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Double Field Theory

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Abstract

The zero modes of closed strings on a torus –the torus coordinates plus dual coordinates conjugate to winding number– parameterize a doubled torus. In closed string field theory, the string field depends on all zero-modes and so can be expanded to give an infinite set of fields on the doubled torus. We use string field theory to construct a theory of massless fields on the doubled torus. Key to the consistency is a constraint on fields and gauge parameters that arises from the $L_0 - \bar{L}_0 = 0$ condition in closed string theory. The symmetry of this double field theory includes usual and 'dual diffeomorphisms', together with a T-duality acting on fields that have explicit dependence on the torus coordinates and the dual coordinates. We find that, along with gravity, a Kalb-Ramond field and a dilaton must be added to support both usual and dual diffeomorphisms. We construct a fully consistent and gauge invariant action on the doubled torus to cubic order in the fields. We discuss the challenges involved in the construction of the full nonlinear theory. We emphasize that the doubled geometry is physical and the dual dimensions should not be viewed as an auxiliary structure or a gauge artifact.

Contents

1 Introduction and summary

T-duality is a striking property of string theory.[1] Closed strings can wrap around non-contractible cycles in spacetime, giving winding states that have no analogue for particle theories. The existence of both momentum and winding states is the key property of strings that allows T-duality: the complete physical equivalence of string theories on dual backgrounds that have very different geometries.

String field theory provides a complete gauge-invariant formulation of string dynamics around any consistent background, and we will use it here to study T-duality. A closed string field theory for a flat spacetime with some spatial directions curled up into a torus was examined long ago by Kugo and Zwiebach [2][2], following earlier work in [4, 5]. In particular, [2] showed how T-duality is realised as a symmetry of the string field theory. The string field theory treats momenta and winding rather

[1] See [1] for a review of T-duality and references.

[2] While this work used a covariantised light-cone formulation of the string field theory, the results are largely applicable to the covariant closed string field theory [3] which we use here.

symmetrically and, as a consequence, expanding the string field gives component fields that depend on *both* momentum and winding number. Fourier transforming to position space then gives component fields that depend on both the spacetime coordinates conjugate to momentum and on new periodic coordinates conjugate to winding number. For a spacetime which is a product of a Minkowski space M with a d-dimensional torus T^d, the component fields are then fields on $M \times T^{2d}$ where the *doubled torus* T^{2d} contains the original spacetime torus T^d together with another torus T^d parameterised by the winding coordinates. In fact, the doubled torus contains the original torus T^d as well as the tori related to it by T-duality. Then T-duality can be viewed as changing which T^d subspace of the doubled torus is to be regarded as part of the spacetime [6].

The complete closed string field theory on a torus is exotic and complicated. To our knowledge, it has not been examined in detail at the component level to try to uncover how spacetime fields realise the magic of T-duality. This is one of the main purposes of the present paper. As a simplification, we restrict ourselves to the 'massless' sector or, more precisely, to the set of fields that would be massless in the uncompactified theory. We thus focus on the gravity, antisymmetric tensor (Kalb-Ramond), and dilaton fields. We include all momenta and winding excitations of these fields by keeping their full dependence on the coordinates of the doubled torus. T-duality exchanges momentum and winding excitations, so that we expect T-duality to be a symmetry of this massless theory. A T-duality symmetric field theory on the doubled torus that can incorporate all T-dual geometries is likely to be novel and perhaps even exotic. Our hope is that this massless theory exists and it is not so complicated as to defy construction. Our results so far are encouraging: we have constructed the theory to cubic order in the fields. No higher derivatives are needed: each term has two derivatives, as in Einstein gravity.

Previous work on double field theory includes that of Tseytlin [7] who used a first-quantized approach with non-covariant actions for left and right-moving string coordinates on the torus. He calculated amplitudes for vertex operators depending on both coordinates, finding partially gauge-fixed cubic interactions for metric perturbations that are consistent with our action. It would be interesting to develop the first-quantised approach further, perhaps using the covariant formulation of [6]. Siegel [8] considered the field theory for the massless sector of closed strings without winding modes, but this restriction is implemented in an $O(d, d, \mathbb{Z})$ covariant fashion through an intriguing formulation of T-duality. An effective field theory on a doubled torus also arose in the study of open strings on a torus with space-filling and point-like D-branes [9].

The gauge symmetry of the theory we build should include diffeomorphisms for each T^d subspace of the doubled torus that can arise as a possible spacetime. We find that this is the case, and the linearised transformations include linearised diffeomorphisms on the doubled torus as well as a doubled version of the antisymmetric tensor field gauge symmetry. The non-linear structure is rather intricate and a simple characterization remains to be found. We find that the Jacobi identities are not satisfied, so the symmetry appears not to be diffeomorphisms on the doubled torus or even a Lie algebra. Gauge invariance requires that the fields and gauge parameters satisfy a constraint that arises from the $L_0 - \bar{L}_0 = 0$ constraint of closed string field theory.

The doubled torus T^{2d} arises naturally in the first-quantized approach to strings on a torus, leading to a number of approaches involving sigma models whose target is the space with doubled torus fibres [6, 7, 10, 11, 12, 13, 14, 15, 16, 17]. T-duality extends to spacetimes that have a torus fibration if the fields are independent of the coordinates of the torus fibres. The Buscher rules [18] for $d = 1$, and their extension to $d > 1$ [19], encode the transformation of such a background under T-duality. In the doubled torus formalism of Refs. [6, 14], the T^d fibres of such a background are replaced with doubled torus fibres T^{2d}. A key feature of this formalism is that T-duality is a manifest geometric symmetry, as the T-duality group acts through diffeomorphisms on the doubled torus fibres. Moreover, the target space with doubled torus fibres incorporates all possible T-dual geometries. The conventional picture emerges only on choosing a T^d subspace of each T^{2d} fibre to be the spacetime torus, and T-duality acts to change which T^d subspace is chosen [6]. The fact that T-duality is a symmetry means that the physics is the same in each case.

If fields have explicit dependence on the torus coordinates, the situation is not well understood. It is expected that fields that depend on the spacetime torus coordinates x should transform into fields that depend on the dual coordinates $\tilde{x}$. Dependence on the dual coordinates is puzzling, but one would expect that while x-dependence affects particles, $\tilde{x}$-dependence should affect winding modes, so that particles and winding modes could experience different backgrounds; see e.g. [20]. Dependence on $\tilde{x}$ has been associated with world-sheet instanton effects [20], and a number of calculations have supported this view [21, 22, 23, 24]. General string backgrounds, however, should involve fields depending on both x and $\tilde{x}$, and it is to be expected that there should be an extension of the T-duality transformation rules to this general case [25, 26]. We find the T-duality transformations that are a symmetry of the double action for fields that depend on both x and $\tilde{x}$. The fields in this action arise naturally from string field theory. In the case with no dependence on the dual coordinates $\tilde{x}$, we use the non-linear relation between these fields and the familiar metric and B-field to find a generalisation of the Buscher rules to the case of fields with general dependence on the torus coordinates x (or any set of coordinates related to these by a duality). The form of these transformations then suggest a natural further generalisation to the case in which the fields have full dependence on x and $\tilde{x}$.

We would like to emphasize that the inclusion of dual coordinates in double field theory is not a gauge redundancy or a reformulation of an underlying non-doubled geometry. The dual coordinates are needed to represent physical degrees of freedom; one cannot eliminate the dependence of fields on the additional coordinates using gauge conditions or solving constraints. This is perhaps less obvious in first quantization than in second quantization. In first quantization the familiar sigma model for closed strings on tori defines a conformal field theory. Using a doubled torus or other additional structures for the sigma model gives a better and more useful description of the *same* conformal field theory. It allows, for example, a natural construction of vertex operators for states with both momentum and winding. The physics, however, is in the conformal field theory, which includes momentum and winding, however they are described. In the string field theory a non-doubled formulation is not even an option. The string field, always defined by the conformal field theory state space, necessarily depends on coordinates conjugate to momentum and dual coordinates conjugate to winding. This dependence is nontrivial. While string field theory is now known to have nonperturbative information (at least in the open sector), our use of closed string field theory here has been more limited. String field theory was useful in the construction of a nontrivial action and gauge transformations that would have been hard to guess or construct directly.

Let us now discuss in some detail the setup and results in the present paper. We shall be interested in closed string theory in D-dimensional flat space with d compactified directions, $\mathbb{R}^{n-1,1} \times T^d$ where $n + d = D$. We shall present our discussion for the critical $D = 26$ bosonic closed string, but much of this applies to closed superstring theories. We use coordinates $x^i = (x^\mu, x^a)$ with $i = 0, ..., D-1$ which split into coordinates x^μ on the n-dimensional Minkoswski space $\mathbb{R}^{n-1,1}$ and coordinates x^a on the d-torus T^d. States are labelled by the momentum $p_i = (k_\mu, p_a)$ and the string windings w^a. For coordinates with periodicity $x^a \sim x^a + 2\pi$, the operators p_a and w^a have integer eigenvalues – these are the momentum and winding quantum numbers. Perturbative states are of the form

$$\sum_I \int dk \sum_{p_a, w^a} \phi_I(k_\mu, p_a, w^a)\, \mathcal{O}^I |k_\mu, p_a, w^a\rangle\,, \tag{1.1}$$

where $\mathcal{O}^I$ are operators built from matter and ghost oscillators and $\phi_I(k, p_a, w^a)$ are momentum-space fields which also depend on the winding numbers. Fourier transforming, dependence on the momenta k_μ, p_a becomes dependence on the spacetime coordinates x^μ, x^a as usual, while dependence on w^a is replaced by dependence on a new periodic coordinate $\tilde{x}_a$ conjugate to winding numbers w^a. Thus the fields ϕ_I above give us coordinate-space fields

$$\phi_I(x^\mu,\, x^a\,, \tilde{x}_a)\,. \tag{1.2}$$

Then $(x^a, \tilde{x}_a)$ are periodic coordinates for the doubled torus T^{2d}. All physical string states must satisfy the level matching condition, *i.e.*, they must be annihilated by $L_0 - \bar{L}_0$:

$$L_0 - \bar{L}_0 = N - \bar{N} - p_a w^a = 0\,. \tag{1.3}$$

This constraint will play a central role in our work. The free string on-shell condition $L_0 + \bar{L}_0 - 2 = 0$ takes a simple form when the background antisymmetric tensor vanishes:

$$M^2 \equiv -(k^2 + p^2 + w^2) = \frac{2}{\alpha'}(N + \bar{N} - 2)\,. \tag{1.4}$$

Here $\alpha' p^2 = \hat{G}^{ab} p_a p_b$ and $\alpha' w^2 = \hat{G}_{ab} w^a w^b$ where $\hat{G}_{ab}$ is the torus metric and $N, \bar{N}$ are the number operators for the left and right moving oscillators. We can view M^2 as the D-dimensional mass-squared and the associated massless states $(M^2 = 0)$ satisfy $N + \bar{N} = 2$.

The mass M in D-dimensions should not be confused with the mass $\mathcal{M}$ in the n-dimensional Minkoswki space obtained after compactification:

$$\mathcal{M}^2 \equiv -k^2 = p^2 + w^2 + \frac{2}{\alpha'}(N + \bar{N} - 2)\,. \tag{1.5}$$

For a rectangular torus the metric is $\hat{G}_{ab} = \delta_{ab} R_a^2/\alpha'$, where R_a is the radius of the circle along x^a. If all the circles are sufficiently large compared with the string length $(R_a^2 \gg \alpha')$, then $w^2 = \sum_a w_a^2 R_a^2/\alpha'$ is large and $p^2 = \sum_a p_a^2 \alpha'/R_a^2$ is small, so that the states that are light compared to the string scale include those which have $w^a = 0$ and $N + \bar{N} = 2$. This is the Kaluza-Klein tower of states obtained by compactifying the theory of massless states in D dimensions. A conventional effective field theory in the n-dimensional Minkowski space would keep states for which $\mathcal{M}^2$ is zero or small, and would give the leading terms in a systematic expansion in $\mathcal{M}^2$. Instead, here we focus on $M^2 = 0$ states and in so doing, we are keeping certain states that, from the lower-dimensional point of view, are heavy while neglecting some which are lighter.[3] It is possible that the theory we are trying to build should be considered as an effective theory in which we keep a set of massless fields, including all of their large-energy excitations, and integrate out everything else. At special points in the torus moduli space there are extra states with $\mathcal{M}^2 = 0$ giving enhanced gauge symmetry, while near these special points these states will have small $\mathcal{M}^2$. These have $(N, \bar{N}) = (1, 0)$ or $(N, \bar{N}) = (0, 1)$ and so have $M^2 = -2/\alpha'$; we will not include these here.

T-duality is an $O(d, d; \mathbb{Z})$ symmetry of the string theory acting linearly on the torus coordinates $x^a, \tilde{x}_a$ and preserving their boundary conditions. This includes a $\mathbb{Z}_2$ symmetry for each direction a that interchanges x^a with $\tilde{x}_a$. For a rectangular torus in which x^a is a coordinate for a circle of radius R_a, $\tilde{x}_a$ is the coordinate for a T-dual circle of radius α'/R_a. Performing a $\mathbb{Z}_2$ on each of the toroidal dimensions takes a theory on the original spacetime $\mathbb{R}^{n-1,1} \times T^d$ with coordinates x^μ, x^a to a theory in the dual spacetime $\mathbb{R}^{n-1,1} \times \widetilde{T}^d$ with coordinates $x^\mu, \tilde{x}_a$.

In the closed string field theory for this toroidal background the string field $|\Psi\rangle$ is a general state of the form (1.1), and so can be viewed as a collection of component fields $\phi_I(x^\mu, x^a, \tilde{x}_a)$. It should be emphasized that the difference between the toroidally compactified theory and the D-dimensional Minkowski space theory is that the toroidal zero modes are doubled; no new oscillators are added. Two off-shell constraints must be satisfied by both the string field and the gauge parameter $|\Lambda\rangle$. We must have

$$(b_0 - \bar{b}_0)|\Psi\rangle = 0, \qquad (b_0 - \bar{b}_0)|\Lambda\rangle = 0\,, \tag{1.6}$$

and the associated level-matching conditions

$$(L_0 - \bar{L}_0)|\Psi\rangle = 0\,, \quad (L_0 - \bar{L}_0)|\Lambda\rangle = 0\,. \tag{1.7}$$

The free field equation is $Q|\Psi\rangle = 0$, where Q is the BRST operator, and it is invariant under gauge transformations $\delta|\Psi\rangle = Q|\Lambda\rangle$. The ket $|\Lambda\rangle$ gives rise to an infinite set of gauge parameters that depend on x^μ, x^a, and $\tilde{x}_a$. On account of (1.3) and (1.7) the string field satisfies

$$(N - \bar{N})|\Psi\rangle = p_a w^a |\Psi\rangle\,, \tag{1.8}$$

and for a component field $\phi_I(x^\mu, x^a, \tilde{x}_a)$ we have

$$(N_I - \bar{N}_I)\,\phi_I = \frac{1}{2}\,\alpha' \Delta \phi_I\,, \qquad \text{with} \quad \Delta \equiv -\frac{2}{\alpha'}\frac{\partial}{\partial x^a}\frac{\partial}{\partial \tilde{x}_a}\,. \tag{1.9}$$

Here the N_I and $\bar{N}_I$ are the eigenvalues of N and $\bar{N}$ on the CFT state for which ϕ_I is the expansion coefficient. Thus string field theory is a theory of constrained fields, but the constraint still allows fields with non-trivial dependence on both x^a and $\tilde{x}_a$ if $d > 1$.

[3]We thank David Gross for emphasizing this point to us.

For $N = \bar{N} = 1$ we have the following fields, all with $M^2 = 0$:[4]

$$h_{ij}(x^\mu, x^a, \tilde{x}_a), \qquad b_{ij}(x^\mu, x^a, \tilde{x}_a), \qquad d\,(x^\mu, x^a, \tilde{x}_a)\,. \tag{1.10}$$

The constraint requires that these fields are all annihilated by the differential operator Δ. The solutions independent of $\tilde{x}$ give the gravity field $h_{ij}(x^\mu, x^a)$, the antisymmetric tensor field $b_{ij}(x^\mu, x^a)$, and the dilaton $d(x^\mu, x^a)$ in D dimensions. The solutions independent of x^a give dual versions of these fields, while again the general case depends on both x^a and $\tilde{x}_a$ (for $d > 1$). Note that e.g. h_{ij} decomposes as usual into $h_{\mu\nu}$, $h_{\mu a}$, h_{ab} and there is no doubling of the tensor indices. At higher levels the fields have the same index structure as for the uncompactified string theory, but now depend on $\tilde{x}$ as well as x^μ, x^a and are subject to the constraint (1.9).

In this paper we focus on the $M^2 = 0$ fields in (1.10). The relevant gauge parameters are a pair of vector fields $\epsilon_i(x^\mu, x^a, \tilde{x}_a)$ and $\tilde{\epsilon}_i(x^\mu, x^a, \tilde{x}_a)$, both of which are annihilated by Δ. Our analysis of the quadratic theory shows that the linearised gauge transformations take the form

$$\begin{aligned} \delta h_{ij} &= \quad \partial_i \epsilon_j + \partial_j \epsilon_i \;+\; \tilde{\partial}_i \tilde{\epsilon}_j + \tilde{\partial}_j \tilde{\epsilon}_i\,, \\ \delta b_{ij} &= -(\tilde{\partial}_i \epsilon_j - \tilde{\partial}_j \epsilon_i) - (\partial_i \tilde{\epsilon}_j - \partial_j \tilde{\epsilon}_i)\,, \\ \delta d \;&= -\tfrac{1}{2}\,\partial \cdot \epsilon + \tfrac{1}{2}\,\tilde{\partial} \cdot \tilde{\epsilon}\,. \end{aligned} \tag{1.11}$$

We use the notation $\tilde{x}_i = (\tilde{x}_a, 0)$ and $\tilde{\partial}_i = (\partial/\partial\tilde{x}_a\,, 0,)$ which makes it clear that only the coordinates on the torus are doubled. The above gauge structure is rather intricate and novel. For parameters and fields that are independent of $\tilde{x}$, these are the standard linearised diffeomorphisms (acting on x^i) with parameter ϵ_i and antisymmetric tensor gauge transformations with parameter $\tilde{\epsilon}_i$. A dilaton ϕ which is a scalar (invariant under these linearised transformations) can be defined by $\phi = d + \frac{1}{4}\eta^{ij} h_{ij}$. Parameters and fields that are independent of x^a live on the dual space with coordinates $x^\mu, \tilde{x}_a$. These are again linearised diffeomorphisms, now acting on $x^\mu, \tilde{x}_a$, and antisymmetric tensor gauge transformations, but the roles of the parameters ϵ_i and $\tilde{\epsilon}_i$ have been interchanged. Now $\tilde{\epsilon}_i$ is the diffeomorphism parameter and ϵ_i the antisymmetric tensor gauge parameter. In this case, the scalar dilaton would be $\tilde{\phi} = d - \frac{1}{4}\eta^{ij} h_{ij}$. While ϕ is invariant under ϵ transformations and $\tilde{\phi}$ is invariant under $\tilde{\epsilon}$ transformations, there is no combination of d and $\eta^{ij} h_{ij}$ that is invariant under both. In the full non-linear theory there is no dilaton that is a scalar under both diffeomorphisms and dual diffeomorphisms, and d is the natural field to use. Nonlinearly, one has a relation of the form $e^{-2d} = e^{-2\phi}\sqrt{-g}$; the dilaton d is invariant under T-duality and its expectation value provides the duality-invariant string coupling constant [2, 34, 14].

In the general case with dependence on both x^a and $\tilde{x}_a$ one has both diffeomorphisms and dual diffeomorphisms, giving an intriguing structure of 'doubled diffeomorphisms'. Moreover, we will show the diffeomorphisms and antisymmetric tensor gauge transformations become closely linked, with the roles of the parameters interchanged by T-duality. The consistency of this free theory hinges crucially on the constraint $\Delta = 0$ satisfied by the fields and gauge parameters. Given the general

[4]There are additional auxiliary fields and gauge trivial fields that do not contribute propagating degrees of freedom.

interest in theories on doubled tori, we analyze the free theory further and find that linearised double diffeomorphisms *cannot* be realised with the h_{ij} field alone: the Kalb-Ramond and dilaton fields must be added. While diffeomorphism symmetry does not fix the field content of the massless sector of closed string theory, 'double diffeomorphisms' does!

We are guided by string field theory to build a remarkable interacting generalisation of the linearised massless theory described above. In doing so we obtain a two-derivative theory with a gauge invariance that is the nonlinear version of the doubled diffeomorphisms found in the quadratic theory. The constraint $\Delta = 0$ remains unmodified and the theory remains a theory of constrained fields. The action is given in (3.25) and the gauge transformations are given in (3.27). The theory also has a discrete $\mathbb{Z}_2$ symmetry (3.26) that arises from the orientation invariance of the underlying closed string theory. It should be emphasized that the quadratic part of the action that we write is exactly that of the string field theory, but the cubic part of the action is not. In constructing this cubic part we drop all terms with more than two derivatives. We also drop the momentum-dependent sign factors due to cocycles that enforce the mutual locality of vertex operators [27, 28, 4, 5, 32]. Gauge invariance works to this order without the inclusion of such terms, although some may be needed to achieve a complete nonlinear construction. The role of sign factors is discussed in Section 5.

The symmetry algebra of closed string field theory is not a Lie algebra (the Jacobi identitites do not hold) as in familiar theories, but rather a homotopy Lie algebra [3]. The structure of the interactions we find in our double field theory leads to a symmetry algebra that appears not to be a Lie algebra, suggesting that some of the homotopy structure of the string field theory survives in the massless theory. As we discuss in Section 5, an explicit projector is needed so that the product of two fields in the kernel of Δ is also in the kernel of Δ. The presence of this projection is part of the reason the brackets that define the composition of gauge parameters do not satisfy a Jacobi identity. Understanding the full symmetry of the theory is a central open problem. Further discussion of open problems and directions for further research can be found in §6.

