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Large Gauge Transformations in Double Field Theory

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Abstract

Finite gauge transformations in double field theory can be defined by the exponential of generalized Lie derivatives. We interpret these transformations as 'generalized coordinate transformations' in the doubled space by proposing and testing a formula that writes large transformations in terms of derivatives of the coordinate maps. Successive generalized coordinate transformations give a generalized coordinate transformation that differs from the direct composition of the original two. Instead, it is constructed using the Courant bracket. These transformations form a group when acting on fields but, intriguingly, do not associate when acting on coordinates.

Contents

1 Introduction

Double field theory is a spacetime description of the massless sector of closed string theory that makes T-duality manifest by doubling the coordinates. In addition to the usual spacetime coordinates x^i, $i = 0, \ldots, D-1$, there are dual 'winding' coordinates $\tilde{x}_i$, which together with the x^i combine into coordinates $X^M = (\tilde{x}_i, x^i)$ transforming in the fundamental representation of the T-duality group $O(D, D)$. This theory has been formulated in [1–4]. Earlier important work can be found in [5–7] and further developments have been discussed in [8–35].

There are various formulations of double field theory. This paper uses the generalized metric formulation [4], in which the fundamental dynamical field is the $O(D, D)$ matrix

$$\mathcal{H}_{MN} \;=\; \begin{pmatrix} g^{ij} & -g^{ik}b_{kj} \\ b_{ik}g^{kj} & g_{ij} - b_{ik}g^{kl}b_{lj} \end{pmatrix} \,, \tag{1.1}$$

that unifies the spacetime metric g_{ij} and the Kalb-Ramond two-form b_{ij} and that transforms covariantly under $O(D,D)$. In addition, the theory features the dilaton d, which is a scalar under $O(D,D)$. This dilaton field is a spacetime density and is related to the scalar dilaton ϕ through the field redefinition $e^{-2d} = \sqrt{-g}e^{-2\phi}$. The double field theory action can be written in terms of a generalized curvature scalar $\mathcal{R}$ that is a function of $\mathcal{H}$ and d [4],

$$S_{\rm DFT} \;=\; \int dx d\tilde{x}\, e^{-2d}\,\mathcal{R}(\mathcal{H},d)\ . \tag{1.2}$$

This curvature scalar is a manifestly $O(D,D)$ invariant expression in terms of $\mathcal{H}$, d and 'doubled' derivatives $\partial_M = (\tilde{\partial}^i, \partial_i)$, and so the $O(D,D)$ invariance of (1.2) is manifest. This theory also features a gauge invariance whose infinitesimal transformations are parametrized by an $O(D,D)$ vector parameter $\zeta^M = (\tilde{\zeta}_i, \zeta^i)$ that combines the diffeomorphism parameter ζ^i and the b field gauge parameter $\tilde{\zeta}_i$. It acts on the physical fields as

$$\begin{aligned}\delta_\zeta \mathcal{H}_{MN} \;&=\; \zeta^P\partial_P\mathcal{H}_{MN} + \big(\partial_M\zeta^P - \partial^P\zeta_M\big)\mathcal{H}_{PN} + \big(\partial_N\zeta^P - \partial^P\zeta_N\big)\mathcal{H}_{MP}\ ,\\ \delta_\zeta d \;&=\; \zeta^M\partial_M d - \frac{1}{2}\partial_M\zeta^M\ .\end{aligned} \tag{1.3}$$

We may define a generalized Lie derivative $\widehat{\mathcal{L}}_\zeta$ acting on $O(D,D)$ tensors with arbitrary index structure. For the generalized metric the above gauge transformation is in fact the generalized Lie derivative: $\delta_\zeta\mathcal{H}_{MN} = \widehat{\mathcal{L}}_\zeta\mathcal{H}_{MN}$. Under these variations $\mathcal{R}$ transforms as a generalized scalar, $\delta_\zeta\mathcal{R} = \zeta^M\partial_M\mathcal{R}$, from which the gauge invariance of (1.2) immediately follows. More precisely, in order to verify this invariance the following 'strong constraint' is required:

$$\partial^M\partial_M \;\equiv\; \eta^{MN}\partial_M\partial_N \;=\; 0\ , \quad \text{with} \qquad \eta^{MN} \;=\; \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}\ . \tag{1.4}$$

The above constraint must hold when acting on arbitrary fields and parameters and all their products (so that $\partial^M\partial_M A = 0$ and $\partial^M A\,\partial_M B = 0$ for any fields or parameters A and B). Here η_{MN} denotes the $O(D,D)$ invariant metric. This constraint actually implies that one can always find an $O(D,D)$ rotation into a T-duality frame in which the coordinates depend only, say, on the x^i.

Satisfying this constraint by setting $\tilde{\partial}^i = 0$, the action (1.2) reduces to the standard low-energy effective action for the NS-NS sector of closed string theory. Moreover, the gauge variations (1.3) reduce for the components in (1.1) precisely to the standard (infinitesimal) general coordinate transformations and b field gauge transformations. We stress that the gauge transformations (1.3) are not infinitesimal diffeomorphisms on the doubled space, because they do not close according to the Lie bracket but rather according to the 'C-bracket' [2, 4, 5],

$$\big[\delta_{\zeta_1},\delta_{\zeta_2}\big] \;=\; -\delta_{[\zeta_1,\zeta_2]_c}\ , \qquad \big[\zeta_1,\zeta_2\big]_c^M \;\equiv\; \zeta_1^N\partial_N\zeta_2^M - \frac{1}{2}\,\zeta_{1N}\partial^M\zeta_2^N - (1\leftrightarrow 2)\ , \tag{1.5}$$

which is the $O(D,D)$ covariant extension of the Courant bracket of generalized geometry.

In this paper we will investigate the finite or large gauge transformations corresponding to the infinitesimal variations (1.3). Since these gauge variations do not represent infinitesimal diffeomorphisms of the doubled space we cannot resort to Gauss and Riemann and postulate the usual coordinate transformation rules of vectors and one-forms. In fact, inspection of (1.3) shows that each index appears to be some 'hybrid' between covariant and contravariant indices. It is thus not clear how finite transformations can be consistently defined.

We find, however, that it is possible to view finite gauge transformations as arising from some suitably defined 'generalized coordinate transformations'. We introduce such coordinate transformations with the features that are expected from the infinitesimal gauge transformations. This implies that they do not satisfy all the properties of diffeomorphisms. For instance, two successive diffeomorphisms give a third diffeomorphism that is simply defined by direct composition of the first two. Two successive 'generalized coordinate transformations' also result in a generalized coordinate transformation, but the resulting transformation is not obtained by the direct composition of the two maps. This is the group manifestation of the fact that the gauge algebra is governed by the Courant bracket (1.5) rather than the Lie bracket.

Given a generalized coordinate transformation $X \to X' = f(X)$, we propose the following associated transformation for an $O(D,D)$ vector A_M:

$$A'_M(X') \;=\; \mathcal{F}_M{}^N A_N(X)\,, \tag{1.6}$$

where the matrix $\mathcal{F}$ is defined by

$$\mathcal{F}_M{}^N \;\equiv\; \frac{1}{2}\Big(\frac{\partial X^P}{\partial X'^M}\frac{\partial X'_P}{\partial X_N} + \frac{\partial X'_M}{\partial X_P}\frac{\partial X^N}{\partial X'^P}\Big)\,. \tag{1.7}$$

Here the indices on coordinates are raised and lowered with the $O(D,D)$ invariant metric, $X_M = \eta_{MN}X^N = (x^i, \tilde{x}_i)$, etc. More generally, a tensor with an arbitrary number of $O(D,D)$ indices transforms 'tensorially', with each index rotated by the matrix $\mathcal{F}$. We show that $\mathcal{F}$ is in fact an $O(D,D)$ matrix. In ordinary geometry we would simply have $\mathcal{F}_M{}^N = \frac{\partial X^N}{\partial X'^M}$. In double field theory the dilaton d provides the scalar density $\exp(-2d)$. We give the transformation law for this density under large coordinate transformations in (2.23).

We will show that the transformation rule in (1.6) and (1.7) implies the infinitesimal transformations (1.3) when we set $X' = X - \zeta(X)$. We have also verified that this transformation satisfies the following consistency requirements: It implies the usual formulae for coordinate transformations that transform only the x^i or only the $\tilde{x}_i$. It leaves the $O(D,D)$ invariant metric in (1.4) invariant, i.e., this metric takes the same *constant* form in all coordinate systems, something required in double field theory but inconsistent in conventional differential geometry. Moreover, the strong constraint (1.4) in one coordinate system implies the strong constraint in all other coordinate systems.

As mentioned above, the generalized coordinate transformations do not compose like ordinary diffeomorphisms. In order to elucidate this point, it is useful to introduce an alternative form of the finite gauge transformations. The rule (1.6) defines the transformed tensor by giving its transformed components at the *transformed* point X'. As in general relativity, this can be seen as a *passive* transformation, but it is useful to also have an *active* form of the gauge transformations which transforms the field components, but not the coordinates. For general relativity this problem has been discussed in the literature, see, e.g., [36, 37], where it is found that gauge transformations connected to the identity can be realized as an exponential of the Lie derivative. Thus, given an ordinary vector field $A_m(x)$ we have the transformed field $A'_m(x)$

given by

$$A'_m(x) \;=\; e^{\mathcal{L}_\xi}\, A_m(x)\,, \tag{1.8}$$

where $\mathcal{L}_\xi$ is the Lie derivative in the representation appropriate for a vector, and all fields and parameters depend on x. It can be shown that this transformation is induced by the following diffeomorphism

$$x'^m \;=\; e^{-\xi^k\partial_k}\, x^m\,. \tag{1.9}$$

In double field theory we can follow the above strategy. Even though the Courant bracket does not define a Lie algebra, generalized Lie derivatives define a Lie algebra under commutators. We can therefore realize a finite gauge transformation by exponentiating the generalized Lie derivative:

$$A'_M(X) \;=\; e^{\widehat{\mathcal{L}}_\xi}\, A_M(X)\,, \tag{1.10}$$

where all fields and parameters depend on X. If finite gauge transformations are defined this way it is simple to use the Baker-Campbell-Hausdorff formula to show that the field transformations form a group and compose according to the Courant bracket. Our key technical result is the determination of the generalized coordinate transformation

$$X'^M \;=\; e^{-\Theta^K(\xi)\partial_K} X^M\,, \qquad \Theta^K(\xi) \;\equiv\; \xi^K + \mathcal{O}(\xi^3)\,, \tag{1.11}$$

so that (1.6) and (1.7) lead to the transformation (1.10), at least to $\mathcal{O}(\xi^4)$. The composition rule for generalized coordinate transformations, calculable from the definition (1.10), will be verified explicitly with $\mathcal{F}$ expanded to quadratic order in ξ. We note in passing that while the exponential (1.10) only makes sense for gauge transformations connected to the identity, the generalized coordinate transformations may be applicable more generally.

Even though the composition rule is non-standard, we are intrigued that the simple generalization of conventional tensor transformations given by (1.6) and (1.7) exists, seems to pass all consistency checks, and is, plausibly, the unique form compatible with (1.10). Surprisingly, while generalized coordinate transformations form a group when acting on fields, they do not satisfy associativity at the level of coordinate maps. Further discussion of this result and other open questions can be found in the concluding section.

2 Finite gauge transformations

In this section we propose finite gauge transformations for double field theory. These transformations are induced by (and written in terms of) generalized coordinate transformations. We begin by discussing these coordinate transformations and compute the derivatives of the maps using a simple parameterization. We show that the strong constraint is preserved by these coordinate transformations and that applying the transformation rule to ∂_M is consistent with the chain rule. Finally, η_{MN} is an invariant tensor so that $\mathcal{F}$ is actually an $O(D,D)$ matrix.

2.1 Coordinate transformations and strong constraint

In this subsection we describe some generalized coordinate transformations of the doubled coordinates. We will use throughout section 2 and 3 – but not in the rest of the paper – a parameterization with a parameter $\zeta^M(X)$ and new coordinates X' given by the exact relation

$$X'^M \;=\; X^M - \zeta^M(X)\,. \tag{2.1}$$

It follows by differentiation that

$$\frac{\partial X'^Q}{\partial X^P} \;=\; \delta_P{}^Q \;-\; \partial_P\zeta^Q\,, \tag{2.2}$$

and in matrix notation we write this as

$$\Big(\frac{\partial X'}{\partial X}\Big)_P{}^Q \;\equiv\; \frac{\partial X'^Q}{\partial X^P} \;=\; \big(\mathbf{1}-a\big)_P{}^Q\,, \quad \text{with} \qquad a_P{}^Q \equiv\; \partial_P\zeta^Q\,. \tag{2.3}$$

Note that when representing coordinate derivatives as matrices we will always associate the first index (row index) with the coordinate in the denominator and the second index (column index) with the coordinate in the numerator. The matrix inverse provides us with the other derivatives

$$\Big(\frac{\partial X}{\partial X'}\Big)_M{}^P \;\equiv\; \frac{\partial X^P}{\partial X'^M} \;=\; \Big(\frac{1}{1-a}\Big)_M{}^P \;=\; \Big(1+a+a^2+a^3+\dots\Big)_M{}^P\,, \tag{2.4}$$

or, more explicitly,

$$\frac{\partial X^P}{\partial X'^M} \;=\; \delta_M{}^P + \partial_M\zeta^P + \,\partial_M\zeta^L\partial_L\zeta^P \,+\, \partial_M\zeta^L\partial_L\zeta^R\partial_R\zeta^P + \mathcal{O}(\zeta^4)\,. \tag{2.5}$$

Let us now consider the strong constraint (1.4). In this setup with large coordinate transformations we assume that ζ^M as well as all X-dependent fields satisfy the strong constraint:

$$\eta^{MN}\partial_N A\,\partial_M B \;\equiv\; \partial^M A\,\partial_M B \;=\; 0\,, \qquad \partial^P\partial_P A \;=\; 0\,. \tag{2.6}$$

In the above, $A(X)$ and $B(X)$ can be ζ^M or any field of the theory, like the dilaton or the generalized metric. The strong constraint implies that the product $a^t a$ vanishes:

$$a^t a \;=\; 0\,. \tag{2.7}$$

Indeed,

$$0 \;=\; \partial_P\zeta_M\,\partial^P\zeta_N \;=\; a_{PM}\,a^P{}_N \;=\; (a^t)_{MP}\,a^P{}_N \;=\; (a^t a)_{MN}\,. \tag{2.8}$$

We claim that if the strong constraint holds for all fields and parameters in coordinate system X it will then hold for coordinate system X'. We begin by proving the following lemma in which two functions A and B of X are differentiated with mixed-type derivatives:

$$\partial^{M\prime}A\,\partial_M B \;=\; 0\,, \qquad \partial^{M\prime}\,\partial_M\,A \;=\; 0\,. \tag{2.9}$$

To see this we note that a primed derivative, with the help of (2.5), can be written as

$$\begin{aligned}
\partial'^M \;\equiv\; \eta^{MN}\partial'_N \;&=\; \eta^{MN}\frac{\partial X^P}{\partial X'^N}\partial_P \\
&=\; \eta^{MN}\big(\delta_N{}^P + \partial_N\zeta^P + \partial_N\zeta^K\partial_K\zeta^P + \partial_N\zeta^K\partial_K\zeta^Q\partial_Q\zeta^P + \cdots\big)\partial_P \\
&=\; \big(\eta^{MP} + \partial^M\zeta^P + \partial^M\zeta^K\partial_K\zeta^P + \partial^M\zeta^K\partial_K\zeta^Q\partial_Q\zeta^P + \cdots\big)\partial_P \\
&=\; \partial^M + \partial^M\zeta^K(\delta_K{}^P + \partial_K\zeta^P + \partial_K\zeta^Q\partial_Q\zeta^P + \cdots)\partial_P\,.
\end{aligned} \tag{2.10}$$

We see that structurally this takes the form

$$\partial'^M \;=\; \partial^M + (\partial^M\zeta^K)\,\mathcal{U}_K{}^P\,\partial_P\,, \tag{2.11}$$

where $\mathcal{U}$ is a matrix function of X whose expression is in fact not important. The lemmas now follow easily:

$$\partial'^M A\,\partial_M B \;=\; \Big(\partial^M A + (\partial^M\zeta^K)\,\mathcal{U}_K{}^P\,\partial_P A\Big)\partial_M B \;=\; 0\,, \tag{2.12}$$

by use of the strong constraint as in (2.6). Similarly,

$$\partial'^M\,\partial_M A \;=\; \Big(\partial^M + (\partial^M\zeta^K)\,\mathcal{U}_K{}^P\,\partial_P\Big)\partial_M A \;=\; 0\,. \tag{2.13}$$

Using the lemma and (2.11) it now follows that

$$\partial'^M A\,\partial'_M B \;=\; 0\,, \qquad \partial'^M\partial'_M A \;=\; 0\,. \tag{2.14}$$

These can be viewed as the statement that the strong constraint holds in the primed coordinates.

