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Unique Equilibrium in the Eaton-Gersovitz Model of Sovereign Debt

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Unique equilibrium in the Eaton-Gersovitz model of sovereign debt

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Abstract

We provide a proof that Markov perfect equilibrium is unique in the standard infinite-horizon incomplete-market model with a default option which, following Eaton and Gersovitz (1981), has become a benchmark for quantitative analyses of sovereign debt (Arellano (2008), Aguiar and Gopinath (2006), Aguiar and Amador (2014)).

1 Motivation

A common view of sovereign debt markets is that they are prone to multiple equilibria: a market panic may inflate bond yields, deteriorate the sustainability of government debt and precipitate a default event, justifying investor fears. Indeed, Mario Draghi's speech in July 2012, announcing that the ECB was "ready to do whatever it takes" to preserve the single currency, and the subsequent creation of the Outright Monetary Transactions (OMT) program, are widely seen as having moved Eurozone sovereign debt markets out of an adverse equilibrium: since then, bond spreads have experienced dramatic falls as fears of default have receded. Prominent models in the academic literature, such as Calvo (1988), Cole and Kehoe (2000) and Lorenzoni and Werning (2013), feature multiple equilibria which justify this common view.

At the same time, in the last decade, a booming quantitative literature in the line of Eaton and Gersovitz (1981)—initiated by Arellano (2008) and Aguiar and Gopinath (2006), and summarized by Aguiar and Amador (2014)—has studied sovereign debt markets using a benchmark infinite-horizon incomplete-market model whose analytical properties are not well understood[1]. In particular, it had so far not been known whether these models might feature multiple Markov

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[1] Another booming literature uses a similar class of models to analyze unsecured consumer credit (Chatterjee, Corbae, Nakajima and Ríos-Rull (2007))

perfect equilibria, and if so, whether the reason behind the multiplicity echoed the common intuition described in the previous paragraph. An example of how the literature viewed this issue is in Hatchondo, Martinez and Sapriza (2009):

> Krusell and Smith (2003) show that, typically, there is a problem of indeterminacy of Markov-perfect equilibria in an infinite-horizon economy. In order to avoid this problem, we analyze the equilibrium that arises as the limit of the finite-horizon economy equilibrium.

In this note, we show that Markov perfect equilibrium *is unique* in the canonical infinite-horizon model with Markov income and permanent autarky punishment. We also extend the uniqueness result to a case where income is independent and identically distributed across periods, but re-entry is possible after a stochastic period of market exclusion—a typical assumption used in the literature. The key to the proof is to rule out the possibility of self-sustaining improvements in the value of borrowing arising from improvements in the terms of borrowing.

The intuition for the impossibility of such a feedback is as follows. If there are to be two different equilibria, in one of them the government must be running a higher debt level. That debt level must be self-sustaining, in that a government starting with this amount of debt must find that the value from continuing to pay justifies not defaulting. But in the alternative, lower-debt equilibrium, the government could have followed an issuance strategy along every path that would have maintained its liabilities at a uniform distance from the higher-debt government, economizing on interest costs and default premia. And yet that government found default to be worthwhile. Since governments are equally well-off once access to financial markets is lost, equilibrium must be unique.

Interestingly, this proof strategy by replication has echoes of that used by Bulow and Rogoff (1989) to rule out reputational equilibria in a similar class of models where sovereign governments retain the ability to save after defaulting. Although the Bulow-Rogoff result is cast in a complete market setting, we show in Section 2.3 how to adapt it to fit an incomplete market framework, drawing parallels to our own argument along the way. The Bulow-Rogoff result does not apply directly to the model we study: the government can be prevented from saving after default, and may face additional, non-reputational costs of default in the form of output losses. This allows some debt to be sustained in equilibrium. But just as Bulow and Rogoff (1989) show that no positive level of debt can be purely self-sustaining in equilibrium, we show that multiple equilibria in the canonical model cannot each be self-sustaining.

Our result is important because it shows that the multiplicity intuition is *not* valid in a benchmark model that is accepted as a good description—both qualitative and quantitative—of sovereign debt markets. It provides an additional analytical result for a model about which few such results exist. And it shows that alternative strategies to compute Markov perfect equilibria should all converge to the same solution. Even though we do not cover all cases of models written in the literature—indeed, the proofs become more involved as the model increases in complexity—our results suggest that it is unlikely that quantitative findings in the literature are driven by a hidden equilibrium selection.

To be clear, our objective is not to deny that sovereign debt markets can be prone to self-fulfilling crises, or that OMT may have ruled out a bad equilibrium. But we hope that a proof of equilibrium uniqueness in this benchmark model may help sharpen the literature's understanding of the assumptions that are needed for such multiple equilibria to exist. We provide our thoughts on this matter in the conclusion.

2 Short-term debt, Markov income, and permanent exclusion

2.1 Model description

We now describe what we call the *canonical* infinite-horizon model with Markov income and permanent autarky punishment (see Aguiar and Amador (2014)). Output y follows a discrete Markov chain with elements in $\mathcal{Y}$, $|\mathcal{Y}| = Y \in \mathbb{N}$ and transition matrix $\pi(y'|y)$. A government has utility over consumption

$$\mathbb{E}\left[\sum_{t=0}^{\infty} \beta^t u(c_t)\right]$$

Each period the government receives y as endowment, and strategically decides to default given $y \in \mathcal{Y}$ and the level of debt b that it has promised to repay. Default is punished by permanent autarky, with output also reduced by an exogenous cost $\phi(y) \in [0, y]$. While in good credit standing, the government decides how much to borrow or lend given a bond revenue schedule. This schedule is determined by competitive, risk-neutral international investors, who demand an expected gross return of R.