In closing this introduction we note that the work here furnishes some new results in closed string field theory. The cubic theory of the massless fields, required to see the full structure of diffeomorphisms, was not worked out before. The formulation of gravity in string theory uses auxiliary fields that must be eliminated using their equations of motion as well as a gauge trivial scalar field that must be carefully gauged away. Field redefinitions are needed to obtain a simple form of the gauge transformations. In the end, the formulation of gravity plus antisymmetric field and a dilaton in string theory is extremely efficient; it uses $e_{ij} = h_{ij} + b_{ij}$ and a duality-invariant scalar d (related to linearized order to the usual dilaton ϕ by $d = \phi - \frac{1}{4}h$). The cubic action we present is much simpler than the cubic action obtained by direct expansion of the familiar action for gravity, antisymmetric tensor, and dilaton. The results in this paper suffice to find the field redefinitions that connect the string field theory and sigma model fields for the massless sector of the closed string to quadratic order in the fields and without derivatives. Earlier work in this direction includes that of [29], which discussed general coordinate invariance in closed string field theory and [30], which studied the constraints that T-duality imposes on the relation between closed string fields and sigma model fields.

2 The Free Theory

In this section we begin by giving an argument supporting our claim that linearised double diffeomorphism invariance requires the massless multiplet of closed string theory. We then review closed string theory on toroidal backgrounds, setting the notation and giving the basic results used in this paper. We then use the free closed string field theory to construct the free double field theory. We study the symmetries in detail and emphasize the differences with the conventional free theory of gravity, antisymmetric field, and dilaton.

2.1 Linearised double diffeomorphism symmetry

In the introduction we introduced $M^2 = 0$ fields depending on $(x^\mu, x^a, \tilde{x}_a)$ with linearised transformations (1.11). These included a field $h_{ij}(x^\mu, x^a, \tilde{x}_a)$ transforming under linearised diffeomorphisms as

$$\delta h_{ij} = \partial_i \epsilon_j + \partial_j \epsilon_i \,. \tag{2.1}$$

and under linearised 'dual diffeomorphisms' as

$$\tilde{\delta} h_{ij} = \tilde{\partial}_i \tilde{\epsilon}_j + \tilde{\partial}_j \tilde{\epsilon}_i \,. \tag{2.2}$$

We will now show why we cannot have a theory of h_{ij} alone that is invariant under such 'double diffeomorphisms'. We will find that introducing a Kalb-Ramond field and a dilaton is essential, and that the constraint $\Delta = 0$ must be satisfied for invariance.

For Einstein's gravity $S = \frac{1}{2\kappa^2} \int \sqrt{-g} R$, and to quadratic order in the fluctuation field $h_{ij}(x) \equiv g_{ij}(x) - \eta_{ij}$ one has

$$(2\kappa^2)\, S_0 = \int dx \Big[\, \frac{1}{4} h^{ij} \partial^2 h_{ij} - \frac{1}{4}\, h \partial^2 h + \frac{1}{2} (\partial^i h_{ij})^2 + \frac{1}{2} h \, \partial_i \partial_j \, h^{ij} \, \Big] \,. \tag{2.3}$$

This action, of course, is invariant under (2.1), but we wish to implement also the dual diffeomorphisms (2.2). For a field $h_{ij}(\tilde{x}, x)$ depending on both x and $\tilde{x}$, the action is an integral over the full $n + 2d$ dimensional doubled space. We will denote this integral as $\int [dx d\tilde{x}]$. The natural action is

$$\begin{aligned} (2\kappa^2)\, S = \int [dx d\tilde{x}] \Big[\, & \frac{1}{4} h^{ij} \partial^2 h_{ij} - \frac{1}{4}\, h \partial^2 h + \frac{1}{2} (\partial^i h_{ij})^2 + \frac{1}{2} h \, \partial_i \partial_j \, h^{ij} \\ & + \frac{1}{4} h^{ij} \tilde{\partial}^2 h_{ij} - \frac{1}{4}\, h \tilde{\partial}^2 h + \frac{1}{2} (\tilde{\partial}^i h_{ij})^2 + \frac{1}{2} h \, \tilde{\partial}_i \tilde{\partial}_j \, h^{ij} \, \Big] \,. \end{aligned} \tag{2.4}$$

For a gravity field $h_{ij}(x^i)$ independent of $\tilde{x}_a$ the action reduces to the linearised Einstein action on the space with coordinates x^i. For a gravity field $h_{ij}(x^\mu, \tilde{x}_a)$ independent of x^a the action reduces to the linearised Einstein action on the dual space with coordinates $x^\mu, \tilde{x}_a$. The first line in (2.4) is invariant under the δ transformations (2.1), the second is invariant under the $\tilde{\delta}$ transformations (2.2).

Let us vary the double action S under $\tilde{\delta}$. The second line is invariant and varying the first gives

$$\begin{aligned} (2\kappa^2)\, \tilde{\delta} S = \int [dx d\tilde{x}] \Big[\; & h^{ij} \partial^2 \tilde{\partial}_i \tilde{\epsilon}_j + \partial_i h^{ij} \, (\partial^k \tilde{\partial}_k) \tilde{\epsilon}_j \\ & - h \, \partial^2 \, \tilde{\partial} \cdot \tilde{\epsilon} \; + h \, (\partial_i \tilde{\partial}^i) \partial_j \tilde{\epsilon}^j \\ & + \partial_i h^{ij} \, \partial^k \, \tilde{\partial}_j \tilde{\epsilon}_k + \, (\partial_i \partial_j h^{ij}) \tilde{\partial} \cdot \tilde{\epsilon} \, \Big] \,. \end{aligned} \tag{2.5}$$

We have organised the right-hand side so that the terms on each line would cancel if the tilde derivatives were replaced by ordinary derivatives. As we can see, no cancellation whatsoever takes place! Grouping related terms we have

$$\begin{aligned}(2\kappa^2)\,\tilde{\delta}S = \int [dxd\tilde{x}]\Big[\, & h^{ij}\partial^2\tilde{\partial}_i\tilde{\epsilon}_j - h^{ij}\partial_i\partial^k\tilde{\partial}_j\tilde{\epsilon}_k + (\partial_i\partial_j h^{ij} - \partial^2 h\,)\,\tilde{\partial}\cdot\tilde{\epsilon} \\ & + (\partial^i h_{ij} - \partial_j h)\,(\partial\cdot\tilde{\partial})\tilde{\epsilon}^j\,\Big]\,.\end{aligned} \tag{2.6}$$

The terms on the second line vanish when the gauge parameter $\tilde{\epsilon}$ satisfies the constraint $\partial\cdot\tilde{\partial} = 0$. Relabeling the indices on the first two terms, we get

$$\begin{aligned}(2\kappa^2)\,\tilde{\delta}S = \int [dxd\tilde{x}]\Big[\, & (\tilde{\partial}_j h^{ij})\partial^k(\partial_i\tilde{\epsilon}_k - \partial_k\tilde{\epsilon}_i) + (\partial_i\partial_j h^{ij} - \partial^2 h\,)\,\tilde{\partial}\cdot\tilde{\epsilon} \\ & + (\partial^i h_{ij} - \partial_j h)\,(\partial\cdot\tilde{\partial})\tilde{\epsilon}^j\,\Big]\,.\end{aligned} \tag{2.7}$$

In order to cancel this variation we need new fields with new gauge transformations. To cancel the first term we can use a Kalb-Ramond field b_{ij} and a new term S_1 in the action:

$$(2\kappa^2)\,S_1 = \int [dxd\tilde{x}]\;(\tilde{\partial}_j h^{ij})\partial^k b_{ik}\,, \qquad \text{with} \quad \tilde{\delta}b_{ij} = -(\partial_i\tilde{\epsilon}_j - \partial_j\tilde{\epsilon}_i)\,. \tag{2.8}$$

To cancel the second term in (2.7) we introduce a dilaton ϕ and a new term S_2 given by

$$(2\kappa^2)S_2 = \int [dxd\tilde{x}](-2)(\partial_i\partial_j h^{ij} - \partial^2 h\,)\phi\,, \quad \text{with} \qquad \tilde{\delta}\phi = \frac{1}{2}\,\tilde{\partial}\cdot\tilde{\epsilon}\,. \tag{2.9}$$

The above are the first steps in the construction of a consistent quadratic theory. More terms are needed, and we will find the full, invariant quadratic action from the closed string field theory in §2.3. The lessons are clear, however. Implementation of linearised doubled diffeomorphisms for h_{ij} requires the addition of further fields, most naturally, a Kalb-Ramond gauge field and a dilaton. Moreover, a second-order differential constraint is required: fields and gauge parameters must be annihilated by $\partial\cdot\tilde{\partial}$. In fact, to this order, it suffices for the gauge parameters to satisfy the constraint.

It is natural to ask if by adding further fields one can find an action that is invariant without the constraint. The offending term on the second line of (2.7) can be cancelled in this way, but then further terms are needed. We have not been able to find a non-trivial theory that is invariant under both δ and $\tilde{\delta}$ transformations without use of the constraint.

2.2 General toroidal backgrounds

An explicit discussion of closed string field theory in toroidal backgrounds was given in the work of Kugo and Zwiebach [2]. Following this work, we review the basic results that will be needed here. We begin with the string action, given by[5]

$$S = -\frac{1}{4\pi}\int_0^{2\pi} d\sigma \int d\tau\big(\sqrt{\gamma}\gamma^{\alpha\beta}\partial_\alpha X^i\partial_\beta X^j G_{ij} + \epsilon^{\alpha\beta}\partial_\alpha X^i\partial_\beta X^j B_{ij}\big)\,. \tag{2.10}$$

[5]Our formulae will keep explicit factors of α'. In the worldsheet action (2.10) G_{ij}, B_{ij}, and the X^i are all dimensionless.

The string coordinates

$$X^i = \{X^a\,, X^\mu\}\,, \tag{2.11}$$

split into string coordinates X^μ for n-dimensional Minkowski space and periodic string coordinates X^a for the internal d-dimensional torus:

$$X^a \sim X^a + 2\pi\,. \tag{2.12}$$

In the above action G_{ij} and B_{ij} are the constant background metric and antisymmetric tensor, respectively. As usual, we define the inverse metric with upper indices:

$$G^{ij}G_{jk} = \delta^i_k\,. \tag{2.13}$$

The background fields are taken to be

$$G_{ij} = \begin{pmatrix} \hat{G}_{ab} & 0 \\ 0 & \eta_{\mu\nu} \end{pmatrix}, \qquad B_{ij} = \begin{pmatrix} \hat{B}_{ab} & 0 \\ 0 & 0 \end{pmatrix}. \tag{2.14}$$

and we define

$$E_{ij} \equiv G_{ij} + B_{ij} = \begin{pmatrix} \hat{E}_{ab} & 0 \\ 0 & \eta_{\mu\nu} \end{pmatrix}, \quad \hat{E}_{ab} \equiv \hat{G}_{ab} + \hat{B}_{ab}\,. \tag{2.15}$$

The Hamiltonian H for this theory takes the form

$$4\pi H = (X'\,, 2\pi P)\,\mathcal{H}(E) \begin{pmatrix} X' \\ 2\pi P \end{pmatrix}, \tag{2.16}$$

where the derivatives of the coordinates $X^{i\prime} = \partial_\sigma X^i$ and the momenta P_i are combined into a $2D$ dimensional column vector and the $2D \times 2D$ matrix $\mathcal{H}$ is given by

$$\mathcal{H}(E) = \begin{pmatrix} G - BG^{-1}B & BG^{-1} \\ -G^{-1}B & G^{-1} \end{pmatrix}. \tag{2.17}$$

The matrix $\mathcal{H}(E)$ satisfies the constraint $\mathcal{H}^{-1} = \eta\mathcal{H}\eta$.

The mode expansions for X^i, P_i and the dual coordinates $\tilde{X}_i$ take the form

$$\begin{aligned} X^i(\tau,\sigma) &= x^i + w^i\sigma + \tau\, G^{ij}(p_j - B_{jk}w^k) + \frac{i}{\sqrt{2}}\sum_{n\neq 0}\frac{1}{n}\big[\alpha^i_n e^{in\sigma} + \bar{\alpha}^i_n e^{-in\sigma}\big]e^{-in\tau}\,, \\ 2\pi P_i(\tau,\sigma) &= p_i + \frac{i}{\sqrt{2}}\sum_{n\neq 0}\big[E^t_{ij}\,\alpha^j_n e^{in\sigma} + E_{ij}\,\bar{\alpha}^j_n e^{-in\sigma}\big]e^{-in\tau}\,, \\ \tilde{X}_i(\tau,\sigma) &= \tilde{x}_i + p_i\sigma + \tau\big[(G - BG^{-1}B)_{ij}w^j + (BG^{-1})_i^{\ j} p_j\big] + \frac{i}{\sqrt{2}}\sum_{n\neq 0}\frac{1}{n}\big[-E^t_{ij}\alpha^j_n e^{in\sigma} + E_{ij}\bar{\alpha}^j_n e^{-in\sigma}\big]e^{-in\tau}. \end{aligned} \tag{2.18}$$

Given (2.12), $x^a \sim x^a + 2\pi$ and w^a and p_a take integer values. Conjugate to the winding charges w^a, there are periodic coordinates $\tilde{x}_a$ satisfying $\tilde{x}_a \sim \tilde{x}_a + 2\pi$. In the above expansions we use

$$\begin{aligned} w^i &= \{w^a, w^\mu\} = \{w^a, 0\}\,, \\ \tilde{x}_i &= \{\tilde{x}_a, \tilde{x}_\mu\} = \{\tilde{x}_a, 0\}\,, \end{aligned} \tag{2.19}$$

which state that there are no windings nor dual coordinates along the Minkowski directions. We have the commutation relations:

$$[\,x^i, p_j\,] = i\,\delta^i_j\,, \quad [\,\tilde{x}_i, w^j\,] = i\,\hat{\delta}^j_i\,, \tag{2.20}$$

where $\hat{\delta}^j_i = \text{diag}\{\hat{\delta}^a_b, 0\}$ so that the second relation is just $[\,\tilde{x}_a, w^b\,] = i\,\hat{\delta}^b_a$. Moreover

$$[\,\alpha^i_m,\, \alpha^j_n\,] = [\,\bar{\alpha}^i_m,\, \bar{\alpha}^j_n\,] = \; m\,G^{ij}\,\delta_{m+n,0}\,. \tag{2.21}$$

Finally, we have the zero-modes given by

$$\begin{aligned} \alpha^i_0 = &\ \frac{1}{\sqrt{2}} G^{ij}\left(p_j - E_{jk}w^k\right), \\ \alpha^i_0 = &\ \frac{1}{\sqrt{2}} G^{ij}\left(p_j + E^t_{jk}w^k\right). \end{aligned} \tag{2.22}$$

Lowering the indices and writing in terms of the dimensionless coordinates x^i and $\tilde{x}_i$ gives

$$\begin{aligned} \alpha_{0i} = &\ -\frac{i}{\sqrt{2}}\left(\frac{\partial}{\partial x^i} - E_{ik}\frac{\partial}{\partial \tilde{x}_k}\right) = -i\sqrt{\frac{\alpha'}{2}}\,D_i\,, \\ \bar{\alpha}_{0i} = &\ -\frac{i}{\sqrt{2}}\left(\frac{\partial}{\partial x^i} + E^t_{ik}\frac{\partial}{\partial \tilde{x}_k}\right) = -i\sqrt{\frac{\alpha'}{2}}\,\bar{D}_i\,, \end{aligned} \tag{2.23}$$

where we introduced derivatives D_i and $\bar{D}_i$ with the dimensions of inverse length and used $p_j = \frac{1}{i}\partial_j$ as well as $w^k = \frac{1}{i}\tilde{\partial}^k$. The derivatives D and $\bar{D}$ can then be written as

$$\boxed{\begin{aligned} D_i = &\ \frac{1}{\sqrt{\alpha'}}\left(\frac{\partial}{\partial x^i} - E_{ik}\,\frac{\partial}{\partial \tilde{x}_k}\right), \\ \bar{D}_i = &\ \frac{1}{\sqrt{\alpha'}}\left(\frac{\partial}{\partial x^i} + E^t_{ik}\,\frac{\partial}{\partial \tilde{x}_k}\right). \end{aligned}} \tag{2.24}$$

We work in Lorentzian signature (both for the worldsheet and spacetime) and D and $\bar{D}$ are independent real derivatives with respect to right- and left-moving coordinates $\tilde{x}_i - E_{ij}x^j$ and $\tilde{x}_i + E^t_{ij}x^j$, respectively. Indeed, $\tilde{X}_i - E_{ij}X^j$ is a function of $(\sigma - \tau)$ and $\tilde{X}_i + E^t_{ij}X^j$ is a function of $(\sigma + \tau)$. For the noncompact directions there are no dual derivatives and we have

$$\frac{\partial}{\partial \tilde{x}_i} = \left\{\frac{\partial}{\partial \tilde{x}_a}\,, 0\right\}. \tag{2.25}$$

As a consequence, while $D_a \neq \bar{D}_a$ we have $D_\mu = \bar{D}_\mu$.

It is useful to introduce operators $\Box$ and Δ, both quadratic in the α_0 and $\bar{\alpha}_0$ operators:

$$\begin{aligned} -\frac{\alpha'}{2}\Box \;\equiv&\; \frac{1}{2}\alpha^i_0\,G_{ij}\alpha^j_0 + \frac{1}{2}\,\bar{\alpha}^i_0\,G_{ij}\bar{\alpha}^j_0\,, \\ -\frac{\alpha'}{2}\Delta \;\equiv&\; \frac{1}{2}\alpha^i_0\,G_{ij}\alpha^j_0 - \frac{1}{2}\,\bar{\alpha}^i_0\,G_{ij}\bar{\alpha}^j_0\,. \end{aligned} \tag{2.26}$$

We note that, in general

$$L_0 - \bar{L}_0 = N - \bar{N} - \frac{\alpha'}{2}\,\Delta\,, \qquad (2.27)$$

so that the level matching condition for fields with $N = \bar{N}$ becomes the constraint $\Delta = 0$. In terms of our derivatives, we get

$$\begin{aligned} \square &= \frac{1}{2}\, D^i\, G_{ij} D^j + \frac{1}{2}\, \bar{D}^i\, G_{ij} \bar{D}^j \;= \frac{1}{2}\big(\, D^i D_i + \bar{D}^j \bar{D}_j\big)\,, \\ \Delta &= \frac{1}{2}\, D^i\, G_{ij} D^j \;- \frac{1}{2}\, \bar{D}^i\, G_{ij} \bar{D}^j \;= \frac{1}{2}\big(\, D^i D_i - \bar{D}^j \bar{D}_j\big)\,. \end{aligned} \qquad (2.28)$$

Writing $D^2 \equiv D^i D_i$ and $\bar{D}^i \bar{D}_i = \bar{D}^2$ we have

$$\boxed{\square = \frac{1}{2}(D^2 + \bar{D}^2)\,, \quad \Delta = \frac{1}{2}(D^2 - \bar{D}^2)\,.} \qquad (2.29)$$

An explicit computation using the expressions for α_0 and $\bar{\alpha}_0$ gives

$$\square = \frac{1}{\alpha'}\Big(G^{ij}\,\frac{\partial}{\partial x^i}\,\frac{\partial}{\partial x^j} + 2\,(BG^{-1})_i{}^j\;\frac{\partial}{\partial \tilde{x}_i}\,\frac{\partial}{\partial x^j} + (G - BG^{-1}B)_{ij}\,\frac{\partial}{\partial \tilde{x}_i}\frac{\partial}{\partial \tilde{x}_j}\Big)\,. \qquad (2.30)$$

Note that the contribution to $\square$ from the non-compact directions is the expected term $\frac{1}{\alpha'}\eta^{\mu\nu}\partial_\mu\partial_\nu$. Recalling that E is a constant, this can be rewritten as

$$\square = \frac{1}{\alpha'}\;\partial^t\;\mathcal{H}(E)\;\partial\,, \quad \text{with} \quad \partial = \begin{pmatrix} \frac{\partial}{\partial \tilde{x}_i} \\ \frac{\partial}{\partial x^j} \end{pmatrix}\,. \qquad (2.31)$$

Another short computation, together with (2.25), shows that the operator Δ takes the form

$$\boxed{\Delta = -\frac{2}{\alpha'}\sum_i \frac{\partial}{\partial \tilde{x}_i}\frac{\partial}{\partial x^i} = -\frac{2}{\alpha'}\sum_a \frac{\partial}{\partial \tilde{x}_a}\frac{\partial}{\partial x^a}\,.} \qquad (2.32)$$

Note that no background fields are required here. We can also write

$$\Delta = -\frac{1}{\alpha'}\;\partial^t\;\eta\;\partial\,, \quad \text{with} \quad \eta = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}\,. \qquad (2.33)$$

While $\square$ is a Laplacian for the metric $\mathcal{H}(E)$, Δ is one for the $O(D,D)$ invariant metric η.

In string field theory the physical state conditions $L_0 + \bar{L}_0 - 2 = 0$ and $L_0 - \bar{L}_0 = 0$ are treated very differently. The former arises from the free string field equation of motion and gives equations of the form $\square\Phi_I = ...$ for the component fields $\Phi_I(x, \tilde{x})$. The latter is imposed as a constraint on the string field, so that the fields with $N = \bar{N}$ are required to satisfy

$$\Delta\Phi_I = 0\,.$$

As usual, we include the standard bc ghost system with ghost oscillators $b_n, c_n, \bar{b}_n, \bar{c}_n$.

