the integer quantum Hall state, the low-lying states in the bulk entanglement spectrum percolate. (b) The corresponding single particle bulk entanglement spectrum for the symmetric partition of the integer quantum Hall state. The model Hamiltonian used here is $H(k) = (\cos k_x + \cos k_y - \mu)\sigma_z + \sin k_x\sigma_x + \sin k_y\sigma_y$. Each block of the partition used is 5×5 sites. Note the massless Dirac dispersion at Γ . (c) An asymmetric partition in which the low-lying states do not percolate. The Hilbert space is the same as that of (a), but the lattice constant is different. The resulting BES is gapped (d).

gapped. In the edge picture, the edge modes are confined within each block and do not percolate [Fig. 2(c)]. For such Chalker-Coddington network models, the spectrum is gapped [15].

On the other hand, the entanglement ground states in partitions (i) and (ii) are vastly different. Since the density matrix ρ_A in partition (ii) is defined on disconnected islands, it factorizes into a direct product and thus must be topologically trivial. In contrast, since ρ_A in partition (i) is defined on the percolating sea, it must have the same topological order as the ground state $|\Psi\rangle$: this is manifest in the extreme limits where B is the null set. Hence, as one shrinks the size of A and concurrently enlarges B to interpolate from partition (i) to (ii), the entanglement ground state $|\psi_0\rangle_A$ must change from carrying the topological order to being a trivial product state of islands. Whether topological order is present in a gapped system is a yes or no question. Therefore, in order to accommodate this change of topology in the entanglement ground state, the bulk entanglement Hamiltonian H_A must close the gap somewhere in between partitions (i) and (ii).

Where does the gap close? We now argue that for the vast majority of topological states (to be precisely defined below), the gap in the bulk entanglement spectrum must close at a symmetric partition, where A and B are related by symmetry such as translation or reflection [see Fig. 1(c) or 2(a)]. In the edge picture, for such symmetric extensive partitions the edge modes have equal left- and right-turning amplitudes at the nodes, and these network models are critical [15].

To understand this claim more generally, we first make the following conjecture: if the bulk entanglement spectrum from an extensive partition is gapped, the largest-weight state $|\psi_0\rangle_A \otimes |\tilde{\psi}_0\rangle_B$ in the Schmidt decomposition possesses the same topological order as the original ground state $|\Psi\rangle$. Intuitively, this conjecture is expected to hold because all other components in the Schmidt decomposition, $|\psi_i\rangle_A \otimes |\tilde{\psi}_i\rangle_B$ with $i \neq 0$, are exponentially suppressed by the entanglement gap. Therefore, it should be possible to “push” the entanglement gap to infinity [25] and thereby deform the original ground state (1) into the entanglement ground state of the extensive partition $|\psi_0\rangle_A \otimes |\tilde{\psi}_0\rangle_B$. This argument suggests that $|\psi_0\rangle_A \otimes |\tilde{\psi}_0\rangle_B$ and $|\Psi\rangle$ are adiabatically connected and, hence, carry the same topological order.

Let us apply the above conjecture to a symmetric partition of a topological state $|\Psi\rangle$, where A and B are symmetry related. Suppose for the sake of argument that the bulk entanglement spectrum is gapped. It then follows from our conjecture that $|\Psi\rangle$ is topologically equivalent to a direct product of two essentially identical states: $|\psi_0\rangle_A \otimes |\tilde{\psi}_0\rangle_B$, where $|\psi_0\rangle_A$ and $|\tilde{\psi}_0\rangle_B$ are related by a translation or reflection. This implies that $|\Psi\rangle$ must have a “doubled” topological order. In contrast, the majority of topological states carry an elementary unit of topological

order that cannot be divided equally into halves. We thus prove by contradiction that the bulk entanglement spectrum of an “irreducible” topological state must be either gapless or have degenerate ground states at a symmetric partition.

To summarize, we have argued that the discrete nature of topological order dictates that one subsystem in an extensive partition inherits the topological order while the other does not. As a consequence, as one varies the partition, phase transition(s) must occur in the bulk entanglement spectrum, which is constructed from a single ground state wave function. Our conclusion is based on general principles as well as the microscopic understanding in terms of percolating edge modes. This is illustrated below using the example of integer quantum Hall states, for which rigorous results will be derived.

Example.—We consider an integer quantum Hall state on a two-dimensional lattice [26] (also known as the Chern insulator), defined by a generic, translationally invariant tight-binding Hamiltonian H . For such free fermion systems, the entanglement Hamiltonian of a subsystem A takes a quadratic form [27]: $H_A = \sum_{r,r' \in A} \mathcal{H}_A(r, r') c_r^\dagger c_{r'}$. The set of eigenvalues of \mathcal{H} is denoted by $\{\epsilon_j\}$, which can be regarded as single-particle levels. The entanglement spectrum of A is then obtained by filling these levels. It then follows that the important, low-lying part of the entanglement spectrum for a fixed density of particles in A comes from the vicinity of the highest occupied level.