We focus on Markov perfect equilibria, where the schedule depends only on the size of the bond issue: $Q(y, b')$ is the total amount raised when the government promises to repay b'. Letting $p = 1$ denote the decision to repay and $p = 0$ denote the decision to default, the value function $V^o(y, b)$ given income y and debt b is given by

$$V^o(y,b) = \max_{p \in \{0,1\}} pV(y,b) + (1-p)V^d(y) \tag{1}$$

where the value of repaying is

$$\begin{aligned} V(y,b) &= \max_{b'} u(c) + \beta \mathbb{E}_{y'|y}\left[V^o(y',b')\right] \\ &\quad \text{s.t.} \quad c + b = y + Q(y,b') \end{aligned} \tag{2}$$

and the value of default is

$$V^d(y) = u(y - \phi(y)) + \beta \mathbb{E}_{y'|y}\left[V^d(y')\right] \tag{3}$$

We assume that when a government is indifferent between repayment and default, it chooses to repay. Thus $p(y,b) = 1$ if, and only if $V(y,b) \geq V^d(y)$. Since international investors receive expected repayment $\mathbb{E}_{y'|y}[p(y',b')]$, and demand return R, the bond price schedule is

$$Q(y,b') = \frac{b'}{R}\mathbb{E}_{y'|y}\left[p\left(y',b'\right)\right] = \frac{b'}{R}\mathbb{P}_{y'|y}\left[V(y',b') \geq V^d(y')\right] \tag{4}$$

Definition. A Markov perfect equilibrium is a set of policy functions $p\,(y,b)$, $c\,(y,b)$, $b'\,(y,b)$ for repayment, consumption and next period borrowing, value functions $V\,(y,b)$, $V^o\,(y,b)$, $V^d\,(y)$ and a bond revenue schedule $Q(y,b')$ such that (1)-(4) are satisfied.

Assumption 1. *u is strictly increasing, continuous, concave, and $u(c) = -\infty$ if $c \leq 0$.*

Assumption 2. *To rule out Ponzi schemes, there exists an upper bound $B > 0$ such that the government must choose $b \leq B$. (B may be chosen high enough to be non-binding in equilibrium.)*

Assumption 3. *$R > 1$ and $\beta R < 1$.*

Under these assumptions, it is possible to prove that $V\,(y,b)$ is continuous, strictly decreasing in b, and declines to $-\infty$ for sufficiently large b.[2] Hence, at each level of income y, there exists a threshold $b^*\,(y)$ such that the government repays if and only if $b \leq b^*\,(y)$; this threshold is the unique value that satisfies the equality

$$V\,(y, b^*\,(y)) = V^d(y)$$

Note that the thresholds $b^*\,(y)$ need not be ordered monotonically in y, since a higher current income typically raises both the value of repaying and the value of default. In the case of i.i.d income with no output costs of default, it is possible to show that $b^*\,(y)$ is increasing in y (see Arellano (2008)) but such monotonicity is not needed for our proof.

Given (4), a bond revenue schedule Q is characterized by the thresholds $\{b^*\,(y)\}_{y\in\mathcal{Y}}$ induced by the policy function:

$$Q(y,b') = \frac{b'}{R}\mathbb{P}_{y'|y}\left[b' \leq b^*\left(y'\right)\right] = \frac{b'}{R}\sum_{\{y':b'\leq b^*(y')\}}\pi\left(y'|y\right) \tag{5}$$

An illustration of equilibrium objects is given in figures 1 and 2.[3]

[2] See Appendix A for all the proofs that are not in the main text.

[3] The computation is for an example with two states $\mathcal{Y} = \{y_L, y_H\}$, with an iid income process, so that the bond price schedule is independent of y. The calibration is $u = \frac{c^{1-\gamma}}{1-\gamma}$ with $\gamma = 2$, $\beta = 0.8$, $R = 1.1$, $y_L = 0.2$, $y_H = 1.2$ and $\pi\,(y_L) = 0.2$. Starting from a risk-free bond revenue schedule, the algorithm iterates on the value function using a grid with 750 points, and updates the bond revenue schedule using the default policy until convergence. See Hatchondo, Martinez and Sapriza (2010) for a discussion of quantitative solution methods in this class of models.

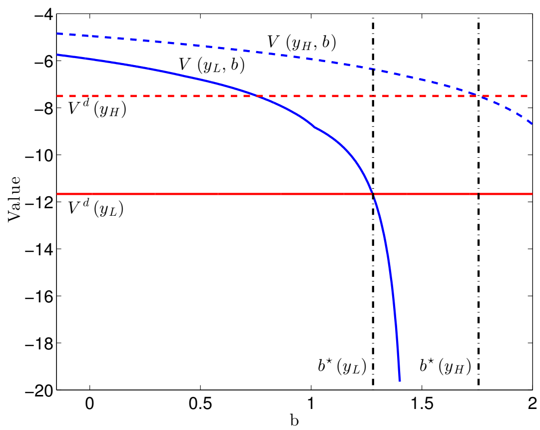

Figure 1: Example value function

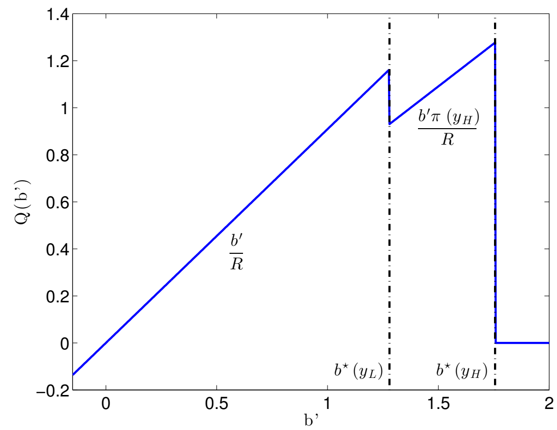

Figure 2: Example $Q(y, b)$

2.2 Uniqueness of equilibrium

Suppose that we have two distinct equilibria (V, Q) and $(\widetilde{V}, \widetilde{Q})$, each with a set of default thresholds $\{b^*(y)\}_{y \in \mathcal{Y}}$ and $\{\tilde{b}^*(y)\}_{y \in \mathcal{Y}}$. At first glance, it seems difficult to derive any relationship between these two equilibria.