Rectangular Tori: Let us consider the case where $\hat{B}_{ab} = 0$ and the metric is diagonal. If R_a denotes the physical radius of the circle $X^a \sim X^a + 2\pi$ we have

$$\hat{E}_{ab} = \hat{G}_{ab} = \frac{R_a^2}{\alpha'}\,\delta_{ab}\,, \qquad \hat{G}^{ab} = \frac{\alpha'}{R_a^2}\,\delta^{ab}\,. \tag{2.34}$$

For the derivatives we find

$$\begin{aligned} D_a = \ & \frac{1}{\sqrt{\alpha'}}\Big(\frac{\partial}{\partial x^a} - \frac{R_i^2}{\alpha'}\delta_{ab}\frac{\partial}{\partial \tilde{x}_b}\Big)\,, \quad \bar{D}_a = \ \frac{1}{\sqrt{\alpha'}}\Big(\frac{\partial}{\partial x^a} + \frac{R_i^2}{\alpha'}\,\delta_{ab}\frac{\partial}{\partial \tilde{x}_b}\Big)\,, \\ \square = \ & \frac{1}{\alpha'}\Big(\,\eta^{\mu\nu}\partial_\mu\partial_\nu + \frac{\alpha'}{R_i^2}\,\delta^{ab}\frac{\partial}{\partial x^a}\frac{\partial}{\partial x^b} + \frac{R_i^2}{\alpha'}\,\delta_{ab}\frac{\partial}{\partial \tilde{x}_a}\frac{\partial}{\partial \tilde{x}_b}\Big)\,. \end{aligned} \tag{2.35}$$

We can introduce coordinates u^a and $\tilde{u}_a$ that have physical lengths (repeated indices not summed)

$$u^a = R_a\, x^a\,, \qquad u^a \sim u^a + 2\pi R_a\,, \qquad \tilde{u}_a = \frac{\alpha'}{R_a}\,\tilde{x}_a\,, \qquad \tilde{u}_a \sim \tilde{u}_a + 2\pi\frac{\alpha'}{R_a}\,. \tag{2.36}$$

For the noncompact directions we can take $u^\mu = \sqrt{\alpha'}x^\mu$. We then get

$$\begin{aligned} D_a = \ & \frac{R_a}{\sqrt{\alpha'}}\Big(\frac{\partial}{\partial u^a} - \delta_{ab}\frac{\partial}{\partial \tilde{u}_b}\Big)\,, \quad \bar{D}_a = \ \frac{R_a}{\sqrt{\alpha'}}\Big(\frac{\partial}{\partial u^a} + \delta_{ab}\frac{\partial}{\partial \tilde{u}_b}\Big)\,, \\ \square = \ & \eta^{\mu\nu}\frac{\partial}{\partial u^\mu}\frac{\partial}{\partial u^\nu} + \delta^{ab}\frac{\partial}{\partial u^a}\frac{\partial}{\partial u^b} + \delta_{ab}\,\frac{\partial}{\partial \tilde{u}_a}\frac{\partial}{\partial \tilde{u}_b}\,. \end{aligned} \tag{2.37}$$

2.3 Quadratic action from string field theory

The closed string field with $N = \bar{N} = 1$ takes the form

$$\begin{aligned} |\Psi\rangle = \int [dp]\ \Big(& -\frac{1}{2}e_{ij}(p)\,\alpha_{-1}^i\bar{\alpha}_{-1}^j\,c_1\bar{c}_1 + e(p)\,c_1c_{-1} + \bar{e}(p)\,\bar{c}_1\bar{c}_{-1} \\ & + i\sqrt{\frac{\alpha'}{2}}\,\big(\,f_i(p)\,c_0^+c_1\alpha_{-1}^i + \bar{f}_j(p)\,c_0^+\bar{c}_1\bar{\alpha}_{-1}^j\big)\,|p\rangle\,. \end{aligned} \tag{2.38}$$

We have used $\int[dp]$ to denote the integral over the continuous momenta p_μ and the sum over the discrete momenta p_a and discrete winding w^a so that, for example, $e(p) = e(p_\mu, p_a, w^a)$. The string field has ghost number two: each term includes two ghost oscillators acting on the ghost-number zero state $|p\rangle$. In the above $c_0^\pm = \frac{1}{2}(c_0 \pm \bar{c}_0)$ and we define $b_0^\pm = b_0 \pm \bar{b}_0$, so that $\{c_0^\pm, b_0^\pm\} = 1$. As required, $b_0^-|\Psi\rangle = 0$ because $b_0^-|p\rangle = 0$ and the ghost oscillator c_0^- does not appear in $|\Psi\rangle$. This expansion of the string field features five momentum-space component fields: $e_{ij}, e, \bar{e}, f$, and $\bar{f}$.

We wish to construct the quadratic action, given by

$$(2\kappa^2)\,S^{(2)} = -\frac{2}{\alpha'}\langle\Psi|\,c_0^-Q|\Psi\rangle\ . \tag{2.39}$$

Here Q is the (ghost-number one) BRST operator of the conformal field theory and $\langle\Psi|$ denotes the BPZ conjugate of the string field $|\Psi\rangle$ in (2.38). The computation of $S^{(2)}$ is straightforward[6] and the

[6] We use the inner product $\langle p'|\,c_{-1}\bar{c}_{-1}c_0^-\,c_0^+c_1\bar{c}_1\,|p\rangle = (2\pi)^{n+2d}\,\delta(p-p')$. The BRST operator is $Q = -\frac{\alpha'}{2}\,c_0^+\square + \alpha_0\cdot\big(\alpha_{-1}c_1 + c_{-1}\alpha_1\big) + \bar{\alpha}_0\cdot\big(\bar{\alpha}_{-1}\bar{c}_1 + \bar{c}_{-1}\bar{\alpha}_1\big) - b_0^+(c_{-1}c_1 + \bar{c}_{-1}\bar{c}_1) + \ldots$.

result is

$$(2\kappa^2)\, S^{(2)} = \int [dx\, d\tilde{x}] \Big[\, \frac{1}{4} e_{ij} \Box e^{ij} \,+\, 2\bar{e}\, \Box\, e - f_i\, f^i - \bar{f}_i\, \bar{f}^i \qquad - \, f^i\, \big(\bar{D}^j e_{ij} - 2\, D_i \bar{e}\big) + \bar{f}^j\, \big(D^i e_{ij} \,+ 2\, \bar{D}_j e\big)\Big] . \tag{2.40}$$

Here $\int [dx d\tilde{x}] \,=\, \int d^n x^\mu d^d x^a d^d \tilde{x}_a$ is an integral over all $n + 2d$ coordinates of $\mathbb{R}^{n-1,1} \times T^{2d}$. The definitions of $\Box, D$, and $\bar{D}$ were given in §2.2. All indices are raised and lowered with the metric G^{ij}. The gauge parameter $|\Lambda\rangle$ for the linearised gauge transformations is

$$|\Lambda\rangle = \int [dp]\, \Big(\frac{i}{\sqrt{2\alpha'}} \lambda_i(p)\, \alpha^i_{-1} c_1 - \frac{i}{\sqrt{2\alpha'}} \bar{\lambda}_i(p)\, \bar{\alpha}^i_{-1} \bar{c}_1 + \mu(p)\, c_0^+ \Big) |p\rangle\, . \tag{2.41}$$

The string field Λ has ghost number one and is annihilated by b_0^-. It encodes two vectorial gauge parameters λ_i and $\bar{\lambda}_i$ and one scalar gauge parameter μ. The consistency of the string field theory requires the level-matching conditions (1.7). As a result, the fields $e_{ij}, d, e, \bar{e}, f_i, \bar{f}_i$ and the gauge parameters $\lambda, \bar{\lambda}, \mu$ must be annihilated by Δ (defined in (2.32)):

$$\Delta e_{ij} = \Delta d = \Delta e = \Delta \bar{e} = \Delta f_i = \Delta \bar{f}_i = 0\, , \qquad \Delta \lambda_i = \Delta \bar{\lambda}_i = \Delta \mu = 0\, . \tag{2.42}$$

The quadratic string action (2.39) is invariant under the gauge transformations

$$\delta |\Psi\rangle = Q |\Lambda\rangle\, . \tag{2.43}$$

Expanding this equation using (2.41) and (2.38) gives the following gauge transformations of the component fields:

$$\begin{aligned} \delta e_{ij} &= D_i \bar{\lambda}_j + \bar{D}_j \lambda_i\, , \\ \delta f_i &= -\tfrac{1}{2} \Box\, \lambda_i + D_i \mu\, , \\ \delta \bar{f}_i &= \;\; \tfrac{1}{2} \Box\, \bar{\lambda}_i + \bar{D}_i \mu\, , \\ \delta e &= -\tfrac{1}{2} D^i \lambda_i + \mu\, , \\ \delta \bar{e} &= \;\; \tfrac{1}{2} \bar{D}^i \bar{\lambda}_i + \,\mu\, . \end{aligned} \tag{2.44}$$

We can now introduce fields d and χ by

$$d = \frac{1}{2}\, (e - \bar{e})\, , \qquad \text{and} \quad \chi = \frac{1}{2}\, (e + \bar{e})\, . \tag{2.45}$$

The gauge transformations of d and χ are

$$\begin{aligned} \delta d &= -\frac{1}{4} (D^i \lambda_i + \bar{D}^i \bar{\lambda}_i)\, , \\ \delta \chi &= -\frac{1}{4} (D^i \lambda_i - \bar{D}^i \bar{\lambda}_i) + \mu\, . \end{aligned} \tag{2.46}$$

We can use μ to make the gauge choice

$$\chi = 0\, .$$

After this choice is made, gauge transformations with arbitrary λ and $\bar{\lambda}$ require compensating μ transformations to preserve $\chi = 0$. These do not affect d or e_{ij}, as neither transforms under μ gauge transformations. It does change the gauge transformations of f and $\bar{f}$, but this is of no concern here as these auxiliary fields will be eliminated using their equations of motion. Therefore, we set $e = d$ and $\bar{e} = -d$ in (2.40) and eliminate the auxiliary fields f_i and $\bar{f}_i$, using

$$f_i = -\frac{1}{2}\big(\bar{D}^j e_{ij} - 2D_i\bar{e}\big)\,, \qquad \bar{f}_j = \frac{1}{2}\big(D^i e_{ij} + 2\bar{D}_j e\big)\,. \tag{2.47}$$

The result is the following quadratic action

$$\boxed{(2\kappa^2)\, S^{(2)} = \int [dxd\tilde{x}]\,\Big[\,\frac{1}{4}e_{ij}\Box e^{ij} + \frac{1}{4}(\bar{D}^j e_{ij})^2 + \frac{1}{4}(D^i e_{ij})^2 - 2\,d\,D^i\bar{D}^j e_{ij} \;-\; 4\,d\Box d\,\Big]\,.} \tag{2.48}$$

The gauge transformations generated by λ are

$$\begin{aligned} \delta_\lambda e_{ij} &= \;\bar{D}_j\lambda_i\,, \\ \delta_\lambda d \;&= -\frac{1}{4}D\cdot\lambda\,, \end{aligned} \tag{2.49}$$

and the gauge transformations generated by $\bar{\lambda}$ are

$$\begin{aligned} \delta_{\bar{\lambda}} e_{ij} &= \;D_i\bar{\lambda}_j\,, \\ \delta_{\bar{\lambda}} d \;&= -\frac{1}{4}\bar{D}\cdot\bar{\lambda}\,, \end{aligned} \tag{2.50}$$

where we use a dot to indicate index contraction: $a\cdot b \equiv a^i b_i$. The action is invariant under the $\mathbb{Z}_2$ symmetry

$$e_{ij} \;\to\; e_{ji}\,, \quad D_i \;\to\; \bar{D}_i\,, \quad \bar{D}_i \;\to\; D_i\,, \quad d \;\to\; d\;\,, \tag{2.51}$$

which, as we will discuss later, is related to the invariance of the closed string theory under orientation reversal. For our present purposes we note that this relates the δ_λ and $\delta_{\bar{\lambda}}$ transformations, so that invariance under this $\mathbb{Z}_2$ and δ_λ implies invariance under $\delta_{\bar{\lambda}}$.

A short computation using (2.29) shows that the variation $\delta = \delta_\lambda + \delta_{\bar{\lambda}}$ of the action (2.48) gives

$$(2\kappa^2)\,\delta S^{(2)} = \int [dxd\tilde{x}]\,\Big[\,\frac{1}{2}\,e^{ij}\Delta(\bar{D}_j\lambda_i - D_i\bar{\lambda}_j) + 2d\Delta(D\cdot\lambda - \bar{D}\cdot\bar{\lambda})\Big]\,. \tag{2.52}$$

As expected, the variation vanishes only if we use the constraint $\Delta = 0$. Note that it is sufficient for the invariance of the quadratic action that the parameters satisfy the constraints $\Delta\lambda = \Delta\bar{\lambda} = 0$. We have attempted to relax the constraints by adding extra fields, but have been unable to find a gauge invariant action without constraints.

The action (2.48) and the associated gauge transformations are completely general. They describe the dynamics of fluctuations about the toroidal background with background field E_{ij}. This background field enters the action through the derivatives, as can be seen from (2.24).

2.4 Comparison with conventional actions

We can compare our general free theory (2.48) with the one discussed in §2.1. For this we scale the coordinates by $\sqrt{\alpha'}$ to give them dimensions of length and the derivatives (2.24) become

$$\begin{aligned} D_i &= \partial_i - \tilde{\partial}_i - B_{ik}\tilde{\partial}^k\,, \\ \bar{D}_i &= \partial_i + \tilde{\partial}_i - B_{ik}\tilde{\partial}^k\,, \end{aligned} \tag{2.53}$$

where we defined

$$\tilde{\partial}_i \equiv G_{ik}\tilde{\partial}^k = G_{ik}\frac{\partial}{\partial\tilde{x}_k}\,. \tag{2.54}$$

Then

$$\Box = \partial^2 + \tilde{\partial}^2 + (B_{ij}\tilde{\partial}^j)^2 - 2B_{ij}\partial^i\tilde{\partial}^j\,, \quad \text{and} \quad \Delta = -2\partial_i\tilde{\partial}^i\,. \tag{2.55}$$

Here G_{ij} is used to raise and lower indices and $\partial^2 = G^{ij}\partial_i\partial_j$, etc. For simplicity we will consider backgrounds with $B_{ij} = 0$. The derivatives and laplacians above become

$$D_i = \partial_i - \tilde{\partial}_i\,, \quad \bar{D}_i = \partial_i + \tilde{\partial}_i\,, \quad \Box = \partial^2 + \tilde{\partial}^2\,, \quad \text{and} \quad \Delta = -2\,\partial_i\tilde{\partial}^i\,. \tag{2.56}$$

We decompose the field e_{ij} into its symmetric and antisymmetric parts:

$$e_{ij} = h_{ij} + b_{ij}\,, \quad \text{with} \quad h_{ij} = h_{ji}\,, \quad b_{ij} = -b_{ji}\,. \tag{2.57}$$

The action (2.48) then gives

$$\begin{aligned} (2\kappa^2)\,S^{(2)} = \int [dxd\tilde{x}]\,\Big[\quad & \frac{1}{4}\,h^{ij}\partial^2 h_{ij} + \frac{1}{2}(\partial^j h_{ij})^2 - 2\,d\,\partial^i\partial^j\,h_{ij} - 4\,d\,\partial^2\,d \\ & + \frac{1}{4}\,h^{ij}\tilde{\partial}^2 h_{ij} + \frac{1}{2}(\tilde{\partial}^j h_{ij})^2 + 2\,d\,\tilde{\partial}^i\tilde{\partial}^j\,h_{ij} - 4\,d\,\tilde{\partial}^2\,d \\ & + \frac{1}{4}\,b^{ij}\partial^2 b_{ij} + \frac{1}{2}(\partial^j b_{ij})^2 \\ & + \frac{1}{4}\,b^{ij}\tilde{\partial}^2 b_{ij} + \frac{1}{2}(\tilde{\partial}^j b_{ij})^2 \\ & + (\partial_k h^{ik})(\tilde{\partial}^j b_{ij}) + (\tilde{\partial}^k h_{ik})(\partial_j b^{ij}) - 4\,d\,\partial^i\tilde{\partial}^j b_{ij} \quad \Big]\,. \end{aligned} \tag{2.58}$$

To appreciate this result, we recall the standard action $S_{\rm st}$ for gravity, Kalb-Ramond, and dilaton fields

$$(2\kappa^2)S_{\rm st} = \int dx\sqrt{-{\rm g}}\,e^{-2\phi}\Big[R - \frac{1}{12}H^2 + 4(\partial\phi)^2\Big]\,. \tag{2.59}$$

We expand to quadratic order in fluctuations using ${\rm g}_{ij} = G_{ij} + h_{ij}$, $\phi = d + \frac{1}{4}G^{ij}h_{ij}$, and ${\rm b}_{ij} = B_{ij} + b_{ij}$, with constant G_{ij} and B_{ij}. It follows that $H_{ijk} = \partial_i b_{jk} + \cdots$, and we find

$$(2\kappa^2)\,S_{\rm st}^{(2)} = \int dx\;L[\,h, b, d; \partial\,]\,, \tag{2.60}$$

where

$$\begin{aligned} L[\,h, b, d; \partial\,] = \; & \frac{1}{4}\,h^{ij}\partial^2 h_{ij} + \frac{1}{2}(\partial^j h_{ij})^2 - 2\,d\,\partial^i\partial^j\,h_{ij} - 4\,d\,\partial^2\,d \\ & + \frac{1}{4}\,b^{ij}\partial^2 b_{ij} + \frac{1}{2}(\partial^j b_{ij})^2\,. \end{aligned} \tag{2.61}$$

Comparing with (2.60) we see that our action (2.58) can be written as

$$(2\kappa^2)\,S^{(2)} = \int [dxd\tilde{x}]\,\Big[\,L[\,h,b,d;\partial\,] + L[\,h,b,-d;\tilde{\partial}\,] \\ + (\partial_k h^{ik})(\tilde{\partial}^j b_{ij}) + (\tilde{\partial}^k h_{ik})(\partial_j b^{ij}) - 4\,d\,\partial^i\tilde{\partial}^j b_{ij}\,\Big]\,. \tag{2.62}$$

While in (2.60) the fields depend only on the spacetime coordinates x^i, here they depend on $\tilde{x}$ also. The lagrangian L appears twice, first with ordinary derivatives ∂ and then with dual derivatives $\tilde{\partial}$, together with $d \to -d$. Finally, in the last line we have unusual terms with mixed derivatives. They introduce novel quadratic couplings between the metric and the Kalb-Ramond field! Finally, there is a new coupling of the dilaton to the Kalb-Ramond field.

We now turn to the symmetries. The linearised version of the standard action (2.60) is invariant under linearised diffeomorphisms:

$$\begin{aligned} \delta h_{ij} &= \partial_i\epsilon_j + \partial_j\epsilon_i\,, \\ \delta b_{ij} &= \ 0\,, \\ \delta d \ &= -\tfrac{1}{2}\partial\cdot\epsilon\,, \end{aligned} \tag{2.63}$$

as well as antisymmetric tensor gauge transformations:

$$\begin{aligned} \delta h_{ij} &= \ \ 0\,, \\ \delta b_{ij} &= -\partial_i\tilde{\epsilon}_j + \partial_j\tilde{\epsilon}_i\,, \\ \delta d \ &= \ 0\,. \end{aligned} \tag{2.64}$$

Note that the scalar dilaton $\phi \equiv d + \frac{1}{4}G^{ij}h_{ij}$ is invariant under linearised diffeomorphisms.

The symmetries of the double field theory (2.58) are (2.49) and (2.50). Defining

$$\epsilon_i \equiv \frac{1}{2}(\lambda_i + \bar{\lambda}_i)\,, \qquad \tilde{\epsilon}_i \equiv \frac{1}{2}(\lambda_i - \bar{\lambda}_i)\,, \tag{2.65}$$

we can rewrite these gauge transformations in a more familiar form. The transformations with parameter ϵ are

$$\begin{aligned} \delta h_{ij} &= \ \ \partial_i\epsilon_j + \partial_j\epsilon_i\,, \\ \delta b_{ij} &= -(\tilde{\partial}_i\epsilon_j - \tilde{\partial}_j\epsilon_i)\,, \\ \delta d \ &= -\tfrac{1}{2}\,\partial\cdot\epsilon\,. \end{aligned} \tag{2.66}$$

These give transformations of the same form as the linearised diffeomorphisms (2.63) together with an exotic gauge transformation of b_{ij} in which dual derivatives $\tilde{\partial}$ act on the parameter. The transformations with parameter $\tilde{\epsilon}$ are

$$\begin{aligned} \tilde{\delta} h_{ij} &= \ \ \tilde{\partial}_i\tilde{\epsilon}_j + \tilde{\partial}_j\tilde{\epsilon}_i\,, \\ \tilde{\delta} b_{ij} &= -(\partial_i\tilde{\epsilon}_j - \partial_j\tilde{\epsilon}_i)\,, \\ \tilde{\delta} d \ &= \ \ \tfrac{1}{2}\,\tilde{\partial}\cdot\tilde{\epsilon}\,. \end{aligned} \tag{2.67}$$

Comparing with (2.64), we see Kalb-Ramond gauge transformations with parameter $\tilde{\epsilon}$ together with gravity transformations that are linearised diffeomorphisms with ∂ replaced by $\tilde{\partial}$. Note that this time the scalar dilaton is $\tilde{\phi} \equiv d - \frac{1}{4}G^{ij}h_{ij}$, since this is invariant under linearised dual diffeomorphisms. Also interesting is that the transformation of d under these dual diffeomorphisms is of the same form as the one in (2.66), but with opposite sign. While the Minkowski space theory has a gauge invariant dilaton $\phi = d + \frac{1}{4}h$, there is none in the toroidal theory. We certainly have $\delta\phi = 0$, but $\tilde{\delta}\phi = \tilde{\partial} \cdot \tilde{\epsilon}$. There is no dilaton that is invariant under both ϵ and $\tilde{\epsilon}$ transformations.

3 Cubic action and gauge transformations

In this section we use closed string field theory to compute the cubic interactions for the string field (2.38) together with the gauge transformations with parameter (2.41) to linear order in the fields. The computation is laborious since there are many terms to consider but the techniques are standard in string field theory.

In the action we have kept only the terms with a total number of derivatives (D or $\bar{D}$) less than or equal to two. In the gauge transformations we have kept the terms linear in the fields and which are relevant to an action with two derivatives. This strategy was expected to lead to an action that is exactly gauge invariant to this order, just as it does for string field actions around flat space. The constraint $\Delta = 0$ does not involve terms with different numbers of derivatives so no complication is expected.

The string field theory product used to define the interactions involves a projector. The string fields satisfy the constraint $L_0 - \bar{L}_0 = 0$ and the projector imposes the constraint $L_0 - \bar{L}_0 = 0$ on the product. Such projector should lead to a projector that imposes the constraint $\Delta = 0$ in our field theory products, and thus in our interactions. We discuss this in detail in section 5. As we show there, however, when the fields satisfy the $\Delta = 0$ constraint no additional projectors are needed for the cubic interactions. The projectors are needed in the gauge transformations, in the terms that involve a product of a field and a gauge parameter. In order to avoid cluttering the notation we will leave them implicit. As explained in section 5 the check of gauge invariance to this order is correctly done naively, ignoring the projectors.

The vertex operators for strings on a torus include cocycles that lead to momentum-dependent sign factors in the exact cubic string field theory interactions, and these sign factors should also appear in our cubic double field theory action. These factors are not expected to affect gauge invariance to cubic order. We present the results of this section without cocycle-induced sign factors, but will discuss these further in section 5.

As a check of our results, we used the gauge transformations obtained in section 3.2 to independently construct the cubic term in the action by the Noether method. The result is exactly the same cubic action that we present here. We have also checked that the gravitational sector of the action agrees with that in the standard action (2.59), expanded to cubic order with the help of [33], for fields independent of $\tilde{x}$ and in a gauge in which the metric perturbation is traceless.