2.2 Large gauge transformations

For a scalar $S(X)$ the coordinate transformation will be taken to be the usual one,

$$S'(X') \;=\; S(X)\,. \tag{2.15}$$

It then follows to first order in ζ that

$$S'(X) - \zeta^M\partial_M S \;=\; S(X) \quad\Rightarrow\quad \delta_\zeta S \;\equiv\; S'(X) - S(X) \;=\; \zeta^M\partial_M S\,. \tag{2.16}$$

For a generalized vector A_M we need a transformation rule that acts on it like for a one-form and a vector simultaneously. Indeed, here our main clue is the infinitesimal transformation

$$\delta A_M \;\equiv\; A'_M(X) - A_M(X) \;=\; \widehat{\mathcal{L}}_\zeta A_M \;=\; \zeta^P\partial_P A_M + \big(\partial_M\zeta^N - \partial^N\zeta_M\big)A_N\,. \tag{2.17}$$

This must be reproduced by the formula we propose once the parameter ζ is taken to be small. We propose the transformation

$$A'_M(X') \;=\; \frac{1}{2}\Big(\frac{\partial X^P}{\partial X'^M}\,\frac{\partial X'_P}{\partial X_N} + \frac{\partial X'_M}{\partial X_P}\,\frac{\partial X^N}{\partial X'^P}\Big)\,A_N(X)\,. \tag{2.18}$$

In here we have defined $X_N \equiv \eta_{NM}X^M$ and $X'_N \equiv \eta_{NM}X'^M$. Expanding to first order in ζ we find with (2.2) and (2.5)

$$\begin{aligned} A'_M(X) - \zeta^P\partial_P A_M(X) \;&=\; \frac{1}{2}\Big(\big(\delta_M{}^P + \partial_M\zeta^P\big)\big(\delta^N{}_P - \partial^N\zeta_P\big) \\ &\qquad + \big(\delta^P{}_M - \partial^P\zeta_M\big)\big(\delta_P{}^N + \partial_P\zeta^N\big)\Big)A_N(X) \\ &=\; \frac{1}{2}\Big(2\delta_M{}^N + 2\partial_M\zeta^N - 2\partial^N\zeta_M\Big)A_N(X) + \mathcal{O}(\zeta^2) \\ &=\; A_M(X) + \big(\partial_M\zeta^N - \partial^N\zeta_M\big)A_N(X) + \mathcal{O}(\zeta^2)\,, \end{aligned} \tag{2.19}$$

which indeed reproduces (2.17). The transformation (2.18) is not fully determined by the constraint that the infinitesimal transformations arise correctly. A number of options allow for this result. Other consistency checks appear to select (2.18) as the only possible choice, as we will discuss in this and the following section.

Before we proceed with the analysis of the transformation (2.18) we introduce some notation. We write

$$A'_M(X') \;=\; \mathcal{F}_M{}^N A_N(X)\,, \tag{2.20}$$

where the matrix $\mathcal{F}$ is defined by

$$\mathcal{F}_M{}^N \;\equiv\; \frac{1}{2}\Big(\frac{\partial X^P}{\partial X'^M}\,\frac{\partial X'_P}{\partial X_N} + \frac{\partial X'_M}{\partial X_P}\,\frac{\partial X^N}{\partial X'^P}\Big)\,. \tag{2.21}$$

More generally, any $O(D,D)$ tensor we require to transform under generalized coordinate transformations such that each index is rotated by the matrix $\mathcal{F}_M{}^N$.

Double field theory also requires the definition of a scalar density. The transformation (1.3) of the dilation d implies that

$$\delta_\zeta e^{-2d} \;=\; \partial_M\big(\zeta^M e^{-2d}\big)\,. \tag{2.22}$$

This is the infinitesimal transformation of a scalar density, and it is the same transformation that we have in ordinary differential geometry. Thus, the finite gauge transformation of this density must be given by

$$e^{-2\,d'(X')} \;=\; \Big|\det\frac{\partial X}{\partial X'}\Big|\,e^{-2\,d(X)}\,. \tag{2.23}$$

Of course, using (2.5) and expanding this to first order in ζ it is easily seen that the variation $\delta_\zeta d = d'(X) - d(X)$ coincides with that given in (1.3). Further exploration of the consistency of (2.23) will be discussed in sections 4.2, 5.2, and 5.3.

The transformation (2.18) can be expanded to all orders in ζ. In the matrix notation we have used for coordinate derivatives we have

$$\mathcal{F}_M{}^N(X',X) \;=\; \frac{1}{2}\Big(\,\frac{\partial X}{\partial X'}\Big(\frac{\partial X'}{\partial X}\Big)^t + \Big(\frac{\partial X'}{\partial X}\Big)^t\,\frac{\partial X}{\partial X'}\,\Big)_M{}^N\,. \tag{2.24}$$

We have added the coordinate arguments in a specific order: the first input is the new coordinate and the second input is the old coordinate. We will only use those arguments when needed explicitly. In index-free notation we write

$$\mathcal{F}(X',X) \;=\; \frac{1}{2}\Big(\,\frac{\partial X}{\partial X'}\Big(\frac{\partial X'}{\partial X}\Big)^t + \Big(\frac{\partial X'}{\partial X}\Big)^t\,\frac{\partial X}{\partial X'}\,\Big)\,. \tag{2.25}$$

Note that $\mathcal{F}$ is in fact an *anticommutator* of partial derivatives:

$$\mathcal{F}(X',X) \;=\; \frac{1}{2}\Big\{\,\frac{\partial X}{\partial X'}\,,\;\Big(\frac{\partial X'}{\partial X}\Big)^t\Big\}\,. \tag{2.26}$$

Using our expansions (2.3) and (2.4) we immediately write

$$\begin{aligned}\mathcal{F}_M{}^N \;&=\; \frac{1}{2}\Big(\Big(\frac{1}{1-a}\Big)(1-a^t) + (1-a^t)\Big(\frac{1}{1-a}\Big)\Big)_M{}^N\\ &=\; \frac{1}{2}\Big(\;1 + (a-a^t) + (a^2 - aa^t) + (a^3 - a^2a^t) + \cdots\\ &\qquad\quad + 1 - a^t + a + a^2 + a^3 + \cdots\Big)_M{}^N\,,\end{aligned} \tag{2.27}$$

where we used the strong constraint $a^t a = 0$ and expanded in the last equation. We also note that by the strong constraint between ζ and any field it follows that if A satisfies the strong constraint, so does A' defined by (2.18). Combining these terms gives us the result

$$\mathcal{F} \;=\; 1 + a - a^t + \sum_{n=2}^{\infty}\big(a^n - \frac{1}{2}a^{n-1}a^t\big)\,. \tag{2.28}$$

Let us finally note that for $aa^t = 0$ the two lines in the second equation of (2.27) are equal, which in turn means that the two terms in the definition of $\mathcal{F}$ coincide, and so (2.25) reduces to one term,

$$aa^t \;=\; 0 \quad \Rightarrow \quad \mathcal{F}(X', X) \;=\; \frac{\partial X}{\partial X'}\Big(\frac{\partial X'}{\partial X}\Big)^t \;=\; \Big(\frac{\partial X'}{\partial X}\Big)^t\,\frac{\partial X}{\partial X'}\,. \tag{2.29}$$

Although this does not hold in general, it does hold for a few special cases that we inspect in section 3.

We now perform a basic consistency check. We should be able to use the transformation (2.18) for partial derivatives, which also have an index down. Therefore, we must have

$$\partial'_M \;=\; \frac{1}{2}\Big(\frac{\partial X^P}{\partial X'^M}\,\frac{\partial X'_P}{\partial X_N} + \frac{\partial X'_M}{\partial X_P}\,\frac{\partial X^N}{\partial X'^P}\Big)\,\partial_N\,. \tag{2.30}$$

On the other hand, partial derivatives must also transform with the chain rule

$$\partial'_M \;=\; \frac{\partial X^N}{\partial X'^M}\frac{\partial}{\partial X^N} \;=\; \frac{\partial X^N}{\partial X'^M}\partial_N\,. \tag{2.31}$$

The two expressions are consistent thanks to the strong constraint. For this note that the first expression can be written as

$$\partial'_M \;=\; \frac{1}{2}\frac{\partial X^P}{\partial X'^M}\,(\delta^N{}_P - \partial^N\zeta_P)\partial_N + \frac{1}{2}(\delta^P{}_M - \partial^P\zeta_M)\,\partial'_P\,. \tag{2.32}$$

By the lemmas (2.9) the term $(\partial^P\zeta_M)\partial'_P$ vanishes acting on any function. Moreover, the term $(\partial^N\zeta_P)\partial_N$ also vanishes. Bringing the right-most non-vanishing term to the left-hand side, we have

$$\frac{1}{2}\,\partial'_M \;=\; \frac{1}{2}\frac{\partial X^P}{\partial X'^M}\,\partial_P\,, \tag{2.33}$$

showing that the usual transformation of derivatives is consistent with (2.30).

Our final check here is that the metric η_{MN} is an invariant tensor. For this we must have

$$\eta_{MN} \;=\; \mathcal{F}_M{}^R\,\mathcal{F}_N{}^S\,\eta_{RS}\,. \tag{2.34}$$

This equation states that $\mathcal{F}_M{}^N$ *is in fact* an $O(D,D)$ matrix.[1] Raising the N index we have

$$\delta_M{}^N \;=\; \mathcal{F}_M{}^R\,\mathcal{F}^N{}_R \;=\; (\mathcal{F}\,\mathcal{F}^t\,)_M{}^N\,, \tag{2.35}$$

and therefore we must check that

$$\mathcal{F}\,\mathcal{F}^t \;=\; \mathbf{1}\,. \tag{2.36}$$

[1]It should be noted, however, that we cannot think of the generalized coordinate transformations as local $O(D,D)$ transformations with an X-dependent $O(D,D)$ matrix $h = \mathcal{F}(X)$. The reason is that in the transformation of the argument we would need $X'^M = \mathcal{F}^M{}_N X^N$, which in general is different from the actual X'.

We thus calculate with (2.27)

$$\mathcal{F}\mathcal{F}^t \;=\; \frac{1}{2}\Big(\frac{1}{1-a}\,(1-a^t) + (1-a^t)\frac{1}{1-a}\Big)\,\frac{1}{2}\Big((1-a)\frac{1}{1-a^t} \;+\; \frac{1}{1-a^t}(1-a)\Big)\,. \tag{2.37}$$

The cross terms give multiples of the unit matrix, but the other two terms are more complicated,

$$\mathcal{F}\mathcal{F}^t \;=\; \frac{1}{2} + \frac{1}{4}\Big(\frac{1}{1-a}\,(1-a^t)(1-a)\frac{1}{1-a^t} + (1-a^t)\frac{1}{1-a}\frac{1}{1-a^t}(1-a)\Big)\,. \tag{2.38}$$

We note that if the order of the second and third factors in the first term was opposite we would have a simple product. The same holds for the first and second factors in the second term. The computation is thus helped by the use of the following commutators:

$$\big[1-a^t, 1-a\big] \;=\; -aa^t\,, \quad \Big[1-a^t, \frac{1}{1-a}\Big] \;=\; \frac{1}{1-a}\,aa^t\,. \tag{2.39}$$

With these (2.38) becomes

$$\mathcal{F}\mathcal{F}^t \;=\; 1 + \frac{1}{4}\Big(-\frac{1}{1-a}\,aa^t\,\frac{1}{1-a^t} + \frac{1}{1-a}\,aa^t\frac{1}{1-a^t}(1-a)\Big)\,. \tag{2.40}$$

The terms in parenthesis cancel: in the second one we can bring the a^t in aa^t to the right, where it kills a. We thus proved that $\mathcal{F}\,\mathcal{F}^t = \mathbf{1}$. This implies the desired gauge invariance of η or, equivalently, its independence of the chosen coordinate system. Moreover, it proves that $\mathcal{F}_M{}^N$ is an $O(D,D)$ matrix.

It is also straightforward to verify that, as expected, $\mathcal{F}$ and $\mathcal{F}^t$ are also inverses of each other in the other direction:

$$\mathcal{F}^t\mathcal{F} \;=\; 1\,. \tag{2.41}$$

Indeed, this time we get

$$\mathcal{F}^t\mathcal{F} \;=\; \frac{1}{2} + \frac{1}{4}\Big((1-a)\frac{1}{1-a^t}\frac{1}{1-a}(1-a^t) + \frac{1}{1-a^t}\,(1-a)(1-a^t)\frac{1}{1-a}\Big)\,. \tag{2.42}$$

The simplest way to evaluate the left-over terms is to expand using $a^ta = 0$. Each of the two summands gives in fact simple expressions:

$$\mathcal{F}^t\mathcal{F} \;=\; \frac{1}{2} + \frac{1}{4}\Big((1-aa^t) \;+\; (1+aa^t)\Big) \;=\; \mathbf{1}\,. \tag{2.43}$$

The coordinate transformation for a generalized tensor with an upper index is obtained from (2.20) by raising the index:

$$A'^M(X') \;=\; \mathcal{F}^M{}_N A^N(X)\,. \tag{2.44}$$

Of course, the indices on $\mathcal{F}$ are raised and lowered with η, so that (2.21) gives

$$\mathcal{F}^M{}_N \;=\; \frac{1}{2}\Big(\frac{\partial X_P}{\partial X'_M}\,\frac{\partial X'^P}{\partial X^N} + \frac{\partial X'^M}{\partial X^P}\,\frac{\partial X_N}{\partial X'_P}\Big)\,. \tag{2.45}$$

Consistent with the invariance of η, it follows that the contraction of upper and lower indices gives a tensor of lower rank, e.g.,

$$A'^M B'_M \;=\; \mathcal{F}^M{}_N A^N \mathcal{F}_M{}^K B_K \;=\; A^N(\mathcal{F}^t\mathcal{F})_N{}^K B_K \;=\; A^N\delta_N{}^K B_K \;=\; A^N B_N\,. \tag{2.46}$$

Let us comment on inverse transformations. If we perform a coordinate transformation $X \to X'$ followed by $X' \to X$ the result should be no coordinate transformation. In the notation of (2.24) we should have

$$\mathcal{F}_M{}^N(X,X')\,\mathcal{F}_N{}^P(X',X) \;=\; \delta_M{}^P\,. \tag{2.47}$$

As we would expect, this is closely related to the $O(D,D)$ properties of $\mathcal{F}$ noted above. We see from (2.25)

$$\begin{aligned}\mathcal{F}(X,X') \;&=\; \frac{1}{2}\Big(\frac{\partial X'}{\partial X}\Big(\frac{\partial X}{\partial X'}\Big)^t + \Big(\frac{\partial X}{\partial X'}\Big)^t\frac{\partial X'}{\partial X}\Big)\\ &=\; \frac{1}{2}\Big(\frac{\partial X}{\partial X'}\Big(\frac{\partial X'}{\partial X}\Big)^t + \Big(\frac{\partial X'}{\partial X}\Big)^t\frac{\partial X}{\partial X'}\Big)^t \;=\; \mathcal{F}(X',X)^t\,.\end{aligned} \tag{2.48}$$

With indices, we write

$$\mathcal{F}_M{}^N(X,X') \;=\; \mathcal{F}^N{}_M(X',X)\,. \tag{2.49}$$

Back on the left-hand side of (2.47) we have

$$\mathcal{F}^N{}_M(X',X)\,\mathcal{F}_N{}^P(X',X) \;=\; (\mathcal{F}^t\mathcal{F})_M{}^P \;=\; \delta_M{}^P\,. \tag{2.50}$$

This confirms that the postulated transformation is consistent with the independent definition of the inverse.