For instance, if for some y we have $b^*(y) > \tilde{b}^*(y)$ and for other y we have $b^*(y) < \tilde{b}^*(y)$, neither bond price schedule Q or $\widetilde{Q}$ will necessarily dominate the other. In one equilibrium, it may be easier to borrow at certain levels of b and harder at others—and with an infinite horizon, the complex differences in default policy may be induced endogenously by the resulting differences in payoffs.

The key observation of this paper is that we can cut through this complexity with a simple inequality for the two value functions V and $\tilde{V}$, related to the maximum difference between the default thresholds. The basis of this inequality is a simple replication strategy we call *mimicking at a distance*. Suppose that $\tilde{b}^*(y)$ exceeds $b^*(y)$ by at most $M > 0$. Then we show that it is always weakly better to start with debt of $b - M$ in the (V, Q) equilibrium than with debt of b in the $(\widetilde{V}, \widetilde{Q})$ equilibrium, and indeed strictly better whenever $b \leq \tilde{b}^*(y)$. This observation, formalized in Theorem 1, will ultimately be the basis of the proof that distinct equilibria are impossible in Theorem 2.

Why? The government with debt $b - M$ in the (V, Q) equilibrium has the option to *mimic* the policy of the government with debt b in the $(\widetilde{V}, \widetilde{Q})$ equilibrium—always defaulting at the same points, and otherwise choosing the same level of debt for the next period minus M. Before it defaults, this government is better off because it pays less to service debt, allowing it to consume more. Debt service, in turn, costs less for two reasons. First, the mimicking government poses weakly less risk of default. This is due to the choice of M: since M is the maximum amount by which the default thresholds $\tilde{b}^*(y)$ exceed the thresholds $b^*(y)$, as long as the government in the (V, Q) equilibrium chooses debt of M less than the government it is mimicking, it is weakly less likely to default. Second, the mimicking government has strictly less debt, meaning that the cost of providing an expected return of $R > 1$ on this debt is lower.

Following this policy, the mimicking government always consumes more until default, implying weakly higher payoffs that become strictly higher as long as it does not default right away.

Theorem 1 (Mimicking at a distance.). *Consider two distinct equilibria (V, Q) and $\left(\widetilde{V}, \widetilde{Q}\right)$, with associated default thresholds $\{b^*(y)\}_{y\in\mathcal{Y}}$ and $\left\{\tilde{b}^*(y)\right\}_{y\in\mathcal{Y}}$. Define*

$$M = \max_y \tilde{b}^*(y) - b^*(y) \tag{6}$$

and assume without loss of generality that $M > 0$. Then, for any y,

$$V(y, b - M) \geq \widetilde{V}(y, b) \tag{7}$$

with strict inequality whenever $b \leq \tilde{b}^*(y)$.

Proof. First, note that for any b' and y, applying (5) we have

$$\begin{aligned} Q\left(y, b' - M\right) &= \frac{(b' - M)}{R} \sum_{\{y' : b' - M \leq b^*(y')\}} \pi\left(y'|y\right) \geq \frac{(b' - M)}{R} \sum_{\{y' : b' \leq \tilde{b}^*(y')\}} \pi(y'|y) \\ &> \left(\frac{b'}{R} \sum_{\{y' : b' \leq \tilde{b}^*(y')\}} \pi(y'|y) \right) - M = \widetilde{Q}(y, b') - M \end{aligned} \tag{8}$$

Thus the amount that a government in equilibrium (V, Q) can raise by issuing $b' - M$ of debt is always strictly larger than the amount that a government in equilibrium $(\widetilde{V}, \widetilde{Q})$ can raise by issuing b' of debt, minus M. The two intermediate inequalities in (8) reflect the two sources of this advantage. First, there are weakly more cases in which $b' - M \leq b^*(y')$ than in which $b' \leq \tilde{b}^*(y')$, and this higher chance of repayment makes it possible to raise more. Second, since $R > 1$, issuing M less debt costs strictly less than M in the current period.

Now we can formally define the *mimicking at a distance* policy. For any income and debt levels y and b, let the history y^0 be such that the income and debt owed at $t = 0$ are y and b. The equilibrium $\left(\widetilde{V}, \widetilde{Q}\right)$ induces an allocation $\left\{\widetilde{c}\left(y^t\right), \widetilde{b}\left(y^{t-1}\right), \widetilde{p}\left(y^t\right)\right\}_{y^t \succeq y^0}$ at all histories following y^0.[4] We construct a policy for the government in the equilibrium (V, Q) starting at y^0 as follows. For every history $y^t \succeq y^0$, let

$$\begin{aligned} b\left(y^{t-1}\right) &= \widetilde{b}\left(y^{t-1}\right) - M \\ p\left(y^t\right) &= \widetilde{p}\left(y^t\right) \\ c\left(y^t\right) &= \begin{cases} \widetilde{c}\left(y^t\right) + Q\left(y_t, \widetilde{b}\left(y^t\right) - M\right) - \left(\widetilde{Q}\left(y_t, \widetilde{b}\left(y^t\right)\right) - M\right) & \text{if} \quad \widetilde{p}\left(y^t\right) = 1 \\ \widetilde{c}\left(y^t\right) & \text{if} \quad \widetilde{p}\left(y^t\right) = 0 \end{cases} \end{aligned} \tag{9}$$

[4] $\widetilde{b}(y^t)$ is defined to be the amount of debt chosen at history y^t to be repaid in period $t + 1$.