3.1 Cubic terms and gauge transformations from CSFT

The string field theory action is non-polynomial and takes the form

$$(2\kappa^2)S = -\frac{2}{\alpha'}\Big[\langle\Psi|c_0^- \,Q|\Psi\rangle + \frac{1}{3}\{\Psi,\Psi\,,\Psi\} + \frac{1}{3\cdot 4}\{\Psi,\Psi\,,\Psi,\Psi\} + \cdots\ \Big]\,. \tag{3.1}$$

Here $\{\Psi,\Psi,\Psi\} = \langle\Psi|c_0^-[\Psi,\Psi]\rangle$ and $\{\Psi,\Psi,\Psi,\Psi\} = \langle\Psi|c_0^-[\Psi,\Psi,\Psi]\rangle$ where $[\cdot,\cdot]$ is the closed string product and $[\cdot,\cdot,\cdot]$ is a triple product of string fields. The higher order terms require the introduction of products of all orders, with relations between them implied by gauge invariance. The closed string products (discussed further in section 5) are graded commutative and therefore symmetric when the entries are even vectors in the state space. The string field Ψ is even. The gauge transformations are

$$\delta_\lambda\Psi = Q\lambda + [\lambda\,,\Psi] + \cdots\,, \tag{3.2}$$

where the dots denote terms with higher powers of the string field Ψ. The computation of the action to cubic order in the string field (2.38) gives:

$$\begin{aligned}(2\kappa^2)\,S = \int[dxd\tilde{x}]\Big[\ &\frac{1}{4}e_{ij}\Box e^{ij} \,+\, 2\bar{e}\,\Box\, e - f_if^i - \bar{f}_i\bar{f}^i -\, f^i\big(\bar{D}^je_{ij} - 2D_i\bar{e}\big) + \bar{f}^j\big(D^ie_{ij} + 2\bar{D}_je\big)\\
&-\frac{1}{8}\,e_{ij}\Big(-(D_ke^{kj})(\bar{D}_le^{il}) \,-(D_ke^{kl})(\bar{D}_le^{ij}) \,-2\,(D^ie_{kl})(\bar{D}^j e^{kl})\\
&\qquad + 2(D^ie_{kl})\,(\bar{D}^le^{kj}) \,+2(D^ke^{il})(\bar{D}^je_{kl})\Big)\\
&+\frac{1}{2}e_{ij}f^i\bar{f}^j - \frac{1}{2}f_if^i\,\bar{e} + \frac{1}{2}\bar{f}_i\bar{f}^i\,e\\
&-\frac{1}{8}\,e_{ij}\Big((D^i\bar{D}^je)\bar{e} - (D^ie)(\bar{D}^j\bar{e}) - (\bar{D}^je)(D^i\bar{e}) + e\,D^i\bar{D}^j\bar{e}\Big)\\
&-\frac{1}{4}\,f^i\Big(e_{ij}\bar{D}^j\bar{e} + \bar{D}^j(e_{ij}\bar{e})\Big) + \frac{1}{4}\,f^i\Big((D_ie)\bar{e} - eD_i\bar{e}\Big)\\
&-\frac{1}{4}\,\bar{f}^j\Big(\,e_{ij}D^ie + D^i(e_{ij}\,e)\Big) + \frac{1}{4}\,\bar{f}^j\Big(\,(\bar{D}_je)\bar{e} - e\,\bar{D}_j\bar{e}\Big)\Big]\,.\end{aligned} \tag{3.3}$$

All fields are assumed to satisfy the constraint $\Delta = 0$. The above action is invariant under the exchanges

$$e_{ij}\ \leftrightarrow\ e_{ji}\,,\quad D_i\ \leftrightarrow\ \bar{D}_i\,,\quad f_i\ \leftrightarrow\ -\bar{f}_i\,,\quad e\ \leftrightarrow\ -\bar{e}\,. \tag{3.4}$$

This discrete symmetry implies we need only concern ourselves with the gauge transformations generated by λ and by μ. Those generated by $\bar{\lambda}$ can be written in terms of the λ ones and the discrete transformations above. For the λ gauge transformations we find, to linear order in the fields,

$$\begin{aligned}\delta_\lambda e_{ij} &= \bar{D}_j\lambda_i \,-\, \frac{1}{4}\Big[\ \lambda^kD_ie_{kj} - (D_i\lambda^k)e_{kj} + D^k(\lambda_ie_{kj}) + \,(D^k\lambda_i)e_{kj} - D^k(\lambda_ke_{ij}) -\, \lambda_kD^ke_{ij}\ \Big]\\
&\quad-\,\frac{1}{4}\Big[\lambda_i\bar{D}_j\bar{e} - (\bar{D}_j\lambda_i)\bar{e}\Big] \,+\, \frac{1}{2}\,\lambda_i\,\bar{f}_j\,,\\
\delta_\lambda e &= -\frac{1}{2}D^i\lambda_i - \frac{1}{4}f^i\lambda_i + \frac{1}{8}\,\big(eD^i\lambda_i + 2(D^ie)\lambda_i\big)\,,\\
\delta_\lambda\bar{e} &= \,\frac{1}{16}\,\big(\bar{e}D^i\lambda_i + 2(D^i\bar{e})\lambda_i\big)\,.\end{aligned} \tag{3.5}$$

We have not written the gauge transformations for the auxiliary fields f_i and $\bar{f}_i$ since they are quite cumbersome and will not be needed. The μ gauge transformations, to linear order in the fields, are

$$\begin{aligned}\delta_\mu e_{ij} &= 0\,,\\ \delta_\mu e &= \mu - \frac{3}{8}\mu e\,,\\ \delta_\mu \bar{e} &= \mu + \frac{3}{8}\mu\bar{e}\,.\end{aligned} \tag{3.6}$$

To preserve the constraint, the variation of any field must be annihilated by Δ. The field-independent terms in the variations meet this requirement as the gauge parameters are in the kernel of Δ. The terms involving a product of a field and a gauge parameter are not guaranteed to satisfy the constraint and a projection is needed. In section 5 we discuss the natural projector $[[\cdot]]$ that satisfies $\Delta[[A]] = 0$ for arbitrary $A(x,\tilde{x})$. All terms linear in the fields in the above gauge transformations include an implicit $[[\ldots]]$ around them. We do not write these brackets here to avoid cluttering.

The closed string field theory predicts a gauge algebra that is quite intricate [3]: the bracket of two gauge transformations is in general a gauge transformation with field dependent structure constants and the gauge algebra only closes on-shell. To lowest nontrivial order we find

$$[\delta_{\lambda_1}, \delta_{\lambda_2}]\Psi = \delta_\Lambda\Psi + (\text{on-shell} = 0\,\text{terms}) \qquad \text{with} \quad \Lambda = [\lambda_2, \lambda_1] + \ldots\,, \tag{3.7}$$

where the dots represent field dependent terms. The product of parameters $[\lambda_2, \lambda_1]$ is antisymmetric under the interchange $1 \leftrightarrow 2$ since the λ's have ghost number one. For gauge parameters λ_1 and λ_2 the computation of the closed string product, keeping the lowest number of derivatives, gives

$$\begin{aligned}\Lambda^i = &\ \frac{1}{2}\Big[(\lambda_2\cdot D)\lambda_1^i - (\lambda_1\cdot D)\lambda_2^i\Big]\\ &+\frac{1}{4}\Big[\ \lambda_1\cdot D^i\lambda_2 - \lambda_2\cdot D^i\lambda_1\Big]\\ &+\frac{1}{4}\Big[\lambda_1^i(D\cdot\lambda_2) - \lambda_2^i(D\cdot\lambda_1)\ \Big] \equiv \{\lambda_2,\lambda_1\}^i\,.\end{aligned} \tag{3.8}$$

In the above, we introduced a bracket $\{\cdot,\cdot\}$ of two vectors, defined by the right hand side. It is the bracket induced by the closed string product and resembles the Lie bracket of vector fields, but does not coincide with it. One can show that ghost number conservation implies that the commutator of two λ transformations does not give a $\bar{\lambda}$ transformation nor does it give a μ-transformation. In (3.8) the projection brackets $[[\ldots]]$ act on the right hand side, since any allowed gauge parameter must be in the kernel of Δ. We have verified the structure of Λ^i in (3.8) by computing explicitly the leading inhomogeneous term in the commutator of two transformations on e_{ij}. The projectors cause no complication.

It is of interest to see if the bracket $\{\lambda_2, \lambda_1\}$ forms a Lie algebra. The first line of (3.8) is the Lie derivative, but the other two lines are exotic. We have found that

$$\{\{\lambda_2,\lambda_1\},\lambda_3\} + \{\{\lambda_3,\lambda_2\},\lambda_1\} + \{\{\lambda_1,\lambda_3\},\lambda_2\} \neq 0\,. \tag{3.9}$$

We have checked that this non-vanishing result occurs even if all products of λ's are in the kernel of Δ. So the failure of the Jacobi identity is not only due to the projectors implicit in the bracket $\{\,,\,\}$. The fact that the Jacobi identity does not hold is a reflection of the homotopy-Lie algebra structure of the string field theory gauge algebra.

Fixing the μ gauge symmetry. We noted in the quadratic theory that the μ symmetry could be used to set $e = d$ and $\bar{e} = -d$ in the action. A similar result holds at the cubic level, as we discuss now. First note that the μ transformations in (3.6) give

$$\begin{aligned} \delta_\mu(e-\bar{e}) &= \ 0 \ - \frac{3}{8}\,\mu(e+\bar{e})\,, \\ \delta_\mu(e+\bar{e}) &= 2\mu - \frac{3}{8}\mu(e-\bar{e})\,. \end{aligned} \tag{3.10}$$

As a result, the following fields

$$\begin{aligned} d &\equiv \frac{1}{2}\,(e-\bar{e}) + \frac{3}{64}\,(e+\bar{e})^2\,, \\ \chi &\equiv \frac{1}{2}\,(e+\bar{e}) + \frac{3}{32}\,(e^2-\bar{e}^2)\,, \end{aligned} \tag{3.11}$$

have transformations in which terms linear in fields vanish:

$$\begin{aligned} \delta_\mu d &= 0\,, \\ \delta_\mu \chi &= \mu\,. \end{aligned} \tag{3.12}$$

We now use μ to set $\chi = 0$. Since

$$\chi = \frac{1}{2}\,(e+\bar{e})\Big(1 + \frac{3}{16}\,(e-\bar{e})\Big)\,, \tag{3.13}$$

the perturbative solution to $\chi = 0$ is

$$e = -\bar{e}\,. \tag{3.14}$$

It then follows from (3.11) that in this gauge

$$d = \frac{1}{2}\,(e-\bar{e})\,, \tag{3.15}$$

and we can take $e = d$ and $\bar{e} = -d$ in evaluating the action and the gauge transformations.

Note that λ gauge transformations now require compensating μ transformations to preserve the gauge $\chi = 0$. Indeed, it follows from

$$(\delta_\mu + \delta_\lambda)\chi = \mu - \frac{1}{4}D\cdot\lambda \,+\, \text{non-linear}\,, \tag{3.16}$$

that we must set $\mu = \frac{1}{4}D\cdot\lambda + \ldots$ and therefore the final λ gauge transformations take the form $\delta_\lambda + \delta_{\mu=\frac{1}{4}D\cdot\lambda+\ldots}$. Since $\delta_\mu e_{ij} = 0$ and $\delta_\mu d = 0$, this only affects the auxiliary fields. Since auxiliary fields will be eliminated, we need not concern ourselves with these compensating gauge transformations.

3.2 Simplifying the gauge transformations

We now turn to simplifying the λ-gauge transformations of e_{ij} and d, dropping all terms of quadratic and higher order in the fields. The field equations for the auxiliary fields f and $\bar{f}$ have non-linear terms, but to the order we are working it suffices to substitute for f and $\bar{f}$ in the gauge transformation (3.5) using the linearised field equations (2.47). From (3.15) we have

$$\delta d = \frac{1}{2}(\delta e - \delta\bar{e})\,. \tag{3.17}$$

In the formulae for δd, and δe_{ij}, we can set $e = d$ and $\bar{e} = -d$. Then (3.5) gives

$$\begin{aligned}
\delta_\lambda e_{ij} = \bar{D}_j\lambda_i \,-\, \frac{1}{4}\Big[\,\lambda^k D_i e_{kj} - (D_i\lambda^k)e_{kj} + \lambda_i D^k e_{kj} +\, 2(D^k\lambda_i)e_{kj} \\
-\,(D\cdot\lambda)e_{ij} - 2\,\lambda_k D^k e_{ij}\,\Big] + \frac{1}{4}\Big[\lambda_i\bar{D}_j d - (\bar{D}_j\lambda_i)d\Big] \,+\, \frac{1}{4}\,\lambda_i\,(D^k e_{kj} + 2\bar{D}_j d)\,, \\
\delta_\lambda d \;= \frac{1}{2}\Big[-\frac{1}{2}D^i\lambda_i - \frac{1}{4}f^i\lambda_i + \frac{1}{8}\,(eD^i\lambda_i + 2(D^i e)\lambda_i) - \frac{1}{16}\,(\bar{e}D^i\lambda_i + 2(D^i\bar{e})\lambda_i)\Big]\,.
\end{aligned} \tag{3.18}$$

Next we look for redefinitions of the fields and gauge parameters. After some manipulation the above transformations can be written as

$$\begin{aligned}
\delta_\lambda e_{ij} =\; & \bar{D}_j\Big(\lambda_i + \frac{3}{4}\lambda_i d\Big) \,+\frac{1}{2}\Big[\,(D_i\lambda^k)e_{kj} -\, (D^k\lambda_i)e_{kj} +\, \lambda_k D^k e_{ij}\,\Big] \\
& +\, D_i\Big(-\frac{1}{4}\lambda^k e_{kj}\Big) - \delta_\lambda(e_{ij}d)\,, \\
\delta_\lambda d \;=\; & -\frac{1}{4}D^i\Big(\lambda_i + \frac{3}{4}\lambda_i d\Big) + \frac{1}{2}(\lambda\cdot D)\,d \\
& -\,\frac{1}{4}\bar{D}^j\Big(-\frac{1}{4}\lambda^k e_{kj}\Big) - \frac{1}{32}\,\delta_\lambda(e_{ij}e^{ij}) - \frac{9}{16}\,\delta_\lambda\, d^2\,.
\end{aligned} \tag{3.19}$$

We redefine the gauge parameter λ_i by taking $\lambda_i + \frac{3}{4}\lambda_i d \;\to\; \lambda_i$, without affecting the remaining terms linear in fields. Moreover, note that the first term on the second line in each of the above transformations can be thought of as linearised transformations with an effective barred parameter $\bar{\lambda} = -\frac{1}{4}\lambda^k e_{kj}$. The $\delta_{\bar{\lambda}}$ transformation with parameter $\bar{\lambda} = -\frac{1}{4}\lambda^k e_{kj}$ leaves the quadratic action invariant, while it changes the cubic action by terms cubic in the fields. In checking the invariance of the quadratic plus cubic action up to terms quadratic in the fields, these $\delta_{\bar{\lambda}}$ transformations constitute a separate symmetry and so need not be included in the λ transformations. We can therefore ignore them and we will do so. We then have

$$\begin{aligned}
\delta_\lambda\Big(\,e_{ij} + e_{ij}d\Big) \;\;=\;\; \bar{D}_j\lambda_i \,+\frac{1}{2}\Big[\,(D_i\lambda^k)e_{kj} -\, (D^k\lambda_i)e_{kj} +\, \lambda_k D^k e_{ij}\,\Big]\,, \\
\delta_\lambda\Big(\,d - \frac{1}{32}\,e_{ij}e^{ij} - \frac{9}{16}\,d^2\Big) = -\frac{1}{4}D\cdot\lambda + \frac{1}{2}(\lambda\cdot D)\,d\,.
\end{aligned} \tag{3.20}$$

We redefine the fields

$$\begin{aligned}
e'_{ij} \;=&\;\; e_{ij} + e_{ij}d\,, \\
d' \;\;=&\;\; d + \frac{1}{32}e_{ij}e^{ij} + \frac{9}{16}d^2\,.
\end{aligned} \tag{3.21}$$

to give primed fields that have simple gauge transformations

$$\begin{aligned} \delta_\lambda e'_{ij} &= \bar{D}_j\lambda_i + \frac{1}{2}\Big[(D_i\lambda^k)e'_{kj} - (D^k\lambda_i)e'_{kj} + \lambda_k D^k e'_{ij} \Big] , \\ \delta_\lambda d' &= -\frac{1}{4}D\cdot\lambda + \frac{1}{2}(\lambda\cdot D)\, d' \quad . \end{aligned} \tag{3.22}$$

After these field redefinitions, it is convenient to drop the primes to simplify notation. We do so in what follows.

3.3 Simplifying the action

We now consider the full action (3.3) and first fix the μ gauge symmetry by setting $e = d$ and $\bar{e} = -d$. We then eliminate the auxiliary fields f and $\bar{f}$ and, after a fair amount of straightforward work, we find

$$\begin{aligned} (2\kappa^2)\, S = \int [dx d\tilde{x}] \Big[\; & \frac{1}{4}e_{ij}\Box e^{ij} + \frac{1}{4}(\bar{D}^j e_{ij})^2 + \frac{1}{4}(D^i e_{ij})^2 - 2\, d\, D^i\bar{D}^j e_{ij} \; - \; 4\, d\,\Box\, d \\ & - \frac{1}{8}\, e_{ij}\Big(-(D_k e^{kl})(\bar{D}_l e^{ij}) - 2\,(D^i e_{kl})(\bar{D}^j e^{kl}) \\ & \qquad + 2(D^i e_{kl})\,(\bar{D}^l e^{kj}) + 2(D^k e^{il})(\bar{D}^j e_{kl})\Big) \\ & + \frac{1}{2} d\,\Big(e_{ij}\,\bar{D}_k\bar{D}^j e^{ik} + e_{ij}\, D_l D^i e^{lj} + \,(D^i e_{ij})^2 + (\bar{D}^j e_{ij})^2\Big) \\ & - \frac{1}{4}e_{ij}\,(D^i\bar{D}^j d)d - \frac{9}{4}e_{ij}(D^i d)(\bar{D}^j d) - \frac{1}{2}\, d^2\,\Box\, d\, . \end{aligned} \tag{3.23}$$

This is the action expected to be invariant under the original gauge transformations (3.18). Since we simplified those gauge transformations by the field redefinitions (3.21) we now perform these same field redefinitions in the action. From (3.21), we set

$$\begin{aligned} e_{ij} &= e'_{ij} - e'_{ij}d' \, . \\ d &= d' - \frac{1}{32}e'_{ij}e'^{ij} - \frac{9}{16}d'^2 \, . \end{aligned} \tag{3.24}$$

to obtain an action in terms of the primed fields. Dropping all primes, the result is

$$\boxed{\begin{aligned} (2\kappa^2) S = \int [dx d\tilde{x}] \Big[\; & \frac{1}{4}e_{ij}\Box e^{ij} + \frac{1}{4}(\bar{D}^j e_{ij})^2 + \frac{1}{4}(D^i e_{ij})^2 - 2\, d\, D^i\bar{D}^j e_{ij} \; - \; 4\, d\,\Box\, d \\ & + \frac{1}{4}\, e_{ij}\Big((D^i e_{kl})(\bar{D}^j e^{kl}) - (D^i e_{kl})\,(\bar{D}^l e^{kj}) - (D^k e^{il})(\bar{D}^j e_{kl})\Big) \\ & + \frac{1}{2} d\,\Big((D^i e_{ij})^2 + (\bar{D}^j e_{ij})^2 + \frac{1}{2}(D_k e_{ij})^2 + \frac{1}{2}(\bar{D}_k e_{ij})^2 + 2e^{ij}(D_i D^k e_{kj} + \bar{D}_j\bar{D}^k e_{ik})\Big) \\ & + 4\, e_{ij} d\, D^i\bar{D}^j d + 4\, d^2\,\Box\, d \; \Big] \, . \end{aligned}} \tag{3.25}$$

The discrete $\mathbb{Z}_2$ symmetry (2.51) we found in the quadratic theory is preserved here. This is essentially manifest for all terms except the e^3 terms, where it takes a small computation to confirm the symmetry. The transformations are written again here for convenience

$$\boxed{\mathbb{Z}_2 \text{ transformations}: \quad e_{ij} \;\to\; e_{ji}\,, \quad D_i \;\to\; \bar{D}_i\,, \quad \bar{D}_i \;\to\; D_i\,, \qquad d \;\to\; d\;.} \tag{3.26}$$

The gauge transformations are those in (3.22)

$$\boxed{\begin{aligned} \delta_\lambda e_{ij} &= \;\bar{D}_j\lambda_i \,+\frac{1}{2}\Big[\,(D_i\lambda^k)e_{kj} -\,(D^k\lambda_i)e_{kj} +\,\lambda_k D^k e_{ij}\,\Big]\,,\\ \delta_\lambda d &= -\frac{1}{4}D\cdot\lambda+\frac{1}{2}(\lambda\cdot D)\,d\;\;. \end{aligned}} \tag{3.27}$$

The discrete symmetry (3.26) of the action S is fundamental to our analysis. It implies that gauge transformations with barred gauge parameters obtained from (3.27) by the discrete symmetry are also invariances of S. The action S then has the appropriate doubled symmetry. For future reference the barred gauge transformations are

$$\begin{aligned} \delta_{\bar\lambda} e_{ij} &= \;D_i\bar\lambda_j \,+\frac{1}{2}\Big[\,(\bar{D}_j\bar\lambda^k)e_{ik} -\,(\bar{D}^k\bar\lambda_j)e_{ik} +\,\bar\lambda_k \bar{D}^k e_{ij}\,\Big]\,,\\ \delta_{\bar\lambda} d &= -\frac{1}{4}\bar{D}\cdot\bar\lambda+\frac{1}{2}(\bar\lambda\cdot\bar{D})\,d\;\;. \end{aligned} \tag{3.28}$$

In all of the above gauge transformations, there is an implict projection $[[\,\cdot\,]]$ to the kernel of Δ for the terms linear in the fields.