Our computations used at various points the strong constraint. This constraint implies unusual relations. For example we have found that

$$-\,1 + \frac{\partial X}{\partial X'} + \Big(\frac{\partial X'}{\partial X}\Big)^t \;=\; \Big(\frac{\partial X'}{\partial X}\Big)^t\frac{\partial X}{\partial X'}\,, \tag{2.51}$$

which is readily checked using (2.3) and (2.4). This relation allows us to write $\mathcal{F}$ differently, but not in any simpler way. Using the above and (2.26) we have, for example,

$$\mathcal{F} \;=\; -1 + \frac{\partial X}{\partial X'} + \Big(\frac{\partial X'}{\partial X}\Big)^t + \frac{1}{2}\Big[\frac{\partial X}{\partial X'}\,,\,\Big(\frac{\partial X'}{\partial X}\Big)^t\Big]\,. \tag{2.52}$$

Using relations like this we have experimented with various other candidate expressions for $\mathcal{F}$, but have not found an equally natural expression that passes all consistency requirements.

3 Special gauge transformations and $O(D,D)$

The purpose of this section is two-fold. We first show, in subsection 3.1, how the standard, finite coordinate transformations of the non-doubled fields arise from the finite transformations generated by $\mathcal{F}$ in the doubled theory. In subsection 3.2 we discuss to what extent finite $O(D,D)$ transformations are contained in the gauge group. Viewing the $O(D,D)$ rotation of coordinates directly as a generalized coordinate transformation leads to a puzzling result: the gauge transformed field and the $O(D,D)$ transformed field differ by one power of the $O(D,D)$ rotation. Resolving this paradox we find that only the geometric subgroup $GL(D,\mathbb{R})\ltimes\mathbb{R}^{\frac{1}{2}D(D-1)}$ can always be realized as special coordinate transformations, but that in the context of a reduction on the torus T^d, the full $O(d,d)$ subgroup of $O(D,D)$ is part of the gauge group.

3.1 General coordinate and b-field gauge transformations

We will now show that the postulated finite coordinate transformations in double field theory reduce for special cases to the standard finite gauge transformations, namely general coordinate transformations and b-field gauge transformations. It turns out that these transformations, c.f. (3.3), (3.6) and (3.8) below, are special transformations $X \to X'$ for which the two terms in (2.18) are actually equal so that $\mathcal{F}$ simplifies to one term, as in (2.29),

$$\frac{\partial X'_M}{\partial X_P}\frac{\partial X^N}{\partial X'^P} \;=\; \frac{\partial X^P}{\partial X'^M}\frac{\partial X'_P}{\partial X_N} \qquad \Rightarrow \qquad \mathcal{F}_M{}^N \;=\; \frac{\partial X'_M}{\partial X_P}\frac{\partial X^N}{\partial X'^P} \;=\; \frac{\partial X^P}{\partial X'^M}\frac{\partial X'_P}{\partial X_N}\;. \tag{3.1}$$

We recall from (2.29) that this holds if $aa^t = 0$, which by (2.3) means

$$(aa^t)_{MN} \;=\; a_M{}^P a_{NP} \;=\; \partial_M\zeta^P\,\partial_N\zeta_P\;. \tag{3.2}$$

If we have either $\zeta^i = 0$ or $\tilde{\zeta}_i = 0$ the $O(D,D)$ invariant sum over P vanishes and (3.1) holds. This will apply below, since we will consider general coordinate and b-field gauge transformations separately.

We start with a vector $A_M(x)$ independent of $\tilde{x}$ and a coordinate transformation

$$x^i \;\to\; x^{i\prime} \;=\; x^{i\prime}(x)\;, \qquad \tilde{x}'_i \;=\; \tilde{x}_i\;. \tag{3.3}$$

Since this transformation leaves $\tilde{x}_i$ invariant, the corresponding parameter $\tilde{\zeta}_i$ is zero, and thus we can apply (3.1). Specializing (2.20) to A_i and using the second form of $\mathcal{F}$ in (3.1) we get

$$A'_i(x') \;=\; \frac{\partial X^P}{\partial x^{i\prime}}\frac{\partial X'_P}{\partial X_N}\,A_N(x) \;=\; \frac{\partial x^p}{\partial x^{i\prime}}\frac{\partial \tilde{x}'_p}{\partial \tilde{x}_n}\,A_n(x) \;=\; \frac{\partial x^p}{\partial x^{i\prime}}\,\delta_p^n\,A_n(x) \;=\; \frac{\partial x^p}{\partial x^{i\prime}}\,A_p(x)\;, \tag{3.4}$$

which is precisely the standard general coordinate transformation of a co-vector. Specializing (2.20) to A^i we get

$$A^{i\prime}(x') \;=\; \frac{\partial X^P}{\partial \tilde{x}'_i}\frac{\partial X'_P}{\partial X_N}\,A_N(x) \;=\; \frac{\partial \tilde{x}_p}{\partial \tilde{x}'_i}\frac{\partial x^{p\prime}}{\partial x^n}\,A^n(x) \;=\; \delta_p^i\,\frac{\partial x^{p\prime}}{\partial x^n}\,A^n(x) \;=\; \frac{\partial x^{i\prime}}{\partial x^n}\,A^n(x)\;, \tag{3.5}$$

which is the general coordinate transformation of a vector.

If we consider now a field depending only on $\tilde{x}$ and a transformation

$$\tilde{x}_i \;\to\; \tilde{x}'_i \;=\; \tilde{x}'_i(\tilde{x})\;, \qquad x^{i\prime} \;=\; x^i\;, \tag{3.6}$$

that transforms only the $\tilde{x}$ we have $\zeta^i = 0$ and so we can again apply (3.1). We get by a completely analogous computation

$$A'_i(\tilde{x}') \;=\; \frac{\partial \tilde{x}'_i}{\partial \tilde{x}_n}\,A_n(\tilde{x})\;, \qquad A^{i\prime}(\tilde{x}') \;=\; \frac{\partial \tilde{x}_p}{\partial \tilde{x}'_i}\,A^p(\tilde{x})\;. \tag{3.7}$$

Therefore, they transform conventionally, where we recall that for dual coordinate transformations the notion of covariant and contravariant indices is interchanged.

Let us now consider b-field gauge transformations, which should follow from

$$\tilde{x}'_i \;=\; \tilde{x}_i - \tilde{\zeta}_i(x)\;, \qquad x^{i\prime} \;=\; x^i\;. \tag{3.8}$$

As $\tilde{\zeta}_i$ depends on x this transformation mixes x and $\tilde{x}$, but still satisfies condition (3.1) since $\zeta^i = 0$. We first compute

$$\frac{\partial X'^M}{\partial X^N} \;=\; \begin{pmatrix} \frac{\partial \tilde{x}'_i}{\partial \tilde{x}_j} & \frac{\partial x^{i\prime}}{\partial \tilde{x}_j} \\ \frac{\partial \tilde{x}'_i}{\partial x^j} & \frac{\partial x^{i\prime}}{\partial x^j} \end{pmatrix} \;=\; \begin{pmatrix} \delta_i{}^j & 0 \\ -\partial_j \tilde{\zeta}_i & \delta_j{}^i \end{pmatrix} \,, \tag{3.9}$$

and the inverse

$$\frac{\partial X^M}{\partial X'^N} \;=\; \begin{pmatrix} \frac{\partial \tilde{x}_i}{\partial \tilde{x}'_j} & \frac{\partial x^i}{\partial \tilde{x}'_j} \\ \frac{\partial \tilde{x}_i}{\partial x^{j\prime}} & \frac{\partial x^i}{\partial x^{j\prime}} \end{pmatrix} \;=\; \begin{pmatrix} \delta_i{}^j & 0 \\ \partial_j \tilde{\zeta}_i & \delta_j{}^i \end{pmatrix} \,. \tag{3.10}$$

We will now show that (3.8) indeed leads to the expected b-field gauge transformations. We apply a finite gauge transformation to the generalized metric

$$\mathcal{H}_{MN} \;=\; \begin{pmatrix} \mathcal{H}^{ij} & \mathcal{H}^i{}_j \\ \mathcal{H}_i{}^j & \mathcal{H}_{ij} \end{pmatrix} \;=\; \begin{pmatrix} g^{ij} & -g^{ik}b_{kj} \\ b_{ik}g^{kj} & g_{ij} - b_{ik}g^{kl}b_{lj} \end{pmatrix} \,, \tag{3.11}$$

which reads

$$\mathcal{H}'_{MN}(X') \;=\; \frac{\partial X^P}{\partial X'^M}\,\frac{\partial X'_P}{\partial X_K}\,\frac{\partial X^Q}{\partial X'^N}\,\frac{\partial X'_Q}{\partial X_L}\,\mathcal{H}_{KL}(X) \,. \tag{3.12}$$

Specializing to the component $\mathcal{H}^{ij}$, we get

$$\mathcal{H}'^{\,ij} \;=\; \frac{\partial X^P}{\partial \tilde{x}'_i}\,\frac{\partial X'_P}{\partial X_K}\,\frac{\partial X^Q}{\partial \tilde{x}'_j}\,\frac{\partial X'_Q}{\partial X_L}\,\mathcal{H}_{KL} \,, \tag{3.13}$$

and we assume that $\mathcal{H}$ depends initially only on x so that by (3.8) $\mathcal{H}'$ has the same coordinate dependence, which we suppress. Inserting the non-vanishing derivatives we get

$$\mathcal{H}'^{\,ij} \;=\; \frac{\partial \tilde{x}_p}{\partial \tilde{x}'_i}\,\frac{\partial x'^p}{\partial x^k}\,\frac{\partial \tilde{x}_q}{\partial \tilde{x}'_j}\,\frac{\partial x'^q}{\partial x^l}\,\mathcal{H}^{kl} \;=\; \delta^i_p\,\delta^p_k\,\delta^j_q\,\delta^q_l\,\mathcal{H}^{kl} \;=\; \mathcal{H}^{ij} \,, \tag{3.14}$$

and comparing with (3.11) we deduce that

$$g^{ij\,\prime} \;=\; g^{ij} \,. \tag{3.15}$$

Thus, as expected, the metric is invariant under b-field gauge transformations. Specializing now to the component $\mathcal{H}^i{}_j$ and inserting the non-vanishing derivatives we get

$$\begin{aligned} \mathcal{H}^i{}_j{}' \;&=\; \frac{\partial X^P}{\partial \tilde{x}'_i}\,\frac{\partial X'_P}{\partial X_K}\,\frac{\partial X^Q}{\partial x^{j\prime}}\,\frac{\partial X'_Q}{\partial X_L}\,\mathcal{H}_{KL} \\ &=\; \frac{\partial \tilde{x}_p}{\partial \tilde{x}'_i}\,\frac{\partial x'^p}{\partial x^k}\,\frac{\partial x^q}{\partial x^{j\prime}}\,\frac{\partial \tilde{x}'_q}{\partial \tilde{x}_l}\,\mathcal{H}^k{}_l + \frac{\partial \tilde{x}_p}{\partial \tilde{x}'_i}\,\frac{\partial x'^p}{\partial x^k}\,\frac{\partial x^q}{\partial x^{j\prime}}\,\frac{\partial \tilde{x}'_q}{\partial x^l}\,\mathcal{H}^{kl} + \frac{\partial \tilde{x}_p}{\partial \tilde{x}'_i}\,\frac{\partial x'^p}{\partial x^k}\,\frac{\partial \tilde{x}_q}{\partial x^{j\prime}}\,\frac{\partial x^{q\prime}}{\partial x^l}\,\mathcal{H}^{kl} \\ &=\; \delta_p{}^i\,\delta_k{}^p\,\delta_j{}^q\,\delta_q{}^l\,\mathcal{H}^k{}_l + \delta_p{}^i\,\delta_k{}^p\,\delta_j{}^q\,(-\partial_l\tilde{\zeta}_q)\mathcal{H}^{kl} + \delta_p{}^i\,\delta_k{}^p\,(\partial_j\tilde{\zeta}_q)\delta_l{}^q\,\mathcal{H}^{kl} \\ &=\; \mathcal{H}^i{}_j - \partial_l\tilde{\zeta}_j\,\mathcal{H}^{il} + \partial_j\tilde{\zeta}_l\,\mathcal{H}^{il} \,. \end{aligned} \tag{3.16}$$

Making use of (3.11) we then find that

$$-\,g^{ik\,\prime}\,b'_{kj} \;=\; -g^{ik}b_{kj} - g^{ik}(\partial_k\tilde{\zeta}_j - \partial_j\tilde{\zeta}_k) \,. \tag{3.17}$$

From this and (3.15) we infer that

$$b'_{ij} \;=\; b_{ij} + \partial_i\tilde{\zeta}_j - \partial_j\tilde{\zeta}_i \,, \tag{3.18}$$

showing that the generalized coordinate transformations reproduce precisely the finite b-field gauge transformations.

3.2 The relation between $O(D,D)$ and gauge symmetries

We ask now to what extent $O(D,D)$ transformations are generalized coordinate transformations. Consider the finite $O(D,D)$ transformation

$$X'^M \;=\; h^M{}_N X^N\,, \quad \text{or} \qquad X' \;=\; h\,X\,, \tag{3.19}$$

which, by definition, acts on a vector field as

$$A'_M(X') \;=\; A_N(X)\big(h^{-1}\big)^N{}_M \quad \text{or} \qquad A'(X'=hX) \;=\; A(X)\,h^{-1}\,. \tag{3.20}$$

As a first naive attempt let us view (3.19) as a generalized coordinate transformation and compute its action on a vector $A_M(X)$. The derivatives are

$$\frac{\partial X'^M}{\partial X^N} \;=\; h^M{}_N\,, \qquad \frac{\partial X^M}{\partial X'^N} \;=\; (h^{-1})^M{}_N\,, \tag{3.21}$$

or in matrix notation

$$\frac{\partial X'}{\partial X} \;=\; h^t\,, \qquad \frac{\partial X}{\partial X'} \;=\; (h^{-1})^t\,. \tag{3.22}$$

We can then use (2.24) to write the gauge transformation, including the $O(D,D)$ metrics that are implicit in (2.21) in the PP contractions and the coordinates with lowered indices:

$$A'_M(X') \;=\; \frac{1}{2}\Big((h^{-1})^t\eta^{-1}h\eta + \eta^{-1}h\eta(h^{-1})^t\Big)_M{}^N A_N(X)\,. \tag{3.23}$$

We have $h\eta h^t = \eta$, from which we conclude for the first term

$$(h^{-1})^t\eta^{-1}h\eta \;=\; (h^{-1})^t\eta^{-1}\eta(h^t)^{-1} \;=\; \big[(h^{-1})^t\big]^2\,, \tag{3.24}$$

and for the second

$$\eta^{-1}h\eta(h^{-1})^t \;=\; \eta^{-1}\eta(h^t)^{-1}(h^{-1})^t \;=\; \big[(h^{-1})^t\big]^2\,. \tag{3.25}$$

Thus, the transformation rule is

$$A'_M(X') \;=\; \Big[\big((h^{-1})^t\big)^2\Big]_M{}^N A_N(X) \;=\; A_N(X)\big[\big(h^{-1}\big)^2\big]^N{}_M\,. \tag{3.26}$$

In index-free notation,

$$A'(X'=hX) \;=\; A(X)(h^{-1})^2\,. \tag{3.27}$$

Comparing with (3.20) we infer that the gauge symmetry gives the square of the matrix we want! This is the finite version of the same phenomenon encountered at the infinitesimal level in [3]. There we saw that the infinitesimal version of the naive ansatz (3.19) leads to a relative factor of two between the transport term and the rest.

The reason that the above does not indicate an inconsistency is that, viewed as a general coordinate transformation, the ansatz (3.19) is not allowed in general by the strong constraint. We will use (3.27) as a guide to modify the generalized coordinate transformation associated to the duality transformation (3.19). While the coordinate transformation will differ from the duality transformation in the way coordinates are rotated, the field transformations can be made to agree, under conditions to be explained below.