We have, at all histories where repayment takes place ($\widetilde{p}\left(y^t\right) = 1$),

$$\begin{aligned} c\left(y^t\right) + b\left(y^{t-1}\right) - Q\left(y_t, b\left(y^t\right)\right) &= c\left(y^t\right) + \widetilde{b}\left(y^t\right) - M - Q\left(y_t, \widetilde{b}\left(y^t\right) - M\right) \\ &= \widetilde{c}\left(y^t\right) + \widetilde{b}\left(y^t\right) - \widetilde{Q}\left(y_t, \widetilde{b}\left(y^t\right)\right) \\ &= y_t \end{aligned}$$

and trivially $c(y^t) = \tilde{c}(y^t) = y_t - \phi(y_t)$ whenever $\tilde{p}(y^t) = 0$. Hence the budget constraint is satisfied at each y^t. Furthermore, using (8) we see that $c(y^t) > \tilde{c}(y^t)$ whenever $\tilde{p}(y^t) = 1$. In short, when there is repayment, the mimicking policy (9) sets consumption $c(y^t)$ equal to consumption $\widetilde{c}(y^t)$ in the other equilibrium, plus a bonus $Q\left(y_t, \widetilde{b}\left(y^t\right) - M\right) - \left(\widetilde{Q}\left(y_t, \widetilde{b}\left(y^t\right)\right) - M\right) > 0$ from lower debt costs. In default, the two policies provide equal consumption.

The mimicking policy, of course, need not be optimal; but since it is feasible, it serves as a lower bound for $V(y, b - M)$. We thus conclude from $c(y^t) \geq \widetilde{c}(y^t)$ that

$$V\left(y, b - M\right) \geq \sum_{y^t \succeq y^0} \beta^t \Pi\left(y^t\right) u\left(c\left(y^t\right)\right) \geq \sum_{y^t \succeq y^0} \beta^t \Pi\left(y^t\right) u\left(\widetilde{c}\left(y^t\right)\right) = \widetilde{V}\left(y, b\right)$$

with strict inequality whenever $\tilde{p}(y^0) = 1$ (or equivalently $b \leq \widetilde{b}^*(y)$), since this implies $c(y^0) > \tilde{c}(y^0)$. □

An illustration of the mimicking policy used in Theorem 1 is given in Figures 3 and 4, which depict time paths in a hypothetical two-state case. In this case, debt starts relatively high and the high-income state y_H keeps recurring, leading the government to deleverage in anticipation of lower incomes in the future. Figure 3 shows the paths of $\widetilde{b}$ (filled circles) and the mimicking policy $b = \widetilde{b} - M$ (hollow circles), while Figure 4 shows the paths of $\widetilde{c}$ (filled circles) and the consumption $c = \widetilde{c} + Q(y, \widetilde{b} - M) - (\widetilde{Q}(y, \widetilde{b}) - M)$ induced by the mimicking policy (hollow circles).

Although c is always greater than $\widetilde{c}$ in Figure 4, the gap $c - \widetilde{c}$ differs substantially across periods. This reflects fluctuations in the two sources of $c - \widetilde{c}$: differences in default premia, and the lower cost of servicing $b = \widetilde{b} - M$ rather than $\widetilde{b}$. First, since both debt levels at $t = 2$ are above the respective default thresholds for y_L, there is no difference at $t = 1$ in the two default premia. At $t = 3$, however, the mimicking policy achieves a debt level below $b^*(y_L)$, while the other policy has debt that remains above $\widetilde{b}^*(y_L)$. Thus the default premium disappears at $t = 2$ for the mimicking policy while still being paid for the other policy, leading to an expansion in the gap $c - \widetilde{c}$. From $t = 4$ onward both policies achieve debt levels below their y_L default thresholds, leading to the disappearance of all default premia. This causes the gap $c - \widetilde{c}$ to compress dramatically starting at $t = 3$.

The central observation is that if it starts with debt $M = \max \widetilde{b}^*(y) - b^*(y)$ below the other government, the mimicking government can keep itself at the fixed distance M, achieving higher consumption along the way.

We now turn to the main result, which uses Theorem 1 to rule out multiple equilibria (V, Q)

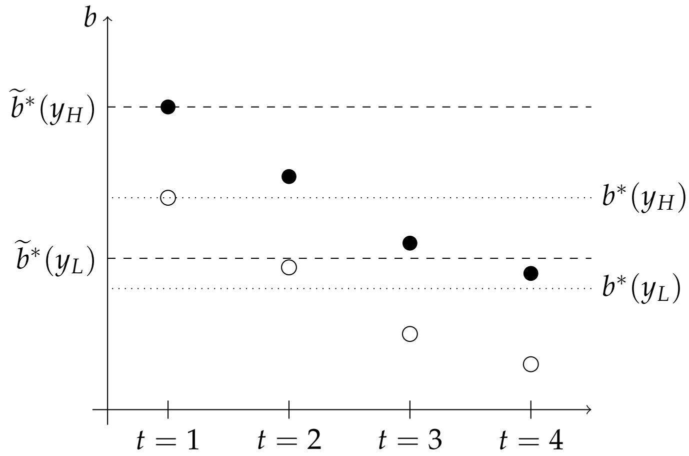

Figure 3: Example paths for b and $\widetilde{b}$.

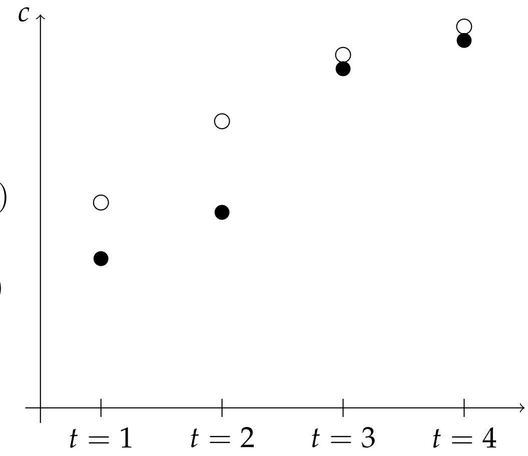

Figure 4: Example paths for c and $\widetilde{c}$.

and $(\widetilde{V}, \widetilde{Q})$ altogether.