As a check of the action S we used the Noether method to construct a cubic term to be added to the quadratic action for which the action is invariant under (3.27), up to terms cubic or higher in the fields. The result was precisely the action S given above. We note that the cubic action can be rewritten in a suggestive way (up to quartic terms) as

$$(2\kappa^2)\,S = \int [dxd\tilde{x}]\;e^{-2d}\,\Big[-\frac{1}{4}K - 2\,e_{ij}D^i\bar{D}^j d + 2(Dd)^2 + 2(\bar{D}d)^2\Big]\,. \tag{3.29}$$

Here $K = K_2 + K_3$, where

$$K_2 = (D^i e_{ij})^2 + (\bar{D}^j e_{ij})^2 + \frac{1}{2}(D^k e_{ij})^2 + \frac{1}{2}(\bar{D}^k e_{ij})^2 + 2e^{ij}(D_iD^k e_{kj} + \bar{D}_j\bar{D}^k e_{ik})\,, \tag{3.30}$$

coincides, up to total derivatives, with the quadratic Lagrangian for e_{ij} and

$$K_3 = -\,e_{ij}\Big(\,(D^i e_{kl})(\bar{D}^j e^{kl}) - (D^i e_{kl})\,(\bar{D}^l e^{kj}) - (D^k e^{il})(\bar{D}^j e_{kl})\Big)\,, \tag{3.31}$$

coincides with the cubic Lagrangian for e_{ij}. This suggests that K may give the leading terms in the expansion of some curvature.

We can now reconsider the algebra of gauge transformations discussed around equation (3.8). Our field redefinitions cause the mixing of the unbarred and barred transformations, so some of the simplicity is lost. Nevertheless the answers are still reasonably compact. The commutation of two gauge transformations with parameters $(\lambda_1, \bar{\lambda}_1)$ and $(\lambda_2, \bar{\lambda}_2)$ is a gauge transformation with parameters $(\Lambda, \bar{\Lambda})$ that to leading order are field independent:

$$\begin{aligned}
\Lambda^i = &\ \frac{1}{2}\Big[\ (\lambda_2 \cdot D + \bar{\lambda}_2 \cdot \bar{D})\, \lambda_1^i - (\lambda_1 \cdot D + \bar{\lambda}_1 \cdot \bar{D})\lambda_2^i\Big] \\
&+ \frac{1}{4}\Big[\ \lambda_1 \cdot D^i \lambda_2 - \lambda_2 \cdot D^i \lambda_1\Big] - \frac{1}{4}\Big[\ \bar{\lambda}_1 \cdot D^i \bar{\lambda}_2 - \bar{\lambda}_2 \cdot D^i \bar{\lambda}_1\Big]\,, \\
\bar{\Lambda}^i = &\ \frac{1}{2}\Big[\ (\lambda_2 \cdot D + \bar{\lambda}_2 \cdot \bar{D})\, \bar{\lambda}_1^i - \ (\lambda_1 \cdot D + \bar{\lambda}_1 \cdot \bar{D})\, \bar{\lambda}_2^i\Big] \\
&- \frac{1}{4}\Big[\ \lambda_1 \cdot \bar{D}^i \lambda_2 - \lambda_2 \cdot \bar{D}^i \lambda_1\Big] + \frac{1}{4}\Big[\ \bar{\lambda}_1 \cdot \bar{D}^i \bar{\lambda}_2 - \bar{\lambda}_2 \cdot \bar{D}^i \bar{\lambda}_1\Big]\ .
\end{aligned} \tag{3.32}$$

The constraint $\Delta = 0$ on the parameters is used in calculating the algebra. The same caveats discussed earlier apply here. The commutator of gauge transformations to all orders in the fields is expected to include field dependent structure constants as well as terms that vanish on-shell. There is an implicit projection $[[\,\cdot]]$ in the above right-hand sides so that $(\Lambda, \bar{\Lambda})$ are in the kernel of Δ. Finally, the brackets $[\,\cdot, \cdot]$ implicit above do not satisfy the Jacobi identity.

3.4 Conventional field theory limits

In this section we examine the gauge transformations in the limits where there is dependence on either just x or just $\tilde{x}$ coordinates and show that we recover the expected results. Interestingly, these two limits require two different sets of field redefinitions and these break the discrete $\mathbb{Z}_2$ symmetry of the theory.

We wish to compare our results with the gauge transformations of the conventional (undoubled) theory of a metric $\mathrm{g}_{ij}(x^k)$, Kalb-Ramond field $\mathrm{b}_{ij}(x^k)$, and a dilaton $\phi(x^k)$. Under diffeomorphisms with parameter ξ^i and antisymmetric gauge transformations with parameter α_i, the first two fields transform as

$$\begin{aligned}
\delta \mathrm{g}_{ij} &= \mathcal{L}_\xi \mathrm{g}_{ij}\,, \\
\delta \mathrm{b}_{ij} &= \mathcal{L}_\xi \mathrm{b}_{ij} + \partial_i \alpha_j - \partial_j \alpha_i\,.
\end{aligned} \tag{3.33}$$

For the dilaton we have

$$\delta\phi = \xi^i \partial_i \phi\,. \tag{3.34}$$

Here $\mathcal{L}_\xi$ is the Lie derivative with respect to ξ^i. For any rank two tensor r_{ij}, the Lie derivative with respect to ξ^i takes the form

$$\mathcal{L}_\xi\, r_{ij} = (\partial_i \xi^k)\, r_{kj} + (\partial_j \xi^k)\, r_{ik} + \xi^k \partial_k r_{ij}\,. \tag{3.35}$$

The above form of the standard diffeomorphisms is all we need to compare with our results. It is interesting, however, to write the transformations more geometrically. We first note that (3.33) can be written as

$$\begin{aligned}\delta \mathrm{g}_{ij} &= \nabla_i \xi_j + \nabla_j \xi_i\,, \\ \delta \mathrm{b}_{ij} &= H_{ijk}\xi^k + \partial_i \tilde{\xi}_j - \partial_j \tilde{\xi}_i\,,\end{aligned} \tag{3.36}$$

where ∇ is the covariant derivative with Levi-Civita connection Γ, H is the field strength

$$H_{ijk} = \partial_i \mathrm{b}_{jk} + \partial_j \mathrm{b}_{ki} + \partial_k \mathrm{b}_{ij}\,, \tag{3.37}$$

and we have defined

$$\xi_i \equiv \mathrm{g}_{ij}\xi^j, \qquad \tilde{\xi}_i \equiv \alpha_i - \mathrm{b}_{ij}\xi^j\,. \tag{3.38}$$

Introducing the field

$$\mathcal{E}_{ij} = \mathrm{g}_{ij} + \mathrm{b}_{ij}\,, \tag{3.39}$$

the transformations can be written as transformations of $\mathcal{E}$:

$$\begin{aligned} &\delta \mathcal{E}_{ij} = \nabla_i \xi_j + \nabla_j \xi_i + H_{ijk}\xi^k + \partial_i \tilde{\xi}_j - \partial_j \tilde{\xi}_i \\ \rightarrow \qquad &\delta \mathcal{E}_{ij} = \hat{\nabla}_i \xi_j + \hat{\nabla}_j \xi_i + \partial_i \tilde{\xi}_j - \partial_j \tilde{\xi}_i\,,\end{aligned} \tag{3.40}$$

where $\hat{\nabla}$ is the derivative for the connection with torsion

$$\hat{\Gamma}^k_{ij} = \Gamma^k_{ij} - \mathrm{g}^{kl} H_{ijl} = \frac{1}{2}\,\mathrm{g}^{kl}\left(\partial_i \mathcal{E}_{lj} + \partial_j \mathcal{E}_{il} - \partial_l \mathcal{E}_{ij}\right). \tag{3.41}$$

The transformation (3.40) encodes nicely the full gauge structure of the fields. For the dilaton transformation (3.34) it is convenient to define a field d by

$$e^{-2\mathrm{d}} \equiv \sqrt{-\mathrm{g}}\, e^{-2\phi}\,. \tag{3.42}$$

Since $\sqrt{-\mathrm{g}}$ is a density we find that $e^{-2\mathrm{d}}$ is also a density:

$$\delta e^{-2\mathrm{d}} = \partial_i (e^{-2\mathrm{d}}\xi^i)\,. \tag{3.43}$$

Returning to the task at hand, we split the fields into constant background fields G, B plus fluctuations

$$\begin{aligned}\mathrm{g}_{ij} &= G_{ij} + h_{ij}\,, \\ \mathrm{b}_{ij} &= B_{ij} + b_{ij}\,.\end{aligned} \tag{3.44}$$

The transformations (3.33) then imply transformations for the fluctuations. A short computation shows that they can be written as

$$\begin{aligned}\delta h_{ij} &= \partial_i \epsilon_j + \partial_j \epsilon_i + \mathcal{L}_\xi h_{ij}\,, \\ \delta b_{ij} &= \partial_i \tilde{\epsilon}_j - \partial_j \tilde{\epsilon}_i + \mathcal{L}_\xi b_{ij}\,,\end{aligned} \tag{3.45}$$

where

$$\epsilon_i = G_{ij}\xi^j, \qquad \tilde{\epsilon}_i = \alpha_i - B_{ij}\xi^j\,. \tag{3.46}$$

Defining as usual the field $\breve{e}_{ij}$ that puts together the two types of fluctuations,

$$\breve{e}_{ij} = h_{ij} + b_{ij}\,, \tag{3.47}$$

we readily find that it transforms as

$$\begin{aligned}\delta\breve{e}_{ij} &= (\partial_i\epsilon_j + \partial_j\epsilon_i) + (\partial_i\tilde{\epsilon}_j - \partial_j\tilde{\epsilon}_i) + \mathcal{L}_\epsilon \breve{e}_{ij} \\ &= (\partial_i\epsilon_j + \partial_j\epsilon_i) + (\partial_i\tilde{\epsilon}_j - \partial_j\tilde{\epsilon}_i) + ((\partial_i\epsilon^k)\breve{e}_{kj} + (\partial_j\epsilon^k)\breve{e}_{ik} + \epsilon^k\partial_k\breve{e}_{ij})\,,\end{aligned} \tag{3.48}$$

where indices are raised and lowered using G_{ij}. This is our final form for the conventional gauge transformations, to be compared with the result arising from the cubic theory we have constructed.

Our analysis requires both unbarred and barred gauge parameters, so we put together the gauge transformations (3.27) and (3.28) to obtain the transformations

$$\begin{aligned}\delta e_{ij} =\; &\bar{D}_j\lambda_i + \frac{1}{2}\Big[\,(D_i\lambda^k)e_{kj} - (D^k\lambda_i)e_{kj} + \lambda_k D^k e_{ij}\,\Big] \\ &+ D_i\bar{\lambda}_j + \frac{1}{2}\Big[\,(\bar{D}_j\bar{\lambda}^k)e_{ik} - (\bar{D}^k\bar{\lambda}_j)e_{ik} + \bar{\lambda}_k\bar{D}^k e_{ij}\,\Big]\,,\end{aligned} \tag{3.49}$$

as well as

$$\delta d = -\frac{1}{4}(D\cdot\lambda + \bar{D}\cdot\bar{\lambda}) + \frac{1}{2}(\lambda\cdot D + \bar{\lambda}\cdot\bar{D})\, d\;\;. \tag{3.50}$$

We can rearrange the former in the suggestive form

$$\begin{aligned}\delta e_{ij} =\; &\bar{D}_j\lambda_i + D_i\bar{\lambda}_j + (D_i\lambda^k)e_{kj} + (\bar{D}_j\bar{\lambda}^k)e_{ik} + \frac{1}{2}\,(\lambda_k D^k + \bar{\lambda}_k\bar{D}^k)e_{ij} \\ &- \frac{1}{2}\,\big(D_i\lambda^k + D^k\lambda_i\big)e_{kj} - \frac{1}{2}\,\big(\bar{D}_j\bar{\lambda}^k + \bar{D}^k\bar{\lambda}_j\big)e_{ik}\,.\end{aligned} \tag{3.51}$$

The first line, as we will see, contains terms that combine naturally to form Lie derivatives. The above transformations are expected to receive corrections of quadratic and higher order in e_{ij}, while those for $\breve{e}_{ij}$ above are exact. The fields e_{ij}, $\breve{e}_{ij}$ are related by non-linear field redefinitions $\breve{e}_{ij} = e_{ij} + O(e^2)$ [30] and next we shall seek such redefinitions to bring the transformations of e_{ij} to the same form as those for $\breve{e}_{ij}$. We then undertake a similar analysis for the T-dual system, and find a different field redefinition is needed.

We now examine the above gauge transformations in two limits. The first is that when fields have no $\tilde{x}^i$ dependence. The second is that when fields have no x^i dependence. It is convenient in both cases to use linear combinations of the gauge parameters:

$$\epsilon_i \equiv \frac{1}{2}(\lambda_i + \bar{\lambda}_i)\,, \qquad \tilde{\epsilon}_i \equiv \frac{1}{2}(\lambda_i - \bar{\lambda}_i)\,. \tag{3.52}$$

We now consider the two possible limits.

3.4.1 Fields with no $\tilde{x}$ dependence.

In this case we can set $\tilde{\partial}$ equal to zero in the derivatives (2.24). It follows then that $D = \bar{D} = \partial$, absorbing $\sqrt{\alpha'}$ into the definition of the coordinates. All indices are raised or lowered with G^{ij} and G_{ij}.

The transformations with parameter ϵ are obtained from (3.51) setting $\lambda_i = \bar{\lambda}_i = \epsilon_i$:

$$\begin{aligned} \delta_\epsilon e_{ij} = &\ \partial_j \epsilon_i + \partial_i \epsilon_j + (\partial_i \epsilon^k) e_{kj} + (\partial_j \epsilon^k) e_{ik} + \epsilon_k \partial^k e_{ij} - \frac{1}{2}(\delta_\epsilon e_i{}^k) e_{kj} - \frac{1}{2}(\delta_\epsilon e^k{}_j) e_{ik} \\ = &\ \partial_j \epsilon_i + \partial_i \epsilon_j + (\partial_i \epsilon^k) e_{kj} + (\partial_j \epsilon^k) e_{ik} + \epsilon_k \partial^k e_{ij} - \frac{1}{2}\delta_\epsilon (e_i{}^k e_{kj})\,. \end{aligned} \tag{3.53}$$

We therefore have

$$\delta_\epsilon \Big(e_{ij} + \frac{1}{2} e_i{}^k e_{kj} \Big) = \ \partial_j \epsilon_i + \partial_i \epsilon_j + (\partial_i \epsilon^k) e_{kj} + (\partial_j \epsilon^k) e_{ik} + \epsilon_k \partial^k e_{ij}\,. \tag{3.54}$$

It follows that the field

$$e^+_{ij} \equiv e_{ij} + \frac{1}{2} e_i{}^k e_{kj}\,, \tag{3.55}$$

transforms as

$$\delta_\epsilon \, e^+_{ij} = (\partial_i \epsilon_j + \partial_j \epsilon_i) + \mathcal{L}_\epsilon e^+_{ij}\,, \tag{3.56}$$

up to terms of order $(e^+_{ij})^2$.

The $\tilde{\epsilon}$-gauge transformations are obtained from (3.51) setting $\lambda_i = -\bar{\lambda}_i = \tilde{\epsilon}_i$:

$$\tilde{\delta}_{\tilde{\epsilon}} \, e_{ij} = \ \partial_j \tilde{\epsilon}_i - \partial_i \tilde{\epsilon}_j - \frac{1}{2}(\delta_{\tilde{\epsilon}} e_i{}^k) e_{kj} - \frac{1}{2}(\delta_{\tilde{\epsilon}} e^k{}_j) e_{ik}\,, \tag{3.57}$$

so that

$$\tilde{\delta}_{\tilde{\epsilon}} \, e^+_{ij} = \ \partial_j \tilde{\epsilon}_i - \partial_i \tilde{\epsilon}_j\,, \tag{3.58}$$

up to terms of order $(e^+_{ij})^2$. The transformations for e^+_{ij} are precisely the standard gauge transformations (3.48), up to higher order terms. This is what we wanted to show.

Note that the full field (background plus fluctuation) with natural gauge transformations is

$$\mathcal{E}_{ij} \equiv E_{ij} + e^+_{ij} + \text{cubic terms} = G_{ij} + B_{ij} + e_{ij} + \frac{1}{2} e_i{}^k e_{kj} + \text{cubic terms}\,, \tag{3.59}$$

so that $\check{e}_{ij} = e_{ij} + \frac{1}{2} e_i{}^k e_{kj}$, up to cubic terms. We now show that $\mathcal{E}$ has the expected gauge transformations. Indeed, for $\tilde{\epsilon}$ transformations (up to terms of quadratic in fields) we have

$$\tilde{\delta}_{\tilde{\epsilon}} \, \mathcal{E}_{ij} = \ \partial_j \tilde{\epsilon}_i - \partial_i \tilde{\epsilon}_j\,. \tag{3.60}$$

The ϵ transformations are a little more intricate. We first compute the Lie derivative of $\mathcal{E}$:

$$\begin{aligned} \mathcal{L}_\epsilon \mathcal{E}_{ij} &= (\partial_i \epsilon^k)\, \mathcal{E}_{kj} + (\partial_j \epsilon^k)\, \mathcal{E}_{ik} + \epsilon^k \partial_k \mathcal{E}_{ij} \\ &= (\partial_i \epsilon^k)\, E_{kj} + (\partial_j \epsilon^k)\, E_{ik} + \mathcal{L}_\epsilon e^+_{ij} \\ &= \partial_i \epsilon_j + \partial_j \epsilon_i - \partial_i (\, B_{jk} \epsilon^k) + \partial_j (B_{ik} \epsilon^k) + \mathcal{L}_\epsilon e^+_{ij} \\ &= \delta_\epsilon \mathcal{E}_{ij} + \tilde{\delta}_{B\epsilon} \mathcal{E}_{ij}\,, \end{aligned} \tag{3.61}$$

where we used (3.56), noted that $\delta_\epsilon \mathcal{E}_{ij} = \delta_\epsilon e^+_{ij}$, and recognised the presence of a $\tilde{\delta}$ transformation with parameter $\tilde{\epsilon}_i = B_{ij} \epsilon^j$. We thus have a symmetry $\bar{\delta}_\epsilon$ for which the transformation of $\mathcal{E}$ is through the Lie derivative (up to terms quadratic in fields):

$$\bar{\delta}_\epsilon \mathcal{E}_{ij} \equiv (\delta_\epsilon + \tilde{\delta}_{B\epsilon}) \mathcal{E}_{ij} = \mathcal{L}_\epsilon \mathcal{E}_{ij}\,. \tag{3.62}$$

The gauge transformation of d is obtained from (3.50). For the δ_ϵ transformations ($\lambda = \bar{\lambda} = \epsilon$) we find, up to terms quadratic in fields,

$$\delta_\epsilon d = -\frac{1}{2}\,\partial \cdot \epsilon + \epsilon \cdot \partial d \,. \tag{3.63}$$

This can be rewritten as

$$\delta_\epsilon e^{-2d} = \partial_i (e^{-2d} \epsilon^i)\,, \tag{3.64}$$

and agrees with (3.43) if d is the same as d, up to terms cubic in the fields. A short calculation shows that $\tilde{\delta}_{\tilde{\epsilon}} d = 0$, as would be expected.

3.4.2 Fields with no x dependence.

The configuration dual to the one considered in §3.4.1 has fields independent of x^a. In order to avoid the complication of indices running over non-compact directions and toroidal directions we will consider the case in which there is no x dependence at all; neither on the non-compact x^μ nor on the toroidal x^a. We will simplify further by only considering the transformations with parameters λ_a and $\bar{\lambda}_a$ and the components e_{ab} of e_{ij}.

For fields that do not depend on x, the derivative ∂_i vanishes and the derivatives (2.24), absorbing $\sqrt{\alpha'}$ into the definition of the coordinates, take the form

$$D_a = -\,\hat{E}_{ac}\,\tilde{\partial}^c\,, \qquad \bar{D}_a = \;\hat{E}_{ca}\,\tilde{\partial}^c\,. \tag{3.65}$$

We expect from T-duality that the theory based on $\tilde{x}$ coordinates sees the dual background $\hat{E}'_{ab} = \hat{G}'_{ab} + \hat{B}'_{ab} = \hat{E}^{-1}_{ab}$. As we will see later, the dual metric $\hat{G}'$ is related to the original metric $\hat{G}$ by

$$\hat{G}'^{-1} = \hat{E}\,\hat{G}^{-1}\,\hat{E}^t = \hat{E}^t\,\hat{G}^{-1}\,\hat{E}\,. \tag{3.66}$$

Note that $\hat{G}'^{-1}$ naturally has lower indices, just like $\hat{E}$ does. For example, from the above we see that $(\hat{G}'^{-1})_{ab} = \hat{E}_{ac}\hat{G}^{cd}\,\hat{E}_{bd}$. Thus we use $\hat{G}'^{-1}$ to *lower* indices of primed objects:

$$A'_a \equiv (\hat{G}'^{-1})_{ab} A'^b\,. \tag{3.67}$$

The fields and gauge parameters appropriate here are field redefinitions of the original fields and gauge parameters whose forms are suggested by T-duality transformations. We introduce the fluctuation field

$$e'^{ab} \equiv -(\hat{E}^{-1})^{ac}\,e_{cd}\,(\hat{E}^{-1})^{db}\,. \tag{3.68}$$

Note that this field has upper indices. In order to compute the gauge transformations we use

$$\delta e'^{ab} = -(\hat{E}^{-1})^{ac}\,\delta e_{cd}\,(\hat{E}^{-1})^{db}\,. \tag{3.69}$$

For the gauge parameters we introduce new primed ones through the relations

$$\lambda_a = -\,\hat{E}_{ab}\,\lambda'^b\,, \qquad \bar{\lambda}_a = \;\hat{E}_{ba}\,\bar{\lambda}'^b\,. \tag{3.70}$$

All other gauge parameters will be taken to vanish.

Consider first the dilaton transformations (3.50), where indices, of course, are contracted with the original metric $\hat{G}^{-1}$. We write this out explicitly

$$\begin{aligned}\delta\, d &= -\frac{1}{4}\hat{G}^{ab}\big(D_a\lambda_b + \bar{D}_a\bar{\lambda}_b\big) + \frac{1}{2}\hat{G}^{ab}\big(\lambda_a D_b + \bar{\lambda}_a\bar{D}_b\big)\, d \\ &= -\frac{1}{4}\hat{G}^{ab}\big(\hat{E}_{ac}\tilde{\partial}^c\,\hat{E}_{bd}\lambda'^d + \hat{E}_{ca}\tilde{\partial}^c\hat{E}_{db}\bar{\lambda}'^d\big) + \frac{1}{2}\hat{G}^{ab}\big(\hat{E}_{ac}\lambda'^c\hat{E}_{bd}\tilde{\partial}^d + \hat{E}_{ca}\bar{\lambda}'^c\hat{E}_{db}\tilde{\partial}^d\big)\, d\,, \\ &= -\frac{1}{4}(\hat{G}'^{-1})_{cd}\big(\tilde{\partial}^c\,\lambda'^d + \tilde{\partial}^c\bar{\lambda}'^d\big) + \frac{1}{2}(\hat{G}'^{-1})_{cd}\big(\lambda'^c\tilde{\partial}^d + \bar{\lambda}'^c\tilde{\partial}^d\big)\, d\,,\end{aligned} \tag{3.71}$$

where we made use of (3.65) and (3.70) to obtain the second line and (3.66) to obtain the last line. This can now be rewritten as

$$\delta d = -\frac{1}{4}\,\tilde{\partial}\cdot[\lambda' + \bar{\lambda}'] + \frac{1}{2}\,[\lambda' + \bar{\lambda}']\cdot\tilde{\partial}d\ \ , \tag{3.72}$$

where indices are contracted with $\hat{G}'^{-1}$. Taking $\bar{\lambda}' = \lambda' = \epsilon'$, this gives

$$\delta d = -\frac{1}{2}\,\tilde{\partial}\cdot\epsilon' + \epsilon'\cdot\tilde{\partial}d\ \ , \tag{3.73}$$

which can be rewritten as

$$\delta e^{-2d} = \tilde{\partial}\cdot(e^{-2d}\epsilon')\,, \tag{3.74}$$

and is of the same form as (3.43). Gauge transformations $\tilde{\delta}$ of the dilaton with $\lambda = -\bar{\lambda}'$ vanish on account of (3.72).