Consider first the geometric subgroup $GL(D,\mathbb{R}) \ltimes \mathbb{R}^{\frac{1}{2}D(D-1)}$ of $O(D,D)$, whose elements do not mix the x and $\tilde{x}$ coordinates. This subgroup, we claim, can be realized as (generalized) coordinate transformations. To prove this claim, we work in a frame in which the fields do not depend on $\tilde{x}$. Consider the dualities defined by a constant $\Lambda \in GL(D,\mathbb{R})$ embedded in $O(D,D)$ as $\Lambda \to h(\Lambda)$, with

$$(h^{-1})^N{}_M(\Lambda) \;=\; \begin{pmatrix} (h^{-1})_j{}^i & (h^{-1})_{ji} \\ (h^{-1})^{ji} & (h^{-1})^j{}_i \end{pmatrix} \;=\; \begin{pmatrix} \Lambda^i{}_j & 0 \\ 0 & (\Lambda^{-1})^j{}_i \end{pmatrix} \,. \tag{3.28}$$

The corresponding $O(D,D)$ transformation (3.20) on a vector $A_M = (A^i, A_i)$ then gives

$$A_i'(x') \;=\; (\Lambda^{-1})^j{}_i\, A_j(x)\,, \qquad A^{i\prime}(x') \;=\; \Lambda^i{}_j\, A^j(x)\,, \tag{3.29}$$

where only the transformation of x is relevant in the argument of the fields. The associated generalized coordinate transformation is

$$x'^i \;=\; \Lambda^i{}_j\, x^j\,, \qquad \tilde{x}_i' \;=\; \tilde{x}_i\,, \qquad \Lambda \in GL(D,\mathbb{R})\,. \tag{3.30}$$

As anticipated above, this is not the coordinate rotation induced by $GL(D,\mathbb{R}) \subset O(D,D)$, which would also transform $\tilde{x}$ (in the dual representation according to (3.28)). Equation (3.30) is a special case of (3.3), so we can use the results of that subsection to find that this coordinate transformation yields

$$A_i'(x') \;=\; (\Lambda^{-1})^j{}_i\, A_j(x)\,, \qquad A^{i\prime}(x') \;=\; \Lambda^i{}_j\, A^j(x)\,, \tag{3.31}$$

resulting in complete agreement with (3.29).

Finally, consider now the constant shift transformations in the duality subgroup $\mathbb{R}^{\frac{1}{2}D(D-1)}$ of $O(D,D)$. These, with constant parameter $e_{ij} = -e_{ji}$, are given by

$$(h^{-1})^N{}_M(e) \;=\; \begin{pmatrix} \delta^i{}_j & -e_{ij} \\ 0 & \delta^j{}_i \end{pmatrix} \,. \tag{3.32}$$

It is easy to check that this acts on the generalized metric by $b_{ij} \to b_{ij} + e_{ij}$. We claim that the associated generalized coordinate transformations are

$$\tilde{x}_i' \;=\; \tilde{x}_i + \frac{1}{2}e_{ij}x^j\,, \qquad x^{i\prime} \;=\; x^i\,. \tag{3.33}$$

Again, this differs (by a factor of two) from the coordinate transformations suggested by the dualities (3.32). Equations (3.33) are a special case of (3.8), applicable for fields that depend only on x, and also result in $b_{ij} \to b_{ij} + e_{ij}$. Summarizing, the full geometric subgroup is part of the gauge group.

Let us now turn to the remaining transformations that complete the geometric subgroup to the full T-duality group $O(D,D)$. Instead of (3.19) we consider the generalized coordinate transformation

$$X'^M \;=\; \left(\sqrt{h}\right)^M{}_N X^N\,. \tag{3.34}$$

The square root of the group element always exists and is itself a group element for the component connected to the identity: we may simply insert a factor of $\frac{1}{2}$ in the exponential representation of h in order to construct $\sqrt{h}$. Since $\sqrt{h}$ is an $O(D,D)$ element the above computation leading to (3.27) proceeds in exactly the same way, but now we obtain

$$A'_M(X') \;=\; \big[\big((\sqrt{h})^{-1}\big)^2\big]^N{}_M\, A_N(X) \;=\; A_N(X)\big(h^{-1}\big)^N{}_M \;. \tag{3.35}$$

More schematically, and without indices, we write

$$A'(X'=\sqrt{h}\,X) \;=\; A(X)\,h^{-1} \;. \tag{3.36}$$

The right-hand side is as required by the $O(D,D)$ transformation (3.20), but the left-hand side is not, because $X' = \sqrt{h}\,X$ rather than $X' = hX$. We conclude that in general the full $O(D,D)$ cannot be seen as part of the gauge group. However, for the special case that the fields depend only on a subset of half of the coordinates that are allowed by the strong constraint the situation changes. In this case we can consider $O(D,D)$ transformations that act only on those coordinates on which the fields do not depend. We then have $A'(X') = A'(X)$ and the two formulas (3.20) and (3.36) coincide. We use this approach now to see that the remaining $O(D,D)$ transformations can be realized as coordinate transformations, consistent with the strong constraint. We already have the group elements (3.28) and (3.32). To generate the full $O(D,D)$ we are missing the elements

$$h^M{}_N(f) \;=\; \begin{pmatrix} \delta_i{}^j & 0 \\ f^{ij} & \delta^i{}_j \end{pmatrix} \qquad \Rightarrow \qquad \big(\sqrt{h}\,\big)^M{}_N(f) \;=\; \begin{pmatrix} \delta_i{}^j & 0 \\ \frac{1}{2}f^{ij} & \delta^i{}_j \end{pmatrix} \;. \tag{3.37}$$

The coordinate transformation (3.34) then reads

$$X'^M \;=\; \big(\sqrt{h}\,\big)^M{}_N X^N \qquad \Rightarrow \qquad \tilde{x}'_i \;=\; \tilde{x}_i \;, \quad x^{i\prime} \;=\; x^i + \frac{1}{2}f^{ij}\tilde{x}_j \;. \tag{3.38}$$

The last equation implies $\tilde{\zeta}_i = 0$, $\zeta^i = -\frac{1}{2}f^{ij}\tilde{x}_j$, and thus the gauge parameters depend only on the $\tilde{x}_i$ on which the above transformation acts. As discussed above, the fields are now assumed to be independent of the dual x^i coordinates, so the strong constraint is satisfied. Therefore we have shown that these particular $O(D,D)$ transformations are special gauge transformations. In other words, in the case of a torus reduction, where the fields are independent of $d < D$ (internal) coordinates, we can view the full $O(d,d)$ subgroup as part of the gauge group. This analysis completes our previous analysis in [3] for the case of finite gauge transformations.

4 Exponentiation of generalized Lie derivatives

In this section we compare the postulated formula (2.20) for generalized coordinate transformations with an alternative definition of finite transformations as the result of exponentiation of generalized Lie derivatives $\widehat{\mathcal{L}}_\xi$, with parameter ξ. We determine how the parameter ξ enters into the generalized coordinate transformation $X \to X'$ to quartic order in ξ and verify the resulting equivalence of the two forms of finite transformations to that order.

4.1 General coordinate transformations

We start by writing a finite coordinate transformation in terms of a parameter $\xi^M(X)$ that generates this transformation as follows

$$X^M \;\rightarrow\; X'^M \;=\; e^{-\xi^P(X)\partial_P}X^M\,. \tag{4.1}$$

In this right-hand side the exponential is meant to be expanded in a power series and the differential operator $\xi^M\partial_M$, written sometimes as ξ, acts to the right on a function to give a function. We can also rewrite (4.1) as an operator equation as follows

$$X' \;=\; e^{-\xi}\,X\,e^{\xi}\,. \tag{4.2}$$

This can be verified with the familiar relation $e^A\,B\,e^{-A} \;=\; B + [A,B] + \frac{1}{2}[A,[A,B]] + \cdots$, recalling that for any function $f(X)$ we have $[\xi, f(X)] \;=\; \xi^M\partial_M f$. Equation (4.2) is to be interpreted as an operator equation, in which the left-hand side is a function that is viewed as an operator acting via multiplication.

The parameter ξ can be related to ζ defined in (5.68), $\zeta^M = \xi^M - \frac{1}{2}\xi^P\partial_P\xi^M + \mathcal{O}(\xi^3)$, but this will not be required in the discussion that follows. The ξ parameterization of the coordinate change will be used henceforth unless noted otherwise. We could write X'^M_ξ to denote the ξ dependence but we will not do so unless it is required to distinguish it from other possible definitions of X'. We write the above diffeomorphism more compactly as

$$X' \;=\; e^{-\xi}X \;=\; \Big(1-\xi+\frac{1}{2}\xi^2-\ldots\Big)X\,, \qquad \xi \;\equiv\; \xi^M\partial_M\,. \tag{4.3}$$

Taking derivatives of X' with respect to X is not complicated and one quickly finds that

$$\frac{\partial X'}{\partial X} \;=\; \mathbf{1} - a + \frac{1}{2}(\xi+a)a - \frac{1}{3!}(\xi+a)^2a + \frac{1}{4!}(\xi+a)^3a + \cdots\,, \quad a_P{}^Q \equiv \partial_P\xi^Q\,. \tag{4.4}$$

In here the ξ operator acts on everything that stands to its right. For example, $\xi a^2 = (\xi a)a + a(\xi a)$. The above right-hand side is a (matrix) function, not a (matrix) differential operator. Letting the ξ act we have

$$\frac{\partial X'}{\partial X} \;=\; \mathbf{1} - a \;+\; \frac{1}{2}\big(\xi a + a^2\big) \;-\; \frac{1}{6}\big(\xi^2a + (\xi a)a + 2a\,\xi a + a^3\,\big) \;+\; \mathcal{O}(\xi^4)\,. \tag{4.5}$$

Equation (4.4) can be written as

$$\frac{\partial X'}{\partial X} \;=\; \Big(e^{-(\xi+a)}\,\mathbf{1}\Big)\,, \tag{4.6}$$

where the full expansion of the exponential acts on the constant matrix $\mathbf{1}$. Since $(\xi+a)\mathbf{1} = a$, one sees immediately that the evaluation of (4.6) yields (4.4). We now claim that we can simply write

$$\frac{\partial X'}{\partial X} \;=\; e^{-(\xi+a)}e^{\xi}\,. \tag{4.7}$$

Here the right-hand side may seem to be a differential operator but it is in fact a function, the function given in (4.6). To prove this define $h(t)$ by

$$h(t) \;\equiv\; e^{-t(\xi+a)}e^{t\xi}\,. \tag{4.8}$$

Taking a derivative of h with respect to t we get $h'(t) = e^{-t(\xi+a)}(-a)e^{t\xi}$, and note that the object in between the exponentials is a function, not a differential operator. We can write this as

$$h'(t) \;=\; e^{-t(\xi+a)}\big(-(\xi+a)\mathbf{1}\big)e^{t\xi}\,. \tag{4.9}$$

Then passing from the n-th derivative to the next goes as follows

$$h^{(n)}(t) = e^{-t(\xi+a)}g_n\, e^{t\xi} \quad\rightarrow\quad h^{(n+1)}(t) = e^{-t(\xi+a)}\big(-(\xi+a)g_n\big)\, e^{t\xi}\,. \tag{4.10}$$

We note that in the expression for $h^{(n+1)}$ the operator ξ acts only on g_n, because the term where it acts on $e^{t\xi}$ cancels against the derivative of $e^{t\xi}$. If g_n is a function, the object in between the exponentials in $h^{(n+1)}$ is also a function. The result now follows because the above establishes that $h^{(n)}(t=0) = (-1)^n(\xi+a)^n\mathbf{1}$, and therefore

$$\frac{\partial X'}{\partial X} \;=\; h(t=1) \;=\; \sum_{n=0}^{\infty}\frac{1}{n!}h^{(n)}(t=0) \;=\; \sum_{n=0}^{\infty}\frac{1}{n!}(-1)^n(\xi+a)^n\mathbf{1} \;=\; e^{-(\xi+a)}\mathbf{1}\,. \tag{4.11}$$

In summary we have shown that

$$\frac{\partial X'}{\partial X} \;=\; e^{-(\xi+a)}e^{\xi} \;=\; \Big(e^{-(\xi+a)}\,\mathbf{1}\Big)\,. \tag{4.12}$$

With this result we can readily write out the inverse matrix

$$\frac{\partial X}{\partial X'} \;=\; e^{-\xi}e^{\xi+a} \;=\; \Big(\mathbf{1}\,e^{-\overleftarrow{\xi}+a}\Big)\,. \tag{4.13}$$

The first equality follows directly from (4.12), the second by a calculation completely analogous to that above. Here, we have introduced the notation $\mathcal{M}(-\overleftarrow{\xi}+a) \equiv -(\xi\mathcal{M}) + \mathcal{M}a$ for the action of this operator on an arbitrary matrix $\mathcal{M}$. The expansion then gives

$$\frac{\partial X}{\partial X'} \;=\; \mathbf{1} + a + \frac{1}{2}a(-\overleftarrow{\xi}+a) + \frac{1}{3!}a(-\overleftarrow{\xi}+a)^2 + \frac{1}{4!}a(-\overleftarrow{\xi}+a)^3 + \mathcal{O}(\xi^5)\,. \tag{4.14}$$

Expanding the $\overleftarrow{\xi}$ action we find

$$\frac{\partial X}{\partial X'} \;=\; 1 + a - \frac{1}{2}\xi a + \frac{1}{2}a^2 + \frac{1}{6}(\xi^2 a - 2(\xi a)a - a\xi a + a^3) + \mathcal{O}(\xi^4)\,. \tag{4.15}$$

It is also of interest to find an expression for $\mathcal{F}$, as defined in (2.25). For this we need a formula for $\big(\frac{\partial X'}{\partial X}\big)^t$. Using equation (4.4) one quickly notes that

$$\begin{aligned}\Big(\frac{\partial X'}{\partial X}\Big)^t \;&=\; \mathbf{1} - a^t + \frac{1}{2}a^t(\overleftarrow{\xi}+a^t) - \frac{1}{3!}a^t(\overleftarrow{\xi}+a^t)^2 + \frac{1}{4!}a^t(\overleftarrow{\xi}+a^t)^3 + \cdots \\ &=\; \Big(\mathbf{1}\,e^{-(\overleftarrow{\xi}+a^t)}\Big) \;=\; e^{-\xi}e^{\xi-a^t}\,,\end{aligned} \tag{4.16}$$

where in the last step we used the second equality in (4.13) with $a \rightarrow -a^t$. At this point it is useful to define a function $\mathcal{E}$ that appears both in (4.13) and (4.16). We take

$$\begin{aligned}\mathcal{E}(k) \;&\equiv\; e^{-\xi}e^{\xi+k} \;=\; \Big(\mathbf{1}\,e^{-\overleftarrow{\xi}+k}\Big)\,, \\ &= 1 + k - \frac{1}{2}\xi k + \frac{1}{2}k^2 + \frac{1}{6}(\xi^2 k - 2(\xi k)k - k\xi k + k^3) + \dots\;,\end{aligned} \tag{4.17}$$

where we made use of (4.13) and its expansion (4.15). We now have

$$\frac{\partial X}{\partial X'} \;=\; \mathcal{E}(a)\,, \qquad \Big(\frac{\partial X'}{\partial X}\Big)^t \;=\; \mathcal{E}(-a^t)\;. \tag{4.18}$$

It follows that

$$\mathcal{F} \;=\; \frac{1}{2}\Big(\mathcal{E}(a)\mathcal{E}(-a^t) + \mathcal{E}(-a^t)\mathcal{E}(a)\Big)\;. \tag{4.19}$$

An expansion to cubic order in ξ is now easily calculated. We find

$$\begin{aligned}\mathcal{F} \;=\; & 1 + (a-a^t) - \frac{1}{2}\xi(a-a^t) + \frac{1}{2}(a-a^t)^2\\ & + \frac{1}{6}\xi^2(a-a^t) - \frac{1}{3}\big(\xi(a-a^t)\big)\,(a-a^t)\\ & - \frac{1}{6}(a-a^t)\xi(a-a^t) + \frac{1}{6}(a-a^t)^3\\ & - \frac{1}{12}\Big((\xi a)a^t - a\xi a^t + a^2a^t - a(a^t)^2\Big) + \mathcal{O}(\xi^4)\;.\end{aligned} \tag{4.20}$$

Comparing with (4.17) we recognize that the first three lines fit precisely the cubic expansion of $\mathcal{E}(a-a^t)$, and so we can write

$$\mathcal{F} \;=\; \mathcal{E}(a-a^t) - \frac{1}{12}\Big((\xi a)a^t - a\xi a^t + a^2a^t - a(a^t)^2\Big) + \mathcal{O}(\xi^4)\,. \tag{4.21}$$

4.2 Ordinary scalar and vector

Before turning to the generalized coordinate transformations let us review for scalars and vectors the derivation of the finite gauge transformations as exponentials of ordinary Lie derivatives corresponding to ordinary diffeomorphisms associated with (4.1).