Theorem 2. *In the canonical model, Markov perfect equilibrium has a unique value function $V(y,b)$ and debt price schedule $Q(y,b)$.*

Proof. Suppose to the contrary that there are distinct equilibria (V,Q) and $\left(\widetilde{V}, \widetilde{Q}\right)$, with associated default thresholds $\{b^*(y)\}_{y\in\mathcal{Y}}$ and $\{\tilde{b}^*(y)\}_{y\in\mathcal{Y}}$. From (5), these thresholds give the price schedule Q, and lemma A.4 shows that the value function V is unique conditional on Q. It suffices, therefore, for us to show that the thresholds are unique.

Without loss of generality, assume that the maximal difference between $\tilde{b}^*$ and b^* is positive and is attained at income level $\bar{y} \in \mathcal{Y}$:

$$\max_y \tilde{b}^*(y) - b^*(y) = \tilde{b}^*(\bar{y}) - b^*(\bar{y}) = M > 0$$

Applying Theorem 1 for $y = \bar{y}$ and $b = \widetilde{b}^*(\bar{y}) = b^*(\bar{y}) + M$, we know that

$$V(\bar{y}, b^*(\bar{y})) > V(\bar{y}, \tilde{b}^*(\bar{y}))$$

But this contradicts the fact that $b^*(\bar{y})$ and $\tilde{b}^*(\bar{y})$ are default thresholds, which requires $V(\bar{y}, b^*(\bar{y})) = V(\bar{y}, \widetilde{b}^*(\bar{y})) = V^d(\bar{y})$. Thus our premise of distinct equilibria cannot stand. □

The intuitive thrust of Theorems 1 and 2 is that distinct debt price schedules cannot both be self-sustaining. No two schedules Q and $\widetilde{Q}$ can simultaneously rationalize their corresponding default thresholds $b^*(\bar{y})$ and $\widetilde{b}^*(\bar{y})$ at the income level $\bar{y}$ of maximum difference. Instead, mimicking at a distance it is strictly better to face Q starting at the lower default threshold $b^*(\bar{y})$, and this is inconsistent with the assumption that the value function at each threshold must be the (common) default value.

An adaptation of Theorem 2 also shows that equilibrium as defined by Zhang (1997) is unique. Zhang restricts borrowing to always be risk-free, with endogenous state-by-state borrowing limits $\phi(y)$ defined such that the value of borrowing at the limit for a state is that of autarky for that state. Assuming that $M = \max_y \{\tilde{\phi}(y) - \phi(y)\} = \tilde{\phi}(\overline{y}) - \phi(\overline{y}) > 0$, a strategy for the government at $\phi(\overline{y})$ of mimicking at a distance the government at $\tilde{\phi}(\overline{y})$ achieves strictly higher value, showing that equilibrium must be unique in this case as well.

2.3 Relation to Bulow and Rogoff

Our replication proof is related to that used by Bulow and Rogoff (1989) to rule out reputational equilibria in sovereign debt models where saving is allowed after default. The Bulow and Rogoff (1989) result only applies directly to environments with complete markets, since their replication strategy specifies state-contingent payments that may not be supported by the span of available assets when markets are incomplete. Our replication strategy makes sure that mimicking is feasible given the asset span, with all benefits from lower interest costs and default premia consumed immediately by the mimicker. One can therefore see our result as an application of Bulow and Rogoff (1989)'s ideas to an incomplete market environment and in a different context.

Other applications of the Bulow-Rogoff argument are possible in incomplete markets, such as the Bulow-Rogoff result itself. Here is the proof in our environment. Suppose that the only punishment for default is a lack of ability to borrow. Consider the maximal debt level attainable in Markov perfect equilibrium, $b^*(y^*) = \max_y b^*(y)$, and suppose $b^*(y^*) > 0$. Whatever repayment strategy is optimal starting at $b^*(y^*)$, it is possible to mimic this strategy at a distance starting at 0, avoiding debt altogether and achieving strictly higher consumption due to savings on interest payments. Since this strategy is feasible after default, it places a lower bound on the value of defaulting when income is y^*; and since it provides greater value than the optimal repayment strategy starting at $b^*(y^*)$, default must be strictly preferable to repayment. This means that debt $b^*(y^*) > 0$ cannot be attained in equilibrium, thereby ruling out any reputational equlibrium with positive debt in this incomplete markets environment.

3 Extensions to the canonical model

There are several natural directions in which the canonical model of Section 2 can be modified. In Section 3.1, we show that determinacy is retained for a particular extension, where the possibility of market reaccess after default is added but the stochastic process for income is simplified to iid. In Section 3.2, we describe how our argument breaks down with other modifications to the basic framework, and speculate on whether multiplicity may be possible in these cases.

3.1 Stochastic market reentry, iid case

In the literature, a typical departure from the canonical model of Section 2 is an assumption that market reaccess is possible after default. This makes the value of default depend on the equilibrium value of borrowing, implying that Theorems 1 and 2 do not directly apply. Nevertheless, for the special case where income follows an iid process, the argument still goes through with some modification.

Suppose now that income y follows an iid process with probability $\pi(y)$, and that it is possible to re-access markets with zero debt after a stochasic period of exclusion, which has independent probability $1-\lambda$ of ending in each period. That is, replace (3) by

$$V^d(y) = u(y - \phi(y)) + \beta\lambda \mathbb{E}_{y'}\left[V^d(y')\right] + \beta(1-\lambda)\,\mathbb{E}_{y'}\left[V^o(y',0)\right] \tag{10}$$

Since the income process is iid, the expected value of reentry $\mathbb{E}_{y'}[V^o(y',0)]$ does not depend on y, and for simplicity we denote it by V^{re}. The iid assumption also implies that the debt price schedule Q depends only on the debt amount b', not the current income y, as (5) reduces to

$$Q(b') = \frac{b'}{R} \sum_{\{y' : b' \le b^*(y')\}} \pi(y') \tag{11}$$

In this setting, Theorem 1 becomes the following.