Let us now turn to the gauge transformations (3.51) of e_{ij} where, again, all indices are raised with G^{ij}. We rewrite this result with lower-indexed fields and derivatives and explicit G^{-1} factors:

$$\begin{aligned}\delta e_{ab} \equiv\ & \bar{D}_b\lambda_a\ + D_a\bar{\lambda}_b + \hat{G}^{cd}(D_a\lambda_d)e_{cb} + \hat{G}^{cd}(\bar{D}_b\bar{\lambda}_d)e_{ac} + \frac{1}{2}\,\hat{G}^{cd}(\lambda_c D_d + \bar{\lambda}_c\bar{D}_d)e_{ab} \\ & -\frac{1}{2}\,\hat{G}^{cd}\big(D_a\lambda_d + D_d\lambda_a\big)e_{cb} - \frac{1}{2}\,\hat{G}^{cd}\big(\bar{D}_b\bar{\lambda}_d +\ \bar{D}_d\bar{\lambda}_b\big)e_{ac}\,.\end{aligned} \tag{3.75}$$

Our task now is to manipulate the right hand side above. We replace D and $\bar{D}$ by the explicit forms in (3.65), write the gauge parameters in terms of the primed gauge parameters in (3.70), simplify using (3.66), and finally evaluate (3.69). This takes some effort, but the result is relatively simple:

$$\begin{aligned}\delta e'^{ab} =\ & \tilde{\partial}^b\lambda'^a\ + \tilde{\partial}^a\bar{\lambda}'^b\ +\ (\tilde{\partial}^a\lambda'_c)e'^{cb} + (\tilde{\partial}^b\bar{\lambda}'_c)e'^{ac}\ + \frac{1}{2}\,(\lambda'_c\tilde{\partial}^c + \bar{\lambda}'_c\tilde{\partial}^c)e'^{ab} \\ & -\frac{1}{2}\,(\hat{G}'^{-1})_{cd}\Big[\big(\tilde{\partial}^a\lambda'^c + \tilde{\partial}^c\lambda'^a\big)\,e'^{db} + e'^{ac}\big(\tilde{\partial}^b\bar{\lambda}'^d + \tilde{\partial}^p\bar{\lambda}'^d\big)\Big]\,.\end{aligned} \tag{3.76}$$

We first take the case when $\lambda' = \bar{\lambda}' = \epsilon'$. We find

$$\delta e'^{ab} =\ \tilde{\partial}^b\epsilon'^a\ + \tilde{\partial}^a\epsilon'^b\ +\ (\tilde{\partial}^a\epsilon'_c)e'^{cb} + (\tilde{\partial}^b\epsilon'_c)e'^{ac}\ + \epsilon'_c\tilde{\partial}^c\,e'^{ab}\ - \frac{1}{2}\,\delta\big(e'^{ac}\,(\hat{G}'^{-1})_{cd}e'^{db}\big)\,, \tag{3.77}$$

which gives

$$\delta\Big(e'^{ab} + \frac{1}{2}\,e'^{ac}\,(\hat{G}'^{-1})_{cd}e'^{db}\Big)\ =\ \tilde{\partial}^b\epsilon'^a\ + \tilde{\partial}^a\epsilon'^b\ +\ (\tilde{\partial}^a\epsilon'_c)e'^{cb} + (\tilde{\partial}^b\epsilon'_c)e'^{ac}\ + \epsilon'_c\tilde{\partial}^c\,e'^{ab}\ \ . \tag{3.78}$$

Note that the tilde derivatives naturally have the index up. The parameters ϵ' naturally have the index down, just like the coordinates, so that an infinitesimal diffeomorphism takes the form $\tilde{x}'_a = \tilde{x}_a + \epsilon'_a$.

For the case $\bar{\lambda}' = -\lambda' = -\tilde{\epsilon}'$, equation (3.76) gives

$$\tilde{\delta} e'^{ab} = \tilde{\partial}^b \tilde{\epsilon}'^a - \tilde{\partial}^a \tilde{\epsilon}'^b - \frac{1}{2}\delta\Big(e'^{ac}\,(\hat{G}'^{-1})_{cd} e'^{db}\Big)\,, \tag{3.79}$$

so that we now have

$$\tilde{\delta}\Big(e'^{ab} + \frac{1}{2}\,e'^{ac}\,(\hat{G}'^{-1})_{cd} e'^{db}\Big) = \tilde{\partial}^b \tilde{\epsilon}'^a - \tilde{\partial}^a \tilde{\epsilon}'^b\,. \tag{3.80}$$

We define

$$\bar{e}^{ab} \equiv e'^{ab} + \frac{1}{2} e'^{ac}\,(\hat{G}'^{-1})_{cd} e'^{db}\,, \tag{3.81}$$

so that, to this order, our gauge transformations take the form

$$\begin{aligned} \tilde{\delta}\bar{e}^{ab} &= \tilde{\partial}^b \tilde{\epsilon}'^a - \tilde{\partial}^a \tilde{\epsilon}'^b\,, \\ \delta\bar{e}^{ab} &= \tilde{\partial}^b \epsilon'^a + \tilde{\partial}^a \epsilon'^b + (\tilde{\partial}^a \epsilon'_c)\bar{e}^{cb} + (\tilde{\partial}^b \epsilon'_c)\bar{e}^{ac} + \epsilon'_c \tilde{\partial}^c\, \bar{e}^{ab}\,. \end{aligned} \tag{3.82}$$

These are the expected transformations.

To give a clearer interpretation we now introduce a field that incorporates the background and the fluctuation. If we denote by $\hat{E}^{ab}$ the inverse background field $\hat{E}^{ab} = \{\hat{E}^{-1}\}$ we now define

$$\mathcal{E}^{ab} \equiv \hat{E}^{ab} + \bar{e}^{ab} + O(e'^3)\,. \tag{3.83}$$

We now show that this has the expected gauge transformations, up to terms of order e'^2. We clearly have

$$\tilde{\delta}\mathcal{E}^{ab} = \tilde{\partial}^b \tilde{\epsilon}'^a - \tilde{\partial}^a \tilde{\epsilon}'^b\,. \tag{3.84}$$

Next, we aim to write $\delta\bar{e}^{ab} = \delta\mathcal{E}^{ab}$ in terms of a Lie derivative. The Lie derivative of $\mathcal{E}^{ab}$ follows from the tensorial transformation

$$\mathcal{E}'^{ab}(\tilde{x}') = \frac{\partial \tilde{x}_c}{\partial \tilde{x}'_a}\frac{\partial \tilde{x}_d}{\partial \tilde{x}'_b}\,\mathcal{E}^{cd}(\tilde{x})\,. \tag{3.85}$$

The result is

$$\mathcal{L}_{\epsilon'}\mathcal{E}^{ab} = (\tilde{\partial}^a \epsilon'_c)\mathcal{E}^{cb} + (\tilde{\partial}^b \epsilon'_c)\mathcal{E}^{ac} + \epsilon'_c \tilde{\partial}^c\, \mathcal{E}^{ab}\,. \tag{3.86}$$

Using (3.83) this gives:

$$\mathcal{L}_{\epsilon'}\mathcal{E}^{ab} = (\tilde{\partial}^a \epsilon'_c)\hat{E}^{cb} + (\tilde{\partial}^b \epsilon'_c)\hat{E}^{ac} + (\tilde{\partial}^a \epsilon'_c)\bar{e}^{cb} + (\tilde{\partial}^b \epsilon'_c)\bar{e}^{ac} + \epsilon'_c \tilde{\partial}^c\, \bar{e}^{ab}\,. \tag{3.87}$$

Noting that $\hat{E}^{ab} = \hat{G}'^{ab} + \hat{B}'^{ab}$, that $\hat{G}'$ raises indices, and using (3.82), we find

$$\delta\mathcal{E}^{ab} = \delta\bar{e}^{ab} = \mathcal{L}_{\epsilon'}\mathcal{E}^{ab} - \tilde{\partial}^a(\epsilon'_c \hat{B}^{cb}) + \tilde{\partial}^b(\epsilon'_c \hat{B}^{ca})\,. \tag{3.88}$$

Since the last two terms give a symmetry of the form (3.84), the theory contains a gauge symmetry

$$\delta\mathcal{E}^{ab} = \mathcal{L}_{\epsilon'}\mathcal{E}^{ab}\,. \tag{3.89}$$

We conclude by comparing the field redefinition used here to that used in the absence of $\tilde{x}$ dependence, namely $e_{ij} \to e^{+}_{ij}$ in (3.55). For this purpose it is convenient to re-express the present field redefinition (3.81)

$$e'^{ab} \to e'^{ab} + \frac{1}{2} e'^{ac} \, (\hat{G}'^{-1})_{cd} e'^{db} \,. \tag{3.90}$$

in terms of the lower-indexed field e_{ab}. For this we use (3.68), which gives $e_{cd} = -\hat{E}_{ce} \, e'^{ef} \, \hat{E}_{fd}$. Thus multiplying (3.90) from the left and from the right by $\hat{E}$, we find

$$e_{ab} \;\to\; e_{ab} - \frac{1}{2} e_{ac} \, (\hat{E}^{-1} \hat{G}'^{-1} \hat{E}^{-1})^{cd} e_{db} \,. \tag{3.91}$$

Using (3.66) we see that

$$e_{ab} \to e^{-}_{ab} \equiv e_{ab} - \frac{1}{2} e_{ac} (\hat{G}^{-1} \, \hat{E}^{t} \hat{E}^{-1})^{cd} \, e_{db} \,. \tag{3.92}$$

The above shows that the field redefinition $e'^{ab} \to \bar{e}^{ab}$ is equivalent to $e_{ab} \to e^{-}_{ab}$. If $\hat{B}_{ab} = 0$ we find $e^{-}_{ab} = e_{ab} - \frac{1}{2} e_{a}{}^{c} \, e_{cb}$, an expression that differs by a crucial sign from $e^{+}_{ij} = e_{ij} + \frac{1}{2} e_{i}{}^{k} \, e_{kj}$.

Note that the field redefinition $e \to e^{+}$ needed to bring the ϵ transformations to the form of x-diffeomorphisms (in the $\tilde{x}$-independent case) differs from the field redefinition $e \to e^{-}$ needed to bring the $\tilde{\epsilon}$ transformations to the form of $\tilde{x}$-diffeomorphisms (in the x-independent case). While our symmetries contain both x-diffeomorphisms and $\tilde{x}$-diffeomorphisms in certain limits, it is not clear how, or even if, they fit together to form diffeomorphisms of the doubled torus.

4 T-duality of the action

We have written a field theory action (3.25) that represents the dynamics of certain fluctuations (e_{ij} and d) about the background E_{ij}. T-duality states that the closed string physics around backgrounds E and E' related by an $O(d,d,\mathbb{Z})$ transformation are identical. In the string field theory context this was proven in [2] by showing that the string field theories formulated around E and E' are equivalent. In fact these theories are related by a homogeneous field redefinition. This field redefinition does not mix fields at different mass levels; on a given field it shuffles momenta and winding, as well as the various polarizations. For this reason, it is to be expected that our construction, which only keeps the $N = \bar{N} = 1$ fields, should have a T-duality symmetry. In this section we prove that T-duality is a property of the action we have constructed. In string field theory, there are cocycle-induced sign factors in the T-duality transformations [2].[7] Our cubic action does not include the momentum dependent sign factors that arise from cocycles and so our T-duality transformations do not include such factors either. As we discuss in Section 5, such sign factors may be needed in some circumstances and could affect the duality transformations.

We also establish that the action is invariant under the background change $B_{ij} \to -B_{ij}$. This discrete symmetry is not part of the group of $O(d,d,\mathbb{Z})$ symmetries, but plays an important role in the theory. We conclude by discussing a natural generalization of the Buscher rules that may describe T-duality transformations of toroidal backgrounds that fail to have $U(1)$ isometries due to explicit

[7] See [32] for a review of the role of cocycles in T-duality in the first-quantized formalism.

dependence on both coordinates and dual coordinates of the tori. Again, this discussion is modulo cocycle-induced sign factors.

4.1 Duality transformations

We begin by reviewing a few properties of duality transformations. The group elements $g \in O(D, D; \mathbb{Z})$ are $2D \times 2D$ matrices of integers that leave the metric η invariant:

$$g^t \eta g = \eta \,, \qquad \eta = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} . \tag{4.1}$$

One readily sees that $\det g = \pm 1$.

Our indices i run over $D = n + d$ values, so that the coordinates x^i split into n non-compact directions x^μ and d compact ones x^a. If $n = 0$ and all dimensions are compact, then the doubled torus has $2D$ periodic coordinates $x^i, \tilde{x}_i$ transforming in the fundamental representation of $O(D, D; \mathbb{Z})$. If there are n non-compact directions, we shall be interested in the $O(d, d; \mathbb{Z})$ subgroup of $O(D, D; \mathbb{Z})$ that preserves x^μ and which acts only on the $2d$ periodic coordinates $x^a, \tilde{x}_a$. It is this $O(d, d; \mathbb{Z})$ subgroup that is a symmetry of the string theory, but it will be useful to represent its action in terms of the $2D \times 2D$ matrix g.

As in §2.2, we write $E = G + B$, with $D \times D$ matrices $E = \{E_{ij}\}$, $G = \{G_{ij}\}$, and $B = \{B_{ij}\}$. We also use $G^{-1} = \{G^{ij}\}$. If we write the $2D \times 2D$ matrix

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} , \tag{4.2}$$

the group action on the background is

$$E' = g(E) = (aE + b)(cE + d)^{-1} . \tag{4.3}$$

We emphasize that we restrict ourselves to matrices g in the $O(d, d; \mathbb{Z})$ subgroup of $O(D, D; \mathbb{Z})$. This means that explicitly we have the $D \times D$ matrices:

$$a = \begin{pmatrix} \hat{a} & 0 \\ 0 & 1 \end{pmatrix} , \quad b = \begin{pmatrix} \hat{b} & 0 \\ 0 & 0 \end{pmatrix} , \quad c = \begin{pmatrix} \hat{c} & 0 \\ 0 & 0 \end{pmatrix} , \quad d = \begin{pmatrix} \hat{d} & 0 \\ 0 & 1 \end{pmatrix} , \tag{4.4}$$

where $\hat{a}, \hat{b}, \hat{c}$, and $\hat{d}$ are $d \times d$ matrices such that

$$\hat{g} = \begin{pmatrix} \hat{a} & \hat{b} \\ \hat{c} & \hat{d} \end{pmatrix} \in O(d, d, \mathbb{Z}) . \tag{4.5}$$

(We use hats for $d \times d$ matrices.) It is straightforward to verify that if $\hat{g} \in O(d, d, \mathbb{Z})$ then $g \in O(D, D, \mathbb{Z})$. The background E is a matrix of the form

$$E = \begin{pmatrix} \hat{E} & 0 \\ 0 & \eta \end{pmatrix} , \quad \text{with} \quad \hat{E} = \hat{G} + \hat{B} = [\hat{E}_{ab}] \quad \text{and} \quad \eta = [\eta_{\mu\nu}] . \tag{4.6}$$

It follows from the transformation in (4.3) that

$$E' = \begin{pmatrix} \hat{E}' & 0 \\ 0 & \eta \end{pmatrix} , \quad \text{with} \quad \hat{E}' = (\hat{a}\hat{E} + \hat{b})(\hat{c}\hat{E} + \hat{d})^{-1} . \tag{4.7}$$

This is the expected transformation of the background metric: the background $\hat{E}$ in the torus is transformed by an element of $O(d, d, \mathbb{Z})$ while the Minkowski background is left unchanged.

It is a familiar result that the non-linear transformation (4.3) of E becomes the linear transformation of the $2D \times 2D$ matrix $\mathcal{H}$ defined in (2.17):

$$\mathcal{H}(E') = g\,\mathcal{H}(E) g^t \,. \tag{4.8}$$

It is useful to introduce the $D \times D$ matrices M (written as $M_i{}^j$) and $\bar{M}$ (written as $\bar{M}_i{}^j$) defined by the relations

$$\begin{aligned} M \equiv d^t - \, E\, c^t \, &= \begin{pmatrix} \hat{d}^t - \hat{E}\, \hat{c}^t & 0 \\ 0 & 1 \end{pmatrix} , \\ \bar{M} \equiv d^t + E^t c^t &= \begin{pmatrix} \hat{d}^t + \hat{E}^t\, \hat{c}^t & 0 \\ 0 & 1 \end{pmatrix} . \end{aligned} \tag{4.9}$$

The matrices M and $\bar{M}$ control the transformation of the metric G obtained from (4.3) by splitting E' into symmetric and antisymmetric parts, $E' = G' + B'$. Indeed, equation (4.10) in [2] gives

$$\begin{aligned} (\hat{d} + \hat{c}\hat{E})^t \; \hat{G}' \; (\hat{d} + \hat{c}\hat{E}) &= \hat{G} \,, \\ (\hat{d} - \hat{c}\hat{E}^t)^t \, \hat{G}' \, (\hat{d} - \hat{c}\hat{E}^t) &= \hat{G} \,. \end{aligned} \tag{4.10}$$

These relations, together with (4.9) quickly lead to

$$\begin{aligned} G^{-1} &= (\bar{M}^t)^{-1} \, G'^{-1} \, \bar{M}^{-1} \,, \\ G^{-1} &= (M^t)^{-1} \, G'^{-1} \, M^{-1} \,. \end{aligned} \tag{4.11}$$

Two more identities from [2] (eqns. (4.19)) are useful to us:

$$\begin{aligned} \hat{b}^t - \hat{E}\hat{a}^t &= -(\hat{d}^t - \hat{E}\hat{c}^t)\hat{E}' \,, \\ \hat{b}^t + \hat{E}^t\hat{a}^t &= \;\; (\hat{d}^t + \hat{E}^t\hat{c}^t)\hat{E}'^t \,. \end{aligned} \tag{4.12}$$

In terms of the $D \times D$ matrices the above relations give,

$$\begin{aligned} b^t - Ea^t &= -ME' \,, \\ b^t + E^t a^t &= \;\; \bar{M}\, E'^t \,. \end{aligned} \tag{4.13}$$

Finally, a perturbation of the background $E + \delta E$ transforms to $E' + \delta E'$ where

$$\delta E' = M^{-1} \delta E (\bar{M}^t)^{-1} \,, \tag{4.14}$$

so that

$$\delta E_{ij} = M_i{}^k \, \bar{M}_j{}^l \, \delta E'_{kl} \;\; . \tag{4.15}$$

4.2 Duality invariance

We will begin with the action (3.25) written around a background E and with fields (e_{ij}, d) collectively denoted by Ψ. Setting $2\kappa^2 = 1$ we have

$$S(E,\Psi) = \int dx^\mu \, d\mathbb{X}\; \mathcal{L}\left[D_k, \bar{D}_l, G^{-1}\,;\; e_{ij}(x^\mu,\mathbb{X}), d(x^\mu,\mathbb{X})\right]. \tag{4.16}$$

Here $\mathbb{X}$ is a $2d$-column vector of coordinates

$$\mathbb{X} \equiv \begin{pmatrix} \tilde{x}_a \\ x^a \end{pmatrix}, \tag{4.17}$$

and $\int d\mathbb{X} \equiv \int d\tilde{x}dx$. The action (4.16) is constructed from lower-index derivatives $D_i, \bar{D}_j$ and the lower-indexed e_{ij} fields (together with d) with all index contractions using the metric G^{-1}. The action depends on the background E through G^{-1} and the derivatives $D, \bar{D}$ (see (2.24)).

We will establish equivalence between the theory on the background E and the theory formulated on a background

$$E' = g(E) \quad \text{with} \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tag{4.18}$$

where g is in the $O(d,d;\mathbb{Z})$ subgroup of $O(D,D;\mathbb{Z})$, as explained in §4.1.

It is notationally convenient to introduce extra coordinates $\tilde{x}_\mu$, so that we have $2D$ coordinates X with

$$X \equiv \begin{pmatrix} \tilde{x}_i \\ x^i \end{pmatrix}, \quad \text{where} \quad \tilde{x}_i = \begin{pmatrix} \tilde{x}_a \\ \tilde{x}_\mu \end{pmatrix}, \quad x^i = \begin{pmatrix} x^a \\ x^\mu \end{pmatrix}. \tag{4.19}$$

We will only consider fields that are independent of the extra coordinates $\tilde{x}_\mu$, so that these coordinates play no role. With the help of these coordinates the action (4.16) can be written as

$$S(E,\Psi) = \int dX\; \mathcal{L}\left[D_k, \bar{D}_l, G^{-1}\,;\; e_{ij}(X), d(X)\right]. \tag{4.20}$$

Here

$$\int dX \equiv \int dx^\mu dx^a d\tilde{x}_a\,, \tag{4.21}$$

with no integration over the trivial coordinates $\tilde{x}_\mu$. Our argument will apply to any action that is of the form (4.20) and with indices contracted in the way we describe below.