The form of H_A is entirely determined from the two-point correlation function of the ground state $C(r, r') = \langle \Psi | c_r^\dagger c_{r'} | \Psi \rangle$, where both sites r and r' lie within subsystem A . Using the exponential decay of $C(r, r')$ at large distance $|r - r'|$, we prove [28] that H_A is short ranged and resembles a physical Hamiltonian, provided that the volume of A does not exceed that of B . In the opposite case, H_A has bands at $\pm\infty$ [28]; nonetheless, the low-lying part of the spectrum still resembles the excitation spectrum of a bulk, physical Hamiltonian.

The ground state of an integer quantum Hall system is indexed by a nonzero Chern number [1]: $C \neq 0$. As a consequence, one cannot choose the phase of Bloch wave functions continuously over the entire Brillouin zone in reciprocal space. Based on this topological obstruction, we prove [28] that the bulk entanglement spectrum from a symmetric extensive partition must be gapless when C is an odd integer (but not necessarily so when C is even). Such an entanglement criticality at the symmetric partition of irreducible quantum Hall states agrees with our conclusion deduced earlier from general considerations.

To gain further insight into the bulk entanglement spectrum, we choose a particular model of a $C = 1$ integer quantum Hall state on a square lattice, and study extensive partitions in which A and B are two sets of blocks as shown in Fig. 2(a). The single-particle levels $\{\epsilon_j\}$ that define the BES of A are plotted as a function of crystal momentum in the 2D Brillouin zone $0 < k_x, k_y < 2\pi/L$ in Fig. 2. We find

numerically that when A and B are asymmetric, the BES is gapped and the entanglement ground state of the (non) percolating subsystem carries Chern number $C = 1$ (0). At the symmetric partition, the bulk entanglement Hamiltonian exhibits a two-dimensional massless Dirac fermion spectrum [Fig. 2(b)], which precisely coincides with a physical Hamiltonian tuned to the quantum Hall transition point at a fixed density and in the absence of disorder.

The correspondence between the entanglement criticality and the quantum Hall plateau transition is remarkable and can be understood as the percolation of chiral edge states, as noted earlier. In this microscopic sense, the phase transition in the bulk entanglement spectrum mirrors the percolation of edge states in an actual quantum Hall transition, both of which are described by the Chalker-Coddington network model and exhibit a massless Dirac fermion spectrum at criticality [15].

The entanglement phase transition can be achieved by tuning the partition in multiple ways. So far, we have changed the size of blocks to vary the degree of asymmetry between A and B . In this procedure, A and B become switched across the symmetric point, which leaves the entanglement spectrum invariant (but not the entanglement ground state). This leads to an interesting duality relating the two sides of the entanglement critical point.

On the other hand, as the size of blocks in subsystem A changes, the Hilbert space of the bulk entanglement Hamiltonian H_A changes accordingly, which is rather different from typical quantum phase transitions. An alternative way of tuning an extensive partition without changing the Hilbert space dimension of A is to start from an array of disconnected blocks, keep the size of blocks fixed, and decrease the distance between blocks in A up to the symmetric point where adjacent blocks touch at corners. Beyond the symmetric point, different blocks touch on edges to make A a percolating sea.

Last but not least, we describe in the Supplemental Material [28] a third procedure to achieve an entanglement phase transition on a fixed subsystem A_s of a symmetric partition. This is achieved by explicitly constructing a continuous family of bulk entanglement Hamiltonians $H_{A_s}(T)$ at a different “entanglement temperature” $0 < T < \infty$.

Discussion.—While our general argument proves the existence of a phase transition in the bulk entanglement spectrum induced by varying the partition, the question remains whether this “entanglement phase transition” exhibits the same critical properties as a phase transition induced by tuning a physical parameter in a physical Hamiltonian. This is indeed the case for integer quantum Hall states, as shown earlier. What about topological phases in general?

Before discussing this issue, it should be noted that the trivial state that appears in the bulk entanglement spectrum

of an asymmetric partition is a direct product of blocks. Thus the entanglement phase transition should be compared with an actual phase transition into such a product state, which can often be realized by imposing an external periodic potential. We expect that the entanglement phase transition obtained from the extensive partition exhibits the same critical properties as actual transitions, if the latter are continuous and fall into a single universality class. Alternatively, both entanglement and actual phase transitions into block-product states can be first order or intervened by an extended critical phase. However, if there is a single transition as the extensive partition is varied between the asymmetric limits, then it must occur at the symmetric partition.

Finally, we end by discussing why a topologically nontrivial ground state is capable of “knowing” its own phase transition. We believe this is due to the peculiar nature of topological phases and topological phase transitions. Without any local order parameter, topological phases are distinct from trivial states only because of nontrivial patterns of entanglement [29]. For this sole reason, a transition is needed to go from one phase to the other. Therefore, it appears to us that a topological phase transition can be generically regarded as percolation of entanglement, which is a generalization of edge state percolation at the quantum Hall transition. In this sense, our work generalizes readily to other systems with gapless edge excitations in any symmetry class and dimension [30]. On the other hand, topological phases such as Z_2 spin liquids do not necessarily have gapless edge states. It remains an interesting and open question whether or not phase transitions in these systems can be understood from the perspective of entanglement percolation.

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