Theorem 3 (Mimicking at a distance, iid case with reentry.)**.** *Consider any two distinct equilibria (V, Q) and $\left(\widetilde{V}, \widetilde{Q}\right)$, with associated default thresholds $\{b^*(y)\}_{y\in\mathcal{Y}}$ and $\left\{\tilde{b}^*(y)\right\}_{y\in\mathcal{Y}}$, such that $\widetilde{V}^{re} \geq V^{re}$.*

Then there exists some y such that $\tilde{b}^*(y) > b^*(y)$, and defining

$$M = \max_y \tilde{b}^*(y) - b^*(y) > 0 \tag{12}$$

we have for any y and b

$$V(y, b-M) - V^d(y) \geq \widetilde{V}(y,b) - \widetilde{V}^d(y) \tag{13}$$

with strict inequality whenever $b \leq \tilde{b}^*(y)$.

Proof. First, we must show that if $\widetilde{V}^{re} > V^{re}$, then there exists some y such that $\tilde{b}^*(y) > b^*(y)$. Suppose to the contrary that $b^*(y) \geq \tilde{b}^*(y)$ for all y, which by (11) implies that $Q(b') \geq \widetilde{Q}(b')$ for all b'. It follows that $V^o(y,0) \geq \widetilde{V}^o(y,0)$ for all y, since the government starting with zero debt and facing the weakly higher debt schedule Q can always replicate the policy of the government facing $\widetilde{Q}$, achieving weakly higher consumption in the process.[5] This implies $V^{re} \geq \widetilde{V}^{re}$, a contradiction; thus $\tilde{b}^*(y) > b^*(y)$ for some y.

[5]Explicitly, it can set $b(y^t) = \tilde{b}(y^t)$, $p(y^t) = \tilde{p}(y^t)$, $c(y^t) = \tilde{c}(y^t) + Q\left(\tilde{b}(y^t)\right) - \widetilde{Q}\left(\tilde{b}(y^t)\right)$, and $c(y^t) \geq \tilde{c}(y^t)$ follows from $Q \geq \widetilde{Q}$.

Meanwhile, if $\widetilde{V}^{re} = V^{re}$, then the statement of the theorem makes no distinction between (V, Q) and $(\widetilde{V}, \widetilde{Q})$. Since the thresholds $\{b^*(y)\}$ and $\{\widetilde{b}^*(y)\}$ must not be identical if the equilibria are distinct, we may suppose without loss of generality that $\widetilde{b}^*(y) > b^*(y)$ for y.

To establish (13) for any y and b, we use the same mimicking at a distance argument as in Theorem 1, although the calculation becomes somewhat more complicated. We continue to set $b(y^t) = \widetilde{b}(y^t) - M$ and $p(y^t) = \widetilde{p}(y^t)$, along with the consumption policy in (9). The payoff in equilibrium $(\widetilde{V}, \widetilde{Q})$, i.e. $\widetilde{V}(y, b)$, is the expected utility in the repayment phase plus the

$$\sum_{\widetilde{p}(y^t)=1} \beta^t \Pi\left(y^t\right) u\left(\widetilde{c}\left(y^t\right)\right) + \sum_{\widetilde{p}(y^t)=0, \widetilde{p}(y^{t-1})=1} \beta^t \Pi\left(y^t\right) \widetilde{V}^d(y_t) \tag{14}$$

while the payoff for the mimicking government, which places a lower bound on $V(y, b - M)$, is

$$\sum_{\widetilde{p}(y^t)=1} \beta^t \Pi\left(y^t\right) u\left(c\left(y^t\right)\right) + \sum_{\widetilde{p}(y^t)=0, \widetilde{p}(y^{t-1})=1} \beta^t \Pi\left(y^t\right) V^d(y_t) \tag{15}$$

Subtracting (14) from (15), and using $c(y^t) \geq \widetilde{c}(y^t)$, we have

$$V(y, b - M) - \widetilde{V}(y, b) \geq \sum_{\widetilde{p}(y^t)=0, \widetilde{p}(y^{t-1})=1} \beta^t \Pi(y^t) \left(V^d(y_t) - \widetilde{V}^d(y_t)\right) \tag{16}$$

Using (10), we compute

$$\begin{aligned} V^d(y') - \widetilde{V}^d(y') &= \beta\lambda \mathbb{E}_{y''}[V^d(y'') - \widetilde{V}^d(y'')] + \beta(1-\lambda)(V^{re} - \widetilde{V}^{re}) \\ \implies \quad V^d(y') - \widetilde{V}^d(y') &= \frac{\beta(1-\lambda)}{1-\beta\lambda}(V^{re} - \widetilde{V}^{re}) \end{aligned} \tag{17}$$

which combined with (16) gives

$$V(y, b - M) - \widetilde{V}(y, b) \geq \frac{\beta(1-\lambda)}{1-\beta\lambda}(V^{re} - \widetilde{V}^{re}) \left(\sum_{\widetilde{p}(y^t)=0, \widetilde{p}(y^{t-1})=1} \beta^t \Pi(y^t) \right)$$

Finally, subtracting $V^d(y) - \widetilde{V}^d(y) = \frac{\beta(1-\lambda)}{1-\beta\lambda}(V^{re} - \widetilde{V}^{re})$ from both sides and using the assumption $\widetilde{V}^{re} \geq V^{re}$ yields

$$\begin{aligned} &(V(y, b - M) - V^d(y)) - (\widetilde{V}(y, b) - \widetilde{V}^d(y)) \\ &\qquad \geq \frac{\beta(1-\lambda)}{1-\beta\lambda}(\widetilde{V}^{re} - V^{re}) \left(1 - \sum_{\widetilde{p}(y^t)=0, \widetilde{p}(y^{t-1})=1} \beta^t \Pi(y^t)\right) \geq 0 \end{aligned} \tag{18}$$

as desired. □

The idea behind this calculation is that at any point, the possibility of future reentry affects both the value of default and the value of repayment. Reentry is *less* important to the value of repayment, however, because it occurs farther in the future and is discounted. Thus the expression in parentheses in (18) is positive. This is why it was necessary to subtract $V^d(y)$ and $\widetilde{V}^d(y)$ in (13): the effect of reentry on V alone is uncertain, but the effect on $V - V^d$ is not.