Consider the general situation of a field Ψ whose infinitesimal gauge transformation is given by the action of an operator $\mathcal{M}_\xi$ linear in the infinitesimal gauge parameter ξ but field independent. We write

$$\Psi'(X) \;=\; \Psi(X) + \mathcal{M}_\xi\Psi(X)\,, \tag{4.22}$$

or, more schematically,

$$\Psi' \;=\; \Psi + \mathcal{M}_\xi\Psi\,. \tag{4.23}$$

In order to construct a finite transformation with finite parameter ξ we define $\Psi(X;t)$ in such a way that $\Psi(X;t=0) = \Psi(X)$ and

$$\Psi(X;t+dt) \;=\; \Psi(X;t) + \mathcal{M}_{dt\,\xi}\Psi(X;t)\,, \tag{4.24}$$

which states that a change of parameter dt is implemented by a gauge transformation with parameter $dt\xi$. One can view $\Psi(X;t)$ as the gauge-transformed field obtained for gauge parameter $t\xi$ and the fully transformed field is $\Psi(X;t=1)$. Because of the linearity of $\mathcal{M}_\xi$ in ξ, the above equation implies that

$$\frac{d\Psi(X;t)}{dt} \;=\; \mathcal{M}_\xi\Psi(X;t)\,. \tag{4.25}$$

Since $\mathcal{M}_\xi$ is field independent, we integrate this immediately and find

$$\Psi(X;t) \;=\; e^{t\mathcal{M}_\xi}\Psi(X;t=0)\,. \tag{4.26}$$

In conclusion, the fully transformed field $\Psi'(X) = \Psi(X;t=1)$ is given by

$$\Psi'(X) \;=\; e^{\mathcal{M}_\xi}\,\Psi(X)\,. \tag{4.27}$$

This is the desired large gauge transformation.

As a warmup let us consider the case of a scalar field. Then the infinitesimal gauge transformation reads

$$\phi'(X) \;=\; \phi(X) + \xi^P\partial_P\phi(X) \;=\; \phi(X) + \mathcal{L}_\xi\phi(X)\,. \tag{4.28}$$

Here $\mathcal{L}_\xi$ denotes the usual Lie derivative, and it is acting on the scalar. The above discussion implies that the large gauge transformation is given by

$$\phi'(X) \;=\; e^{\mathcal{L}_\xi}\,\phi(X) \;=\; e^{\xi}\,\phi(X)\;, \tag{4.29}$$

since $\xi = \xi^M\partial_M$ coincides with the Lie derivative acting on a scalar. We now want to show that this result follows from the basic transformation law

$$\phi'(X') \;=\; \phi(X)\,, \tag{4.30}$$

for the coordinate transformation (4.1). As written in (4.3) we have

$$X' \;=\; e^{-\xi}X \quad\rightarrow\quad e^{\xi}X' \;=\; X\,. \tag{4.31}$$

The last equation requires a little explanation. The ξ operator must act through the chain rule, as it involves X-derivatives. The result is a function, as all derivatives must act on something. We now use that for a general (analytic) function f

$$(e^{\xi}f)(X') \;=\; e^{\xi}f(X')e^{-\xi} \;=\; f(e^{\xi}X'e^{-\xi}) \;=\; f(X)\,, \tag{4.32}$$

using (4.2) and the logic that led to it. Thus, e^{ξ} acts on any (analytic) function by turning X' into X. Therefore,

$$e^{\xi}\phi'(X') \;=\; \phi'(X)\,, \tag{4.33}$$

where here and henceforth we omit the parenthesis around $e^{\xi}\phi'$. Therefore, using also the scalar property (4.30) we have

$$\phi'(X) \;=\; e^{\xi}\phi'(X') \;=\; e^{\xi}\,\phi(X) \;=\; e^{\mathcal{L}_\xi}\,\phi(X)\,, \tag{4.34}$$

just as we had in (4.29). This is what we wanted to show.

For a scalar density $\Phi(X)$ such as e^{-2d} in (2.23) we have infinitesimally

$$\Phi'(X) \;=\; \Phi(X) + \partial_M\big(\xi^M\,\Phi\big) \;=\; \Phi(X) + \mathcal{L}_\xi\Phi(X)\,, \tag{4.35}$$

where here $\mathcal{L}_\xi$ denotes a Lie derivative on the density. This derivative, so defined to act on a density, satisfies the algebra

$$\big[\mathcal{L}_{\xi_1}\,,\mathcal{L}_{\xi_2}\big] \;=\; -\mathcal{L}_{[\xi_1,\xi_2]}\,. \tag{4.36}$$

This is the same algebra of diffeomorphisms that Lie derivatives satisfy acting on arbitrary tensors.

Let us now consider an ordinary vector field, whose infinitesimal coordinate transformation takes the form[2]

$$A'_M \;=\; A_M + \xi^K \partial_K A_M + (\partial_M \xi^K) A_K \;=\; A_M + (\mathcal{L}_\xi A)_M \,. \tag{4.37}$$

All fields here are evaluated at the common argument X. In index free notation we have

$$A' \;=\; A + \xi A + a\, A \;=\; A + \mathcal{L}_\xi A \,, \tag{4.38}$$

which shows that, on the vector, we can view $\mathcal{L}$ as the matrix operator

$$\mathcal{L}_\xi \;=\; \xi + a \,. \tag{4.39}$$

It then follows that the large coordinate transformation of the vector is given by

$$A' \;=\; e^{\mathcal{L}_\xi} A \;=\; e^{\xi + a} A \,. \tag{4.40}$$

We now compare with the large gauge transformation derived from the usual coordinate transformation of a vector,

$$A'_M(X') \;=\; \frac{\partial X^N}{\partial X'^M} A_N(X) \quad \to \quad A'(X') \;=\; \frac{\partial X}{\partial X'} A(X) \,. \tag{4.41}$$

Following (4.33) we write

$$A'(X) \;=\; e^{\xi} A'(X') = e^{\xi}\, \frac{\partial X}{\partial X'} A(X) \,, \tag{4.42}$$

or, leaving out the common argument,

$$A' \;=\; e^{\xi}\, \frac{\partial X}{\partial X'}\, A \,. \tag{4.43}$$

The above partial derivatives were calculated in (4.13). Using them we have

$$A' \;=\; e^{\xi} \big(e^{-\xi} e^{\xi + a} \big)\, A \;=\; e^{\xi + a}\, A \,, \tag{4.44}$$

in agreement with (4.40).

4.3 Generalized vector and reparameterized diffeomorphisms

The case of a generalized scalar is no different from the ordinary scalar. For generalized vectors, however, the situation is quite different. The infinitesimal transformation of a generalized vector is given by the generalized Lie derivative,

$$A'_M \;=\; A_M + \xi^K \partial_K A_M + \big(\partial_M \xi^K - \partial^K \xi_M \big) A_K \;=\; A_M + (\widehat{\mathcal{L}}_\xi A)_M \,. \tag{4.45}$$

[2] Ordinary and generalized vectors will be denoted by the same symbol A_M and are recognized by the context.

In index-free notation we have

$$A' = A + \xi A + \big(a - a^t\big) A = A + \widehat{\mathcal{L}}_\xi A\,, \tag{4.46}$$

which shows that on a generalized vector we can view the generalized Lie derivative as the operator

$$\widehat{\mathcal{L}}_\xi = \xi + a - a^t\,. \tag{4.47}$$

It follows that the large gauge transformation of the vector is then given by

$$A'(X) = e^{\widehat{\mathcal{L}}_\xi} A = e^{\xi + (a - a^t)} A\,. \tag{4.48}$$

We now must compare with the transformation (2.20) we postulated. Following the steps that are by now familiar, we have

$$A'_M(X) = e^\xi A'_M(X') = e^\xi \mathcal{F}_M{}^N A_N(X)\,, \tag{4.49}$$

or in matrix notation

$$A'(X) = e^\xi\, \mathcal{F}\, A\,. \tag{4.50}$$

Equality with (4.48) would require

$$\mathcal{F} = e^{-\xi} e^{\xi + (a - a^t)} \quad ? \tag{4.51}$$

Using the definition (4.17) we are thus asking if

$$\mathcal{F} = \mathcal{E}(a - a^t) \ \ ? \tag{4.52}$$

In here, $\mathcal{F}$ is calculated using the diffeomorphism $X' = \exp(-\xi)X$ and its definition (2.25). The result to cubic order was given in (4.20) and (4.21). We found there that the above relation holds up to quadratic order, but not to cubic order:

$$\mathcal{F} = \mathcal{E}(a - a^t) - \Delta\mathcal{F}\,, \tag{4.53}$$

where the correction $\Delta\mathcal{F}$ is given by

$$\Delta\mathcal{F} = \frac{1}{12}\Big((\xi a) a^t - a\,\xi a^t + a^2 a^t - a(a^t)^2\Big) + \mathcal{O}(\xi^4)\,. \tag{4.54}$$

This is an apparent failure of consistency. But there is some freedom in double field theory that is not available in ordinary field theory. We can use that freedom to alter the parameterization of the diffeomorphism in such a way that the vector field transformations work out. In doing so we must be careful not to spoil the already achieved agreement for the scalar field.

The diffeomorphism we have been considering so far is

$$X'^M_\xi \equiv e^{-\xi^P \partial_P} X^M\,, \tag{4.55}$$

where we have added the subscript ξ to emphasize the role of this parameter. Now we consider a different diffeomorphism

$$X'^M_\Theta \equiv e^{-\Theta^P \partial_P} X^M = X^M - \Theta^M + \frac{1}{2}\Theta^P \partial_P \Theta^M - \frac{1}{3}\Theta^P \partial_P\, \Theta^K \partial_K \Theta^M + \mathcal{O}(\Theta^4)\,. \tag{4.56}$$

We are to design the new diffeomorphism – or equivalently to fix $\Theta(\xi)$ – so that $\mathcal{F}_\Theta$, given by

$$\mathcal{F}_\Theta \;\equiv\; \frac{1}{2}\Big(\frac{\partial X}{\partial X'_\Theta}\Big(\frac{\partial X'_\Theta}{\partial X}\Big)^t + \Big(\frac{\partial X'_\Theta}{\partial X}\Big)^t\frac{\partial X}{\partial X'_\Theta}\Big)\,, \tag{4.57}$$

satisfies the requisite relation

$$\mathcal{F}_\Theta = \mathcal{E}(a - a^t)\,, \tag{4.58}$$

that guarantees that $\mathcal{F}_\Theta$ generates the same transformation as the exponential of the generalized Lie derivative. In here we will achieve the above equality up to terms of order ξ^3 and in the appendix we extend the result to order ξ^4.

We now consider the case when Θ equals ξ to leading order but has higher order corrections. Since $\Delta\mathcal{F}$ is cubic in ξ we have no use for quadratic corrections and we write

$$\Theta^M \;=\; \xi^M - \delta_3^M(\xi) + \ldots \tag{4.59}$$

The subscript in δ indicates that this term is cubic in ξ. We will also assume that in δ_3^M the index M is carried by a derivative. Schematically,

$$\Theta^M \;=\; \xi^M + \sum_i \rho_i\, \partial^M \chi_i\,, \tag{4.60}$$

with ρ_i and χ_i functions of ξ. Because of the strong constraint, the action of $\Theta^P\partial_P$ on fields (like ξ, or Θ, but not X), reduces to the action of $\xi^P\partial_P$:[3]

$$\Theta^P\partial_P(\,\text{fields}\,) \;=\; \xi^P\partial_P(\,\text{fields}\,)\,. \tag{4.61}$$

Applied to (4.56) this gives

$$\begin{aligned} X'^M_\Theta \;&\equiv\; e^{-\Theta^P\partial_P}X^M \;=\; X^M - \Theta^M + \frac{1}{2}\xi^P\partial_P\Theta^M - \frac{1}{3!}\xi^P\partial_P\xi^K\partial_K\Theta^M + \mathcal{O}(\xi^4) \\ &=\; X^M - \Theta^M + \frac{1}{2}\xi\Theta^M - \frac{1}{3!}\xi\xi\Theta^M + \mathcal{O}(\xi^4)\,. \end{aligned} \tag{4.62}$$

On a scalar the new diffeomorphism results in the same large coordinate transformation. Since

$$X'_\Theta \;=\; e^{-\Theta^P\partial_P}X \quad\to\quad X \;=\; e^{\Theta^P\partial_P}X'\,, \tag{4.63}$$

we have, as before (see the discussion starting with (4.31) and leading to (4.33)),

$$\phi'(X) \;= e^{\Theta^P\partial_P}\phi'(X') \;=\; e^{\Theta^P\partial_P}\phi(X) \;=\; e^{\xi^P\partial_P}\phi(X) \;=\; e^{\hat{\mathcal{L}}_\xi}\phi(X)\,. \tag{4.64}$$

We now aim to compute $\mathcal{F}_\Theta$. First, using (4.59) and (4.62), we can write the relation between the two X's as

$$X'^M_\Theta \;=\; X'^M_\xi + \delta_3^M + \mathcal{O}(\xi^4)\,. \tag{4.65}$$

We then have

$$\begin{aligned} \frac{\partial X'_\Theta}{\partial X} \;&=\; \frac{\partial X'_\xi}{\partial X} + \Delta_3\,, \qquad (\Delta_3)_Q{}^M \;\equiv\; \partial_Q\delta_3{}^M \;=\; (\partial\delta_3)_Q{}^M\,, \\ \frac{\partial X}{\partial X'_\Theta} \;&=\; \frac{\partial X}{\partial X'_\xi} - \Delta_3\,. \end{aligned} \tag{4.66}$$

[3] We note that this modification is consistent with the transformation of a density like the dilaton, which is unmodified compared to ordinary geometry, see (2.23), because by the strong constraint the extra term $\partial_M\xi^M$ in the transformation rule is also unchanged when replacing ξ by Θ.

We use the definition (4.57) to find that

$$\mathcal{F}_\Theta \;=\; \mathcal{F}_\xi + \Delta_3^t - \Delta_3\,. \tag{4.67}$$

In this light we have from (4.53)

$$\mathcal{F}_\Theta \;=\; \mathcal{E}(a-a^t) - \Delta\mathcal{F} + \Delta_3^t - \Delta_3\,. \tag{4.68}$$

We are to design the new diffeomorphism so that $\mathcal{F}_\Theta$ is equal to $\mathcal{E}(a-a^t)$, and therefore we must find a $\Theta(\xi)$ for which

$$\Delta_3^t - \Delta_3 \;=\; \Delta\mathcal{F}\,. \tag{4.69}$$

We claim that Θ is given by

$$\Theta^M \;=\; \xi^M + \frac{1}{12}\,(\xi\xi^L)\partial^M\xi_L + \mathcal{O}(\xi^4)\,, \tag{4.70}$$

or equivalently

$$\delta_3 \;=\; -\frac{1}{12}\,(\xi\xi^L)\partial^M\xi_L\,. \tag{4.71}$$

We confirm this quickly. The definition of Δ_3 in (4.66) gives

$$\Delta_3 \;=\; \partial\delta_3 \;=\; -\frac{1}{12}\big((\xi a)a^t + a^2a^t\big) - \frac{1}{12}(\xi\xi^L)\partial\partial\xi_L\,, \tag{4.72}$$

where the matrix indices are carried by the partial derivatives $\partial\partial$ in the second term. Moreover

$$\Delta_3^t - \Delta_3 \;=\; \frac{1}{12}\Big((\xi a)a^t + a^2a^t - a\xi a^t - a(a^t)^2\Big)\,. \tag{4.73}$$

This coincides exactly with $\Delta\mathcal{F}$ in (4.54). Thus equation (4.69) holds and we have completed the verification of (4.58) to order ξ^3:

$$\mathcal{F}_\Theta = \mathcal{E}(a-a^t) + \mathcal{O}(\xi^4)\,. \tag{4.74}$$

In Appendix A we carry the computation to quartic order and show that the above δ_3 suffices to generate the terms that must be cancelled. Thus the above actually holds with $\mathcal{O}(\xi^5)$. We expect that there will be a need for a correction δ_5 to quintic order.

5 Composition of generalized coordinate transformations

In this section we study the composition of gauge transformations. As we will argue, our previous results that relate large gauge transformations to exponentials of Lie derivatives guarantee the existence of a composition law. This is true both for the ordinary and for the generalized case. It will also become clear here that in the generalized case the composition of the underlying coordinate transformations is exotic.