There is a natural action of $O(D,D)$ on the $2D$ coordinates X but, as before, we only consider the $O(D,D;\mathbb{Z})$ transformations in the $O(d,d;\mathbb{Z})$ subgroup that preserves x^μ and $\tilde{x}_\mu$ and respects the periodicities of the $x^a, \tilde{x}_a$. Such a transformation takes X to X' where

$$X' = \begin{pmatrix} \tilde{x}' \\ x' \end{pmatrix} = gX = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \tilde{x} \\ x \end{pmatrix} = \begin{pmatrix} a\tilde{x} + bx \\ c\tilde{x} + dx \end{pmatrix}. \tag{4.22}$$

Then $O(d,d;\mathbb{Z})$ transformations act as diffeomorphisms of the doubled torus T^{2d}, the subgroup of the large diffeomorphisms $GL(2d;\mathbb{Z})$ preserving η. Our ansatz for the transformation of e, d in the general case follows from the transformations found in [2]:

$$\begin{aligned} e_{ij}(X) &= M_i{}^k\, \bar{M}_j{}^l\, e'_{kl}(X')\,, \\ d(X) &= d'(X')\,. \end{aligned} \tag{4.23}$$

Using this to write the fields e, d in terms of e', d' in in (4.20) gives

$$S(E, \Psi(\Psi')) = \int dX' \; \mathcal{L}\left[D_i, \bar{D}_j, G^{-1}\,; \; M_i{}^k \, \bar{M}_j{}^l \, e'_{kl}(X'), d'(X')\right], \tag{4.24}$$

where we have used

$$\int dX \equiv \int dx^\mu dx^a d\tilde{x}_a = \int dx^\mu d(x')^a d(\tilde{x}')_a = \int dX' \,, \tag{4.25}$$

since the Jacobian of the transformation is unity.

The transformation (4.22) of X implies that the (lower-indexed) derivatives acting on the new fields can be rewritten in terms of primed derivatives based on E' as follows

$$\begin{aligned} D = &\quad MD' \,, \\ \bar{D} = &\quad \bar{M}D' \,, \end{aligned} \tag{4.26}$$

as we will show below. Then the action becomes

$$S(E, \Psi(\Psi')) = \int dX' \; \mathcal{L}\left[MD', \bar{M}D', G^{-1}\,; \; M_i{}^k \, \bar{M}_j{}^l \, e'_{kl}(X'), d'(X')\right]. \tag{4.27}$$

If we can show that this is equal to

$$S(E', \Psi') = \int dX' \; \mathcal{L}\left[D', \bar{D}', \, G'^{-1}\,; \; e'_{ij}(X'), d'(X')\right], \tag{4.28}$$

then we will have

$$S(E', \Psi') = S(E, \Psi(\Psi')) \,, \tag{4.29}$$

establishing the desired physical equivalence.

To show this, we need to keep track of which indices transform with an M and which with an $\bar{M}$. For this argument, we introduce a notation in which lower indices i transform with an M and lower indices $\bar{i}$ transform with a $\bar{M}$, while upper indices transform with the inverses of these matrices. Then (4.26) implies that the derivatives are $D_i, \bar{D}_{\bar{j}}$ while (4.23) implies that $e_{i\bar{j}}$ has a first index which is unbarred and a second which is barred. The two forms for the transformation of the metric in (4.11) imply that we can write G^{-1} with with two unbarred indices as G^{ij} or with two barred ones as $G^{\bar{i}\bar{j}}$. For any action in which all unbarred indices are contracted amongst themselves using G^{ij} and all the barred indices are contracted amongst themselves using $G^{\bar{i}\bar{j}}$, equation (4.11) implies that all factors of M and $\bar{M}$ will cancel. This gives the equality of (4.27) and (4.28), as required. The index contractions in the cubic action (3.25) indeed obey this rule. We see from the string field (2.38) that the first index in e_{ij} is tied to an unbarred oscillator while the second is tied to a barred oscillator. It is clear from the commutation relations (2.21) that contractions always relate two un-barred or two barred operators, but cannot ever mix them. The same is true for the derivatives D and $\bar{D}$ that arise from unbarred and barred zero modes, as shown in (2.23). It follows that the action derived from the string field theory obeys the stated contraction rules, and so must be T-dual in this way.

To complete the above proof we must derive (4.26). Consider the action of derivatives with respect to x and $\tilde{x}$ on functions of X'. As a preliminary, short calculations using (4.22) give

$$\begin{aligned}\frac{\partial}{\partial x}F(X') &= \Big(b^t\frac{\partial}{\partial\tilde{x}'}+d^t\frac{\partial}{\partial x'}\Big)F(X')\,,\\ \frac{\partial}{\partial\tilde{x}}F(X') &= \Big(a^t\frac{\partial}{\partial\tilde{x}'}+c^t\frac{\partial}{\partial x'}\Big)F(X')\,.\end{aligned}\tag{4.30}$$

We then have for D_i

$$\begin{aligned}DF(X') &= \frac{1}{\sqrt{\alpha'}}\Big(\frac{\partial}{\partial x}-E\,\frac{\partial}{\partial\tilde{x}}\Big)F(X')\\ &= \frac{1}{\sqrt{\alpha'}}\Big(b^t\frac{\partial}{\partial\tilde{x}'}+d^t\frac{\partial}{\partial x'}-E\Big[a^t\frac{\partial}{\partial\tilde{x}'}+c^t\frac{\partial}{\partial x'}\Big]\Big)F(X')\\ &= \frac{1}{\sqrt{\alpha'}}\Big((d^t-Ec^t)\frac{\partial}{\partial x'}+(b^t-Ea^t)\frac{\partial}{\partial\tilde{x}'}\Big)F(X')\,.\end{aligned}\tag{4.31}$$

Making use of (4.13)

$$DF(X') = M\;\frac{1}{\sqrt{\alpha'}}\Big(\frac{\partial}{\partial x'}-E'\frac{\partial}{\partial\tilde{x}'}\Big)F(X') = M\;D'F(X')\,,\tag{4.32}$$

as we wanted to show. We repeat for the derivative $\bar{D}_i$:

$$\begin{aligned}\bar{D}F(X') &= \frac{1}{\sqrt{\alpha'}}\Big(\frac{\partial}{\partial x}+E^t\,\frac{\partial}{\partial\tilde{x}}\Big)F(X')\\ &= \frac{1}{\sqrt{\alpha'}}\Big(b^t\frac{\partial}{\partial\tilde{x}'}+d^t\frac{\partial}{\partial x'}+E^t\Big[a^t\frac{\partial}{\partial\tilde{x}'}+c^t\frac{\partial}{\partial x'}\Big]\Big)F(X')\\ &= \frac{1}{\sqrt{\alpha'}}\Big((d^t+E^tc^t)\frac{\partial}{\partial x'}+(b^t+E^ta^t)\frac{\partial}{\partial\tilde{x}'}\Big)F(X')\,.\end{aligned}\tag{4.33}$$

Making use of (4.13)

$$\bar{D}F(X') = \bar{M}\;\frac{1}{\sqrt{\alpha'}}\Big(\frac{\partial}{\partial x'}+E'^t\frac{\partial}{\partial\tilde{x}'}\Big)F(X') = \bar{M}\;\bar{D}'F(X')\,,\tag{4.34}$$

as we wanted to show. This completes our proof of (4.26), and therefore our proof of T-duality.

4.3 Inversion

We now give some explicit formulae relevant to the $\mathbb{Z}_2$ duality transformation that simultaneously exchanges all tori coordinates x^a and $\tilde{x}_a$. This duality transforms the toroidal background with

$$\hat{g} = \begin{pmatrix}\hat{a} & \hat{b}\\ \hat{c} & \hat{d}\end{pmatrix} = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} \in O(d,d,\mathbb{Z})\,.\tag{4.35}$$

Explicitly, $\hat{b}_{ab}=\delta_{ab}$ and $\hat{c}^{ab}=\delta^{ab}$, introducing metrics that can naturally raise and lower indices in what follows. Using (4.7) we find that the toroidal part of the background is transformed to:

$$\hat{E}' = \hat{E}^{-1}\,,\qquad\qquad E' = \begin{pmatrix}\hat{E}^{-1} & 0\\ 0 & \eta\end{pmatrix}.\tag{4.36}$$

We also have from (4.10)

$$\hat{E}^t\,\hat{G}'\hat{E} = \hat{E}\,\hat{G}'\hat{E}^t = \hat{G}\,. \tag{4.37}$$

Taking inverses and solving for $\hat{G}'^{-1}$ we find

$$\hat{G}'^{-1} = \hat{E}\,\hat{G}^{-1}\,\hat{E}^t = \hat{E}^t\,\hat{G}^{-1}\,\hat{E}\,. \tag{4.38}$$

This gives equation (3.66) which was used to investigate the gauge transformations of the theory in which fields depend only on $\tilde{x}$. We also have from (4.9)

$$M = \begin{pmatrix} -\hat{E} & 0 \\ 0 & 1 \end{pmatrix}\,, \quad \bar{M} = \begin{pmatrix} \hat{E}^t & 0 \\ 0 & 1 \end{pmatrix}\,, \tag{4.39}$$

so that

$$M_a{}^b = -\hat{E}_{ac}\delta^{cb}, \qquad \bar{M}_a{}^b = \hat{E}_{ca}\delta^{cb}\,, \tag{4.40}$$

using $\hat{c}^{ab} = \delta^{ab}$. The transformation for the field e was given in (4.23) and takes the form $e_{ab}(X) = M_a{}^c\,\bar{M}_b{}^d\,e'_{cd}(X')$, since the matrices M and $\bar{M}$ are block diagonal. We then find

$$e_{ab}(X) = -\hat{E}_{ac}\,e'^{cd}(X')\,\hat{E}_{db}\,, \tag{4.41}$$

where $e'^{cd} = e'_{ab}\delta^{ac}\delta^{bd}$, giving an e' with upper indices, which was the natural convention used in §3.4.2. If we solve for e' we immediately obtain (3.68). The above results justify the starting point of the analysis in §3.4.2.

4.4 The discrete symmetry $B \to -B$

It is well known that the background change $B_{ij} \to -B_{ij}$ in a toroidally compactified theory is a symmetry of the closed string theory. Since B_{ij} couples electrically to the string, this symmetry is a consequence of the orientation invariance of the theory. We now show that the discrete symmetry discovered in the action guarantees the invariance of the physics under $B_{ij} \to -B_{ij}$.

We begin with our action

$$S(E,\Psi) = \int dX\;\mathcal{L}\Big[D,\bar{D},G^{-1}\,;\,e_{ij}(X),d(X)\Big]\,. \tag{4.42}$$

The replacement $B \to -B$ makes $E \to G - B$, which means

$$E \to E^t\,. \tag{4.43}$$

This does not affect the metric G, but the action formulated with background E^t has the derivatives changed. Given (2.24), we have

$$S(E^t,\Psi) = \int dX\;\mathcal{L}\Big[\frac{1}{\sqrt{\alpha'}}\Big(\frac{\partial}{\partial x} - E^t\frac{\partial}{\partial\tilde{x}}\Big),\;\frac{1}{\sqrt{\alpha'}}\Big(\frac{\partial}{\partial x} + E\frac{\partial}{\partial\tilde{x}}\Big),\;G^{-1}\,;\,e_{ij}(X),d(X)\Big]\,. \tag{4.44}$$

We now redefine the fields as

$$\begin{aligned} e_{ij}(X) &= e'_{ji}(X')\,, \\ d(X) &= d'(X')\,, \end{aligned} \tag{4.45}$$

with

$$X' = \begin{pmatrix} \tilde{x}' \\ x' \end{pmatrix} = \begin{pmatrix} -\tilde{x} \\ x \end{pmatrix} . \tag{4.46}$$

The effect of this change on the derivatives is to reverse the sign of the terms carrying a tilde coordinate, so by now

$$S(E^t, \Psi(\Psi')) = \int dX' \, \mathcal{L}\left[\bar{D}', \, D', G^{-1} \, ; \, e'_{ji}(X'), \, d'(X')\right], \tag{4.47}$$

where we also used $dX = dX'$. The above action has exactly the replacements associated with the discrete symmetry (3.26) that leaves the action invariant, so

$$S(E^t, \Psi(\Psi')) = \int dX' \, \mathcal{L}\left[D', \, \bar{D}', G^{-1} \, ; \, e'_{ij}(X'), d'(X')\right] = S(E, \Psi') . \tag{4.48}$$

This shows the physical equivalence of the actions formulated around E and around E^t.

4.5 Field redefinitions, Buscher rules, and generalised T-duality

If the fields e_{ij} and d depend on the spacetime coordinates $x^i = (x^\mu, x^a)$ but are independent of the dual coordinates $\tilde{x}_a$, then there is a conventional low-energy effective theory. The effective field theory for these fields obtained using string field theory must be equivalent to the standard string low-energy effective field theory (2.59) with higher-derivative α' corrections. The standard theory is written in terms of the total field $\mathcal{E}_{ij}$ which defines g_{ij} and b_{ij} fields that have the standard diffeomorphism and anti-symmetric tensor gauge transformations (3.36). The map from string field theory to the standard effective field theory has been studied in [30], but has not been found explicitly. We have shown in (3.59) that

$$\mathcal{E}_{ij} = \mathrm{g}_{ij} + \mathrm{b}_{ij} = E_{ij} + e'_{ij} + \frac{1}{2} e'_i{}^k e'_{kj} + \text{cubic corrections} . \tag{4.49}$$

Here e'_{ij} is the field used for the double field theory and is related to the string field theory variable e_{ij} arising in (2.38) by (3.21), so that $e'_{ij} = \; e_{ij} + e_{ij} d$, making it clear that the dilaton d mixes in. In the following, we will use only e'_{ij} and drop the primes.

The full non-linear relation will include α' corrections involving derivatives of e_{ij} and d and string loop corrections. It is also subject to field redefinition ambiguities [30]. To zeroth order in α', however, the relation should contain no derivatives, on dimensional grounds and because it is used to match two two-derivative actions. Then at zeroth order in α' and at string tree level there must be some algebraic function $f(e,d)$, so that

$$\mathcal{E}_{ij} \equiv E_{ij} + f_{ij}(e,d) \, , \qquad f_{ij}(e,d) = e_{ij} + \frac{1}{2} e_i{}^k e_{kj} + \text{cubic corrections} \, , \tag{4.50}$$

with $\mathcal{E}_{ij}$ transforming as in (3.40). Moreover, this relation should apply both for the compactified and uncompactified theory. The field $\mathcal{E}_{ij}$ combines the background and fluctuations geometrically.

When the fields are independent of the torus coordinates x^a as well as $\tilde{x}_a$, the $U(1)^d$ torus action is an isometry leaving the fields invariant and T-duality acts through the Buscher rules [18, 19]. The full metric g_{ij} and Kalb-Ramond field b_{ij} depend on x^μ and are independent of $x^a, \tilde{x}_a$ and so transform according to the extension [19] of the Buscher rules for the torus:

$$\mathcal{E}' = g(\mathcal{E}) = (a\mathcal{E} + b)(c\mathcal{E} + d)^{-1} . \tag{4.51}$$

These transformations are expected to receive α' corrections and possibly string loop corrections, but to zeroth order in α' and string tree level, they are the complete non-linear transformations. This can now be compared with the T-duality transformations of e_{ij}, d found above. As the coordinates x^μ do not transform, the dilaton is invariant and (4.23) gives

$$\begin{aligned} e_{ij}(x) &= M_i{}^k \bar{M}_j{}^l e'_{kl}(x) , \\ d(x) &= d'(x) . \end{aligned} \tag{4.52}$$

For infinitesimal e_{ij}, we have $\mathcal{E}_{ij} = E_{ij} + e_{ij} + O(e^2)$ and using (4.15) we see that the expansion of (4.51) gives a linear transformation of e_{ij} which is precisely (4.52) plus quadratic corrections. The requirement that the relation (4.50) should map the linear transformation (4.52) to the fractional linear transformation places stringent constraints on the function f_{ij}, as discussed in [30]. A simple explicit function that is compatible with these two forms of the T-duality transformations was found in [30], but requiring such compatibility does not fix the function uniquely.

Let us now return to the case in which the fields depend on the torus coordinates x^a as well as x^μ (but not $\tilde{x}_a$), so that massive Kaluza-Klein modes with momenta on the torus exist. The dependence on x^a means that the $U(1)^d$ torus action does not preserve the fields and so the usual Buscher rules do not apply. Nonetheless, our linear transformations (4.23) for e, d still apply, and the full fields with geometric gauge transformations are still given by (4.50). The function f_{ij} in (4.50) still converts linear duality transform transformations into fractional linear transformations. As a result, we learn that the linear transformation of e_{ij}, d implies the non-linear transformations of $\mathcal{E}, d$ given by

$$\begin{aligned} \mathcal{E}'(X') &= g(\mathcal{E}(X)) = (a\mathcal{E}(X) + b)(c\mathcal{E}(X) + d)^{-1} , \\ d'(X') &= d(X) . \end{aligned} \tag{4.53}$$

Here, the argument X refers to $(x^\mu, x^a, \tilde{x}_a = 0)$ and $X' = gX$. For inversion in all d circles, X' is given by $(x'^\mu, x'^a, \tilde{x}'_a) = (x^\mu, 0, x^a)$ so that the T-dual of a configuration with dependence on (x^μ, x^a) is one with dependence on $(x^\mu, \tilde{x}_a)$, as expected. Conversely, the T-dual of a configuration with dependence on $(x^\mu, \tilde{x}_a)$ is one with dependence on (x^μ, x^a).

We now turn to the general case in which the fields e_{ij} and d depend on $\tilde{x}_a$ as well as x^μ, x^a. Then the T-duality transformations of e_{ij}, d are still given by (4.23). In this case it is not so clear how we should define the total field $\mathcal{E}(X)$ as we no longer have a conventional field theory description to guide us. Moreover, as we saw in §3.4.2, different field redefinitions are useful in different contexts. A natural definition, however, is to take $\mathcal{E}_{ij} \equiv E_{ij} + e_{ij} + f_{ij}(e, d)$ with the *same* algebraic function f that arose above in the map from string field theory to the effective field theory, so that we recover the results above in the case in which there is no dependence on $\tilde{x}_a$. If we do so, the fact that f_{ij} maps our linear T-duality transformations to fractional linear ones implies that the transformation of this $\mathcal{E}_{ij}$ is again given by (4.53) but now with general dependence on the coordinates $(x^\mu, x^a, \tilde{x}_a)$. This is a simple and manifestly $O(d, d; \mathbb{Z})$ compatible candidate for the generalisation of the Buscher rules to the case with general dependence on $(x^\mu, x^a, \tilde{x}_a)$.

The fact that the double field theory action satisfies

$$S(E, e, d) = S(E', e', d') , \tag{4.54}$$

implies that the transformation of both the background E and the fields e, d is a symmetry of the action. If the action can be rewritten in terms of the total field $\mathcal{E}$ so that $S(E, e, d) = S(\mathcal{E}, d)$, then it will be manifestly independent of the split into a background field E and a fluctuation e and will be invariant under the T-duality transformations (4.53)

$$S(\mathcal{E}, d) = S(\mathcal{E}', d') \,. \tag{4.55}$$

The matrix $\mathcal{H}(E)$ defined in (2.17) for the background field E has a natural generalisation for the total field $\mathcal{E} = \mathrm{g} + \mathrm{b}$. We define the $2D \times 2D$ matrix $\mathcal{H}(\mathcal{E})$ by

$$\mathcal{H}(\mathcal{E}) = \begin{pmatrix} \mathrm{g} - \mathrm{b}\mathrm{g}^{-1}\mathrm{b} & \mathrm{b}\mathrm{g}^{-1} \\ -\mathrm{g}^{-1}\mathrm{b} & \mathrm{g}^{-1} \end{pmatrix} . \tag{4.56}$$

It follows from (4.8) that the background transformation (4.53) induces a simple linear transformation for $\mathcal{H}(\mathcal{E})$

$$\mathcal{H}(\mathcal{E}'(X')) = g\, \mathcal{H}(\mathcal{E}(X))\, g^t \,. \tag{4.57}$$

From (4.57), the inverse $\mathcal{G}(\mathcal{E}) \equiv (\mathcal{H}(\mathcal{E}))^{-1}$ transforms as

$$\mathcal{G}(\mathcal{E}) = \, g^t\, \mathcal{G}(\mathcal{E}'(X'))\, g \,. \tag{4.58}$$

This can be written suggestively using $X' = gX$:

$$\mathcal{G}(X) = \left(\frac{\partial X'}{\partial X}\right)^t \mathcal{G}'(X') \left(\frac{\partial X'}{\partial X}\right) . \tag{4.59}$$

where $\mathcal{G}'(X') = \mathcal{G}(\mathcal{E}'(X'))$ and $\mathcal{G}(X) = \mathcal{G}(\mathcal{E}(X))$. This shows that $\mathcal{G}$ behaves as a covariant tensor under $O(D, D)$ transformations. Indeed, $\mathcal{G}$ defines a duality invariant line element

$$ds^2 = dX^t\, \mathcal{G}(\mathcal{E}(X))\, dX \,. \tag{4.60}$$

The metric $\mathcal{G}$ and its relation to the generalised metric in generalised geometry is discussed in §6.

5 Constraint, cocycles, and null subspaces

In this section we discuss some of the subtle issues that arise in our construction. We have referred to these at various points in the earlier sections. We begin with an examination of the constraint that requires fields and gauge parameters to be in $\ker(\Delta)$, namely, the kernel of the second-order differential operator Δ. We define the natural linear projection $[[\,\cdot\,]]$ that takes an arbitrary double field to this kernel. We then turn to a discussion of cocycles and sign factors. It is possible that the nonlinear completion of the theory will involve these sign factors. Finally, we conclude with a discussion of null spaces that arise from the restriction to double fields that have no winding in some suitable T-dual frame, resulting in a conventional field theory for that non-winding sector.

5.1 The constraint and projectors

The constraint $L_0 - \bar{L}_0 = 0$ is applied to all fields and gauge parameters. The product of two fields satisfying the constraint will not satisfy it in general, and for this reason the string product $[\cdot,\cdot]$ includes an explicit projection onto states that satisfy $L_0 - \bar{L}_0 = 0$. It also has an insertion of b_0^- that ensures that the string product is annihilated by b_0^-. Schematically, we have

$$[\Psi_1, \Psi_2] \equiv \int \frac{d\theta}{2\pi} e^{i\theta(L_0 - \bar{L}_0)} b_0^- [\Psi_1, \Psi_2]' = \delta_{L_0 - \bar{L}_0, 0}\, b_0^- [\Psi_1, \Psi_2]' , \tag{5.1}$$

where the primed bracket $[\cdot\,,\,\cdot]'$ inserts the states in the three-punctured sphere that defines the vertex. The b_0^- insertion implies that the string product has an intrinsic ghost number of minus one: $\text{gh}([A, B]) = \text{gh}(A) + \text{gh}(B) - 1$. This inclusion of the projection in the string product in covariant closed string field theory leads to the failure of a Jacobi identity and this then requires further higher order interactions resulting in a non-polynomial theory. Concretely, one finds [3]

$$\begin{aligned} 0 = & Q[B_1, B_2, B_3] + [QB_1, B_2, B_3] + (-1)^{B_1}[B_1, QB_2, B_3] + (-1)^{B_1+B_2}[B_1, B_2, QB_3] \\ & + (-)^{B_1}[B_1, [B_2, B_3]] + (-1)^{B_2(1+B_1)}[B_2, [B_1, B_3]] + (-1)^{B_3(1+B_1+B_2)}[B_3, [B_1, B_2]] . \end{aligned} \tag{5.2}$$

If the string product $[\cdot\,,\cdot\,,\cdot]$ satisfied a Jacobi-like identity, the terms on the second line would add up to zero. Since they do not, one requires an elementary triple product represented by $[\cdot\,,\cdot\,,\cdot]$ and used to define a quartic elementary interaction. This triple product (as well as all higher ones) must also include a projection to states that satisfy $L_0 - \bar{L}_0 = 0$. The failure of Q to be a derivation of this product is equal to the violation of the Jacobi identity. The above relation is part of the defining relations of the L_∞ homotopy Lie-algebra [3, 31].