As with Theorem 2, we can now prove

Theorem 4. *In the model with iid income and stochastic market re-entry, Markov perfect equilibrium has a unique value function $V(y,b)$ and debt price schedule $Q(b)$.*

Proof. This closely resembles the proof of Theorem 2. As before, it suffices to prove that the default thresholds $\{b^*(y)\}_{y\in\mathcal{Y}}$ and $\{\tilde{b}^*(y)\}_{y\in\mathcal{Y}}$ are unique.

Assume without loss of generality that $\widetilde{V}^{re} \geq V^{re}$. Invoking Theorem 3, we know that the maximum difference M between $\widetilde{b}^*$ and b^* is positive. Suppose that it is attained at some $\bar{y}$.

$$\max_y \tilde{b}^*(y) - b^*(y) = \tilde{b}^*(\bar{y}) - b^*(\bar{y}) = M > 0$$

Applying inequality (16) from Theorem 3 for $y = \bar{y}$ and $b = \widetilde{b}^*(\bar{y})$, we have

$$V(\bar{y}, b^*(\bar{y})) - V^d(\bar{y}) > \widetilde{V}(\bar{y}, \widetilde{b}^*(\bar{y})) - \widetilde{V}^d(\bar{y})$$

But by the definition of $b^*(\bar{y})$ and $\widetilde{b}^*(\bar{y})$, we must have $V(\bar{y}, b^*(\bar{y})) = V^d(\bar{y})$ and $\widetilde{V}(\bar{y}, \widetilde{b}^*(\bar{y})) = \widetilde{V}^d(\bar{y})$, so that this inequality reduces to $0 > 0$, an impossibility. $\square$

3.2 Other extensions

It is natural to ask whether we can generalize Theorem 4 to the general Markov process for income considered in Theorem 2. As it stands, the approach in Theorems 3 and 4 does not admit any straightforward generalization: it relies on the fact that in the iid case, the expected value from reentry does not depend on the current state. That way, when faced with multiple equilibria, we can pick the equilibrium with the higher expected value from reentry, and then compare equilibria in a direction such that this higher value only strengthens our argument.

With a general Markov process, there is no longer any unambiguous ranking across equilibria of the expected value from future reentry, which now depends (perhaps in a complicated way) on the current level of income. This disrupts our argument. But the proof's failure in this case does not point the way to any simple counterexample either, and indeed it suggests that any such counterexample must be rather counterintuitive. For instance, it is most natural to envision multiple equilibria such that the bond prices from one equilibrium dominate those from the other: in our notation, $\widetilde{Q}(y,b) \geq Q(y,b)$ for all y and b. This corresponds to the usual intuition—an intuition that we rejected in simpler cases—that cheaper debt might be self-sustaining, as it raises the benefits from participating in financial markets and makes governments more eager to avoid default. In this case, however, there *is* an unambiguous ranking of the value from reentry (reentry is better in the equilibrium where it is cheaper to borrow), and our strategy from Theorem 4 can be invoked to rule out the multiplicity.

At the very least, therefore, if multiplicity exists in the general Markov case with reentry we know that it must be a surprising kind of multiplicity—among any two equilibria, each must offer cheaper borrowing in some places and more expensive borrowing in others. For this reason, we suspect that our argument's inapplicability here is more of a technical issue than a harbinger of hidden multiplicity. But we currently cannot rule out the possibility of nonuniqueness in this environment.

Another possible extension is to allow for haircuts on existing debt, where governments must pay back some fraction of their debt even after defaulting. Although this is an important case, it is not easily addressed using our approach. If default does not completely purge the existing stock of debt, the value from default depends on the current level of debt, whereas our argument makes essential use of the value from default being constant. (See, for instance, the last two sentences in the proof of Theorem 2.)

Finally, another important strand of the literature considers long-term debt, as in Hatchondo and Martinez (2009). Here our argument breaks down completely: a simple mimicking strategy is no longer viable when the bond price schedule is so complex, with bond prices influenced by the likelihood of endogenous default in the arbitrarily distant future. In a related continuous time environment, Lorenzoni and Werning (2014) exhibit multiple equilibria: in their model, an adverse shift in the bond price schedule forces the government into a path of increasing debt, which justifies the initial shift. Although their analysis does not adapt directly to the Hatchondo and Martinez (2009) model, it does suggest that multiple equilibria may be present.

4 Conclusion

We have showed that the Eaton-Gersovitz model of sovereign debt with default does not admit multiple equilibria. It is useful to reflect upon the features of the environment which make this result hold, by comparing it to two classes of alternative environments which do feature multiple equilibria.

In the model of Cole and Kehoe (2000), the government observes whether the current-period bond auction has been successful before it decides whether or not to repay previous-period creditors. When bond auctions fail, repayment must be done out of current-period resources instead of being smoothed over many periods, which the risk-averse government finds more costly. This timing assumption creates a coordination problem among creditors, giving rise to multiple equilibria. The literature sometimes refers to this phenomenon as "rollover multiplicity". The model we study rules it out, since uncertainty over the bond auction outcome is not allowed conditional on the current state and level of issuance.