5.1 Facts on composition

To begin we consider two diffeomorphisms

$$\begin{aligned} X' &= e^{-\xi_1(X)} X\,, \\ X'' &= e^{-\xi_2(X')} X'\,. \end{aligned} \tag{5.1}$$

We also consider a diffeomorphism from X to X''

$$X'' = e^{-\xi_{12}(X)} X\,. \tag{5.2}$$

If this diffeomorphism is induced by the composition of the previous two diffeomorphisms we have $X'' = e^{-\xi_2(X')}e^{-\xi_1(X)}X$ and therefore

$$e^{-\xi_{12}(X)} = e^{-\xi_2(X')}\, e^{-\xi_1(X)}\,. \tag{5.3}$$

In order for the argument of ξ_2 to become X we multiply the right hand side by unity, expressed as $e^{-\xi_1(X)}e^{\xi_1(X)}$:

$$e^{-\xi_{12}(X)} = e^{-\xi_1(X)}\, \big(e^{\xi_1(X)}e^{-\xi_2(X')}\, e^{-\xi_1(X)}\big)\,. \tag{5.4}$$

Recall from (4.32) that $e^{\xi_1} f(X')e^{-\xi_1} = f(X)$, for any regular function $f(X)$. Using this we find the relations

$$\begin{aligned} e^{-\xi_{12}(X)} &= e^{-\xi_1(X)}\, e^{-\xi_2(X)}\,, \\ e^{\,\xi_{12}(X)} &= e^{\,\xi_2(X)}\, e^{\,\xi_1(X)}\,, \end{aligned} \tag{5.5}$$

where the second line is obtained by taking the inverse of the first line. We are now in a position to use the Baker-Campbell-Hausdorff (BCH) relation to write an explicit expression for ξ_{12}:

$$\xi_{12} = \xi_2 + \xi_1 + \frac{1}{2}\big[\xi_2, \xi_1\big] + \frac{1}{12}\big(\big[\xi_2, \big[\xi_2, \xi_1\big]\big] + \big[\xi_1, \big[\xi_1, \xi_2\big]\big]\big) + \ldots \tag{5.6}$$

For our applications here we will just need this formula to quadratic order in ξ:

$$\xi_{12} = \xi_1 + \xi_2 - \frac{1}{2}\big[\,\xi_1\,,\,\xi_2\,\big] + \mathcal{O}(\xi^3)\,. \tag{5.7}$$

A useful alternative picture of the situation involves the Lie derivative operator $\mathcal{L}_\xi$. The key properties of this operator are its linearity in ξ and the commutator $[\mathcal{L}_{\xi_1}, \mathcal{L}_{\xi_2}] = -\mathcal{L}_{[\xi_1,\xi_2]}$.[4] We can combine the exponentials of two such operators as follows

$$e^{\mathcal{L}_{\xi_1(X)}}e^{\mathcal{L}_{\xi_2(X)}} = e^{\mathcal{L}_{\,\xi_{12}(X)}}\,, \tag{5.8}$$

where we claim that ξ_{12} is the one determined above. To see this, let the above operator equation act on a scalar field S. On a scalar the Lie derivative acts just like the vector operator: $\mathcal{L}_\xi S = \xi S$, and we therefore get

$$e^{\mathcal{L}_{\xi_1(X)}}e^{\xi_2(X)}S = e^{\xi_2(X)}e^{\mathcal{L}_{\xi_1(X)}}S = e^{\xi_2(X)}e^{\xi_1(X)}S = e^{\xi_{12}(X)}S\,, \tag{5.9}$$

[4] Let us note that here and below (see eq. (5.9)), we employ the convention that the (generalized) Lie derivatives act on the fields first. The opposite convention according to which the Lie derivatives are operators acting equitably on everything on the right leads to a different sign in the commutator of Lie derivatives.

consistent with (5.5). We can also explicitly combine the operators on the left-hand side of (5.8) using BCH:

$$e^{\mathcal{L}_{\xi_1(X)}}e^{\mathcal{L}_{\xi_2(X)}} \;=\; e^{\mathcal{L}_{\xi_1}+\mathcal{L}_{\xi_2}+\frac{1}{2}[\mathcal{L}_{\xi_1},\mathcal{L}_{\xi_2}]+\cdots} \;=\; e^{\mathcal{L}_{\xi_1}+\mathcal{L}_{\xi_2}-\frac{1}{2}\mathcal{L}_{[\xi_1,\xi_2]}+\cdots} \;=\; e^{\mathcal{L}_{\xi_1+\xi_2-\frac{1}{2}[\xi_1,\xi_2]+\cdots}}\,. \qquad (5.10)$$

This, of course, gives the same determination of ξ_{12}.

In the generalized case the coordinate transformations become subtle to handle, but the analogy to Lie derivatives holds. Thus, in view of (5.8) we now consider the corresponding exponentials of generalized Lie derivatives,

$$e^{\widehat{\mathcal{L}}_{\xi_1(X)}}e^{\widehat{\mathcal{L}}_{\xi_2(X)}} \;=\; e^{\widehat{\mathcal{L}}_{\xi^c_{12}(X)}}\,. \qquad (5.11)$$

As for the BCH relation, the only difference with ordinary Lie derivatives is that the commutator of generalized Lie derivatives gives a generalized Lie derivative with parameter equal to (minus) the *Courant* bracket of the parameters. It follows that the parameter ξ^c_{12} written above is in fact given by the same formula (5.6) that gives ξ_{12} but this time using the Courant bracket:

$$\xi^c_{12} \;=\; \xi_2+\xi_1+\frac{1}{2}\big[\xi_2,\xi_1\big]_c+\frac{1}{12}\Big(\big[\xi_2,\big[\xi_2,\xi_1\big]_c\big]_c+\big[\xi_1,\big[\xi_1,\xi_2\big]_c\big]_c\Big)+\ldots\,. \qquad (5.12)$$

It is important to clarify the notation in the above formula. In the generalized theory, due to the strong constraint, it is not synonymous to speak of the components A^M of a vector, or the vector operator $A^M\partial_M$. The above equation must be thought of as an equation for *components*:

$$(\xi^c_{12})^M \;=\; \xi_2^M+\xi_1^M+\frac{1}{2}\big[\xi_2,\xi_1\big]_c^M+\ldots\,. \qquad (5.13)$$

This distinction is relevant: while the vector components $(\xi^c_{12})^K$ and $(\xi_{12})^K$ are not equal, we claim that the strong constraint implies the equality of the corresponding vector operators

$$(\xi^c_{12})^K\partial_K \;=\; (\xi_{12})^K\partial_K\,. \qquad (5.14)$$

We can write this as $\xi^c_{12}=\xi_{12}$ when no confusion is possible, but recalling that the vectors do not have the same components. In order to prove (5.14) we first recall that the C-bracket (1.5) differs from the Lie bracket by a vector whose index is carried by a derivative:

$$[A,B]^K_c \;=\; [A,B]^K+(\cdots\partial^K\cdots)\,, \qquad (5.15)$$

where the expressions indicated by dots are not presently relevant. It then follows by the strong constraint that

$$[A,B]^M_c\partial_M \;=\; [A,B]^M\partial_M\,. \qquad (5.16)$$

Next we verify the following relation for Lie brackets:

$$\big[\,\chi\,,(\cdots\vec{\partial}\cdots)\big] \;=\; (\cdots\vec{\partial}\cdots)\,. \qquad (5.17)$$

This states that the Lie bracket of an arbitrary vector with a vector whose index is carried by a derivative is again a vector whose index is carried by a derivative. This is readily verified computing the M-th component of the following commutator:

$$\big[\,\chi\,,\rho\,\vec{\partial}\,\eta\,\big]^M \;=\; \chi^K\partial_K(\rho\,\partial^M\eta)-(\rho\partial^K\eta)\partial_K\chi^M \;=\; \chi^K\partial_K(\rho\,\partial^M\eta)\,, \qquad (5.18)$$

and, as claimed, the right hand side is a vector whose index is carried by a derivative. We can now see that a double nested C-commutator also reduces to a Lie commutator:

$$\begin{aligned}\big[\,\chi_1\,,[\chi_2,\chi_3]_c\,\big]_c^M\partial_M \;&=\; \big[\,\chi_1\,,[\chi_2,\chi_3]_c\,\big]^M\partial_M \\ &=\; \big[\,\chi_1\,,[\chi_2,\chi_3]\,\big]^M\partial_M + \big[\,\chi_1\,,(\cdots\vec{\partial}\cdots)\,\big]^M\partial_M\,,\end{aligned} \tag{5.19}$$

using (5.16) and then (5.15). We now use (5.17) to conclude that, as claimed

$$\big[\,\chi_1\,,[\chi_2,\chi_3]_c\,\big]_c^M\partial_M \;=\big[\,\chi_1\,,[\chi_2,\chi_3]\,\big]^M\partial_M + (\cdots\partial^M\cdots)\partial_M \;=\; \big[\,\chi_1\,,[\chi_2,\chi_3]\,\big]^M\partial_M\,. \tag{5.20}$$

It is now easy to make an inductive argument to show that

$$\big[\,\chi_1\,,[\chi_2,[\chi_3\,\cdots[\chi_{n-1}\,,\chi_n]_c]_c\ldots]_c\,\big]_c^M\partial_M \;=\; \big[\,\chi_1\,,[\chi_2,[\chi_3\,\cdots[\chi_{n-1}\,,\chi_n]\,]\ldots]\,\big]^M\partial_M\,. \tag{5.21}$$

Indeed, in such an argument one may assume that all commutators are Lie except for the most nested one. Then one uses (5.15) for this commutator to get the desired term with all brackets of Lie type and an extra term where that most nested commutator is replaced by a vector with index carried by a derivative. Then successive application of (5.17) gives the desired result. Having shown this, and given the form of ξ_{12}^c in (5.12), we see that (5.14) is true.

5.2 General argument for composition

For the ordinary vector we wrote

$$A'(X') \;=\; \mathcal{G}(X',X)A(X)\,, \tag{5.22}$$

and with the diffeomorphism

$$X' \;=\; e^{-\xi}X\,, \tag{5.23}$$

we found that

$$G(X',X) \;=\; \frac{\partial X}{\partial X'}\,, \tag{5.24}$$

can be written as

$$\mathcal{G}(X',X) \;=\; e^{-\xi}\,e^{\,\xi+a}\,. \tag{5.25}$$

Moreover, with these results we also found that transformation (5.22) implies

$$A'(X) \;=\; e^{\mathcal{L}_\xi}A(X) \;=\; e^{\,\xi+a}A\,. \tag{5.26}$$

As indicated above, acting on vectors, $\mathcal{L}_\xi = \xi + a$. Therefore, when the operators in (5.8) are acting on a vector we have

$$e^{\xi_2+a_2}\,e^{\xi_1+a_1} \;=\; e^{\xi_{12}+a_{12}}\,. \tag{5.27}$$

We now verify that the composition property of $\mathcal{G}$,

$$\mathcal{G}(X'',X')\,\mathcal{G}(X',X) \;=\; \mathcal{G}(X'',X)\,, \tag{5.28}$$

is a consequence of (5.27).[5] Given (5.25) the above equation requires that

$$e^{-\xi_2(X')}\,e^{\,\xi_2(X')+a_2'}\;e^{-\xi_1(X)}\,e^{\,\xi_1(X)+a_1} \;=\; e^{-\xi_{12}(X)}\,e^{\,\xi_{12}(X)+a_{12}}\,. \tag{5.29}$$

[5]Eqn. (5.28) follows directly from (5.24), but such derivation is not available for the generalized case.

Let us show that this gives us (5.27). Acting with $e^{\xi_{12}(X)}$

$$e^{\xi_{12}(X)}e^{-\xi_2(X')}\,e^{\xi_2(X')+a_2'}\;e^{-\xi_1(X)}\,e^{\xi_1(X)+a_1} \;=\; e^{\xi_{12}(X)+a_{12}}\,. \tag{5.30}$$

Then using (5.3)

$$e^{\xi_1(X)}\,e^{\xi_2(X')+a_2'}\,e^{-\xi_1(X)}\;e^{\xi_1(X)+a_1} \;=\; e^{\xi_{12}(X)+a_{12}}\,. \tag{5.31}$$

The first three factors on the left-hand side give[6] the factor $e^{\xi_2(X)+a_2}$. Thus (5.31) becomes

$$e^{\xi_2(X)+a_2}\;\;e^{\xi_1(X)+a_1} \;=\; e^{\xi_{12}(X)+a_{12}}\,, \tag{5.32}$$

which is identical to (5.27).

We can now turn to the generalized case. The composition law on a scalar is no different from that in ordinary geometry and holds as in that case. For the scalar density we have that the Lie derivatives $\mathcal{L}_\xi$ considered in (4.35) lead to

$$e^{\mathcal{L}_{\xi_1(X)}}e^{\mathcal{L}_{\xi_2(X)}} \;=\; e^{\mathcal{L}_{\,\xi_{12}(X)}}\,. \tag{5.33}$$

But for the scalar density (or the scalar) Lie derivatives take the same form as generalized Lie derivatives, so we have

$$e^{\widehat{\mathcal{L}}_{\xi_1(X)}}e^{\widehat{\mathcal{L}}_{\xi_2(X)}} \;=\; e^{\widehat{\mathcal{L}}_{\,\xi_{12}(X)}}\,. \tag{5.34}$$

Moreover, acting on a scalar density (or a scalar) any contribution to ξ^M of the form $\cdots\partial^M\cdots$ will vanish on $\widehat{\mathcal{L}}_\xi$. Thus by virtue of (5.14) we can replace ξ_{12} by ξ_{12}^c in the above, finding that on a scalar density we have

$$e^{\widehat{\mathcal{L}}_{\xi_1(X)}}e^{\widehat{\mathcal{L}}_{\xi_2(X)}} \;=\; e^{\widehat{\mathcal{L}}_{\,\xi_{12}^c(X)}}\,. \tag{5.35}$$

The integration of the infinitesimal transformation of the scalar density leads to (2.23) and by the above argument, such transformation must be consistent with composition, as expressed in the generalized case by the equation above. We verify this explicitly at the end of section 5.3.

Let us now consider the large transformation of the vector field is represented by the relation

$$A'(X') \;=\; \mathcal{F}(X',X)A(X)\,, \tag{5.36}$$

with

$$\mathcal{F}(X',X) \;=\; \frac{1}{2}\Big(\frac{\partial X}{\partial X'}\frac{\partial X'^{\,t}}{\partial X} + \frac{\partial X'^{\,t}}{\partial X}\,\frac{\partial X}{\partial X'}\Big)\,. \tag{5.37}$$

We have already shown that

$$X' \;=\; e^{-\Theta(\xi)}X\,, \quad \Theta(\xi) = \xi + \mathcal{O}(\xi^3)\,, \tag{5.38}$$

leads to

$$\mathcal{F}(X',X) \;=\; e^{-\xi}\,e^{\xi+k}\,, \qquad k \;=\; a - a^t\,, \tag{5.39}$$

at least to $\mathcal{O}(\xi^5)$. Moreover, this and (5.36) imply that

$$A'(X) \;=\; e^{\widehat{\mathcal{L}}_\xi}A(X) \;=\; e^{\xi+k}A\,. \tag{5.40}$$

[6]Note that a_2' goes to a_2 because $(a_2')_Q{}^P = \partial_Q'\xi_2(X')^P = h(X')_Q{}^P$ is ultimately a function of X' so that $e^{\xi_1}(a_2')_Q{}^Pe^{-\xi_1} = e^{\xi_1}h(X')_Q{}^Pe^{-\xi_1} = h(X)_Q{}^P = \partial_Q\xi_2(X)^P = (a_2)_Q{}^P$.