Consider now the states with $N = \bar{N} = 1$. Projection down to the physical space with $\Delta = 0$ is most easily discussed in momentum space. Consider a field ϕ (with $N = \bar{N} = 1$) with definite momenta and winding numbers $(w^a, p_a) = (m^a\,, n_a)$ with $a = 1, 2, \ldots, d$. Then

$$\Delta\phi = 0 \quad \leftrightarrow \quad \sum_a n_a m^a \equiv n\, m = 0 . \tag{5.3}$$

We combine the winding m^a and the momentum n_a of ϕ into a $2d$-column vector v:

$$v = \begin{pmatrix} m \\ n \end{pmatrix} \in \mathbb{Z}^{2d} , \tag{5.4}$$

and define the inner product with respect to the $O(d, d)$ invariant metric $\hat{\eta}$

$$v \circ v' \equiv v^T \hat{\eta}\, v' = \big(m, n\big) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} m' \\ n' \end{pmatrix} = mn' + nm' . \tag{5.5}$$

Since $v \circ v = 2n_a m^a$, the $\Delta = 0$ constraint on the vector takes the form

$$\Delta\phi = 0 \quad \leftrightarrow \quad v \circ v = 0 . \tag{5.6}$$

In words, the vector v is null with respect to η. A field with definite momentum and winding must satisfy this condition to be allowed. A general superposition of such allowed fields is also allowed, since Δ is a linear operator. If we have fields ϕ and ϕ' with null momenta v and v', the product $\phi\phi'$ has momentum $v+v'$ which is not null in general. The product will only satisfy the constraint if the momenta are orthogonal:

$$\Delta(\phi\phi') = 0 \quad \leftrightarrow \quad (v+v')\circ(v+v') = 0 \quad \leftrightarrow \quad v\circ v' = 0\,. \tag{5.7}$$

The constraint is not satisfied by products since Δ is a second-order differential operator. We enforce the constraint as follows. Given a general field $A(x^\mu, x^a, \tilde{x}_a)$, a Fourier series for the compact dimensions yields

$$A(x^\mu, x^a, \tilde{x}_a) = \sum_{v\in\mathbb{Z}^{2d}} \hat{A}(x^\mu, v)\, e^{iv^T\mathbb{X}} = \sum_{v\in\mathbb{Z}^{2d}} \hat{A}(x^\mu, v)\, e^{im^a\tilde{x}_a + in_a x^a}\,. \tag{5.8}$$

Since $\Delta = -\frac{2}{\alpha'}\partial_a\tilde{\partial}^a$ we find

$$\Delta A = \frac{1}{\alpha'}\sum_{v\in\mathbb{Z}^{2d}} v\circ v\;\hat{A}(x^\mu, v)\;e^{iv^T\mathbb{X}}\,. \tag{5.9}$$

A canonical projection of a general A into a field $[[A]]$ that satisfies the $\Delta=0$ constraint is defined by

$$[[A]] \equiv \sum_{v\in\mathbb{Z}^{2d}} \delta_{v\circ v,0}\;\hat{A}(x^\mu, v)\, e^{iv^T\mathbb{X}}\,. \tag{5.10}$$

The role of the Kronecker delta is to retain only the Fourier components of the field whose momenta are null. It is now clear that

$$\Delta[[A]] = 0\,. \tag{5.11}$$

It is also clear from the definition that with constants α and β and functions A and B we have

$$[[\alpha A + \beta B]] = \alpha\,[[A]] + \beta\,[[B]]\,. \tag{5.12}$$

The operation $[[\,\cdot\,]]$ is a linear map from the space of functions on the doubled torus to the kernel of Δ. It is a projector because applying it twice has the same effect as applying it once. The operation $[[\,\cdot\,]]$ implements the $\Delta=0$ constraint in the same way that the Kronecker delta in (5.1) implements the level matching constraint for a general string field. For constrained fields $A(x,\tilde{x})$ and $B(x,\tilde{x})$, the product $[[A(x,\tilde{x})B(x,\tilde{x})]]$ projects onto those Fourier modes $\hat{A}(x,v)\hat{B}(x,v')$ with v, v' both null and orthogonal, $v\circ v' = 0$.

The closed string product includes the projector $\delta_{L_0-\bar{L}_0,0}$ because the product of two allowed states should give an allowed state. We must therefore use the projection $[[\,\cdot\,]]$ in the gauge transformations (3.27) to ensure that the gauge variations are allowed variations of the fields. This means that, properly written, the gauge transformations are

$$\begin{aligned} \delta_\lambda e_{ij} &= \bar{D}_j\lambda_i + \frac{1}{2}\Big[\Big[(D_i\lambda^k)e_{kj} - (D^k\lambda_i)e_{kj} + \lambda_k D^k e_{ij}\Big]\Big]\,, \\ \delta_\lambda d &= -\frac{1}{4}D\cdot\lambda + \frac{1}{2}\big[\big[(\lambda\cdot D)\,d\big]\big]\,. \end{aligned} \tag{5.13}$$

Happily, there is no need to use the projection $[[\,\cdot\,]]$ in the cubic action. The action is correct as written in equation (3.25). This is not difficult to explain. Let A be a field that satisfies $\Delta A = 0$ and B be a field that does not. Then, we claim that

$$\int A\,[[\,B\,]] = \int A\,B\,, \tag{5.14}$$

where the integral is over x^μ, x^a, and $\tilde{x}_a$. It follows from the above that the projection of B to the kernel of Δ is not needed. This is clear in momentum space. The integration implies that any Fourier mode of B with momentum v can only couple to a Fourier mode of A with momentum $(-v)$. Then any Fourier mode of B with momentum v that is not allowed cannot contribute since $(-v)$ is also not allowed and thus cannot be found in A, as A satisfies the constraint. Next consider the cubic term in the action. Since this arises from the string field theory term $\langle\Psi, [\Psi\,, \Psi]\rangle$ the structure we obtain must be a sum of terms of the form $\int \phi_1\,[[\phi_2\phi_3]]$. Since ϕ_1 satisfies the constraint, equation (5.14) shows that the projector is not needed for the product $\phi_2\phi_3$. Therefore, we do not need to include additional projectors in the quadratic and cubic terms in the action. Similar remarks apply to the check of gauge invariance of the action. The above gauge transformations induces terms of the form $\int \phi_1\,[[\lambda\phi_2]]$. Again, the projector is not needed to the order to which we are working, and we can proceed naively.

This convenient simplification may disappear for terms in the action quartic in fields. The terms that arise from the elementary quartic interaction $\langle\Psi, [\Psi\,, \Psi\,, \Psi]\rangle$ of the closed string field theory action would have the form $\int \phi_1[[\phi_2\phi_3\phi_4]]$. Again, because of (5.14) the projector is not needed and this term equals $\int \phi_1\phi_2\phi_3\phi_4$. On the other hand, terms that arise from integrating out other fields will have a projector of the form $\int \phi_1\phi_2\,[[\phi_3\phi_4]]$. This projector cannot be eliminated. It is clear that given four fields, the projector can be inserted in three inequivalent ways – the number of ways in which the fields can be partitioned into groups of two. It seems tempting to believe that terms with these three inequivalent positions of the projector may be related to terms with no projector through identities in the spirit of (5.2).

5.2 Cocycles

We now address another important issue. Closed string vertex operators in toroidal backgrounds have cocycles – operators that are included to ensure standard commutation properties [27, 28]. If $\mathcal{V}^0_{v_\alpha}$ denotes the naive vertex operator for a state with momenta and winding specified by v_α one finds

$$\mathcal{V}^0_{v_2}(z_2,\bar{z}_2)\;\mathcal{V}^0_{v_1}(z_1,\bar{z}_1) \;=\; e^{i\pi\, v_1\circ v_2}\;\mathcal{V}^0_{v_1}(z_1,\bar{z}_1)\;\mathcal{V}^0_{v_2}(z_2,\bar{z}_2)\,. \tag{5.15}$$

The phase factor can be equal to minus one, in which case we have the unpleasant fact that vertex operators for bosons anticommute. A cocycle operator is included multiplicatively to define vertex operators $\mathcal{V}_{v_\alpha}$ that always commute. These cocycles affect the signs of correlation functions. As a result, there are extra signs that are introduced in the three-string vertex [4, 5, 2]. Up to field redefinitions, the sign factor that affects the amplitude $\langle\mathcal{V}_{v_\alpha}\mathcal{V}_{v_\beta}\mathcal{V}_{v_\gamma}\rangle$ is

$$\epsilon_{\alpha\beta\gamma} = e^{i\pi(n^\alpha m^\alpha + n^\gamma m^\beta)}\,. \tag{5.16}$$

Of course $v_\alpha + v_\beta + v_\gamma = 0$. The sign factor can be shown to be cyclic invariant. Under exchange of α and β labels, for example, this sign factor changes as follows:

$$\epsilon_{\beta\alpha\gamma} = \epsilon_{\alpha\beta\gamma}\, e^{i\pi v_\alpha \circ v_\beta} \ . \tag{5.17}$$

The sign factor (5.16) is nontrivial: it cannot be removed by redefinitions of the states corresponding to the vertex operators. For our case of interest the situation is somewhat simpler. We have $n^\alpha m^\alpha = 0$ because all states satisfy the $\Delta = 0$ constraint. As a result, (5.16) becomes

$$\epsilon_{\alpha\beta\gamma} = e^{i\pi\, n^\gamma m^\beta} \ . \tag{5.18}$$

Despite appearances to the contrary this sign factor is fully symmetric under exchange of labels. This can be understood as follows. First recall the simple fact that given three null vectors that add up to zero, the vectors are mutually orthogonal. This shows that $v_\alpha \circ v_\beta = v_\alpha \circ v_\gamma = v_\beta \circ v_\gamma = 0$ and the sign factor associated with exchanges vanishes. A symmetric sign factor could be trivial, but we have not been able to show that (5.18) is trivial. Note, however, that any three (constrained) states coupled by the three string vertex have momenta which are orthogonal and therefore the associated cocycle-free vertex operators commute (see (5.15)).

The cubic action we have written did not include cocycle-induced sign factors. If present, such signs also appear in the gauge transformations and in the duality transformations. It is known that string field theory gauge invariance to $O(\Lambda\Psi^2)$ holds with or without such sign factors and this may explain why our construction succeeded so far without any sign factors. It is to next order that the sign factors are claimed to be needed for gauge invariance [4]. The cocycle-induced sign factors are non-trivial and required for the full string field theory, but their role may be different for the double field theory we are focussing on. We hope to return to this question in the future.

5.3 Spaces large and small

For fields with arbitrary dependence on the coordinates $\mathbb{X}$ of T^{2d}, the momenta can be arbitrary v's in the *full* momentum lattice $\mathbb{Z}^{2d}$ introduced in §5.1. The constraint

$$v \circ v = 0\,, \tag{5.19}$$

restricts us to the null subspace of $\mathbb{Z}^{2d}$, which we refer to as the *large* space. The equation $v \circ v = 0$ defines a light-cone in $\mathbb{R}^{2d}$ with metric $\hat\eta$, and the large space consists of the points on this light cone with integer coordinates. For general string states with $N \neq \bar N$, this light-cone is replaced by the hyperboloid

$$\tfrac{1}{2}\, v \circ v = N - \bar N\,. \tag{5.20}$$

A $2d$ dimensional space with metric of signature (d, d) can have totally null d-dimensional subspaces (called totally isotropic subspaces in the mathematics literature) in which the indefinite metric restricts to zero, so that all tangent vectors to the subspace are null and mutually orthogonal. We shall be interested in totally null subspaces $T^d \subset T^{2d}$. Writing the metric as $ds^2 = 2dx^a d\tilde x_a$, we see that the d-torus with coordinates x^a and the dual torus with coordinates $\tilde x_a$ are both totally null, and any T^d

obtained from these by acting with $O(d,d;\mathbb{Z})$ will also be totally null. If we let the coordinates of a null subspace be y^a and those of the complement be $\tilde{y}_a$, then the metric $\hat{\eta}$ is $ds^2 = 2dy^a d\tilde{y}_a$ and

$$\Delta = -\frac{2}{\alpha'}\sum_a \frac{\partial}{\partial \tilde{y}_a}\frac{\partial}{\partial y^a}\,. \tag{5.21}$$

For fields that are independent of $\tilde{y}$, the constraint $\Delta = 0$ is automatically satisfied. Moreover, all products of fields satisfy the constraint $\Delta = 0$ and no projection $[[\,\cdot\,]]$ is necessary. Nor are cocycles needed, as all momenta v for such fields are null and mutually orthogonal, so all vertex operators are mutually local. Then the restriction of the full double field theory to fields dependent only on the coordinates y^a of such a null subspace (together with x^μ) should give a conventional local field theory without cocycles or projectors. For the T^d with coordinates x^a, this should be the conventional field theory with action (2.59) (after field redefinitions, and compactified on T^d), while for other choices it should be a dual theory related to this by an $O(d,d;\mathbb{Z})$ transformation. However, these theories can be written in a duality covariant way, by taking the double field theory and restricting the momentum space to a *small* space where all vectors v are not only null, but also mutually orthogonal. With this restriction, the double field theory has no cocycles, constraints or projectors. It would be very interesting to obtain the full nonlinear version of our action under this simplifying assumption. The result may be related to the work of Siegel [8] who constructed a realization of T-duality in the massless sector under the assumption that all momenta are orthogonal.

6 Comments and open questions

A striking feature of string field theory on a torus is that general solutions involve fields on the doubled torus instead of conventional spacetime fields. As a result, the theory is very different from that suggested by conventional effective field theories that, like supergravity limits of superstrings, miss key stringy features. The theory on a torus is a case which is nontrivial enough to be interesting yet is simple enough to be tractable. One of our goals here has been to seek a subsector of this theory that is almost as simple as a conventional field theory but which is rich enough to include much of the magic of string theory.

We have begun the construction of an intriguing double field theory of massless fields h_{ij}, b_{ij}, d depending on both x and $\tilde{x}$. We have used string field theory to find the action to cubic order and showed that its variation under gauge transformations, found to linear order in the fields, vanishes to the requisite order. By including both winding and momenta we do not have a regime where all excitations have parametrically small energy and the theory may not arise as a simple decoupling limit of string theory. If we view our construction as an effective field theory for a natural set of excitations (some of which may have large energy), the string field theory suggests that an action and gauge transformations should exist to all orders in the field, although the explicit calculation of these becomes much harder at higher orders. The unusual features of string field theory include the explicit projectors to the kernel of $L_0 - \bar{L}_0$, cocycle-induced sign factors in the vertices, and the homotopy Lie algebra structure of the string products. These are all expected to play a role in the double field theory, although they have been largely avoided at the cubic level. Of course, the $\Delta = 0$ constraint on fields and gauge parameters, which arises from the $L_0 - \bar{L}_0 = 0$ constraint in string field theory, has played a central role. It was absolutely crucial, even for linearised gauge invariance.

It has long been known that the $L_0 - \bar{L}_0 = 0$ constraint is fundamental and all attempts to formulate closed string field theory without imposing this off-shell condition on the fields and parameters have so far failed. Such a formulation could exist, but a very significant conceptual advance may be needed to find it. In our massless theory the level-matching constraint became $\Delta = 0$. Our attempts to relax this constraint failed, but we hope that understanding the constraint in the simpler setting of the double field theory may shed light on the constraint in the full string field theory.

It is natural to speculate on the full non-linear form of the theory. We noted that the free theory includes gauge parameters that suffice to describe "double-diffeomorphisms" or linearised diffeomorphisms of the doubled space $\mathbb{R}^{n-1,1} \times T^{2d}$. The nonlinear extension shows that the symmetry of the theory appears to be considerably more intricate. In addition to linearised diffeomorphisms, the gauge parameters generate doubled gauge transformations of the antisymmetric tensor field so that there is an interesting mixture of the two symmetries. Second, there is the projection of the gauge parameters to the kernel of Δ. The full symmetry has an algebra that appears to be different from that of diffeomorphisms on the doubled space $\mathbb{R}^{n-1,1} \times T^{2d}$, but does include the diffeomorphisms of various undoubled subspaces $\mathbb{R}^{n-1,1} \times T^{d}$ obtained by keeping only the x^a coordinates, or keeping only a set of coordinates obtained from the x^a by T-duality. As we have noted at various points, the full symmetry of the theory may turn out to be that of a homotopy-Lie algebra, or some related structure. It would be interesting to see what field theory structures arise to define the higher products inherent in such algebra. In a homotopy Lie algebra we have field dependent structure constants and a gauge algebra that only closes on-shell. These features are coherently organised and described by the products. While diffeomorphisms define a conventional Lie algebra, the larger symmetry of our theory most likely does not. Perhaps the most important open question related to the action is that of cocycles. The construction of the quartic terms in the action will have to face this issue, as well as the possibility that explicit projectors to the kernel of Δ will be needed.

The string field theory treats the background E and the fluctuation e rather differently, but gives a treatment to all orders in an arbitrary fluctuation e. In section 4.5, we introduced a total field $\mathcal{E}(X)$ combining both background and fluctuation, showing that it had the right geometric gauge transformations when independent of $\tilde{x}$. The Buscher transformation of $\mathcal{E}$ was extended to the case with dependence on both x and $\tilde{x}$, providing a generalisation of T-duality of the kind proposed in [25]. Rewriting the double field theory in terms of $\mathcal{E}$ would give a version of the theory independent of the split into background and fluctuation, and thus with some degree of background independence. Although arbitrary geometries would be allowed, our formulation would remain very much tied to the topology $\mathbb{R}^{n-1,1} \times T^{d}$; other topologies would have different zero-mode structures.

Some of the structures in our work also arise in generalised geometry, but with important differences. Generalised geometry [35, 36] treats structures on a D dimensional manifold M on which there is a natural action of the group $O(D, D)$. This typically involves doubling the tangent space of a manifold M (replacing the tangent bundle T with $T \oplus T^*$). Tensor indices then run over twice the usual range, but there is dependence only on the D coordinates of M. If M is equipped with a metric and B-field $\mathcal{E}_{ij} = \mathrm{g}_{ij} + \mathrm{b}_{ij}$, these can be usefully combined into the $2D \times 2D$ matrix $\mathcal{H}(\mathcal{E})$ given by (4.56). The inverse matrix $\mathcal{G}(\mathcal{E}) = \mathcal{H}^{-1}$ is the *generalised metric* [36]. It is a $2D \times 2D$ matrix but depends only on the D coordinates of M. Generalised geometry is then the study of conventional geometry with a metric and B-field on M, packaged in a useful way.

In our work, by contrast, we restrict to D-dimensional manifolds $M = \mathbb{R}^{n-1,1} \times T^d$ and find that string theory leads us to $\mathbb{R}^{n-1,1} \times T^{2d}$, with a doubling of the torus coordinates but no doubling of the range of tensor indices. We found it notationally useful to double the coordinates of $\mathbb{R}^{n-1,1}$ also, to give a space M_{doubled} with dimension $2D$. Our fields depend non-trivially on the doubled torus coordinates $(x^a, \tilde{x}_a)$ and on the Minkowski coordinates x^μ, but do not depend on the extra dual Minkowski coordinates $\tilde{x}_\mu$. It follows that we can use the double field theory fluctuations to define the field $\mathcal{E} = E_{ij} + e_{ij} + \ldots$ in (4.49) that depends nontrivially on $(x^\mu, x^a, \tilde{x}_a)$. We then define an $\mathcal{H}(\mathcal{E})$ by (4.56) and its inverse $\mathcal{G}(\mathcal{E})$. If $\mathcal{E}$ depends only on the coordinates (x^μ, x^a) of M, then $\mathcal{G}(\mathcal{E})$ is a generalised metric on M, but here we generalise to allow dependence on $\tilde{x}_a$ also. Since $\mathcal{G}(\mathcal{E})$ is a $2D \times 2D$ matrix function on the $2D$ dimensional space M_{doubled} that depends on $(x^\mu, x^a, \tilde{x}_a)$, it is a candidate for a conventional metric on M_{doubled}. We have seen in (4.60) that the line element $ds^2 = dX^t\, \mathcal{G}(\mathcal{E}(X))\, dX$ is invariant under T-duality transformations, which act as large diffeomorphisms of T^{2d}. The metric $\mathcal{G}$ is constrained, because $\mathcal{H}$ is: $\eta\mathcal{H} = \mathcal{H}^{-1}\eta$, and is further restricted by the requirement that Δ annihilate e_{ij} and d. Then $\mathcal{G}$ is a natural and interesting object that could play an important role in the formulation of double field theory.

Our work has been concrete and explicit. It has long been known that the toroidal coordinates in closed string theory should be doubled due to the presence of winding modes and we have given a precise sense to this, showing that the dual coordinates enter on an equal footing with the spacetime coordinates and that fields depend on both spacetime and dual coordinates. We have seen that double field theory exists as a free theory and when we include the lowest-order interactions. A number of key features have been identified precisely. The symmetry structure is novel and remains to be fully understood and the full nonlinear theory remains to be found. We have seen that doubled fields can be used to define a kind of geometry on the doubled space that reduces to conventional spacetime geometry on the original torus or to a dual geometry on the dual torus. This geometry is fully *dynamical* – it depends on *all* of the coordinates of the doubled space, it evolves according to field equations and is subject to constraints. This leads to the conclusion that the full doubled geometry is physical: the dual dimensions should not be viewed as an auxiliary structure or a gauge artifact. It is therefore reasonable to expect that doubled geometry will feature prominently in the eventual understanding of the nature of space and the role of geometry in string theory.

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