In the model of Calvo (1988), multiplicity arises because of the way the bond revenue-raising process works. In the Calvo model, a government borrows an exogenous amount b at date 0 and inherits a liability of $R_b b$ at date 1. It then uses a mix of distortionary taxation and debt repudiation to finance a given level of government spending. Since a higher interest rate R_b tilts the balance towards more repudiation at date 1, and since investors need to break even when lending to the government, there exist two rational expectations equilibria: one with high R_b and high repudiation, and one with low R_b and low repudiation. This is sometimes called "Laffer curve multiplicity" in reference to the shape of the bond revenue curve that arises in this model (the function that gives bond revenue b as a function of promised repayment $R_b b$ has an inverted-V shape). In the model we study, the government directly announces the amount it will owe tomorrow, allowing it to avoid the downward-sloping part of the bond revenue curve.[6] Lorenzoni and Werning (2013) make a forceful argument that such an assumption requires a form of commitment to fiscal adjustment that governments are unlikely to have in practice, and they develop several dynamic variants of Laffer curve multiplicity where debt crises take place in slow motion.

Rollover and Laffer curve multiplicity are attractive and widely studied mechanisms, but there is a prevalent view that multiplicity runs even deeper—that it is a general feature of infinite-horizon models with sovereign debt. This paper rejects that view, showing that a simple and widely adopted model can produce a unique equilibrium. The common intuition that markets can tip between good and bad equilibria, despite its evident practical appeal, need not hold in every instance—rather, it depends on the detailed way in which those markets are modeled. We view the result in this paper as an invitation for continued study of those details, and for a renewed focus on reconciling theory with the widespread perception that debt markets are plagued by multiple equilibria.

References

Aguiar, Mark and Gita Gopinath, "Defaultable Debt, Interest Rates and the Current Account," *Journal of International Economics*, June 2006, *69* (1), 64–83.

_ **and Manuel Amador**, "Sovereign Debt," *Handbook of International Economics*, 2014, *volume 4* (forthcoming).

Arellano, Cristina, "Default Risk and Income Fluctuations in Emerging Economies," *The American Economic Review*, 2008, *98* (3), 690–712.

Bulow, Jeremy and Kenneth Rogoff, "Sovereign Debt: Is to Forgive to Forget?," *The American Economic Review*, March 1989, *79* (1), 43–50.

Calvo, Guillermo A., "Servicing the Public Debt: The Role of Expectations," *The American Economic Review*, 1988, *78* (4), 647–661.

[6]Interestingly, the setup of the original Eaton and Gersovitz (1981) model does not let the government choose on the bond revenue curve *a priori*, although their analysis focuses on equilibria in which it effectively does.

Chatterjee, Satyajit, Dean Corbae, Makoto Nakajima, and José-Víctor Ríos-Rull, "A Quantitative Theory of Unsecured Consumer Credit with Risk of Default," *Econometrica*, November 2007, *75* (6), 1525–1589.

Cole, Harold L. and Timothy J. Kehoe, "Self-Fulfilling Debt Crises," *The Review of Economic Studies*, January 2000, *67* (1), 91–116.

Eaton, Jonathan and Mark Gersovitz, "Debt with Potential Repudiation: Theoretical and Empirical Analysis," *The Review of Economic Studies*, April 1981, *48* (2), 289–309.

Hatchondo, Juan Carlos and Leonardo Martinez, "Long-Duration Bonds and Sovereign Defaults," *Journal of International Economics*, September 2009, *79* (1), 117–125.

_ , _ , and Horacio Sapriza, "Heterogeneous Borrowers in Quantitative Models of Sovereign Default*," *International Economic Review*, November 2009, *50* (4), 1129–1151.

_ , _ , and _ , "Quantitative properties of sovereign default models: Solution methods matter," *Review of Economic Dynamics*, October 2010, *13* (4), 919–933.

Krusell, Per and Anthony A. Smith, "Consumption–Savings Decisions with Quasi–Geometric Discounting," *Econometrica*, January 2003, *71* (1), 365–375.

Lorenzoni, Guido and Ivan Werning, "Slow Moving Debt Crises," Working Paper 19228, National Bureau of Economic Research July 2013.

_ and _ , "Slow Moving Debt Crises," *Manuscript*, 2014.

Zhang, Harold H., "Endogenous Borrowing Constraints with Incomplete Markets," *The Journal of Finance*, December 1997, *52* (5), 2187–2209.

A Proofs

Lemma A.1. *For any y, $V(y,b)$ is strictly decreasing in b*

Proof. Fix y and consider $\widetilde{b} > b$. Let $\widetilde{b'}$ be an optimal choice for next-period borrowing at $\widetilde{b}$. Then, since $\widetilde{b'}$ is a feasible choice at b, we have

$$\begin{aligned} V(y,b) &\geq u\left(y + Q\left(y,\widetilde{b'}\right) - b\right) + \beta \mathbb{E}_{y'}\left[V^o(y',\widetilde{b'})\right] \\ &> u\left(y + Q\left(y,\widetilde{b'}\right) - \widetilde{b}\right) + \beta \mathbb{E}_{y'}\left[V^o(y',\widetilde{b'})\right] = V\left(y,\widetilde{b}\right) \end{aligned}$$

□

Lemma A.2. *V is continuous in b*

Proof. Let c be an optimal choice at b. Then $c-\epsilon$ is a feasible choice at $b+\epsilon$ so

$$V(b+\epsilon)-V(b) \quad \geq \quad u(c-\epsilon)-u(c)$$

which shows that $\lim_{\epsilon\to 0} V(b+\epsilon) \geq V(b)$. From lemma A.1, $V(b+\epsilon) \leq V(b)$ so $\lim_{\epsilon\to 0} V(b+\epsilon) \leq V(b)$. □

Lemma A.3. *For any y, $\lim_{b\to\infty} V(y,b) < V^d(y)$*

Proof. Since the no-Ponzi condition requires that $b' \leq B$, we have $\max_{b'} Q(y,b') \leq B/R < \infty$. Thus when $b > y + B/R$, $V(y,b) = -\infty$. □

Lemma A.4. *For given Q, V is unique.*

Proof. Standard. □