Note that on generalized vectors $\widehat{\mathcal{L}}_\xi = \xi + k$. It now follows that the composition (5.11) of exponentials of generalized Lie derivatives, applied to generalized vectors, gives

$$e^{\xi_2(X)+k_2} e^{\xi_1(X)+k_1} \;=\; e^{\xi_{12}^c(X)+k_{12}^c}\,. \tag{5.41}$$

We claim that composition of $\mathcal{F}$ holds in the following sense:

$$e^{-\xi_2(X')}\, e^{\xi_2(X')+k_2'}\, e^{-\xi_1(X)}\, e^{\xi_1(X)+k_1} \;=\; e^{-\xi_{12}^c(X)}\, e^{\xi_{12}^c(X)+k_{12}^c}\,. \tag{5.42}$$

This means that

$$\mathcal{F}(X'',X')\,\mathcal{F}(X',X) = \mathcal{F}(X'',X)\,, \tag{5.43}$$

where the $\mathcal{F}$ on the right-hand side is built from $X'' = e^{-\Theta(\xi_{12}^c)}X$. To prove (5.42) we first multiply it by $e^{\xi_1(X)}e^{\xi_2(X')}$ to get

$$e^{\xi_1(X)}\, e^{\xi_2(X')+k_2'}\, e^{-\xi_1(X)}\, e^{\xi_1(X)+k_1} \;=\; e^{\xi_1(X)}e^{\xi_2(X')}e^{-\xi_{12}^c(X)}\, e^{\xi_{12}^c(X)+k_{12}^c}\;. \tag{5.44}$$

Consider the first three factors on the above right-hand side. Given (5.14) we can replace ξ_{12}^c by ξ_{12} (since here they are operators) and then use (5.3) to find that these factors give the unit matrix:

$$e^{\xi_1(X)}e^{\xi_2(X')}e^{-\xi_{12}^c(X)} \;=\; e^{\xi_1(X)}e^{\xi_2(X')}e^{-\xi_{12}(X)} \;=\; \mathbf{1}\,. \tag{5.45}$$

On the left-hand side of (5.44) we see that the first and third factor implement the change $X' \to X$ on the second factor. All in all (5.44) becomes

$$e^{\xi_2(X)+k_2}\, e^{\xi_1(X)+k_1} \;=\; e^{\xi_{12}^c(X)+k_{12}^c}\,. \tag{5.46}$$

This is indeed identical to (5.41), as we wanted to show. Note that the above right-hand side is also equal to $e^{\xi_{12}(X)+k_{12}^c}$, but k_{12}^c is built from the *components* $(\xi_{12}^c)^M$, and therefore cannot be traded for k_{12} which is build from the components $(\xi_{12})^M$.

5.3 Testing composition

In this section we test explicitly the composition rules. This provides a confirmation of the arguments presented above and is simply a welcome check on the formalism. While the confirmation to be done is certainly not novel in the ordinary geometry case, the notation to be introduced will help the treatment of the generalized case.

For the three parameters ξ_1, ξ_2, and ξ_{12} we introduce the matrices a_1, a_2, and a_{12} as the analogs of the matrix $a(\xi)_P{}^Q = \partial_P \xi^Q$:

$$(a_1)_M{}^N \;\equiv\; \partial_M \xi_1^N\,, \quad (a_2)_M{}^N \;\equiv\; \partial'_M \xi_2^N(X')\,, \quad (a_{12})_M{}^N \;\equiv\; \partial_M \xi_{12}^N(X)\,. \tag{5.47}$$

The composition law (5.28) then requires

$$\mathcal{G}(\xi_2)\,\mathcal{G}(\xi_1) \;=\; \mathcal{G}(\xi_{12})\,, \tag{5.48}$$

where we have denoted the $\mathcal{G}$ in terms of the parameter that generates the corresponding transformation. This equation must determine ξ_{12}, and we expect that this is the ξ_{12} obtained before.

Recalling that $\mathcal{G} = \frac{\partial X}{\partial X'}$ and making use of (4.15) we can write, to quadratic order,

$$\mathcal{G}(\xi_1) \;=\; \mathbf{1} + a_1 - \frac{1}{2}\xi_1 a_1 + \frac{1}{2}a_1 a_1 + \cdots \;. \tag{5.49}$$

Using (5.49) for the other two ξ's, we quickly find that (5.48) requires, to quadratic order,

$$\mathbf{1} + a_1 + a_2 - \frac{1}{2}\xi_1 a_1 - \frac{1}{2}\xi_2 a_2 + a_2 a_1 + \frac{1}{2}\big(a_1 a_1 + a_2 a_2\big) \;=\; \mathbf{1} + a_{12} - \frac{1}{2}\xi_{12} a_{12} + \frac{1}{2}a_{12} a_{12} \;. \tag{5.50}$$

To linear order this requires $a_{12} = a_1 + a_2$. Writing $a_{12} = a_1 + a_2 + \delta$ one readily determines δ and concludes that the above equation is satisfied if

$$a_{12} \;=\; a_1 + a_2 - \frac{1}{2}[a_1, a_2] + \frac{1}{2}(\xi_1 a_2 + \xi_2 a_1) \;. \tag{5.51}$$

We now calculate a_{12}, with ξ_{12} given in (5.7) and will show that indeed the above a_{12} arises. We begin with

$$(a_{12})_M{}^N \;=\; \partial_M \xi_{12}^N \;=\; \partial_M \big(\xi_1^N + \xi_2^N - \frac{1}{2}\xi_1^P \partial_P \xi_2^N + \frac{1}{2}\xi_2^P \partial_P \xi_1^N\big) \;. \tag{5.52}$$

In here we will have to evaluate the derivative $\partial_M \xi_2^N(X)$ which is closely related to a_2. The relation to quadratic order is readily found,

$$\partial \xi_2(X) \;=\; e^{\xi_1}\, \partial' \xi_2(X') e^{-\xi_1} = a_2 + \xi_1 a_2 + \mathcal{O}(\xi^3) \;. \tag{5.53}$$

Evaluating (5.52) with the help of this relation we find

$$a_{12} \;=\; a_1 + a_2 + \xi_1 a_2 - \frac{1}{2}a_1 a_2 - \frac{1}{2}\xi_1 a_2 + \frac{1}{2}a_2 a_1 + \frac{1}{2}\xi_2 a_1 \;, \tag{5.54}$$

and we recover precisely (5.51), completing the proof to second order.

In the generalized setting, the transformation of a gauge field is now implemented by $\mathcal{F}$, instead of $\mathcal{G}$. Our expansion of $\mathcal{F}$ to quadratic order is read off from (4.20):

$$\mathcal{F}(\xi_1) \;=\; 1 + k_1 - \frac{1}{2}\xi_1 k_1 + \frac{1}{2}k_1 k_1 \,, \qquad \text{with} \qquad k_1 \equiv a_1 - a_1^t \,, \quad a_1 \equiv \partial \xi_1 \,. \tag{5.55}$$

This time the composition rule requires that

$$\mathcal{F}(\xi_2)\mathcal{F}(\xi_1) \;=\; \mathcal{F}(\xi_{12}^c) \,. \tag{5.56}$$

Written out to quadratic order it gives the requirement

$$\mathbf{1} + k_1 + k_2 - \frac{1}{2}\xi_1 k_1 - \frac{1}{2}\xi_1 k_2 + k_2 k_1 + \frac{1}{2}(k_1 k_1 + k_2 k_2) \;=\; \mathbf{1} + k_{12}^c - \frac{1}{2}\xi_{12} k_{12}^c + \frac{1}{2}k_{12}^c k_{12}^c \;. \tag{5.57}$$

This equation will be satisfied if ξ_{12}^c is such that

$$k_{12}^c \;=\; k_1 + k_2 - \frac{1}{2}[k_1, k_2] + \frac{1}{2}(\xi_1 k_2 + \xi_2 k_1) \;. \tag{5.58}$$

It remains to show that this is consistent with

$$\begin{aligned} (\xi_{12}^c)^M \;&=\; \xi_1^M + \xi_2^M - \frac{1}{2}\big[\xi_1, \xi_2\big]_{\mathrm{C}}^M \\ &=\; \xi_1^M + \xi_2^M - \frac{1}{2}\big[\xi_1, \xi_2\big]^M + \frac{1}{4}\xi_{1P}\partial^M \xi_2^P - \frac{1}{4}\xi_{2P}\partial^M \xi_1^P \;, \end{aligned} \tag{5.59}$$

where we used (1.5). The new a^c_{12} here is equal to the old a_{12} in (5.51), plus the contributions from the last two terms above,

$$(a^c_{12})_M{}^N \;=\; \partial_M \xi^N_{12} \;=\; (a_{12})_M{}^N + \frac{1}{4}\partial_M\big(\xi_{1P}\partial^N\xi^P_2 - \xi_{2P}\partial^N\xi^P_1\big)\,. \tag{5.60}$$

Therefore,

$$a^c_{12} \;=\; a_{12} + \frac{1}{4}\,(a_1a^t_2 - a_2a^t_1) + \frac{1}{4}\big(\xi_{1\,P}(\partial\partial)\xi^P_2 - \xi_{2\,P}(\partial\partial)\xi^P_1\big)\,. \tag{5.61}$$

In the last couple of terms the matrix indices are carried by the partial derivatives. Now, when we form $k^c_{12} = a^c_{12} - (a^c_{12})^t$ those terms cancel and we find

$$k^c_{12} \;=\; a^c_{12} - (a^c_{12})^t \;= a_{12} - a^t_{12} + \frac{1}{2}(a_1a^t_2 - a_2a^t_1)\,. \tag{5.62}$$

Let us now expand the right-hand side of (5.58) to see if it agrees with the above k^c_{12}:

$$\begin{aligned}
&k_1 + k_2 - \frac{1}{2}[k_1,k_2] + \frac{1}{2}(\xi_1k_2 + \xi_2k_1)\\
&\quad= a_1 - a^t_1 + a_2 - a^t_2 - \frac{1}{2}[a_1 - a^t_1, a_2 - a^t_2] + \frac{1}{2}\xi_1(a_2 - a^t_2) + \frac{1}{2}\xi_2(a_1 - a^t_1)\\
&\quad= a_1 + a_2 - \frac{1}{2}[a_1,a_2] + \frac{1}{2}(\xi_1a_2 + \xi_2a_1)\\
&\qquad\quad - \Big(a^t_1 + a^t_2 - \frac{1}{2}[a^t_2,a^t_1] + \frac{1}{2}(\xi_1a^t_2 + \xi_2a^t_1)\Big) + \frac{1}{2}[a_1,a^t_2] + \frac{1}{2}[a^t_1,a_2]\\
&\quad= a_{12} - a^t_{12} + \frac{1}{2}(a_1a^t_2 - a^t_2a_1 + a^t_1a_2 - a_2a^t_1)\,,
\end{aligned} \tag{5.63}$$

where we made use of (5.51) to identify the terms that comprise a_{12} and a^t_{12}. We now note that

$$(a^t_1a_2)_{PQ} \;=\; (a_1)^M{}_P\,(a_2)_M{}^Q \;=\; \partial^M\xi_{1P}\,\partial'_M\xi^Q_2 \;=\; 0\,, \tag{5.64}$$

using the strong constraint in the form (2.9). For the same reason $a^t_2a_1 = 0$. As a result the last right-hand side in (5.63) indeed equals k^c_{12}, as given in (5.62). This proves the desired result.

To conclude, we explain how the composition law for generalized coordinate transformations is consistent with the large transformation of a scalar density, as postulated in (2.23). The consistency requires that

$$\det\Big(\frac{\partial X}{\partial X''}\Big)\Big|_{X''=e^{-\xi_{12}}X} \;=\; \det\Big(\frac{\partial X}{\partial X''}\Big)\Big|_{X''=e^{-\Theta(\xi^c_{12})}X}\,. \tag{5.65}$$

On the left-hand side we have the composition of determinants computed directly by matrix multiplication as if the generalized coordinate transformations composed directly; on the right hand side we have the determinant of the true composite generalized transformation. To verify this equality we recall the general identity

$$\det\,(1 + A) \;=\; \exp\Big[\mathrm{tr}\big(A - \frac{1}{2}A^2 + \frac{1}{3}A^3 + \cdots\big)\Big]\,, \tag{5.66}$$

and from (4.15), when $X' = e^{-\xi}X$,

$$\frac{\partial X}{\partial X'} \;=\; 1 + A\,,\quad \text{with}\quad A \;=\; a - \frac{1}{2}\xi a + \frac{1}{2}a^2 + \frac{1}{6}(\xi^2a - 2(\xi a)a - a\xi a + a^3) + \mathcal{O}(\xi^4)\,. \tag{5.67}$$

Equation (5.65) holds if the change

$$\xi^M \to \xi^M + \cdots \partial^M \cdots \tag{5.68}$$

leaves the computation of the determinant invariant. This is because ξ_{12} and ξ_{12}^c differ by such terms, and $\Theta(\xi)$ differs from ξ by such terms. As we can see above, the determinant is expressed in terms of traces of $A, A^2, A^3, \ldots$. We see immediately that $\operatorname{tr} a = \partial\cdot\xi$ is invariant under (5.68). So is $\operatorname{tr} \xi a = \xi^M \partial_M \partial \cdot \xi$, and

$$\operatorname{tr} a^2 \;=\; \partial_M \xi^N \, \partial_N \xi^M \;=\; \partial_M (\xi^N + \cdots \partial^N \cdots)\, \partial_N (\xi^M + \cdots \partial^M \cdots) \tag{5.69}$$

The general term in any power of A is made of a sequence of a factors and ξ operators, and their trace will be invariant under (5.68):

$$\operatorname{tr}\,[aa \ldots (\xi a) \ldots a] \;=\; \partial_M \xi^N \partial_N \xi^P \partial_P \ldots \xi^R (\xi^Q \partial_Q) \partial_R \xi^S \, \partial_S \ldots \xi^W \partial_W \xi^M \,, \tag{5.70}$$

since each ξ index must be contracted with a derivative index (there are no a^t's in here). All in all this makes it manifest that (5.65) holds and that our formula for the large transformation of a density is consistent.

6 Conclusions and open questions

We have presented a proposal for finite gauge transformations in double field theory. These transformations arise, in this viewpoint, from something we call generalized coordinate transformations. While in ordinary geometry a vector field transforms with one power of the matrix of derivatives of the coordinate maps, in double field theory a vector $A_M(X)$ transforms as

$$A'(X') \;=\; \mathcal{F}(X', X) A(X) \,, \tag{6.1}$$

with

$$\mathcal{F}(X', X) \;=\; \frac{1}{2}\Big(\frac{\partial X}{\partial X'} \frac{\partial X'^{\,t}}{\partial X} + \frac{\partial X'^{\,t}}{\partial X} \frac{\partial X}{\partial X'} \Big) \,. \tag{6.2}$$

Apart from passing a number of consistency conditions a key property of the above expression is its relation to finite gauge transformations defined more directly through the exponentiation of generalized Lie derivatives $\widehat{\mathcal{L}}_\xi$:

$$A'(X) \;=\; e^{\widehat{\mathcal{L}}_\xi} A(X) \;=\; e^{\,\xi + k} A \,, \qquad k = a - a^t \,, \quad a = \partial \xi \,. \tag{6.3}$$

To establish that this transformation is equivalent to the transformation (6.1) we had to show that there is a generalized coordinate transformation $X' = f_\xi(X)$ in terms of ξ for which the evaluation of $\mathcal{F}$ results in

$$\mathcal{F}(X', X) \;=\; e^{-\xi} \, e^{\,\xi + k} \,, \tag{6.4}$$

for this indeed implies the equivalence of (6.1) and (6.3). One may have thought that the coordinate transformation $X' = e^{-\xi} X$ would do the job, but it turns out that this only leads to (6.4) holding to order ξ^2. The generalized coordinate transformation can be somewhat more exotic while preserving familiar results due to some flexibility afforded by use of the strong constraint. We showed that in fact

$$X' \;=\; e^{-\Theta(\xi)}X\,, \quad \text{with} \quad \Theta^M \;=\; \xi^M + \frac{1}{12}\,(\xi\xi^L)\partial^M\xi_L + \mathcal{O}(\xi^5)\,, \tag{6.5}$$

leads to (6.4) up to and including $\mathcal{O}(\xi^4)$ terms. Note that Θ^M equals ξ^M to leading order and that the cubic correction is a vector whose index is carried by a derivative. This correction affects the coordinate transformation but also results in $\Theta^M\partial_M = \xi^M\partial_M$ on fields (but not on X). It remains an open problem to show that there exists a $\Theta(\xi)$ that implies (6.4) to all orders in ξ. It would also be of interest to understand the geometrical role of Θ.

Generalized Lie derivatives define a Lie algebra. Indeed, we have [4]

$$\big[\widehat{\mathcal{L}}_{\xi_1}\,,\widehat{\mathcal{L}}_{\xi_2}\big] \;=\; -\widehat{\mathcal{L}}_{[\xi_1,\xi_2]_c}\,, \tag{6.6}$$

with $[\cdot,\cdot]_c$ the C-bracket, and the Jacobi identity holds:

$$\big[\big[\widehat{\mathcal{L}}_{\xi_1},\widehat{\mathcal{L}}_{\xi_2}\big],\widehat{\mathcal{L}}_{\xi_3}\big] + \big[\big[\widehat{\mathcal{L}}_{\xi_2},\widehat{\mathcal{L}}_{\xi_3}\big],\widehat{\mathcal{L}}_{\xi_1}\big] + \big[\big[\widehat{\mathcal{L}}_{\xi_3},\widehat{\mathcal{L}}_{\xi_1}\big],\widehat{\mathcal{L}}_{\xi_2}\big] \;=\; 0\,. \tag{6.7}$$

This happens because the C-bracket Jacobiator of (ξ_1,ξ_2,ξ_3) is a trivial parameter and generalized Lie derivatives of trivial parameters are zero. For both of the above properties one must use the strong constraint. It is then a direct consequence of (6.3) that the finite transformations form a *group*. The Baker-Campbell-Hausdorff formula allows us to combine exponentials to get

$$e^{\widehat{\mathcal{L}}_{\xi_1(X)}}e^{\widehat{\mathcal{L}}_{\xi_2(X)}} \;=\; e^{\widehat{\mathcal{L}}_{\xi^c(\xi_2,\xi_1)}}\,, \tag{6.8}$$

where

$$\xi^c(\xi_2,\xi_1) \;=\; \xi_2+\xi_1+\frac{1}{2}\big[\xi_2,\xi_1\big]_c+\frac{1}{12}\big(\big[\xi_2,\big[\xi_2,\xi_1\big]_c\big]_c+\big[\xi_1,\big[\xi_1,\xi_2\big]_c\big]_c\big)+\ldots\,. \tag{6.9}$$

The group associativity property is guaranteed to hold acting on fields, namely

$$\big(e^{\widehat{\mathcal{L}}_{\xi_1(X)}}e^{\widehat{\mathcal{L}}_{\xi_2(X)}}\big)e^{\widehat{\mathcal{L}}_{\xi_3(X)}} \;=\; e^{\widehat{\mathcal{L}}_{\xi_1(X)}}\big(e^{\widehat{\mathcal{L}}_{\xi_2(X)}}e^{\widehat{\mathcal{L}}_{\xi_3(X)}}\big)\,. \tag{6.10}$$

This results in

$$\exp\big(\,\widehat{\mathcal{L}}_{\xi^c(\xi_3,\xi^c(\xi_2,\xi_1))}\big) \;=\; \exp\big(\,\widehat{\mathcal{L}}_{\xi^c(\xi^c(\xi_3,\xi_2),\xi_1)}\big)\,, \tag{6.11}$$

and implies that the parameters of the left-hand side and right-hand side are equal up to a trivial parameter that does not generate a Lie derivative. A short computation shows that, in fact,

$$\xi^c\big(\,\xi_3\,,\,\xi^c(\xi_2,\xi_1)\big) \;=\; \xi^c\big(\,\xi^c(\xi_3,\xi_2)\,,\,\xi_1\,\big)-\frac{1}{6}J(\xi_1,\xi_2,\xi_3)+\cdots\,, \tag{6.12}$$

where $J(\xi_1,\xi_2,\xi_3) = [\xi_1,[\xi_2,\xi_3]_c]_c + \text{cycl.}$ is the C-bracket Jacobiator that indeed is a trivial parameter, see eq. (8.29) in [2].

In terms of generalized coordinate transformations we have two maps $m_1 : X \to X'$ and $m_2 : X' \to X''$,

$$\begin{aligned} X' \;&=\; e^{-\Theta(\xi_1)(X)}X\,,\\ X'' \;&=\; e^{-\Theta(\xi_2)(X')}X'\,. \end{aligned} \tag{6.13}$$

We are now to find the relevant map $m_{21} : X \to X''$. The direct composition map is *not* the one we get. It would lead to a parameter built from ξ_2 and ξ_1 and the Lie bracket, not the C-bracket. What we get is the map $m_{21} = m_2 \star m_1$ defined by

$$X'' \;=\; e^{-\Theta\left(\xi^c(\,\xi_2\,,\,\xi_1\,)(X)\right)}\,X\,. \tag{6.14}$$

It may seem paradoxical that the direct composition $m_2 \circ m_1$ of maps does not define the map relevant in double field theory, but this is unavoidable and consistent. Is it possible to write the exotic composition law we have here in terms of the maps rather than in terms of the generating ξ parameters? Should the coordinates be viewed in a different way that makes the composition law look more natural?

The exotic composition rule has important consequences for associativity. Consider a third map $m_3 : X'' \to X'''$,

$$X''' \;=\; e^{-\Theta(\xi_3)(X'')}X''\,. \tag{6.15}$$

Given the three maps m_1, m_2 and m_3, we can form a map $X \to X'''$ in two different ways,

$$m_3 \star (m_2 \star m_1)\;, \qquad (m_3 \star m_2) \star m_1\;. \tag{6.16}$$

The first map leads to

$$X''' \;=\; \exp\big(-\Theta(\xi^c(\xi_3, \xi^c(\xi_2, \xi_1)))\big)X\;, \tag{6.17}$$

and the second map leads to

$$X''' \;=\; \exp\big(-\Theta(\xi^c(\xi^c(\xi_3, \xi_2), \xi_1))\big)X\;. \tag{6.18}$$

Due to (6.12) the two maps above are not equal. Indeed, a trivial parameter like the Jacobiator contributes to the transformation of X, see e.g. (4.62). Let us stress that this phenomenon would occur also without the modification from ξ to Θ and that, moreover, this modification does not compensate for the difference between (6.17) and (6.18). Therefore, even though the generalized coordinate transformations build a group when acting on fields, the composition rule $\star$ for coordinate maps does not form a group. In this respect we note that recently there have been proposals that in string theory there is a plausible role for string coordinates that are non-commutative or even non-associative [38–41], and it would be interesting to investigate if the unconventional group structure encountered here can be naturally interpreted in that context. These are important open questions, and any progress could help us learn about the underlying geometry of string theory.

In double field theory the strong constraint guarantees that, at least locally, we may always rotate into a frame where the fields depend only on half of the (doubled) coordinates. It is not yet known how to construct a non-trivial patching of local regions of the doubled manifold leading to more general 'non-geometric' configurations. The notion of a 'T-fold', for instance, is based on the idea that field configurations on overlaps can be glued with the use of T-duality transformations [42]. In order to address questions of this type in double field theory we need a clear picture of the finite gauge transformations, and in this paper we hope to have taken a step in this direction.

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A Modifying the parameterization of the diffeomorphism

The purpose of this section is to verify that Θ, as given in (4.70), is actually correct to quartic order. That is, no quartic term is needed and in fact

$$\Theta^M \;=\; \xi^M - \delta_3^M + \mathcal{O}(\xi^5) = \; \xi^M + \frac{1}{12}\,(\xi\xi^L)\partial^M\xi_L + \mathcal{O}(\xi^5)\,, \tag{A.1}$$

will be sufficient to guarantee that

$$\mathcal{F}_\Theta = \mathcal{E}(a-a^t) + \mathcal{O}(\xi^5)\,. \tag{A.2}$$

We begin by considering the discrepancy $\Delta\mathcal{F}$ between $\mathcal{F}$ and $\mathcal{E}(a-a^t)$ to quartic order in ξ. We write

$$\mathcal{F} \;=\; \mathcal{E}(a-a^t) - \Delta\mathcal{F}\,, \tag{A.3}$$

where $\Delta\mathcal{F}$ is calculated by expansion of (4.19) and was calculated to leading cubic order before. This time we find

$$\begin{aligned}\Delta\mathcal{F} \;=\;& \frac{1}{12}\Big((\xi a)a^t + a^2a^t - a\xi a^t - a(a^t)^2\Big)\\ &-\frac{1}{24}\Big[\,(\xi a)aa^t - aa^t\xi a^t + (\xi^2a)a^t - a(\xi^2a^t)\Big]\\ &-\frac{1}{12}\Big[\,a^2\xi a^t - a(\xi a^t)a^t + a^2(a^t)^2 - \frac{1}{2}a^3a^t - \frac{1}{2}a(a^t)^3\Big]\,.\end{aligned} \tag{A.4}$$

The first line contains the contributions cubic in ξ, while the other two lines contain the contributions quartic in ξ. Recall the expression for X' and that for X'_Θ in (4.62)

$$\begin{aligned}X'^M_\xi \;&\equiv\; X^M - \xi^M + \frac{1}{2}\,\xi\xi^M - \frac{1}{3!}\xi^2\xi^M + \mathcal{O}(\xi^4)\,,\\ X'^M_\Theta \;&\equiv\; X^M - \Theta^M + \frac{1}{2}\xi\Theta^M - \frac{1}{3!}\xi^2\Theta^M + \frac{1}{4!}\xi^3\Theta^M + \mathcal{O}(\xi^5)\,.\end{aligned} \tag{A.5}$$

Using (A.5) we can write the relation between the two X's as

$$X'^M_\Theta \;=\; X'^M_\xi + \delta_3^M + \hat\delta_4^M + \dots\,, \quad \text{with} \;\; \hat\delta_4^M \;=\; -\frac{1}{2}\xi\delta_3^M\,. \tag{A.6}$$

Now define, for $i=3,4$, the derivatives

$$(\Delta_3)_Q{}^M \;=\; \partial_Q{\delta_3}^M\,, \qquad (\Delta_4)_Q{}^M \;=\; \partial_Q{\hat\delta_4}{}^M\,. \tag{A.7}$$

With this notation,

$$\frac{\partial X'_\Theta}{\partial X} \;=\; \frac{\partial X'_\xi}{\partial X} + \Delta_3 + \Delta_4\,. \tag{A.8}$$

A short calculation shows that

$$\Delta_4 \;=\; -\frac{1}{2}(\xi + a)\Delta_3\;. \tag{A.9}$$

Now we need a formula to find the inverse of the above coordinate derivatives. Given the matrix M expanded in powers of ξ as

$$M \;=\; 1 + A_1 + A_2 + A_3 + A_4 + \mathcal{O}(\xi^5)\;, \tag{A.10}$$

with matrix inverse M^{-1}, we find that for the perturbed matrix

$$M' \;=\; M + \Delta A_3 + \Delta A_4 + \mathcal{O}(\xi^5)\,, \tag{A.11}$$

the inverse matrix is given by

$$M'^{-1} \;=\; M^{-1} - \Delta A_3 - \Delta A_4 + (\Delta A_3)A_1 + A_1(\Delta A_3) + \mathcal{O}(\xi^5)\,. \tag{A.12}$$

Applied to (A.8) this gives

$$\begin{aligned}\frac{\partial X}{\partial X'_\Theta} \;&=\; \frac{\partial X}{\partial X'_\xi} - \Delta_3 - \Delta_4 + (\Delta_3(-a) + (-a)\Delta_3)\\ &=\; \frac{\partial X}{\partial X'_\xi} - \Big(\Delta_3 + \Delta_4 + \Delta_3\, a + a\Delta_3\Big)\;.\end{aligned} \tag{A.13}$$

We then find that

$$\mathcal{F}_\Theta \;=\; \mathcal{F}_\xi + \Delta\,, \tag{A.14}$$

where

$$\Delta \;=\; \Delta_3^t - \Delta_3 \; + \Delta_4^t - \Delta_4 - (\Delta_3 a + a\Delta_3) + \frac{1}{2}(\Delta_3 a^t + a\Delta_3^t + \Delta_3^t a + a^t\Delta_3)\,. \tag{A.15}$$

In this light we have from (A.14) and (A.3)

$$\mathcal{F}_\Theta \;=\; \mathcal{E}(a - a^t) - \Delta\mathcal{F} + \Delta\;. \tag{A.16}$$

So in order to get $\mathcal{F}_\Theta = \mathcal{E}(a - a^t)$ we need a $\Theta(\xi)$ for which

$$\Delta \;=\; \Delta\mathcal{F}\,. \tag{A.17}$$

Let us now confirm that our choice for Θ, defined by (A.1) with

$$\delta_3 \;=\; -\frac{1}{12}\,(\xi\xi^L)\partial^M\xi_L\,, \tag{A.18}$$

indeed produces the desired result. The definition (A.7) gives

$$\Delta_3 \;=\; -\frac{1}{12}\big((\xi a)a^t + a^2 a^t\big) - \frac{1}{12}(\xi\xi^L)\partial\partial\xi_L\,, \tag{A.19}$$

where the matrix indices on the last term are carried by the partial derivatives $\partial\partial$. Moreover,

$$
\begin{aligned}
\Delta_4 &= -\frac{1}{2}(\xi+a)\Delta_3 \\
&= \frac{1}{24}\Big[(\xi a)aa^t + (\xi^2 a)a^t + (\xi a)(\xi a^t) + 2a(\xi a)a^t + a^2\xi a^t + a^3 a^t\Big] \\
&\quad + \frac{1}{24}(\xi+a)\big((\xi\xi^P)\partial\partial\xi_P\big) \,.
\end{aligned}
\tag{A.20}
$$

Using the above we can calculate all the ingredients of Δ,

$$
\begin{aligned}
\Delta_3^t - \Delta_3 &= \frac{1}{12}\Big((\xi a)a^t + a^2 a^t - a\xi a^t - a(a^t)^2\Big) \,, \\
-(\Delta_3 a + a\Delta_3) + \ &\frac{1}{2}(\Delta_3 a^t + a\Delta_3^t + \Delta_3^t a + a^t\Delta_3) \,, \\
&= \frac{1}{12}\big(a(\xi a)a^t + a^3 a^t - a^2(a^t)^2\big) \\
&\quad - \frac{1}{24}\big((\xi a)(a^t)^2 + a^2\xi a^t\big) + \frac{1}{24}\big(\, a(\xi\xi^P)\partial\partial\xi_P - (\xi\xi^P)(\partial\partial\xi_P)a^t\big) \,, \\
\Delta_4^t - \Delta_4 &= -\frac{1}{24}\Big[\, (\xi a)aa^t - aa^t\xi a^t + (\xi^2 a)a^t - a(\xi^2 a^t)\Big] \\
&\quad - \frac{1}{24}\Big[\, 2a(\xi a)a^t - 2a(\xi a^t)a^t + a^2\xi a^t - (\xi a)(a^t)^2 + a^3 a^t - a(a^t)^3\Big] \\
&\quad - \frac{1}{24}\big(a(\xi\xi^P)\partial\partial\xi_P - (\xi\xi^P)(\partial\partial\xi_P)a^t\big) \,,
\end{aligned}
\tag{A.21}
$$

where ξ^2 terms on the last line cancelled because the 'matrix' $\partial\partial$ is symmetric. Adding up the above to find Δ we get

$$
\begin{aligned}
\Delta &= \frac{1}{12}\Big((\xi a)a^t + a^2 a^t - a\xi a^t - a(a^t)^2\Big) \\
&\quad - \frac{1}{24}\Big[\, (\xi a)aa^t - aa^t\xi a^t + (\xi^2 a)a^t - a(\xi^2 a^t)\Big] \\
&\quad + \frac{1}{12}\Big[a(\xi a)a^t + a^3 a^t - a^2(a^t)^2 - \frac{1}{2}(\xi a)(a^t)^2 - \frac{1}{2}a^2\xi a^t\Big] \\
&\quad - \frac{1}{12}\Big[\, a(\xi a)a^t - a(\xi a^t)a^t + \frac{1}{2}a^2\xi a^t - \frac{1}{2}(\xi a)(a^t)^2 + \frac{1}{2}a^3 a^t - \frac{1}{2}a(a^t)^3\Big] \,.
\end{aligned}
\tag{A.22}
$$

Combining the last two lines we get

$$
\begin{aligned}
\Delta &= \frac{1}{12}\Big((\xi a)a^t + a^2 a^t - a\xi a^t - a(a^t)^2\Big) \\
&\quad - \frac{1}{24}\Big[\, (\xi a)aa^t - aa^t\xi a^t + (\xi^2 a)a^t - a(\xi^2 a^t)\Big] \\
&\quad - \frac{1}{12}\Big[\, a^2\xi a^t - a(\xi a^t)a^t + a^2(a^t)^2 - \frac{1}{2}a^3 a^t - \frac{1}{2}a(a^t)^3\Big] \,.
\end{aligned}
\tag{A.23}
$$

This coincides exactly with $\Delta\mathcal{F}$ in (A.4). Thus equation (A.17) holds and we have completed the verification that $\mathcal{F}_\Theta = \mathcal{E}(a - a^t)$ up to terms quintic in ξ.

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